

## NATURAL TRANSFORMATIONS OF ORGANISMIC STRUCTURES

■ ION C. BAIANU  
Cavendish Laboratory,  
University of Cambridge,  
England

The mathematical structures underlying the theories of organismic sets,  $(M, R)$ -systems and molecular sets are shown to be transformed naturally within the theory of categories and functors. Their natural transformations allow the comparison of distinct entities, as well as the modelling of dynamics in “organismic” structures.

Organismic sets were built by Nicolas Rashevsky into a representation of biological organisms and societies on a relational basis (Rashevsky, 1954; 1960; 1966a,b; 1967a,b,c; 1968a,b,c; 1969a,b,c; 1970a,b,c; 1971a,b,c), and a wide range of biological and social phenomena were explained within this framework. Attempts to produce axioms for this theory using logical functors and categories (as well as an extension to more general structures), were made with the purpose of analysing mathematically the basic underlying structures (Baianu and Marinescu, 1968; Baianu, 1970; 1971a,b; 1973a). Amongst the direct consequences of our axioms were the realisation that with each organismic set there is an associated algebraic theory,  $S$ , which defines the organismic set in terms of its observable quantities. The algebraic properties of these “observables” were then circumscribed by a general theorem (p. 552 of Baianu, 1970). Furthermore, these structures were shown to be *multistable*, in a precise sense [D8, D9 and equations (16) in *loc. cit.*], without the need for assuming a “potential” function. This may possess certain advantages for obtaining general, multistability conditions (Propositions 1 to 3 in *loc. cit.*), although it was shown by Rashevsky (1971a) that the two descriptions are *topologically* equivalent. The difference arises from the fact that the “potential” description of stability and our observable-related definition are not *algebraically* equivalent. We shall investigate in some detail the conditions under

which organismic sets and their observable structures are transformed naturally, thus paving the way for a direct, structural approach to their variability and dynamics. The necessary mathematical constructions were partly introduced in previous work on dynamical similarity between systems (Baianu, 1971b; Baianu, 1972; Baianu and Scripcariu, 1973) and in conjunction with a functorial representation of  $(M, R)$ -systems (Baianu and Marinescu, 1974).

$(M, R)$ -systems were introduced by Robert Rosen (1958a,b) as simple mathematical models of metabolic and genetic systems in cells. An  $(M, R)$ -system comprises a "metabolic" part,  $M$ , and a "replicating" part,  $R$ . The components of  $R$  produce replicas of the corresponding  $M$ -components if adequate inputs are provided. The partial equivalence between  $(M, R)$ -systems and automata, or "sequential machines", was then shown (Rosen, 1964a) and the general, algebraic properties of  $(M, R)$ -systems were derived (Rosen, 1966a,b; Arbib, 1966; Baianu, 1973). Their topological properties (such as connectedness and dynamical realizations) were investigated together with the biological implications of these mathematical constructions (Rosen, 1961; 1963a,b; 1964b; 1966b; 1971; 1973). The dynamical realizations of  $(M, R)$ -systems led directly to the question of their state-space transformations and revealed their "evolutionary potentialities" (Rosen, 1973). Closely related to this important problem, we had already described a functorial representation of  $(M, R)$ -systems (completed as early as 1971) and considered their natural transformations (Baianu and Marinescu, 1974; submitted in final form in 1971). In a subsequent paper on dynamical similarity (Baianu and Scripcariu, 1973; generated in relation to Rosen's work on analogous systems (1968)), we have modelled the development of organismic sets in connection with relational, nuclear transplant experiments and suggested that natural transformations are operational both in the theory of organismic sets and the theory of  $(M, R)$ -systems (p. 481, *loc. cit.*). Therefore, we shall also discuss the role of natural transformations in the latter theory and investigate the possible reasons for this common feature. It turns out that molecular set theory (Bartholomay, 1960; 1965; 1971), in its initial form, also has scope for the application of natural transformations, thereby suggesting that these originate at the molecular level.

