

Zermelian Extensibility

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The two diametrically opposed tendencies of the thinking mind, the ideas of creative progress and summary completion, which form also the basis of Kant’s “antinomies”, find their symbolic representation as well as their symbolic reconciliation in the transfinite number series, which rests upon the notion of well-ordering and which, though lacking in true completion on account of its boundless progressing, possesses relative way stations, namely those “boundary numbers”, which separate the higher from the lower model types. .

Zermelo (1930), p431.

In his *Grundlagen einer Allgemeinen Mannigfaltigkeitslehre*, Cantor introduces two principles of infinity. The first principle lets one create potential infinities—never ending series of larger transfinite numbers. The second tells us that to every potentially infinite series there corresponds a complete infinity. However, integral to Cantor’s picture is the idea that these two processes of creation and completion can be continued forever. This line of thinking has been very influential in the philosophy of set theory, but has also remained fraught. In this article I explore, in the framework of higher-order logic, one particularly flatfooted articulation of the idea drawing inspiration from some remarks of Zermelo.

The paper is organized as follows. I begin, in section 1, by offering a higher-order (as opposed to a modal) analysis of Aristotle’s notion of a potential infinity, and by relating this to Cantor’s two principles of generation. In section 2 I turn to the idea of indefinite extensibility, as it applies to the notion of well-order and a ZF relation. Following some remarks of Zermelo I formulate purely logical principles, in higher-order logic, stating that every well-order or ZF relation can be extended, and that any sequence of well-orders

or ZF relations has a completion. An alternative modal articulation of these ideas is shown to be subject to a modal version of the Burali-Forti paradox. Section 3 discusses the incompatibility between these higher-order indefinite extensibility principles and a higher-order well-ordering principle, and argues that certain weakenings, such as the idea that the sets of any ZF relation are well-orderable, are both consistent and supported in Cantor’s early writings by the idea that some things form a set when they can be listed (even if only in a potentially infinite list). In section 4 higher-order logics that contain these extensibility principles are defined, and are shown to be consistent using elementary methods (i.e. without forcing). In the final section, I turn to the more general question of whether one needs to posit special first-order entities — like sets, transfinite numbers, and so on — to represent the structure of the higher-order. It is argued that we do not need special purpose entities to represent the higher-order, and that the Zermelian logics developed are especially useful for developing this view for they remove dangling questions concerning the size of the universe that would have to be answered if the universe could be well-ordered. The appendix explores a Cantor-inspired set theory in which to be a set is just to be well-orderable.

1 Higher-Order Formulations of the Potentially Infinite

Aristotle famously drew a distinction between potential infinities and completed infinities, rejecting the latter but not the former. The best way I can think of to spell this out employs second-order or plural resources.¹ An example of a potential infinity might include a series of stretches of time of increasing length, t_1, t_2, t_3, \dots , each properly including its predecessors. Every initial segment of this series can be counted or listed and its members are in good standing by Aristotle’s lights, but according to Aristotle there is no single individual, an infinite stretch of time, that includes them all as parts; this would be a completed infinity. This situation is perfectly consistent provided we reject the mereological principle that any things whatsoever compose a whole. The stretches of time together are potentially infinite, but none of them alone, nor any finite number of them together are potentially infinite. At a first parse, *being potentially infinite* is an irreducibly plural, or irreducibly third-order, property. To posit a completed infinity is to posit a single individual that contains, or stands in some other similar relation, all those things at

¹Rosen (2021).

once. In Aristotle the concept of potential infinity is usually applied to things that are ordered by some sort of part-whole relation, $<_t$, and so is naturally represented by a higher-order predicate that can combine with a binary predicate to form a sentence: that every one of our times bears $<_t$ to another. In general this only guarantees infinitude when the relation in question is a *strict order*, i.e., a transitive irreflexive relation. We thus define the relevant higher-order predicate as follows, writing \forall_e for the first-order quantifier:

$$\text{PotInf } R := \forall_e x (\text{Dom } Rx \rightarrow \exists_e y Rxy) \wedge \text{SO } R$$

writing $\text{Dom } Rx$ for $\exists_e y (Rxy \vee Ryx)$ (“ x is in the domain or R ”) and SO for $\forall_e xyz (Rxy \wedge Ryz \rightarrow Rxz) \wedge \forall_e x \neg Rxx \wedge \exists_e xy Rxy$ (“ R is a non-empty strict order”).

We can capture the potential infinity of the stretches of time t_1, t_2, \dots by ascribing this predicate to $<_t$, the parthood relation restricted to those times. In doing so we are not saying that there is a *series* t_1, t_2, \dots that is potentially infinite. For otherwise Aristotle’s position seems to be incoherent: if one can have potentially infinite series, like t_1, t_2, t_3, \dots , don’t we also have an actually infinite individual, namely the infinite series itself? The higher-order nature of our formulation is thus essential here. It is hard to see how a first-order predicate, ‘is a potential infinity’, could take its place, for if the potentially infinite required, in addition to the times t_1, t_2, \dots , a further individual—a *potential infinity*—to be the logical subject of this predicate, whether it be a series or something else, it seems we are committed to something relevantly like a completed infinity.

We will take this idea as our starting point and not—as many have assumed—an irreducibly modal analysis of the potentially infinite.² In order to theorize in a sufficiently general way, it is consequently necessary to introduce ‘higher-order’ quantifiers that can generalize into the position occupied by predicates as well as names. As we wrote \exists_e for quantifiers binding into name position, we write $\exists_{(e)}$ and $\exists_{(ee)}$ for quantifiers binding into unary and binary predicate position, and so on (in general, $\exists_{(\sigma_1 \dots \sigma_n)}$ for quantification into the position of an n -ary relation between things of types $\sigma_1, \dots, \sigma_n$). Thus, writing Fa for ‘Socrates is wise’ we can generalize both into the position of the name ‘Socrates’ and into the position of the predicate ‘wise’, so that not only $\exists_e x Fx$ but also $\exists_{(e)} X Xa$ follows from Fa by existential generalization. As I will understand it, $\exists_{(e)}$ is a device for generalizing in predicate position and not a first-order quantifier over properties, sets or classes. $\exists_{(e)} X Xa$ does not entail that Socrates instantiates some property, for *Socrates is wise* does

²See, for instance, Lear (1980) and Linnebo and Shapiro (2019).

not entail that there are any properties, and by the transitivity of entailment, nothing that *Socrates is wise* entails does either, including the two immediate consequences of Fa , $\exists x Fx$ and $\exists X Xa$, mentioned above.³ Nonetheless, I will follow the convention of *pronouncing* $\exists_{(e)}$ and $\exists_{(ee)}$ as ‘some property’ and ‘some relation’ respectively, to avoid overly formal prose.

Applying this to our previous remarks: the existence of the potentially infinite, expressed with a second-order existential, may commit us to infinitely many individuals, but not to an infinite individual. Given that we are understanding $\exists_{(ee)} R \text{PotInf } R$ as expressing existential generality in the position of a binary predicate, it does not commit us to any completed infinite entities like sequences, infinite relations or infinite domains of such relations. $\exists_{(ee)} R \forall_e x \exists_e y Rxy$ is entailed by $\forall_e x \exists_e y. x <_t y$, the claim that *for every stretch of time there is a strictly longer stretch*, by existentially generalizing into the position that the binary predicate ‘is strictly longer than’ occupies. We have just argued that this claim does not entail that there are any individuals other than finite stretches of time, so neither do any weaker claims it entails.

Cantor famously embraced completed infinities. In his *Grundlagen* (Cantor (1883)) he states two principles of generation for ‘creating’ infinities. The first principle of generation—which I will simply call *Successor*—ensures that one can always create a potential infinity by adding one to a sequence: “the principle of adding a unity to an already formed and existing number”.⁴ The second principle of generation—which I will call *Limit*—ensures that from any potential infinity one can always create a completed infinity: “if any definite succession of defined integers is put forward of which no greatest exists a new number is created by means of this second principle of generation, which is thought of as the limit of those numbers; that is, it is defined as the next number greater than all of them”. Cantor’s principles do not apply to stretches of time but to special mathematical objects—‘transfinite numbers’, or ‘ordinals’, ordered by a relation $<_\Omega$ —which are governed by these principles. These transfinite numbers are totally ordered by $<_\Omega$ —that is they are strict orders in the previously defined sense such that $\forall_e xy (Rxy \vee Ryx \vee x = y)$. We write this $\text{Tot } <_\Omega$. Writing Ord for $\text{Dom } <_\Omega$ we might axiomatize Cantor’s theory by adding to the principle that the ordinals are totally ordered:

Successor

$$\forall_e x (\text{Ord } x \rightarrow \exists_e y (\text{Ord } y \wedge \text{Suc } xy))$$

³For more on this way of understanding higher-order generalizations see, for instance, Prior (1971), Williamson (2003), Trueman (2020), Bacon (forthcomingb).

⁴Ewald (1996) p907.

Limit

$$\forall_{(e)}X((\forall_e y(Xy \rightarrow \text{Ord } y) \rightarrow \exists_e y(\text{Ord } y \wedge \text{LUB } Xy))$$

Where

$$x \leq_\Omega y := x = y \vee x <_\Omega y$$

$$\text{UB } Xy := \forall_e x(Xx \rightarrow x \leq_\Omega y)$$

$$\text{LUB } Xy := \text{UB } Xy \wedge \forall_e z(\text{UB } Xz \rightarrow y \leq_\Omega z)$$

$$\text{Suc } xy := \text{LUB } \lambda z(z <_\Omega y)x$$

The ordinals are closely related to the notion of a well-order, a notion also introduced into mathematics by Cantor. A totally ordered relation, R , is well-orderd iff R there is always an R *least* individual among any $\text{Dom } R$ individuals:

$$\text{WO } R := \text{Tot } R \wedge \forall_{(e)}X(\exists yXy \wedge \forall_e y(Xy \rightarrow \text{Dom } Ry) \rightarrow \exists_e x(Xx \wedge \forall_e y(Xy \rightarrow x = y \vee Rxy)))$$

It is easy to prove from Cantor's principles that the transfinite numbers are well-ordered.⁵ Arguably for Cantor, the notion of a well-order is prior to that of a transfinite number: in later work, a transfinite number is an abstraction from the notion of a well-order—transfinite numbers represent the order-types of well-orders.⁶

Unfortunately Cantor's two principles of generation, if left unrestricted, lead to the Burali-Forti paradox.⁷

Cantor was aware of the paradox prior to Burali-Forti⁸, and describes it quite clearly in a letter to Hilbert (here is is applying his principle to the alephs

⁵Given some ordinals, X , consider the property of not being strictly greater than any X : $\lambda z \neg \exists_e y(Xy \wedge y <_\Omega z)$. Its least upper-bound will be a minimal element of X .

