How is spontaneous symmetry breaking possible? Understanding Wigner's theorem in light of unitary inequivalence

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May 31, 2013

Abstract

We pose and resolve a puzzle about spontaneous symmetry breaking in the quantum theory of infinite systems. For a symmetry to be spontaneously broken, it must not be implementable by a unitary operator in a ground state's GNS representation. But Wigner's theorem guarantees that any symmetry's action on states is given by a unitary operator. How can this unitary operator fail to implement the symmetry in the GNS representation? We show how it is possible for a unitary operator of this sort to connect the folia of unitarily inequivalent representations. This result undermines interpretations of quantum theory that hold unitary equivalence to be necessary for physical equivalence.

1 Introduction

The precise mathematical definition of spontaneous symmetry breaking (SSB) in quantum theory is somewhat up for grabs. But all hands agree that, in the case of infinitely many degrees of freedom, unitarily inequivalent representations are needed.

In physics more generally, SSB occurs when a ground state is not invariant under a symmetry of the laws. This means that a symmetry transformation will take a ground state to another (formally distinct) ground state. But in quantum field theory, an (irreducible) Hilbert space representation of the commutation relations can include only a single vacuum

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state. So a spontaneously broken symmetry must map between different unitarily inequivalent representations.

Thus for SSB to occur in infinite quantum theory, the broken symmetry must not be implemented by a unitary operator on the ground states' irreducible representations. (That is to say, the symmetry must not intertwine the two representations; otherwise they would be unitarily equivalent.) This is generally agreed to be a necessary condition (Emch and Liu, 2005) and is sometimes taken to be both necessary and sufficient (Earman, 2003; Strocchi, 2008).

Put this way, it can be difficult to see how a symmetry can possibly be spontaneously broken. The difficulty arises from an apparent conflict with Wigner's unitary-antiunitary theorem, a foundational result that applies to all quantum theories. John Earman has stated the puzzle thus:

> [A spontaneously broken] symmetry of the Lagrangian is not unitarily implementable, i.e. its action is not faithfully represented by a unitary operator on Hilbert space. But how can this be, since Wigner's theorem has taught us that a symmetry in QM is represented by a unitary transformation (or, as in the case of time reversal, an anti-unitary transformation)? (Earman, 2003, 338)

Since a symmetry ought to preserve all the empirical predictions of a quantum state, it must not change the transition probabilities between pure states, which are represented by the inner products between vectors in a Hilbert space representation. Wigner's theorem shows that any mapping that preserves these probabilities for all vector states in a Hilbert space must be a unitary mapping.

This fact has occasionally been taken to imply that broken symmetries don't preserve transition probabilities. Fonda and Ghirardi (1970, 446), for example, write that "[T]here are field theories whose Lagrangian is invariant under a certain transformation of the fields whereas there exists no corresponding unitary operator implementing the transformation...We face a situation in which even though a mapping of physically realizable rays is defined, the transformation does not conserve the probability..." Similarly, Arageorgis (1995, 302) has suggested that Wigner's theorem implies no mapping between the states of unitarily inequivalent representations can preserve transition probabilities.

But to the contrary, all symmetries preserve transition probabilities, even broken ones. Besides being physically intuitive, this can be proven rigorously even in paradigm cases of

SSB. But as Earman notes, it isn't immediately clear how to reconcile this with the possibility of SSB. If a symmetry preserves transition probabilities, its action on state vectors must be given by a unitary operator. This operator must fail to implement the symmetry on the ground states' irreducible representations, but how is this possible?

Our task here is to explain how. We'll begin by explaining some general features that apply in all cases of quantum SSB. We will show that in such cases, Wigner's theorem applies. The puzzle therefore appears. To resolve it, we'll explain why the existence of a unitary symmetry in Wigner's sense does not entail the unitary equivalence of the Hilbert space representations it connects. In particular, the unitary operator whose existence is guaranteed by Wigner's theorem lacks some of the properties one would naively expect it to have, permitting it to coexist with unitary inequivalence. There remains a sense in which the symmetry is not (strictly speaking) unitarily implementable. Combined with the orthodox view that symmetries preserve empirical predictions, these facts undermine the notion that unitary equivalence is a necessary condition for physical equivalence.