1. *Natural Transformations in the Theory of Organismic Sets.* Let  $e_i$  be an element of an organismic set  $S_0$ . Then,  $e_i$  performs a set of activities  $S_i^{(a)}$  which result in a set of products  $S_i^{(p)}$ . If there are  $N$  elements in the organismic set  $S_0$ , there will be a total set of activities  $S^{(a)} = \{S_i^{(a)} | i = 1, 2, \dots, N\}$  which may result in a total set of products  $S^{(p)} = \{S_i^{(p)} | i$

$= 1, 2, \dots, N\}$ . Therefore, we can consider the one-to-one mapping

$$\theta: S_0 \rightarrow S^{(a)} \times S^{(p)}$$

which assigns to each  $e_i$  the sets  $S_i^{(a)}$  and  $S_i^{(p)}$  of its activities and products, respectively. However, since at any given moment only a certain subset of  $S_i^{(a)}$  and  $S_i^{(p)}$  are actually realized by the element  $e_i$ , and since there will be a certain order in the performance of its activities, one has to consider cartesian products of the sets  $S_i^{(a)}$  and  $S_i^{(p)}$ :

$$S_i^{(a)} X S_i^{(p)} = S_{ap}^{i, 11},$$

$$S_i^{(a)} X S_i^{(a)} X S_i^{(p)} X S_i^{(p)} = S_{ap}^{i, 22}, \dots,$$

$$\underbrace{(S_i^{(a)} X S_i^{(a)} X \dots)}_{m\text{-times}} X \underbrace{(S_i^{(p)} X \dots X S_i^{(p)})}_{r\text{-times}} = S_{ap}^{i, mr}.$$

With these cartesian products of sets, an *algebraic theory*  $S$  emerges. Its objects are  $(\phi, \phi) = 1, S_{ap} = S^{(a)} X S^{(p)}, \dots, S_{ap}^{mr}$  and its morphisms are  $1 \rightarrow S_{ap}, \dots, S_{ap}^{mr} \rightarrow S_{ap}$ . If the organismic set  $S_0$  is considered as a discrete category (whose only morphisms are  $e_i \xrightarrow{1e_i} e_i$ , for  $i = 1, 2, \dots, N$ ), then there will be some functors,  $S_0 \xrightarrow{F} S$  which associate to each element  $e_i$ , at each moment, its activities and the products made as a result of these activities.

The functors  $F$  play the significant role of *observables* of the organismic set  $S_0$ . One can readily find that, within the limits of our construction, there are some covariant functors  $F^{-1}: S \rightarrow S_0$  which perform the reverse formal action, by associating to an ordered product of sets of activities and biological products an abstract element  $e_i \in S_0$ . Therefore, these functors  $F^{-1}$  define the organismic set  $S_0$  in experimental, or quantitative, terms.

The measurements are carried out on  $S$  and the inferences drawn from these are transferred to  $S_0$ . It is, therefore, natural to consider the possible relationships between the actions of the functors  $F^{-1}$  and their possible transformations, either as a result of possible changes in the dynamics of the biological system represented by  $S_0$ , or as a result of changing the measurement operations—underlying the functors  $F^{-1}$ . More precisely, the category of functors  $S \rightarrow S_0$  (which commute with products) and natural transformations between these functors, is the *associated algebraic* category of  $S$  and is denoted by  $S_0^S$ .

The physical, or “chemical” measurements on  $S_0$  are, then, abstracted in

the following commutative diagram

$$\begin{array}{ccc}
 & S & \\
 F \nearrow & & \searrow \psi \\
 S_0 & \xrightarrow{\chi} & R,
 \end{array} \tag{1}$$

where  $R$  is the algebraic theory whose objects are  $1, R, R^2 = R \times R, \dots, R^n = R \times R \times \dots \times R$  ( $n$ -times), with  $R$  being the set of real numbers (further details of this construction are given on p. 551 of Baianu, 1970). The functors  $\chi: S_0 \rightarrow R$ , play the role of *general observables* (as shown on p. 552 of Baianu, 1970) and are *natural* in the sense of Eilenberg and MacLane (1945), that is, condition (1) holds for *any* functor  $F$  defined as above, and *any* categories  $S_0, S$  and  $R$  of the type discussed here.