⁶p86 of and pp111-112 Cantor (1915).

⁷Cantor does in fact restrict them—in the *Grundlagen* he has a 'principle of limitation' which is presumably supposed to weaken the second principle, but it is a little bit hard to interpret: it states "a new integer could be made with the help of one of the two other principles of creation only if the totality of all previous numbers had the power of a defined number-class which was already in existence over its entire extent." One natural interpretation of this requires that X , in the antecedent of the second principle, be no bigger than the power of any ordinal in it (i.e. add $\exists_e y(Xy \wedge X \cong \lambda Z \forall_e z(Zz \rightarrow z \leq_\Omega y)$ to the antecedent of the second principle; \cong denotes equinumerosity between types (e) and $((ee))$, see 24). This restricted principle has a model where $<_\Omega$ is interpreted by the ordering on a strongly inaccessible cardinal (including, as trivial case, ω).

⁸The fact that Cantor introduces the 'principle of limitation' in the *Grundlagen* strongly suggests that he was aware of a problem in with the unrestricted use of the second principle of generation in 1883 (see Menzel (1984)).

rather than the transfinite ordinals, but the argument is essentially the same in any case):

The totality of alephs is one that cannot be conceived as a determinate well-defined, *finished* set. If this were the case, then this totality would be *followed* in size by a *determinate aleph*, which would therefore both *belong* to this totality (as an element) and *not belong*, which would be a contradiction. (Letter from Cantor to Hilbert, 26 Sept 1897, translated in Ewald (1996).)

After Hilbert points out that the alephs are perfectly determinate and well-define, Cantor insists that it is the notion of being *finished* that is of central importance:

One must only understand the expression ‘finished’ correctly. I say of a set that it can be thought of as *finished* [...] if it is possible without contradiction [...] to think of *all its elements as existing together*, and so to think of the set itself as a *compounded thing for itself*, or [...] if it *possible* to imagine the set as *actually existing* with the totality of its elements.

This is not so different from Aristotle’s notion of an actual infinity—the finite stretches of times cannot be brought together into a single infinite stretch. Cantor’s position is that the ordering $<_{\Omega}$ is not a finished order and so is a potential infinity, in the higher-order sense defined earlier, that cannot be completed (unlike the proper initial segments of $<_{\Omega}$).

But in what sense can’t $<_{\Omega}$ be completed? It is potentially infinite (PotInf $<_{\Omega}$ is essentially what Successor says), but then so is the ordering of the finite numbers, $<_{\omega}$, and this is completable by Cantor’s lights. If there is an individual a that is not a transfinite number we can always extend $<_{\Omega}$ by adding one of these individuals to the end— $x <_{\Omega+1} y$ iff $x <_{\Omega} y$ or $\text{Ord } x$ and $y = a$ —and even if we run out of individuals we can always extend the order “up to isomorphism” by removing the least element⁹, call it 0, from the bottom and moving it to the top.

Cantor’s answer to this question is that the ordinals cannot be ‘enumerated’, in Cantor’s extended sense. Cantor sometimes takes the enumerability of some things to simply mean they can be well-ordered. However the transfinite numbers are provably well-ordered.¹⁰ Here it seems he takes enumerability

⁹Whose existence is guaranteed by the fact that $<_{\Omega}$ is well-ordered

¹⁰The are well-ordered by the relation $\alpha < \beta := \exists_{(ee)} RS(\text{OrdType } \alpha R \wedge \text{OrdType } \beta S \wedge R < S$.

to mean something that can be assigned a transfinite number: that is to say, to be enumerable is to be well-ordered by one of those well-orders having one of these special individuals representing its order-type.¹¹ Taken as an explanation for why certain well-orders are unfinished and cannot be assigned an ordinal representative, and others can, this is patently circular. And so it remains mysterious why some potentially infinite series correspond to completed infinities—can be assigned a transfinite number as order type—and others cannot.¹²

More troubling for this explanation is the fact that there is nothing *inconsistent* about a theory that assigns $<_{\Omega}$ a transfinite number: if the proper initial segments of $<_{\Omega}$ can consistently be assigned individuals as order-types, we can create a new way of representing order-types by individuals that does assign $<_{\Omega}$ an order-type by, as it were, ‘making room in Hilbert’s hotel’—i.e. shifting the individuals representing the finite order types up by one, and assign $<_{\Omega}$ what used to be playing the role of 0. So it’s just not true that Ω can’t be enumerated—we could assign it an individual representing its ordered type if we wanted—it’s simply that Ω *isn’t* enumerated by Cantor’s particular way of doing it. (When we move from order-type to cardinality we find other authors adopting different representations: Frege, for instance, put forward a different (and consistent—see Boolos (1987)) theory of cardinality to Cantor based on Hume’s principle: according to this any things whatsoever are be assigned an individual as number, including the totality of all things.)

This state of affairs is somewhat unsatisfactory. For according to this theory, we have a whole host of perfectly good well-orders that are not represented by any of Cantor’s transfinite numbers. As noted above, not only do we have the ordering on the ordinals, $<_{\Omega}$, but we also have $<_{\Omega+1}$, the ordering obtained from the ordinals by removing the 0 from the beginning, and appending it to the end, and this can be continued indefinitely. In fact, the well-orders continue for quite a bit longer from $<_{\Omega}$ onwards than they do before it. There are strictly more well-orderings that cannot be assigned ordinals than there are well-orders that can, a fact that is essentially the result of turning the Burali-

¹¹Quite possibly he conflated these two things in the *Grundlagen* because he was not aware of the Burali-Forti paradox, so that this is not really a change of meaning but a precisification on one of the two things it could have been taken to mean.

¹²Another principle sometimes discussed in this context is the principle of limitation of size: that things equinumerous with the entire universe are too big to form a set. I cannot find it articulated like this in Cantor, but Hallett (1991) p176 describes this as a ‘spiritual descendent’ of Cantor’s theory. This principle that one can assign something a transfinite number only if it is smaller than the universe is less obviously circular, but still smacks of something that is motivated only by the fact that it avoids inconsistency.

Forti paradox into a theorem of higher-order logic. In what sense, then, is Ω absolutely infinite? How is $<_{\Omega}$, for instance, any different from well-orders of the first transfinite order-type ω , which can be continued $\omega + 1$, and so on. Why does Cantor stop representing well-orders by special individuals at the order-type Ω and not $\Omega + 1$? The problem with any (consistent) theory that attempts to represent the order-types of well-orders by individuals is that they inevitably confront a charge of arbitrariness.

2 Higher-Order Formulations of Indefinite Extensibility

Why postulate special individuals to represent the different order-types of well-orders at all if not all the well-orders can be represented? Why not just formulate principles and reason with the well-orders themselves and forget about trying to find special individuals to represent their order types?

Let us return to our general definition of a potential infinity. In some sense the the completion of a potential infinity, R , is an individual, a , that can be placed ‘above’ each of the individuals in the domain of R . a is not itself in the domain of R , or else R would not be a potential infinity and a would not be above all the $\text{Dom } R$ elements (nothing can be above itself). We must think of R being extended by a to make another well-order, R^{+a} , that includes a in its domain as lying above each of the elements of R . Although we have only talked of individuals as completed infinities— a in this case—there’s also an extended sense in which R^{+a} itself can be thought of a completion of R . Note, of course, that in this general sense R can be completed in multiple different ways. Cantor appears to get into trouble by assuming that there is a particular relation, $<_{\Omega}$, (defined on special individuals, the transfinite numbers), when it seems that any well-order, $<_{\Omega}$ included, can be extended.

I want to put explore the idea that the notion of a well-order is ‘indefinitely extensible’ in a way that goes beyond what both Aristotle and Cantor have put forward. The only way I can think to formulate the indefinite extensibility of the notion of a well-order is not to formulate it in terms of some particular well-order, $<_{\Omega}$, which must already have some particular order-type, but by higher-order quantification over well-orders. These are, in a loose sense, higher-order analogues of Cantor’s two principles of generation. We have the principle that every well-order can be extended by one, and for every sequence of well-orders ordered by the initial segment relation has a well-order containing them as initial segments. To make these easier to state we introduce some abbreviations. Being an initial segment is a higher-order relation defined as

follows:

$$R \leq S := \forall x(\text{Dom } Rx \rightarrow \forall y(Ryx \leftrightarrow Syx))$$

R is a proper initial segment of S , written $R < S$, when $R \leq S$ but $S \not\leq R$. We will later apply these definitions to arbitrary relations. We will say that a higher-order property of relations, X of type $((ee))$, is *linearly ordered* by \leq iff

$$\text{Lin } X := \forall_{(ee)} RS(XR \wedge XS \rightarrow R \leq S \vee S \leq R)$$

We'll say that a relation R is an upperbound of X iff

$$\text{UB } RX := \forall_{(ee)} S(XS \rightarrow S \leq R)$$

With these abbreviations in place we may formulate the two principles as follows.

Successor (Higher-Order)

$$\forall_{(ee)} R(\text{WO } R \rightarrow \exists_{(ee)} S(\text{WO } S \wedge R < S))$$

Limit (Higher-Order)

$$\forall_{((ee))} X((\text{Lin } X \wedge \forall_{(ee)} R(XR \rightarrow \text{WO } R)) \rightarrow \exists_{(ee)} T(\text{WO } T \wedge \text{UB } RX))$$

If these principles are consistent they imply that every well-order can be extended by one, even $<_\Omega$. One can repeat this indefinitely to obtain a potentially infinite well-order that can be completed by the second principle.¹³

Indeed, we seem to find this conception of indefinite extensibility in Zermelo. In 1908 Zermelo introduced and axiomatized the *iterative* conception of sets, according to which they are built up in stages V_0, V_1, \dots in a well-ordered sequence much like Cantor's theory of transfinite ordinals. One can always add a stage, $V_{\alpha+1}$, by taking the powerset of the previous stage V_α (the set containing all its subsets), and given a sequence of stages V_α ordered by inclusion one can take their 'limit' by unioning them together.¹⁴

¹³In fact, the second principle is a theorem of a minimal higher-order order logic—if X is a collection of well-orders linearly ordered by \leq , $Rxy := \exists_{(ee)} S(XS \wedge Sxy)$ will complete them. However, it is useful to state in its own right because later we will consider variants of the principle.

¹⁴Like with Cantor's principles this is inconsistent if applied unrestrictedly; in order to maintain consistency, like Cantor's principle of limitation (see footnote 7), this principle is restricted to sequences of stages that can be indexed by a set that already exists; this latter idea is essentially due to Fraenkel.

Zermelo was theorizing in a higher-order language with a single non-logical binary predicate \in .¹⁵ In this language Zermelo axiomatized the iterative conception of sets theory with a finite list of axioms, the conjunction of which we will call ZF^\in . By replacing the membership predicate in this axiom with another binary predicate, R , we can formulate the claim that R satisfies Zermelo’s conditions ZF^R . In this way we obtain a purely logical predicate, ZF , allowing us to talk about ZF relations in general.

In Zermelo (1930), Zermelo distances himself from the idea that there is a special ZF relation, \in , about which set theory is concerned. Just like $<_\Omega$, Zermelo maintains that any ZF relation— \in included—can be extended to more inclusive ZF relations. Instead of fixating on one particular ZF relation he takes up the the investigation of ZF relations in general: a project that can be undertaken in the purely logical language of higher-order logic, without any set-theoretic primitives.

Zermelo’s picture was that every ZF relation is properly included in a larger one, and any collection of ZF relations ordered by inclusion are included in some ZF relation—there is no special relation \in , which is itself indefinitely extensible. This is what Zermelo says:¹⁶

Let us now put forth the general hypothesis that every categorically determined domain can also be conceived of as a “set” in one way or another; that is, that it can occur as an element of a (suitably chosen) normal domain. It then follows that there corresponds to any normal domain a higher one [...] Likewise, a categorically determined domain of sets arises through union and fusion from every infinite sequence of different normal domains [...] where one always contains the other as a canonical segment.

For Zermelo a ‘normal domain’ for Zermelo is the domain of a ZF -relation R , and a ‘domain’ a collection contained in a normal domain. This allows Zermelo to avoid the problems associated with inconsistent multiplicities. As Geoffrey Hellman puts it, according to Zermelo “set theory should be seen, not as the theory of a unique, all-embracing structure, but instead as a theory

¹⁵Zermelo follows the terminology of Whitehead and Russell (1910-1913), who are more explicit about the fact that second-order quantifiers bind into predicate position. Zermelo by contrast talks informally, using Russell’s term ‘propositional function’ when higher-order quantification is intended.

¹⁶Below I suppress several qualifications Zermelo makes regarding differences between ZF relations that purely concern urelements, as they are not relevant to our present discussion of pure sets.

of an endless infinity of intimately related structures.”¹⁷

Of course, underlying this is the thought that well-orders themselves are indefinitely extensible. Of the transfinite numbers, Zermelo writes that they rest

upon the notion of well-ordering and which, though lacking in true completion on account of its boundless progressing, possesses relative way stations, namely those “boundary numbers” [i.e. inaccessible], which separate the higher from the lower model types.

If we flatfootedly formalize Zermelo’s two remarks in higher-order logic we obtain the following pair of principles: every ZF relation is a proper initial segment of some other ZF relation, and (ii) whenever you have some ZF relations ordered under initiality there is a ZF-relation containing them all as initial segments. Writing $R < S$ for $R \leq S \wedge S \not\leq R$:

Progress $\forall_{(ee)} R (\text{ZF } R \rightarrow \exists_{(ee)} S (\text{ZF } S \wedge R < S))$

Completion $\forall_{((ee))} X (\text{Lin } X \wedge \forall_{(ee)} R (XR \rightarrow \text{ZF } R) \rightarrow \exists_{(ee)} T (\text{ZF } T \wedge \text{UB } TX))$

The idea that the ordinals are indefinitely extensible leads to a variant pair of principles about well-orders: every well-order of inaccessible order type is a proper initial segment of another such relation, and any collection of well-orders that are linearly ordered by \leq are initial segments of some well-order of inaccessible order type.¹⁸

Progress^{WO} $\forall_{(ee)} R (\text{Inaccessible } R \rightarrow \exists_{(ee)} S (\text{Inaccessible } S \wedge R < S))$

Completion^{WO} $\forall_{((ee))} X (\text{Lin } X \wedge \forall_{(ee)} R (XR \rightarrow \text{WO } R) \rightarrow \exists_{\rho} T (\text{Inaccessible } T \wedge \text{UB } TX))$

Of course, this harkens back to Cantor’s principles of generation, that one can add one to any transfinite number, and given any sequence of transfinite numbers we can find the least number which is greater than them all.

Zermelo’s remarks capture an attractive, but somewhat elusive idea. Many philosophers have been seduced by this picture of the set theoretic hierarchy

¹⁷Hellman (1989) p56. Hellman does not think Zermelo is successful in resolving the tension, essentially because Hellman thinks the only way to make sense of the relevant higher-order quantification is in terms of singular quantification over proper classes.

¹⁸The notion of *inaccessible* can be defined in pure higher-order logic. It can be obtained by λ ing out $<_{\Omega}$ from Cantor’s theory of ordinals: i.e. *Inaccessible* R means $\text{Tot } R$, and the two principles of generation, with the principle of limitation included (see footnote 7), hold of R .

as indefinitely extensible, but have had trouble articulating the idea precisely. Common to these formulations is the assumption that there is a distinguished relation, \in , or in the case of the ordinals $<_{\Omega}$, and it is *this* relation that is said to be indefinitely extensible, or not as the case may be. (This is picture, it should be noted, is on its face importantly different from Zermelo’s; for Zermelo it is the higher-order property of being a ZF relation or being a well-order that are indefinitely extensible.)

Let’s consider (briefly) two major attempts to express the indefinite extensibility of *particular* relations, such as \in and $<_{\Omega}$. According to some authors, one must give up on the idea that we can quantify unrestrictedly. Each quantifier gives us a restricted view of the totality of ordinals, as it were, and from no viewpoint can we see them all at once.¹⁹ But to say that a given quantifier is restricted we do so by another quantificational claim: we mean there is *something* not in its range. This is only true if this new quantificational claim ranges more widely than the the original one, and if it too is restricted this can only be articulated by a yet wider quantifier. Zermelo was staunchly against this sort of relativism:

In general, the concept of “allness”, or “quantification”, must lie at the foundation of any mathematical consideration as a basic logical category incapable of further analysis. If we were to restrict the allness in a particular case by means of special conditions, then we would have to do so using quantifications, which would lead us to a regressus in infinitum. Zermelo (1931).

Whether this regress is troublesome remains to be seen, but it does bring to salience a difficulty. How should the quantifier relativist state their positive view that every first-order quantifier is restricted? Presumably they should do this by quantifying into the position of a first-order quantifier—but if this higher-order quantifier is also restricted, it fails to have the required force. And if the higher-order ‘quantifier quantifier’ is unrestricted then one can define an unrestricted first-order quantifier: absolutely everything is F when F satisfies every first-order universal quantifier.²⁰

Other philosophers, following Charles Parsons, have suggested that the indefinite extensibility of \in or $<_{\Omega}$ must be glossed in inherently modal terms.²¹ These authors typically produce modal formulations of the idea that \in or $<_{\Omega}$

¹⁹See the papers in Rayo and Uzquiano (2006).

²⁰See Williamson (2003) for a discussion of related problems.

²¹Parsons (1983). This idea is developed further in Linnebo (2013), and in several subsequent papers of his; a different formalization of the modal idea based on tense logic is given in Studd (2013).

may always be extended—the analogue of Cantor’s first principle of generation, Successor. Applied to the ordinals this can be stated as follows:

Successor (Modal)

$$\Box\forall_e x(\text{Ord } x \rightarrow \Diamond\exists_e y(\text{Ord } y \wedge x <_\Omega y))$$

writing Ord for $\text{Dom } <_\Omega$. But surely just as essential to the picture of the ordinals as extendible by the successor operation is this idea of their being extendible by taking limits, captured in Cantor’s second principle of generation, Limit. After all, the unmodalized version of the above principle—obtained by deleting all modal operators—is just $\text{PotInf } <_\Omega$. Not only can this idea be perfectly consistently captured without modal operators, it fails to distinguish $<_\Omega$ from the ordering of the finite ordinals, $<_\omega$, which we do not think of as being indefinitely extensible. What would the modal analogue of the second principle amount to then? Intuitively, whenever we have a property of ordinals picking out ordinals across worlds (not just a single world) it should be possible for there to be an ordinal that is necessarily at least as big as each of them.