The main algebraic properties of  $S_0^S$  were previously found (the theorem on p. 552 of Baianu, 1970), and these stem from the naturality of condition (1).

In addition to these properties, the functors  $F^{-1}: S \rightarrow S_0$  are also natural in the above sense, that is,

$$\begin{array}{ccc}
 & S & \\
 F^{-1} \swarrow & & \searrow \psi \\
 S_0 & \xrightarrow{\chi} & R
 \end{array} \tag{2}$$

is also commutative for *any*  $F^{-1}, \psi, \chi, S_0, S, R$  defined as above.

When the functors  $F^{-1}$  are varied under the action of *natural transformations* (for a definition see Eilenberg and MacLane, 1945, or D3 of Baianu and Marinescu, 1968), condition (2) is still satisfied by an appropriate choice of observables, and the induced functor is  $\psi = \chi_0 F^{-1}$ . Let us examine in detail the effect of natural transformations in the associated algebraic category of  $S$ , denoted here by  $S_0^S$ . If  $F^{-1}, G^{-1}: S \rightarrow S_0$  are two objects in  $S_0^S$  and  $\eta: F^{-1} \rightarrow G^{-1}$  is a *natural transformation* in  $S_0^S$ , then, by definition, the diagrams of the following type are commutative:

$$\begin{array}{ccc}
 F^{-1}(S_{ap}^{mr}) & \xrightarrow{F^{-1}(f)} & F^{-1}(S_{ap}^1) \\
 \eta_{mr} \downarrow & & \downarrow \eta_1 \\
 G^{-1}(S_{ap}^{mr}) & \xrightarrow{G^{-1}(f)} & G^{-1}(S_{ap}^1)
 \end{array}, \tag{3}$$

for any  $f: S_{ap}^{mr} \rightarrow S_{ap}^1$  in the algebraic theory  $S$ . We note that the mappings  $\eta_{mr}, \eta_1, F^{-1}(f)$  and  $G^{-1}(f)$  are defined in the normal manner, and, therefore, operate *directly on elements of sets*. On the other hand, functors

(such as  $F^{-1}$  and  $G^{-1}$ ) operate *on whole sets*, without “looking inside”—at the elements of these sets. Thus, while functors are used to produce whole-set transformations, the natural transformations introduced above allow a direct comparison, element-by-element, of the results of the action of these functors. We can think of the functors  $D:S_0 \rightarrow S_0$  as *dynamic transformations* of the biological system represented by an organismic set  $S_0$ , and consider the natural transformations as a means of recovering the consequences of these dynamic transformations at the level of the elements of  $S_0$ . The actual observation, or measurement, of a dynamic transformation is naturally performed through the operations abstracted in diagrams (1) and (2).

The formalism developed here does *not* involve special, *physical* assumptions concerning the nature of these dynamic transformations, or the specific structure of the transformed organismic sets. This formalism requires, however, the introduction of “measuring systems”, or “meters” in diagram (1) (as defined in D9 on p. 62 of Comorosan and Baianu, 1969, or on p. 673 of Rosen, 1977). Therefore, it can be extended far beyond the framework of the present constructions (as suggested by Robert Rosen in a personal communication), into the theory of observations and system descriptions (Rosen, 1978). This possible extension will be considered in subsequent publications. The only underlying restrictions of this formalism are *logical* in their nature, as specified by the axioms of the elementary theory of abstract categories (Lawvere, 1963, Baianu, 1970). These restrictions can be relaxed even further in the form of the axioms of the elementary theory of “supercategories” (pp. 547–548 of Baianu, 1970), hence removing the necessity of strict commutativity, or naturality, as required by conditions (1) and (2), above. This *extension* may be required in a general theory of measurement (Rosen, 1978).