Limit (Modal)

$$\Box\forall_{(e)} X(\Box\forall_e y(Xy \rightarrow \text{Ord } y) \rightarrow \Diamond\exists_e x(\text{Ord } x \wedge \Box\forall_e y(Xy \rightarrow y \leq_\Omega x))$$

Unfortunately these two principles are straightforwardly inconsistent, assuming that the ordinals are necessarily well-ordered. The two modal principles are very straightforward modalizations of Cantor’s principles of generation, and they are no better than them — the inconsistency is obtained by modalizing the Burali-Forti paradox. Plugging Ord into X in the second principle we get the possibility of an ordinal, x , that is necessarily at least as big as every ordinal, but then the first principle implies the possibility of an ordinal strictly bigger than x .

Thus the modal extensibilists must, like Cantor, give up on Limit in its unrestricted form. Like Cantor, then, they are committed to a similar kind of arbitrariness concerning when we stop numbering well-orders. This time we are not just concerned with relations that well-order in extension, but ‘modal well-orders’. Roughly speaking, a relation such that, if you laid out its extensions at each possible world, they would all be well-orders related to one another by the initial segment relation. More precisely, a modal well-order is a relation R that: (i) is necessarily a well-order, (ii) cannot shrink across modal space—if it relates x to y it does so necessarily, and (iii) can only expand in

the upwards direction of the order—if y is already in the domain of R , and R doesn't relate x to y then it necessarily doesn't relate x to y .

$$\begin{aligned} \text{WO}^\square R &:= \square \text{WO } R \wedge \\ &\quad \square \forall_e xy (Rxy \rightarrow \square Rxy) \wedge \\ &\quad \square \forall_e xy (\neg Rxy \wedge \text{Dom } Ry \rightarrow \square \neg Rxy) \end{aligned}$$

Like well-orders, one modal well-order can be said to be an initial segment of another. Because modal well-orders can expand at different rates and yet still amount to the same ordering of individuals across all of modal space, we say that one modal well-order is equivalent to an initial segment of another when it is necessarily possible that it is isomorphic to an initial segment of the other (here \preceq is the notion of one order being isomorphic to an initial segment of another which we will define shortly):

$$R \preceq^\square S := \square \diamond R \preceq S$$

While every modal well-order that doesn't 'expand' can possibly be assigned an individual as order type, these are only a very special kind modal well-order.²² Some modal well-orders are not possibly assigned an order type.²³ As before, the point at which the modal well-orders stop being possibly represented will depend on ideosyncratic features of the choice of representation, and as before, there will be far more unrepresented modal well-orders than represented ones, whatever representation one uses.

3 The Well-Ordering Principle

The reader with an eye for paradoxes might wonder whether Zermelo has not committed himself to the Burali-Forti paradox. Zermelo seems to recognise the tension, and likens the principles to a Kantian antinomy of 'progress' and 'completion', from the epigraph. Zermelo does not, however, formalize or otherwise develop his remarks.

²²A non-expanding relation is a persistent relation that additionally doesn't acquire new relata—see the notion of 'Rigidity' defined in Bacon and Dorr (forthcoming).

²³Here is the reason, in sketch. If there is a way of assigning possible individuals to represent possible modal well-orders, then those individuals can be ordered according to whether the modal well-orders they represent bear the \preceq^\square to each other. This ordering itself is a modal well-ordering that cannot be represented by that particular representation, although it may be assigned a representation by other representations of the modal well-orders.

Let's begin by noting that if we were to replace \leq , the relation of *being an initial segment of*, with the relation of *being isomorphic to an initial segment of*, we do indeed get an inconsistency from any of our principles.²⁴ Let X be the property of being a well-order. It is easily show that well-orders are linearly ordered (indeed well-ordered) by \preceq , and so by the variant of Limit, there must be a well-order R , containing an initial segment isomorphic to any well-order.²⁵ But then by the variant of Successor, there is a well-order R^+ that has a proper initial segment isomorphic to R , and thus any well-order relation is isomorphic to a proper initial segment of R^+ . This includes R^+ itself, a contradiction! The inconsistency extends to variants of Zermelo's principles with \preceq replacing \leq : this time we must appeal to a theorem proved by Zermelo himself, that ZF relations are linearly ordered by \preceq .

However, the parallel argument involving \leq does not go through, for the totality of ZF relations and well-orders are not linearly ordered by the initial segment relation. Nor, without further assumptions, do we have a guarantee that we can find a maximal chain of well-orders under the initial segment relation. (Indeed, that on the official reading, the higher-order version of Limit is in fact a theorem of a minimal higher-order logic, for if X contains well-orders that are linearly ordered under \leq then the relation $Sxy := \exists_{(ee)}R(XR \wedge Rxy)$ is a well-order extending each relation in X .²⁶)

One could obtain a contradiction if we instead assumed a higher-order version of the well-ordering principle (an equivalent of the higher-order axiom of choice, see Shapiro (1991)).

The Well-Ordering Principle ^{σ} $\exists_{(\sigma\sigma)}R(\text{WO } R \wedge \forall_{\sigma}x \text{ Dom } Rx)$

Completion tells us that every chain of ZF relations ordered by \leq has an upperbound that is also a ZF relation, and so by Zorn's lemma—a consequence

²⁴We define this relation as follows

$$\begin{aligned} \text{Bij } RXY &:= \forall_e x (Xx \rightarrow \exists_e !y (Yy \wedge Rxy)) \wedge \forall_e y (Yy \rightarrow \exists_e !x (Xx \wedge Rxy)) \\ R \cong S &:= \exists T (\text{Bij } T(\text{Dom } R)(\text{Dom } S) \wedge \forall xyx'y' (Txx' \wedge Tyy' \rightarrow (Rxy \leftrightarrow Sx'y'))) \\ R \preceq S &:= \exists T (T \leq S \wedge R \cong S) \end{aligned}$$

²⁵Note that our definitions of linear order are suitable for preorders, and do not build in antisymmetry. \preceq is not antisymmetric, since one have distinct but isomorphic relations, whereas \leq is antisymmetric.

²⁶When read in terms of \preceq , the situation is reversed: Limit becomes the key ingredient of the Burali-Forti paradox, Successor is not quite a theorem, but every infinite well-order R can be strictly extended under \prec by removing the least element of the order and affixing it to the end.

of The Well-Ordering Principle^{((ee))}—the ordering of ZF relations under \leq has a maximal element, contradicting Progress.

The inconsistency of our higher-order versions of Successor and Limit can also be proven by Zorn’s lemma, but in this case there is also a more direct method that is quite instructive. Suppose that R is a well-order of all the individuals. By Successor, there is a well-order strictly extending R .²⁷ But there cannot be a strict extension, because all of the individuals have been used up in the ordering of R —we cannot reuse an individual appearing in R ’s domain, for a well-order cannot contain a cycle. What is instructive about this method of proof is that it illustrates the difference between the two ways of ordering well-orders, \leq and \preceq . While no well-order has R as a proper initial segment, we can of course find a well-order that has as a proper initial segment something isomorphic to R : simply remove R ’s least element—an operation that will leave R ’s order type alone, provided R is infinite—and tag it to the end of R to make a strictly longer well-order under \preceq .

We thus have a completely flatfooted articulation of the indefinite extensibility of the notion of well-order and ZF relation, that does not require one to deny the ability to quantify unrestrictedly, or to posit special mathematical modalities according to which the length of the set theoretic hierarchy is contingent. Indeed, our extensibility principles are consistent with the Fregean principle of extensionality, which rule out any sort of contingency whatsoever (this implication of extensionalism does, however, render it implausible as a more general principle of higher-order logic).

What solace might Cantor or Zermelo draw from these consistency results? Cantor calls the principle that it is ‘it is always possible to bring any well-defined set into the form of a well-ordered set’ a *law of thought*—‘a law which seems to me fundamental and momentous and quite astonishing by reason of its general validity.’²⁸ Zermelo too accepted Cantor’s principle, although justified it from what he took to be a more basic principle, the axiom of choice, saying that whenever you have some pairwise disjoint non-empty sets, there is a set which contains exactly one element for each of those sets. Both of these principles are principles about *sets*.

The higher-order well-ordering principle is stronger than this, since it implies that even properties whose extensions do not form a set can be enumerated. Recall that Cantor, like Aristotle, maintained that some things are absolutely infinite—they are together inconsistent in the sense that they can-

²⁷Zermelo’s principles of Progress^{WO} and Completion^{WO} also imply every well-order can be extended.

²⁸*Grundlagen* §3.1, translated in Ewald (1996) p886.

not be brought together into a whole ‘finished’ set, they are together were ‘beyond enumeration’.²⁹ Earlier we suggested that Cantor sometimes applied the expressions ‘can be finished’, ‘enumerable’, ‘countable’ and so on, to some things when they could be listed, albeit possibly by a potentially infinite list (i.e. a list that has no end).³⁰ If some things cannot be brought into a set, then they cannot be listed, not even by a list that goes on forever. On this reading an absolutely infinite ‘inconsistent’ multiplicity is thus one that cannot be well-ordered, so we may reject the well-ordering principle for things which do not form a set.³¹

Since the Zermelian approach to indefinite extensibility is purely logical, there is a question about whether we can make sense of this set-theoretic criteria for when properties can be well-ordered in this setting. The idea that only properties defined by sets need be well-orderable takes for granted a particular property of sethood and corresponding membership relation \in . From Zermelo’s perspective there is nothing special about any particular ZF relation, and so any principle that relies on a particular membership relation could be deemed parochial. What we would like is a principle of pure higher-order logic that ensures choice holds for *any* ZF relation.

Note, firstly, there are principles of pure logic that imply set-theoretic choice. In the same way that we defined the higher-order predicate of relations, ZF, in terms of the Zermelo-Fraenkel axioms, we can introduce the notion of a ZFC relation, satisfying also the axiom of choice. Given the higher-order well-ordering principle there is no difference between these relations:

Theorem 3.1. *Given the Higher-Order Well-Ordering Principle, every ZF relation is a ZFC relation.*

For suppose that S is a global well-order, and that R is a ZF relation. We will write ‘element ^{R} ’ for R ‘set ^{R} ’ for $dom(R)$. Suppose that x is a set ^{R} of non-empty disjoint sets ^{R} . Then I can define the relevant choice set ^{R} by taking the set ^{R} consisting of the S -least elements ^{R} of each element ^{R} of x .

The converse to this theorem need not hold. The claim that every ZF relation is a ZFC relation does not imply the higher-order choice principles. In fact, in the models described in section 4 every ZF relation is a ZFC relation

²⁹See Hallett (1991) p150, Lavine (1994) Chapter III§4, Newstead (2009) p546.

³⁰See his criticism of Aristotle in the *Grundlagen*, Ewald (1996) p889, also quoted below. Note that the word ‘countable’ is not being used in the modern sense of being equinumerous with the natural numbers.

³¹It should be noted, however, that Cantor in later writing sometimes does use the stronger form of choice—for instance in his proof that if a multiplicity does not have an \aleph number, then it is a inconsistent multiplicity, and does not form a set.

but there is no global well-ordering of the universe. Thus we might consider adding the following principle of higher-order logic to our existing Cantorian principles:

Local Choice $\forall_{(ee)}R(\text{ZF } R \rightarrow \text{ZFC } R)$

Whether the picture we have outlined would ultimately be acceptable to Cantor, or indeed Zermelo, is unclear. However the view does substantiate several distinctively Cantorian ideas. First, we straightforwardly obtain from our principles the thesis that every potential infinity can be completed — $\forall_{(ee)}R(\text{WO } R \wedge \text{PotInf } R \rightarrow \exists_{(ee)}S(\text{WO } S \wedge R \leq S \wedge \neg \text{PotInf } S)$. Cantor endorses principles like this when outlining his disagreements with Aristotle, although he later has to walk them back on account of the apparent incompleteness of the potentially infinite series of ordinals.³² Against Aristotle, he writes “determinate countings can be carried out just as well for infinite sets as for finite ones, provided that one gives the sets a determinate law that turns them into well-ordered sets. That without such a lawlike succession of the elements of a set it cannot be counted—this lies in the nature of the concept of counting.”³³ This suggests that, at least in the naïve exegesis of his view, the distinction between the countable and absolutely infinite multiplicities is simply the distinction between being well-orderable and not. Of the latter, Cantor mysteriously writes in the *Grundlagen* that we can “never achieve even an approximate conception of the absolute”.³⁴ Whatever this might mean, the present view vindicates something in the vicinity: absolutely infinite totalities, such as the totality of all individuals, cannot be approximated by a potentially infinite series, for no well-ordered list spans every individual.

4 Logics of Zermelian Extensibility

In this section I outline some higher-order logics that formalize, in purely logical terms, certain Zermelian theses about indefinite extensibility, and prove they are consistent relative to a more standard mathematical theory.

Let’s begin by being a bit more precise about the language we have been working in. As previously described, there is a type of singular terms, e , and whenever $\sigma_1, \dots, \sigma_n$ are types we also have a type, $(\sigma_1 \dots \sigma_n)$ of n -ary relations between entities of these respective types. Terms are formed inductively:

³²As noted in footnote 8, he was surely aware of the Burali-Forti paradox at the time of the *Grundlagen*.

³³Ewald (1996) p889.

³⁴Ewald (1996) p916.

we have logical constants \rightarrow , \perp and \forall_σ of types $((\))$, $(\)$ and $((\sigma))$ respectively, and an infinite stock of variables of each type. Given a term R of type $(\sigma_1 \dots \sigma_n \dots \sigma_{n+m})$ and terms a_1, \dots, a_n of types $\sigma_1, \dots, \sigma_n$ we can form a complex term of type $(\sigma_{n+1} \dots \sigma_{n+m})$ by application: $Ra_1 \dots a_n$. And given a term A of type $(\sigma_1 \dots \sigma_n)$ and variable x_0 of type σ_1 we can form a complex predicate of type $(\sigma_0 \dots \sigma_n)$ by $\lambda x_0. A$. Common symbols, like \wedge and \exists_σ , are introduced by abbreviation—of particular note is the higher-order identity relation $=_\sigma$ of type $(\sigma\sigma)$, which is defined as $\lambda xy. \forall_{(\sigma)} X (Xx \rightarrow Xy)$.

A *logic* is just a set of sentences in this logical language that contains the axioms PC, UI and $\beta\eta$ below, and is closed under the rules MP and Gen.³⁵ Following Bacon (2018) I'll call the smallest such theory **H**.

PC All instances of propositional tautologies.

MP From A and $A \rightarrow B$ infer B

Gen From $A \rightarrow B$ infer $A \rightarrow \forall_\sigma \lambda x. B$ when x does not occur free in A .

UI $\forall_\sigma F \rightarrow Ft$ (where t is a term of type σ)

$\beta\eta$ $A \rightarrow B$ whenever A and B are $\beta\eta$ equivalent terms of type t .³⁶

We can now consider logics obtained by adding to **H** principles discussed in the previous sections. They are not all independent of one another. For instance the higher-order Limit principle in fact is already a theorem of **H**, as noted in footnote 13. The higher-order Successor principle follows from Progress^{WO}. We will thus focus on the following logics.

- **HZ** is the higher-order logic axiomatized by Progress and Completion.
- **HZ^{WO}** is axiomatized by Progress^{WO} and Completion^{WO}.
- We will use **(LC)** and **(Ext)** respectively to denote Local Choice and the principle of Extensionalism, so that, e.g., **HZ(LC)** is **HZ** plus Local Choice, and **HZ(Ext)** is **HZ** plus Extensionalism.

Extensionalism is the principle stating that proposition, properties and relations are individuated by coextensiveness.

Extensionalism $\forall_{(\sigma_1 \dots \sigma_n)} RS (\forall_{\sigma_1} x_1 \dots \forall_{\sigma_n} x_n (Rx_1 \dots x_n \leftrightarrow Sx_1 \dots x_n) \rightarrow R =_{(\sigma_1 \dots \sigma_n)} S)$

³⁵If we wanted we could consider languages with non-logical constants, and distinguish between a logic and a theory, but in the present language every theory is a logic.

³⁶For the notion of $\beta\eta$ -equivalence see, for instance, Bacon (forthcominga) chapter 3.

It is not a plausible principle: it implies that all operators are truth-functional, yet there appears to be plenty of genuine contingency. However, it is quite strong and implies many principles of higher-order logic that do appear to be desirable.³⁷ So the consistency of any of the above with Extensionalism implies the consistency with these principles.

We now proceed to establish the following theorem:

Theorem 4.1. $\text{HZ}(\text{LC})(\text{Ext})$ and $\text{HZ}^{\text{WO}}(\text{LC})(\text{Ext})$ are consistent relative to the consistency of $\text{ZFC} +$ “there are infinitely many inaccessible”.

Our strategy is to develop a fairly standard ‘Henkin model’ for these logics.³⁸ A Henkin model is a type of set-theoretic entity, so the metalanguage with which we will reason about these structures will be the language of first-order set theory (i.e. first-order logic with a single non-logical predicate \in) with the axioms and axiom schemas of first-order ZFC and the assumption that there are infinitely many inaccessible. It’s worth here pausing on the fact that in this section we do appear to operate with a distinguished relation \in , contrary to the philosophical vision being pursued. But this appearance can ultimately be dispensed with: once we have proven the consistency of, say, $\text{HZ}(\text{LC})(\text{Ext})$ in this theory, we may obtain a finitary proof of the conditional ‘if $\text{ZFC} +$ “there are infinitely many inaccessible” is consistent, then so is $\text{HZ}(\text{LC})(\text{Ext})$ ’.

We have argued above that HZ (and by extension any stronger logic) implies the negation of The Well-Ordering Principle, and of the higher-order axiom of choice. Thus the models we construct here must invalidate these higher-order choice principles. Nonetheless, our construction of a Henkin model invalidating these higher-order choice principles will be elementary. This is in stark contrast to the situation in standard set theory, where one must undertake much more involved model theoretic constructions in order to invalidate choice, such as Cohen’s method of forcing.³⁹

Let’s start with the promised notion of a Henkin model.

Definition 4.1 (Henkin Structure). *A Henkin structure D is a type indexed collection of sets, D^σ for each type σ , subject to the constraint:*

$$D^{(\sigma_1 \dots \sigma_n)} \subseteq P(D^{\sigma_1} \times \dots \times D^{\sigma_n})$$

³⁷See the principles discussed in section 2 of Bacon and Dorr (forthcoming).

³⁸Henkin (1950)

³⁹If it seems surprising that we can invalidate choice without forcing, it’s worth remembering that in the context of ZFCU—where we allow impure sets—we can construct models in which choice fails straightforwardly for sets containin urelements. See Fraenkle (1922).

A structure is full iff this inclusion is always an identity.

Define $\text{ff} = \emptyset$ and $\text{tt} := \{()\}$.

Note we could adopt a convention of identifying a unary product of D with itself, or else we $D^{(\sigma)}$ is a set of sets of singleton sequences from D^σ have to continually distinguish $\{a\}$ and $\{(a)\}$.

A variable assignment for a Henkin structure is a function, g , defined on variables mapping variables of type σ to elements of D^σ .

Definition 4.2 (Henkin Model). *A model is a pair $(D, \llbracket \cdot \rrbracket)$ where D is a Henkin structure and $\llbracket \cdot \rrbracket$ is a function taking a term of higher-order logic and a variable assignment as arguments such that*

- $\llbracket M \rrbracket^g \in D^\sigma$ for every term M of type σ .
- $\llbracket x \rrbracket^g = g(x)$ for every variable x .
- $\llbracket MN \rrbracket^g = \{(a_1, \dots, a_n) \mid (\llbracket N \rrbracket^g, a_1, \dots, a_n) \in \llbracket M \rrbracket^g\}$
- $\llbracket \lambda x.M \rrbracket^g = \{(a_1, \dots, a_n) \mid (a_2, \dots, a_n) \in \llbracket M \rrbracket^{g[x \mapsto a_1]}\}$
- $\llbracket \forall_\sigma \rrbracket^g = \{D^\sigma\}$
- $\llbracket \wedge \rrbracket^g = \{(\text{tt}, \text{tt})\}$
- $\llbracket \neg \rrbracket^g = \{(\text{ff})\}$

A sentence, A , is true in a model relative to g iff $\llbracket A \rrbracket^g = \text{tt}$

Henkin (1950) shows in essence that every sentence of $\mathbf{H}(\text{Ext})$ is true in every Henkin model.