Using a “cost” function,  $K_i$ , of the activities performed by each element  $e_i$  of  $S_0$  in making the products essential for the survival of  $S_0$ , one can “optimize” the relational structure of  $S_0$ —in the sense of minimizing the total cost,  $K = \sum_{i=1}^N K_i$ . These “optimal” values are to be found among the solutions of the following system of equations:

$$\frac{\partial K_i(a_i, p_i; t)}{\partial a_i} \cdot \delta p_i = 0 \tag{4}$$

$$\frac{\partial K_i(a_i, p_i; t)}{\partial p_i} \cdot \delta a_i = 0 \tag{5}$$

for which actual products are made, that is,  $\delta p_i = 1$  (if no product is made

$\delta p_i = 0$ , and the “cost” function experiences a singularity). The boundary conditions which help to define the minima of  $K_i(a_i, p_i; t)$  are the restrictions imposed by the environment on  $S_0$ , as well as the conditions  $K_i \geq 0$  for  $i = 1, 2, \dots, N$ .

Alternatively, one can simply note that if each element  $e_i$  were completely unrelated to the other elements of  $S_0$  (“completely specialised” organismic set) this would increase the total cost function and make the existence of  $S_0$  improbable. Therefore, there must exist some relations between the elements of  $S_0$  which can decrease the cost function in comparison with the case considered above. This induces some relational “forces” which stabilize the organismic set  $S_0$ . Furthermore, Nicolas Rashevsky (1968c) conjectured that, in fact, the development of an organismic set leads to the maximum possible number of different relations being defined on  $S_0$ . That is, once the organismic set is formed, it can only develop by decreasing its “potential for transformation”  $V$ , as represented in a simplified picture in Figure 1.  $V$  can be more precisely defined as the

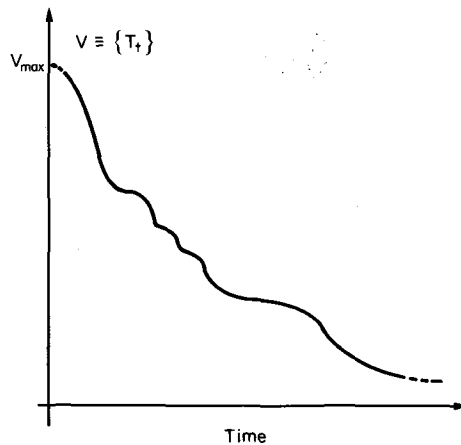


Figure 1

cardinal of the set of all possible transformations,  $\{T_t\}$ , at an instant  $t$ , of the relational structure of  $S_0$ . Since at least some of the transformations suffered by  $S_0$  are irreversible, this potential decreases as the organismic set develops. This formulation is rather similar to the well known “epigenetic landscape” metaphor, although the latter is directly linked to a true potential function.

The transformations of the relational structure of  $S_0$  during its development could be followed using natural equivalences and we showed that the variety of the relations defined on  $S_0$  is, indeed, maximised in the direct

limit of relational structures generated during the development of  $S_0$  (p. 482 and fig. 1 of Baianu and Scripcariu, 1973).

The natural transformations of the observables of an organismic set introduced above extend the previous algebraic approach beyond the limits imposed by the *nuclear equivalence* (p. 476 in *loc. cit.*). Consequently, we can develop an “evolutionary” theory of transformations of organismic sets, in the sense that we are able to compare by means of natural transformations *different* organismic sets and/or their possible developments. A quantitative approach was proposed above and this is only a preliminary attempt to set up numerical procedures in parallel with the theory of organismic sets. The subsequent development of the theory will also have to consider the consequences of “fuzziness” (Baianu and Marinescu, 1968; Baianu, 1980) for the transformations of organismic sets.