Note that not every Henkin structure can be extended to a Henkin model. The structure may fail to contain the interpretations of the logical constants (the final three conditions) or be closed under the operations that correspond to application and λ -abstraction (the second and third). If the structure is closed under these operations, then constraints in definition 4.2 can be reinterpreted as an inductive *definition* of $\llbracket \cdot \rrbracket$. If a Henkin structure D can be extended to a model $(D, \llbracket \cdot \rrbracket)$ then that model is unique, so we will often simply refer to these Henkin structures as models.

Below is a method for constructing Henkin models.

Definition 4.3 (Permutations). *Let $\pi : D^e \rightarrow D^e$ be a permutation. π may be extended to arbitrary elements of the full premodel based on D^e as follows.*

- $\pi^e = \pi$

- $\pi^t = \text{id}$
- $\pi^{(\sigma_1 \dots \sigma_n)} R = \{(\pi a_1, \dots, \pi a_n) \mid (a_1, \dots, a_n) \in R\}$

An element $a \in D^\sigma$ is fixed by π iff $\pi a = a$

By a straightforward induction, one can show that $(\pi^{-1})^\sigma$ is an inverse of π^σ , so that π^σ is a permutation of D^σ for each type σ . Since there is no difference between $(\pi^{-1})^\sigma$ and $(\pi^\sigma)^{-1}$, will henceforth omit superscripts from permutations, letting them be determined by context. It follows that a is fixed by π iff it is fixed by π^{-1} , since if $\pi a = a$ then $\pi^{-1}\pi a = \pi^{-1}a$ and so $a = \pi^{-1}a$.

A full Henkin model equipped with the full set of permutations is a substitution structure in the sense of Bacon (2019). Substitution structures admit the following useful notions:

Definition 4.4 (Metaphysical definability). *Let D be a full Henkin structure. Let $X \subseteq \bigcup_\sigma D^\sigma$ be some collection of relations. Say that $a \in D^\sigma$ is metaphysically defined from X , or ‘fixed by X ’, iff every permutation that fixes every element of X also fixes a .*

Definition 4.5 (Directness). *Let D be a full Henkin structure. Say that $X \subseteq \bigcup_\sigma D^\sigma$ is directed iff, for any R and S in X , there exists at $T \in X$ such that T fixes R and S .*

The next result tells us that from any full Henkin structure, D , and any directed collection of its elements, X , we can form another Henkin structure D/X that can support a model. In fact, the application for which we need this theorem, the Henkin model D/X can be described a bit more simply. However, the general technique is very useful for generating models with various properties where second-order choice fails, so I state the more general theorem.

Definition 4.6. *Let D be a full Henkin structure, and X a directed collection of elements from D . Then the structure D/X is defined by setting $(D/X)^\sigma := \{a \in D_0^\sigma \mid a \text{ is fixed by some element of } X\}$.*

Theorem 4.2. *D/X is a Henkin model.*

Proof. We will show by induction that for every term M , $\llbracket M \rrbracket^g$ is defined for every assignment g , and that there exist a $R \in X$ such that for every π fixing R and every assignment g , $\llbracket M \rrbracket^{\pi \circ g} = \pi(\llbracket M \rrbracket^g)$.⁴⁰

⁴⁰A little more precisely, we are showing by induction on complexity that there is a partial function satisfying the clauses of definition 4.2 for all expressions of that complexity. The union of these partial functions clearly satisfies the conditions for all expressions.

It is easily checked that $\llbracket \forall_\sigma \rrbracket$, $\llbracket \wedge \rrbracket$ and $\llbracket \neg \rrbracket$ are all fixed by every permutation and are independent of the assignment. Clearly for variables, $\pi \llbracket x \rrbracket^g = \pi(g(x)) = \llbracket x \rrbracket^{\pi \circ g}$. It remains to show that $\llbracket \cdot \rrbracket$ can be extended inductively to application terms and λ -terms.

Let us suppose that M and N satisfy the inductive hypothesis, witnessed respectively by R and S in X . Suppose that T m-defines both R and S , so that for any π fixing T , $\llbracket M \rrbracket^{\pi \circ g} = \pi(\llbracket M \rrbracket^g)$ and $\llbracket N \rrbracket^{\pi \circ g} = \pi(\llbracket N \rrbracket^g)$. We will show that if π fixes T , then for every assignment g , $\llbracket MN \rrbracket^{\pi \circ g} = \pi(\llbracket MN \rrbracket^g)$. That is, we must show $\{(a_1, \dots, a_n) \mid (\llbracket N \rrbracket^{\pi \circ g}, a_1, \dots, a_n) \in \llbracket M \rrbracket^{\pi \circ g}\} = \{(\pi a_1, \dots, \pi a_n) \mid (\llbracket N \rrbracket^g, a_1, \dots, a_n) \in \llbracket M \rrbracket^g\}$. We begin with the right to left inclusion. Any tuple in the right-hand-side is of the form $(\pi a_1, \dots, \pi a_n)$ where $(\llbracket N \rrbracket^g, a_1, \dots, a_n) \in \llbracket M \rrbracket^g$. Then by the way $\pi \llbracket M \rrbracket^g$ is defined, $(\pi \llbracket N \rrbracket^g, \pi a_1, \dots, \pi a_n) \in \pi \llbracket M \rrbracket^g$. Since $\pi \llbracket M \rrbracket^g = \llbracket M \rrbracket^{\pi \circ g}$ and $\pi \llbracket N \rrbracket^g = \llbracket N \rrbracket^{\pi \circ g}$, we have that $(\llbracket N \rrbracket^{\pi \circ g}, \pi a_1, \dots, \pi a_n) \in \llbracket M \rrbracket^{\pi \circ g}$ giving us the right-to-left inclusion. For the other inclusion, we may use the previously noted fact that π^{-1} also fixes T , so that we may apply the inductive hypothesis, using π^{-1} as the permutation, and $\pi \circ g$ as the assignment, to obtain the identities $\llbracket M \rrbracket^g = \pi^{-1} \llbracket M \rrbracket^{\pi \circ g}$ and $\llbracket N \rrbracket^g = \pi^{-1} \llbracket N \rrbracket^{\pi \circ g}$. Now we reason as before: if $(\llbracket N \rrbracket^{\pi \circ g}, a_1, \dots, a_n) \in \llbracket M \rrbracket^{\pi \circ g}$, then $(\pi^{-1} \llbracket N \rrbracket^{\pi \circ g}, \pi^{-1} a_1, \dots, \pi^{-1} a_n) \in \pi^{-1} \llbracket M \rrbracket^{\pi \circ g}$, and using the two identities we obtained from the inductive hypothesis, $(\llbracket N \rrbracket^g, \pi^{-1} a_1, \dots, \pi^{-1} a_n) \in \llbracket M \rrbracket^g$. So $(\pi \pi^{-1} a_1, \dots, \pi \pi^{-1} a_n)$, that is (a_1, \dots, a_n) , belongs to the right-hand-side.

We may now show that $\llbracket MN \rrbracket^g$ is defined for every assignment g — i.e. that the third clause from definition 4.2 defines an element of D . Let g be an arbitrary assignment. Using directedness find an $R \in X$ that m-defines T and $g(x)$ for every x appearing in MN . Now by the above $\llbracket MN \rrbracket^{\pi \circ g} = \pi \llbracket MN \rrbracket^g$. But $\llbracket MN \rrbracket^{\pi \circ g} = \llbracket MN \rrbracket^g$ since $\pi g(x) = g(x)$ for every x appearing in MN .

Now suppose the inductive hypothesis holds for M . So there is some $T \in X$ such that that for every permutation π fixing T and every assignment g , $\llbracket M \rrbracket^{\pi \circ g} = \pi \llbracket M \rrbracket^g$. We will show that for every π fixing T and assignment g , $\llbracket \lambda x.M \rrbracket^{\pi \circ g} = \pi \llbracket \lambda x.M \rrbracket^g$. If a tuple is in the right-hand-side it is of the form $(\pi a_1, \dots, \pi a_n)$ where $(a_2, \dots, a_n) \in \llbracket M \rrbracket^{g[x \mapsto a_1]}$. So as before $(\pi a_2, \dots, \pi a_n) \in \pi \llbracket M \rrbracket^{g[x \mapsto a_1]}$ which $= \llbracket M \rrbracket^{\pi \circ (g[x \mapsto a_1])}$ by the inductive hypothesis, which $= \llbracket M \rrbracket^{(\pi \circ g)[x \mapsto \pi a_1]}$. So $(\pi a_1, \dots, \pi a_n) \in \llbracket \lambda x.M \rrbracket^{\pi \circ g}$ as required. As before we may also reverse this reasoning, by using the fact that π^{-1} also fixes T .

The argument that $\llbracket \lambda x.M \rrbracket^g$ is well-defined is identical to the argument for MN . □

We now describe two examples that can be used to generate models of

HZ and HZ^{WO} . Let κ be a limit of inaccessibles. Let D be the full Henkin structure obtained by setting $D = \kappa$.

Example 4.1. We let $<_\alpha$ be the ordering of the ordinals restricted to the ordinal $\alpha < \kappa$. The set $X_1 = \{<_\alpha \mid \alpha < \kappa\}$ forms a directed set, since $<_\alpha$ m -defines $<_\beta$ whenever $\alpha \geq \beta$.