2. *Natural Transformations in the Theory of (M, R)-Systems.* Let us consider the simplest (M, R)-system

$$A \xrightarrow{f} B \xrightarrow{\Phi_f} H(A, B) \xrightarrow{\beta_b} H[B, H(A, B)] \tag{6}$$

where  $A$  is the set of inputs of the metabolic component  $M$ ,  $B$  is the set of outputs of the metabolic component  $M$ ,  $f$  is the mapping from  $A$  to  $B$  representing the activity of  $M$  and  $H(A, B)$  is the set of all mappings from  $A$  to  $B$ .  $H[B, H(A, B)]$  is also a set. Its elements are the mappings of the type  $\phi_f : B \rightarrow H(A, B)$ . These “genetic” maps associate to an output from the metabolic component, a replica of  $M$ , that is, the mapping  $f$ . In addition to these maps, the mapping  $\beta_b$  is defined as the assignment of the genetic map  $\phi_f$  to the corresponding  $f$ , and  $\beta_b$  is readily shown to be invertible (Rosen, 1966). Let us call  $\beta_b$  a “duplication” map. Then, let us consider the  $M$ -component, the  $R$ -component and the duplication map as “black boxes”, or components, of a block diagram, in the manner shown in Figure 2.

This pictorial representation obviates the fact that the complexity of the components of the simplest (M, R)-system increases towards the right-hand-side of the block diagram. Thus, the duplication component is able to

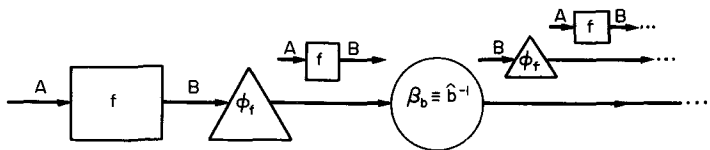


Figure 2

construct a genetic component which, in its turn, constructs the metabolic component. A more precise idea of the implicit mathematical constructions which generate mappings at different levels of sophistication was already proposed on a logical basis by Löfgren (1968). It can also be achieved in a categorical construction (Baianu and Marinescu, 1974). As shown by Rosen (1958b), the metabolic components can be re-organised into a category  $M$ . Let any two sets of  $M$  be  $X$  and  $Y$ , and let  $t: X \rightarrow Y$  be a mapping of  $M$ . If  $Set$  is the category of all sets and mappings of sets then one can define a special functor  $h^X: M \rightarrow Set$  as follows:

$$h^X(Y) = H(X, Y) \quad \text{for any set } Y \text{ in } M; \tag{7}$$

$$h^X(t) = m: H(X, X) \rightarrow H(X, Y) \quad \text{for any } t: X \rightarrow Y \tag{8}$$

$$h^X(g)(t) = g \circ t: H(X, X) \rightarrow H(X, Y'), \quad \text{for any } g: Y \rightarrow Y' \text{ in } M, \tag{9}$$

where  $X$  is a certain fixed object in  $M$ ;  $h^X$  carries  $Y$  into  $H(X, Y)$  without acting on the elements of  $Y$ . A family of functors of the type  $h^X$ , obtained by varying  $X$  in  $M$ , can produce all sets of the form  $H(X, Y)$ . Alternatively, we can use the *canonical* functor  $h: M \rightarrow [M, Set]$  defined by the assignments:

$$S \rightsquigarrow h^X \quad \text{and} \quad t \rightsquigarrow h^H \rightarrow h^Y \tag{10}$$

where  $t: X \rightarrow Y$  and  $[M, Set]$  is a category of functors from  $M$  to  $Set$ . Also, let us consider the *embedding*  $I: M \rightarrow Set$  which carries any  $X$  of  $M$  into the same set  $X$  of  $Set$  and any mapping  $t: X \rightarrow Y$  of  $M$  into the same mapping of  $Set$ .