For the second example we let $D = V_\kappa$ (in fact this same domain could be used in the first example).

Example 4.2. Let \in_α be the membership relation restricted to the sets of rank α (i.e. $\in \cap V_\alpha$). $X_2 = \{\in_\alpha \mid \alpha < \kappa\}$ is directed, since \in_α m -defines

Theorem 4.3. For $i = 1, 2$, a relation $R \subseteq D^{\sigma_1} \times \dots \times D^{\sigma_n}$ is in $(D/X_i)^{\sigma_1 \times \dots \times \sigma_n}$ iff, for some $\alpha < \kappa$, every permutation that is the identity restricted to α (resp. V_α) fixes R .

Proof. A permutation π fixes $<_\alpha$ iff $\pi \upharpoonright_\alpha$ is the identity. This is because there are no non-trivial automorphisms of well-orders. Similarly, π fixes \in_α iff $\pi \upharpoonright_\alpha$ is the identity, because there are no non-trivial automorphisms of V_α .

We prove the latter by \in -induction. Assume that $\pi y = y$ for all $y \in x$. The members of πx are of the form πy for $y \in x$, so $\pi x = x$ by extensionality. The former can be proved similarly by transfinite induction. \square

It follows that the models obtained from X_1 and X_2 are essentially the same. In fact, given choice in the metalanguage κ and V_κ have the same size.

It will be convenient in what follows to say that an element of D/X_1 (or D/X_2) is ‘pinned down’ by λ iff every permutation that is identity on λ (V_λ respectively) fixes that element.

Theorem 4.4. $M = (D/X_1, \llbracket \cdot \rrbracket)$ is a model of $\text{HZ}^{\text{WO}}(\text{LC})(\text{Ext})$. $M' = (D/X_2, \llbracket \cdot \rrbracket)$ is a model of $\text{HZ}(\text{LC})(\text{Ext})$.

Proof. As noted $\text{H}(\text{Ext})$ is validated in any Henkin model. It remains to show Progress, Completion and Local Choice are true in M . We treat these in order.

Progress: Suppose that $R \in (D/X_1)^{(e,e)}$ is a (well-order of inaccessible order type) M . Then R must in fact be a well-order (indeed an inaccessible well-order), for there must exist some $\alpha < \kappa$ such that R is fixed by every permutation that is the identity on α . Thus $\text{Dom}(R) \subseteq \alpha$. Moreover, every subset of α is in $(D/X_1)^{(e)}$ for a similar reason, so that the second-order quantifiers in the claim that R is a (well-order of inaccessible order type) M are essentially unrestricted, so R is in fact a well-order of inaccessible order type. Since κ is a limit of inaccessibles, there is an inaccessible, $\lambda < \kappa$, of greater

order type than R , and using choice we may pick an $R' \subseteq \kappa \times \kappa$ containing R with $Dom(R') \subseteq \lambda$ such that R has order type λ . R' is pinned down by λ because it is a subset of λ , so R' is in $(D/X_1)^{(e,e)}$. Moreover, as with R , we can see that R' (well-order of inaccessible order type) ^{M} iff it is a well-order of inaccessible order type (which it is). Thus we have shown that any inaccessible well-order of M is a proper initial segment of another inaccessible well-order of M . So M is a model of $\text{Progress}^{\text{WO}}$.

Completion: Now suppose that $X \in (D/X_1)^{(e,e)}$ is a set of (well-orders) ^{M} that are (linearly ordered by the initial segment relation) ^{M} , and that X is pinned down by λ . We show that for every $R \in X$, $dom(R) \subseteq \lambda$. Suppose not, and $a \in dom(R) \setminus \lambda$. Let π be a transposition that fixes λ and swaps a with some element b also outside of λ . Since π fixes X and $R \in X$, $\pi R \in X$, and since X is linearly ordered then either πR is an initial segment of R or R is an initial segment of πR . Thus $dom(R) \subseteq dom(\pi R)$ or $dom(\pi R) \subseteq dom(R)$. Of course, $a \in dom(R)$ and $\pi a = b \in dom(\pi R)$, so that either a and b both belong to $dom(R)$ or to $dom(\pi R)$. Without loss of generality suppose the former. Since R is linear either $(a, b) \in R$ or $(b, a) \in R$, in which case $(b, a) \in \pi R$ or $(a, b) \in \pi R$ respectively, and either case is impossible given than one is an initial segment of the other (and both are asymmetric orders).

So $\bigcup X \subseteq \lambda$, and is a well-order. Since $\lambda < \kappa$ there is an inaccessible, γ , between λ and κ and we may extend $\bigcup X$ to a well-order, S , of inaccessible order-type whose domain is γ and is thus pinned down by γ .

Local Choice: Suppose that $R \in (D/X_1)^{(e,e)}$ is a ZF^M relation, and is pinned down by $\lambda < \kappa$. By the same sort of reasoning the domain of R is contained in λ , and thus R is a $\text{ZF}(C)$ relation iff it is a $\text{ZF}(C)^M$ relation, by the fact that the second-order quantifiers in M range over all subsets of the domain of R . Since R is thus a ZF relation, it is isomorphic to V_γ for some inaccessible γ by Zermelo's theorem (Zermelo (1930)), and since V_γ is a ZFC relation (by the axiom of choice), R is a ZFC relation too, and finally a ZFC^M relation.

The proof $(D/X_2, [\cdot])$ is a model of $\text{HZ}(\text{LC})(\text{Ext})$ is essentially the same so I do repeat it here. □

5 Can the First-order Reflect the Higher-order?

The principles formulated and discussed in the previous sections are formulated in pure higher-order logic, and do not concern any sort of abstract mathematical objects. Cantor, Frege and many others since have posited special abstract individuals that, to some extent, reflect the structure of the higher-order and

constitute the subject matter of mathematics. For Cantor these included transfinite numbers and cardinals which are abstracted from well-orders and sets respectively. But the more general question is: to what extent can special purpose mathematical objects represent the structure of the higher-order?

In contemporary philosophy of mathematics this question has most commonly taken the form ‘when do some things form a set?’, but the general issue is of longstanding significance. According to our analysis, Aristotle’s rejection of completed infinities is an instance of this general issue—the higher-order claim $\exists_{(ee)} R \text{PotInf } R$ entails the existence of an infinite series of individuals standing in the part-whole relation to one another, but not of any single infinite individual reflecting in its part-whole structure of the higher-order entity $<_p$. Cantor too was preoccupied with the question of when ‘a many can be thought of as one’—when a higher-order property corresponds to a single individual, a set, when a higher-order well-order can be assigned an individual representing its order-type, and so on.⁴¹ We found Cantor’s answer to the set formation question—some things form a set when they can be *enumerated*—to be circular if by ‘enumerated’ we mean well-ordered and assigned a special sort of individual, a transfinite number, to represent its order-type. While this claim plausibly *comes out* true in the standard framework of ZF—relative to a particular membership relation—given the availability of other ZF relations that answer the set-formation question differently, we are still lacking an explanation for why some properties define a set, and not others.⁴² Other ZF relations may count more or fewer properties as defining a set, and we have no guarantee that there is a single ‘maximal’ ZF relation that counts as set-making those properties defining a set according to some ZF relation or other.⁴³ In section 3 we considered a more straightforward interpretation of ‘enumerable’, found in Cantor’s earlier writings: being well-orderable. According to this theory some things form a set when one can list them, in a well-ordered fashion, in such a way that every element will appear somewhere in the list—every potential infinity can be considered as a finished thing. This answer seems more in

⁴¹‘By an aggregate we are to understand any collection into a whole M of definite and separate objects of our intuition or of our thought.’ §1 p85 of Cantor (1915). Or, similarly, ‘many, which can be thought of as one, i.e., a totality of definite elements that can be combined into a whole by a law.’ (translation from Boolos (1971) p215).

⁴²In second-order ZF, if some pure sets, X , are well-ordered by R which is isomorphic to the ordinal $\leq \alpha$, then there is a one-to-one correspondence, S , between the ordinals less than or equal to α and X ; since the former is a set, the axiom of replacement lets us infer that the range of S is too. One has to be a little careful here, since the standard version of ZFC doesn’t posit any impure sets, so that even singletons of non-sets will fail to form sets. Here I simply interpret the question of when some pure sets form a single set.

⁴³Even assuming higher-order choice.

keeping with the interpretation of the absolutely infinite as not well-orderable suggested at the end of section 3. But unfortunately the resulting set theory is also inconsistent. The argument is not entirely obvious, so I relegate it to an appendix, but roughly speaking the von Neumann ordinals are provably well-ordered in this set theory and we can derive the Burali-Forti paradox.

Let us then consider a different idea: we do not need to posit special purpose individuals to represent the higher-order and to be the subject matter of mathematics, we can simply reason about the higher-order directly and then find some way to interpret mathematics in higher-order logic, by replacing each mathematical statement with a suitable sentence of pure higher-order logic. The most flatfooted account would be to paraphrase a mathematical statement A —let’s say, a sentence of ZF—with the purely logical sentence $\forall_{(ee)}R(\text{ZF } R \rightarrow A[R/\in])$.⁴⁴

If this is to be viable, our higher-order logic must at least contain the principle that there’s at least one ZF relation, for otherwise these paraphrases would be vacuously true (and thus, so would their negations!). There’s one sense in which this higher-order commitment is quarantined from the first-order realm. The resulting logic is conservative over the logical sentences of first-order logic that we already had (non-mathematical) reason to believe in: apart from the theorems of classical first-order logic, it implies, for each n , the claim that there are at least n things, $\exists_n x x = x$, which we already had reason to believe in—there are at least n space-time regions. But once we move beyond purely logical matters, and start asking questions involving non-logical predicates we confront many awkward questions. For instance, once we have renounced special purpose abstract objects to reflect the higher-order, shouldn’t (or at least, couldn’t) everything be concrete? But $\exists_{(ee)}R \text{ZF } R$ implies that there are far more individuals than, for instance, regions of space-time, according to standard theories of space-time.⁴⁵ Arguably there must be more individuals than there are concrete things more generally.⁴⁶ And so, at least in this respect, the higher-order perspective is in the same boat as the standard view about mathematical objects—they exist, and there are lots of them. There are, however, some important differences. One central difficulty for standard platonism concerns how we secure reference to particular mathematical ob-

⁴⁴ZF \in guarantees $\exists_{(ee)}R \text{ZF } R$, and Zermelo’s quasi-categoricity theorem ensures the paraphrase always has the same truth value as the original sentence.