With the above definitions, the genetic maps  $\phi_f$  are now defined by the natural transformations  $\Phi: I \rightarrow h^X$ , for  $X$  varying in  $M$ . Note that the sets of all natural transformations from  $I$  to  $h^X$ ,  $(h^X, I)$ , have the same cardinal as the corresponding sets,  $X$ . This has the following consequence:

**THEOREM.** *The sets  $(h^A, h^B)$  are isomorphic to  $H(B, A)$  and the sets  $(I, H^X)$  are isomorphic to  $H(X, X)$  for any  $X$  in  $M$ , if  $I$  is a representable functor (Popescu, 1973).*

The proof of this theorem results directly from the following commutative diagram of  $Set$ :

$$\begin{array}{ccc}
 H(X, X) & \xrightarrow{h^X(t)} & H(X, Y) \\
 h^X \uparrow & & \uparrow h^Y \\
 X & \xrightarrow{t} & Y
 \end{array} \tag{11}$$



The “representability” condition is equivalent to a computability requirement for the sequential machine associated with the  $(M, R)$ -system.

Apart from enabling us to consider  $(M, R)$ -systems with *variable* structure, replication and duplication maps, this construction is readily extended to produce a new type of  $(M, R)$ -systems: their “metabolic”, “genetic” and “duplication” components can have algebraic, topological or algebraic- and topological-structures, and form categories of the corresponding structures. Both developments have consequences for dynamical realizations of  $(M, R)$ -systems (Rosen, 1973) and are useful tools for the investigation of transformations of  $(M, R)$ -systems.

Interestingly, the canonical functor carries inductive limits into colimits and  $M$  has a left-adequate extension in *Set* (Baianu and Marinescu, 1974), and this is generally true for any small category, *regardless of the structure* of its objects.

The above property is similar to the general theorem for the algebraic category associated with an organismic set, discussed in the previous sections (see also Theorem 1 on p. 215 of Baianu, 1973). This suggests that in spite of their different *formal* aspects, the theories of organismic sets and  $(M, R)$ -systems have common, *underlying* structures and *similar* content. However, this does *not* mean that they are *mathematically* equivalent. In both theories, natural transformations are meaningful and can adequately describe dynamical transformations of organismic sets, or  $(M, R)$ -systems. The possible reasons for this similarity are discussed in the next section.

3. *Natural Transformations in the Theory of Molecular Sets.* Originally, the dynamic transformations of molecular sets were represented by means of mappings between Cartesian products of sets (Bartholomay, 1960, 1965, 1971). A simple case is the unimolecular chemical transformation:

$$T: A \times I \rightarrow B \times I \quad (12)$$

where  $A$  is the original sample set,  $I = [0, \tau]$  is a finite segment of the real time axis,  $A \times I$  denotes the indexing of each  $A$ -type molecule by the time instant at which each such molecule  $a \in A$  actually transforms into a  $B$ -type molecule [equation (3) of Bartholomay, 1971], and  $B \times I$  denotes the set of newly formed  $B$ -type molecules indexed by their corresponding instants of formation. Furthermore, it was proposed that any chemical component molecular set  $A$  of a biochemical subsystem of a biological system should be regarded as a variable quantity or a “molecular set variable” (m.s.v.) which spans certain allowable molecular sets. The functional dependence of m.s.v. on time was suggested to be a kind of “relation” from the time axis

to the class of molecular subsets as “range”. This was called the “wide-sense kinetics” and the concentration or cardinality of the system was considered of secondary importance. An appropriate *development* of the kinetics of molecular sets required probabilistic rules concerning the transition from one possible value (or state) of m.s.v. to the next allowable one, and while the concentration or size of a molecular set component may be constant, the set itself was shown to change continuously its *composition*. This is, in certain cases, the biologically significant factor in the operation of the system (Bartholomay, 1971).

The connection with the previous sections is now transparent. Although the specific case of unimolecular chemical transformations was already discussed in detail (pp. 361–369, *loc. cit.*), it can provide further insight into the possible quantitative developments of the present relational theories as it will be shown below.

It is interesting to note that Anthony F. Bartholomay himself pointed out that “any theoretical resynthesis of organismic sets, categories and supercategories from molecular-set-theoretic foundations would make available to the more macroscopic biological systems a... basic topological frame of reference; namely, molecular-set-topology as opposed to ordinary set topology”. Such a synthesis was attempted here through the consideration of natural transformations in the theories of organismic sets and  $(M, R)$ -systems.

As shown in Section 1, the introduction of *general observables* in the theory of organismic sets removed the apparent asymmetry between physics on the one hand, and biology and sociology, on the other hand (Rashevsky, 1969a, pp. 163–169). Naturally, the measurements on any  $S_0$  are carried out through physical, or “chemical” experiments and, therefore, involve molecular set transformations. The dynamic transformations of an organismic set  $S_0$ —which are here represented by the natural transformations defined in diagram (3)—have, therefore, certain underlying m.s.v. transformations. The relationships between the former and the latter are not at all trivial. A direct approach towards unravelling some of these relationships is now proposed.

Let us consider the transformations of a molecular set  $A$  to be described by the endomorphisms  $A \xrightarrow{f} A$  and denote the set of all possible transformations of  $A$  by  $H(A, A)$ . The molecular sets can be organized as a category  $\mathcal{M}$ , and the functors  $h^X$  introduced in the previous section can be used here, as well, to describe the dynamics of molecular sets.

Specifically,  $h^A: \mathcal{M} \rightarrow \text{Set}$  is defined by

$$h^A(X) = H(A, X) \quad \text{for any } X \text{ in } \mathcal{M}$$

and  $h^A(t) = m: H(A, A) \rightarrow H(A, B)$  for  $t: A \rightarrow B$  in  $\mathcal{M}$ , where  $B$  is the molecular set of reaction products of type "B".

The *molecular set variable* (m.s.v.) is represented by the morphisms  $v$  in the following natural diagram.

$$\begin{array}{ccc}
 & & A \times I \\
 & \nearrow \mathcal{F} & \downarrow v \\
 A & & \\
 & \searrow h^A & \\
 & & H(A, A)
 \end{array} \tag{13}$$

where  $v$  are induced by the presence of the inclusion mappings

$$A \xrightarrow{\mathcal{F}} A \times I \text{ and the commutativity conditions: } h^A = v \circ \mathcal{F}.$$

The unimolecular *chemical reaction* is thus represented by the natural transformation  $h^A \xrightarrow{\eta} h^B$ , as seen from the commutative diagram:

$$\begin{array}{ccc}
 h^A(A) = H(A, A) & \xrightarrow{\eta^A} & H(B, A) = h^B(A) \\
 \downarrow h^{A(t)} & & \downarrow h^{B(t)} \\
 H(A, B) & \xrightarrow{\eta^B} & H(B, B) = h^B(B)
 \end{array} \tag{14}$$

if the "states" of the molecular sets  $A_u = \{a_1, \dots, a_n\}$  and  $B_u = \{b_1, \dots, b_{n-v_0}\}$  are identified with certain endomorphisms in  $H(A, A)$  and  $H(B, B)$ , respectively.

This definition is readily generalised for multi-molecular reactions by considering the canonical functor defined in Section 2:

$$h: \mathcal{M} \rightarrow [\mathcal{M}, \text{Set}]$$

which assigns to each molecular set  $A$ , the functor  $h^A$  and to each chemical transformation  $A \xrightarrow{t} B$  the natural transformation  $h^A \xrightarrow{\eta} h^B$ . In addition, one may need to consider product categories, or categories of products (as shown on p. 216 of Baianu, 1973).

Clearly, the genetic maps introduced in the previous section can be considered to result from molecular set transformations, as seen from a comparison of the above constructions with diagrams (10) and (11).

Natural transformations provide, therefore, a unified, categorical framework both for  $(M, R)$ -systems and molecular set theory.

The Central Theorem discussed in the previous section also applies to molecular set transformations and this implies that the chemical transformations in this theory should be represented by computable functions, although not necessarily in numerical form.