⁴⁵‘Far more than’ is a notion which can be spelled out in terms of higher-order quantification; see footnote 24.

⁴⁶Note that certain plenitudinous views about material constitution do posit enough concrete individuals for the truth of $\exists_{(ee)}R \text{ZF } R$. See for instance Dorr et al. (2021).

jects, like \emptyset , when we do not have any sort of causal contact with them.⁴⁷ The present view posits lots of mathematical objects, but is compatible with the view that these objects are all indistinguishable from one another and cannot be referred to uniquely (except, perhaps, by radically indeterminate names). Any mathematical role that can be played by any one of these individuals can be played by any other.

Nonetheless, once we have granted that there is at least one ZF relation a host of further question seem to dangle. The mere existential is compatible with there being there being, say, exactly five ZF relations up to isomorphism, or with there being some other number. Questions like these arise because it seems there must be a brute fact about how many mathematical objects there are. And these questions seem awkward because they have a feeling of arbitrariness to them: if the number of individuals is the fifth inaccessible, one might wonder why it wasn't the fourth, or the sixth? Corresponding questions about physical objects feel less troublesome—there the physical sciences offer guidance, and the answers are at any rate contingent so there are no apparent brute necessities concerning how many things there are.

The Zermelian logics we have developed in this paper seem well placed to remove these dangling questions, not by answering them but rejecting their presuppositions. The ZF relations are indefinitely extendible, so there is no question of a biggest ZF relation. And there is no answer to the question 'how many things are there' when the universe is not well-orderable. We can still make comparisons of size between the universe and other things, but the 'size' of the universe does not occupy some arbitrary position on a linear scale, like the ordinals.⁴⁸

Another option would be to adopt a modal paraphrase of mathematical statements, as championed in Hellman (1989).⁴⁹ Now a ZF sentence A can be paraphrased as $\Box\forall_{(ee)}R(\text{ZF } R \rightarrow A[R/\in])$. Here we no longer have to posit *any* distinctively mathematical objects, or brute facts about how many mathematical individuals there are. We only need to posit the possibility of

⁴⁷Benacerraf (1965).

⁴⁸There is another sense in which the question 'how many things are there' might not have an answer: the cardinality of the universe may not be expressible in pure higher-order logic (after all, only countably many cardinalities can be expressed in a countable language). But this does not mean that the question has no answer—it just means that one cannot express it in a certain language. If the universe was well-orderable, then the question could be answered in more expressive languages containing primitive cardinality quantifiers for the size of the universe (and if the universe wasn't well-orderable then such a cardinality quantifier couldn't be introduced semantically, and such languages would presumably not exist).

⁴⁹Following Putnam (1967).

a sufficient number of things, concrete or otherwise. Hellman suggests that the \Box here should be interpreted as a *logical* modality. (Understanding the possibility in this claim in terms of Kripke’s notion of ‘metaphysical’ possibility introduces a number of distracting questions that I’d rather circumvent—needless to say, the matter is more fraught.) Taking the linguistic notion of logical consistency as our guide to logical possibility, the assumption that there could have been a ZF relation is modest. The consistency of first-order ZF is certainly still an assumption here, but it is completely uncontentious among set theorists. More contentious is the idea that there is a propositional notion of logical possibility at all. While we have several reasonable accounts of the notion of a logically consistent sentence, some might argue that there is nothing that stands to reality as logical consistency stands to language. While I myself agree with the general concern that one must tread carefully when introducing a propositional notion in the vicinity of a linguistic one, I believe that the notion of logical necessity *can* be put in good standing in higher-order logic, with suitable axioms ensuring that the operator notion behaves like the logical one. It would take me too far afield to develop a proper defence of this here.⁵⁰ It is worth noting, however, that in certain logics for logical necessity, the claim $\Diamond\exists_{(ee)}RZF R$ is in fact a theorem, and so the non-vacuity of mathematical claims fall out of the logic of logical necessity rather than as a special mathematical posit.⁵¹

⁵⁰I have undertaken this elsewhere [ANON].

⁵¹In [ANON] the authors consider adding to a certain higher-order logic, Classicism, a schema that would have $\Diamond\exists_{(ee)}RZF R$ as an instance if first-order ZFC were consistent (as widely assumed). At present my favoured theory of logical necessity is a schema containing all sentences of the form $\Diamond A$ where A is a closed sentences of higher-order logic that has a certain sort of set theoretic model. There is a concern here that to specify the instances of the schema one has to already assume a platonistic set-theory, or at least the non-vacuity of a suitable higher-order paraphrase of a platonistic set theory. Perhaps that is the order of understanding needed for a contemporary logician already familiar with ZFC to arrive at the statement of the schema, but it need not be the order of things in any deeper sense. Zermelo once objected to Skolem’s reformulation of his single higher-order separation axiom as a first-order schema on the grounds that to know what a wff, and thus an instance, is, one has to already have the notion of a finite number in order to know which sentences are formable from the constants by a finite number of applications of the formation rules, and this was not available prior to understanding set-theory itself. These days, few would agree with Zermelo that first-order ZFC (or ever first-order Peano arithmetic) is an inadequate foundation for the theory of finite numbers on these grounds.

6 Appendix: The Inconsistency of a Cantorian Criteria of Set Formation

In this appendix I show that the theory one gets by formalizing Cantor's account of set formation is subject to the Burali-Forti paradox. However here the argument is somewhat less obvious, so it should be presented in detail. The theory inspired by the *Grundlagen* may be axiomatized as follows

Extensionality $\forall_e xy(\forall_e z(z \in x \leftrightarrow z \in y) \rightarrow x =_e y)$.

Well-Ordered Comprehension $\forall_{(ee)} R(\text{WO } R \rightarrow \exists_e x \forall_e y(y \in x \leftrightarrow \text{Dom } Ry)$

Let us define a *von Neumann* ordinal as a transitive set that is well-ordered by \in .⁵²

$$\text{Ord } \alpha := (\forall x(x \in \alpha \rightarrow x \subseteq \alpha) \wedge \text{WO } \lambda xy(x \in y \wedge y \in \alpha))$$

We can then show that the von Neumann ordinals are well-ordered by \in , and thus form a set by Well-Ordered Comprehension. The argument that the von Neumann ordinals are well-ordered is not at all new, but it needs to be checked that it can be carried out in the present set theory.

We begin by showing that von Neumann ordinals are linearly ordered. Let Lemma (a) be the claim that if α and β are ordinals and α is a proper subset of β then $\alpha \in \beta$.⁵³ let Lemma (b) be the claim that if α and β are ordinals then the set of things belonging to both, $\alpha \cap \beta$, exists and is an ordinal.⁵⁴

We now see that von Neumann ordinals linearly ordered, for suppose that α and β are ordinals, and $\alpha \neq \beta$. So $\alpha \cap \beta$ is an ordinal by Lemma (b), and is a proper subset of α or of β . Without loss of generality, we assume the former. Then by Lemma (a) $\alpha \cap \beta \in \alpha$. Now $\alpha \cap \beta$ cannot also be a proper subset of β .

⁵²von Neumann (1923).

⁵³It is proved by noting that by Extensionality there is at least one member of β not in α , and so there must be a least such element, x , under membership since β is well-ordered. If $y \in x$ then $y \in \beta$, since β is transitive, and so $y \in \alpha$ or else x would not be the \in -least element of β not in α . Conversely if $y \in \alpha$ then $y \in \beta$, since $\alpha \subseteq \beta$. Since β is linearly ordered by \in , either $y = x$, $x \in y$ or $y \in x$. y can't be the same as x , since $x \notin \alpha$. Nor can x belong to y , because otherwise x would again belong to α by the transitivity of α and the fact that y belongs to α . Thus $y \in x$. So x and α have the same elements, and are identical by Extensionality. Since $x \in \beta$, $\alpha \in \beta$.

⁵⁴If all X s belong to both α and β , then there is an \in -least X in α , since α is well-ordered by \in . Similarly, if x and y belong to both α and β , then either $x = y$, $x \in y$ or $y \in x$ by the fact that α is well-ordered. So by Well-Ordered Comprehension, there is a set of things belonging to both α and β . It is of course well-ordered by \in , as we have just seen. And it is transitive, by the transitivity of both α and β , so $\alpha \cap \beta$ is an ordinal.

For otherwise, by Lemma (a) it is an element of β , and it is already an element of α , in which case $\alpha \cap \beta \in \alpha \cap \beta$ contradicting the fact that the elements of α (which includes $\alpha \cap \beta$) are well-ordered by membership. So $\beta \subseteq \alpha \cap \beta$ — the other inclusion is clear, so $\beta = \alpha \cap \beta \in \alpha$. In the case that $\alpha \cap \beta$ is a proper subset of β we reason analogously, and conclude $\alpha \in \beta$. So Ord is linearly ordered.

Von Neumann ordinals are also well-ordered by \in . Suppose that all X s are ordinals and α is X . If α is not already the \in -least X , then there is at least one $\beta \in \alpha$ that is X , and so a \in -least $\beta \in \alpha$ that is X . If γ is also an X ordinal, then $\gamma \notin \beta$, for otherwise $\gamma \in \alpha$ by the transitivity of α , contradicting the assumption that β was the \in -least element of α that was X . So either $\gamma = \beta$ or $\beta \in \gamma$, by the fact that \in linearly orders the X s.

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