As discussed in Section 2 for dynamic realizations of  $(M, R)$ -systems, the introduction of specific algebraic, or topological, structures defined over molecular sets will not modify the basic construction in terms of natural transformations and can lead to a refined form of the theory of molecular sets in which dynamic realisation and stability problems are directly considered in terms of transitions between quantum states of molecules. An *observation* process of a molecular set would, thus, involve the preparation of an m.s.v.,  $A$ , into a selected state, or field of states,  $A_u^*$ , and could be described as a morphism.

$$\alpha: H(A, A) \rightarrow R. \quad (15)$$

Similarly, for the chemical products “ $B$ ” of a reaction,

$$\gamma: H(B, B) \rightarrow R \quad (16)$$

is an observable of the m.s.v.  $B$  which is measured in some specially prepared state, or field of states,  $B_u^*$ . Intuitively, we expect that

$$\begin{array}{ccc} H(A, A) & \xrightarrow{-c-} & H(B, B) \\ \alpha \swarrow & & \searrow \gamma \\ & R & \end{array} \quad (17)$$

commutes and is *natural* in  $\alpha$ ,  $\gamma$  and certain “reactions”,  $c$ - which are the result of the “preparation” procedure (in a quantum-mechanical sense), so that  $c: A_u^* \rightarrow B_u^*$  is “uniquely” defined within an uncertainty range  $\delta$  of  $\Delta\alpha \cdot \Delta\gamma$  in  $R$  (Note the “fuzziness” introduced by this description in terms of the quantum-mechanical uncertainty principle).

A chemical transformation in which the *composition* of m.s.v.’s is allowed to vary induces certain natural transformations

$$\begin{aligned} \nu: \alpha &\rightarrow \alpha^* \\ \omega: \gamma &\rightarrow \gamma^* \end{aligned} \quad (18)$$

with  $\alpha, \alpha^*: Set \rightarrow R$  and  $\gamma, \gamma^*: Set \rightarrow R$  being functors defined as in relationships (15)–(17). For multi-molecular reactions, one obtains the commutative diagram of functors:

$$\begin{array}{ccc}
 Set & \xrightarrow{\alpha, \alpha^*, \gamma, \gamma^*, \dots} & R \\
 \uparrow h^A & & \uparrow L \\
 \uparrow h^B & & \vdots \\
 \mathcal{M} & \xrightarrow{h} & [\mathcal{M}, Set]
 \end{array} \tag{19}$$

with  $L$  playing the role of a *generalized* observable, in the sense of Section 1 (diagram 1). Since the biochemical products in the sets  $S_{ap}$  of Section 1 are the end results of such multi-molecular reactions, and may consist of one or more molecular sets, one can consider the natural transformations defined by diagram (3) to arise as a result of composition of natural transformations of m.s.v.'s and their cartesian products [as those in diagrams (18)], and functors [as those in diagrams (19)].

The converse problem of decomposing the natural transformations of organismic sets in terms of natural transformations of m.s.v.'s is by no means trivial and, therefore, it will be difficult, in general, to derive molecular mechanisms from theorems concerning organismic sets alone.

Therefore, the composition of molecular set transformations—to give natural transformations of organismic sets, or  $(M, R)$ -systems—appears more promising at this stage. A somewhat similar conclusion was derived from an investigation of dynamic similarities between transplanted nuclei of the same clone during the development of the ovum (Baianu and Scripcariu, 1973), where the scope and limitations of simple, natural equivalences for the representation of developmental processes were discussed.

The case of organismic sets of zero order—whose elements are genes—was already considered from a molecular and categorical viewpoint (Baianu, 1977) and it would be interesting to derive the consequences of the present approach (using natural transformations) for the dynamic transformations of non-linear, genetic nets and the associated “reaction rates”. This, in turn, should be translated into corresponding transformations of  $(M, R)$ -systems and their dynamical realizations (Rosen, 1971).

Thus, a unified view of the theories of organismic sets,  $(M, R)$ -systems and molecular sets appears, indeed, realizable on the basis of natural transformations of organismic structures.

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