2 Quantum SSB

We begin by recalling some general properties of quantum theories on the algebraic approach. At the broadest level of generality, a quantum theory is described by a C^*-algebra $\mathfrak{A}$ obeying the canonical commutation or anticommutation relations (CCRs or CARs respectively) in their bounded form. This is either an algebra of observables or (as in the present case) a field algebra. The self-adjoint operators in $\mathfrak{A}$ stand for physical quantities and are often called observables. The states of the algebra are the possible assignments of expectation values to the operators in $\mathfrak{A}$. These are given by normed linear functionals $\omega : \mathfrak{A} \to \mathbb{C}$. The expectation value of A in state ω is written $\omega(A)$.

A clear connection exists between this algebraic formalism and the better-known Hilbert space formalism. The abstract algebraic states and observables can be concretely realized by a Hilbert space representation of $\mathfrak{A}$ (also called a representation of the CCRs/CARs). Such a representation is a mapping π from $\mathfrak{A}$ into the algebra of bounded operators $B(\mathcal{H})$ on a Hilbert space $\mathcal{H}$. The representation map is not usually a bijection; Hilbert space representations will include more operators, and in particular more observables, than the C^*-algebra. Some of the states ω of $\mathfrak{A}$ will be representable by density operators on $\mathcal{H}$ that agree with their expectation values for all A in $\mathfrak{A}$. Collectively these states are called the *folium* of the representation π.

Every algebraic state ω has a unique "home" representation in which it is given by a cyclic vector. This is established by the

GNS Theorem. *For each state ω of $\mathfrak{A}$, there is a representation π of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$, and a vector $\Omega \in \mathcal{H}$ such that $\omega(A) = \langle \Omega, \pi(A)\Omega \rangle$, for all $A \in \mathfrak{A}$, and the vectors $\{\pi(A)\Omega : A \in \mathfrak{A}\}$ are dense in $\mathcal{H}$. (Call any representation meeting these criteria a* GNS representation.*) The GNS representation is unique in the sense that for any other representation $(\mathcal{H}', \pi', \Omega')$ satisfying the previous two conditions, there is a unique unitary operator $U : \mathcal{H} \longrightarrow \mathcal{H}'$ such that $U\Omega = \Omega'$ and $U\pi(A) = \pi'(A)U$, for all A in $\mathfrak{A}$* (see Kadison and Ringrose, 1997, 278–279).

The definition of "same representation" presumed in this statement of uniqueness is called *unitary equivalence*. We call representations π and π' *unitarily inequivalent*, and treat them as distinct,[1] if there is no unitary operator U between their Hilbert spaces which relates the representations by

$$U\pi(A) = \pi'(A)U. \tag{1}$$

When equation (1) does hold, we say that the unitary U *intertwines* the representations π and π'.

A useful source on the representation of symmetry in this framework is Roberts and Roepstorff (1969). They posit (very reasonably) that any symmetry of a quantum system must at a minimum consist of two bijections, α from the algebra of physical quantities $\mathfrak{A}$ onto itself and α' from the space of states of $\mathfrak{A}$ onto itself. These must preserve all expectation values, so that

$$\alpha'(\omega(\alpha(A))) = \omega(A). \tag{2}$$

They then show that any such α is a $*$-automorphism of $\mathfrak{A}$, a bijection $\alpha : \mathfrak{A} \longrightarrow \mathfrak{A}$ which preserves its algebraic structure and commutes with the adjoint mapping $(\cdot)^*$. Furthermore, α' can be defined in terms of this $*$-automorphism as it acts on states. Clearly $\alpha'(\omega)$ is given

[1] Although we always treat inequivalent representations as formally or mathematically distinct, note that they may not always be physically inequivalent. As we shall see, they are sometimes related by symmetries, which are normally assumed to preserve all the physical facts.

by $\omega \circ \alpha^{-1} = \omega(\alpha^{-1}(A))$. We therefore have a justification, from physical principles, of the oft-cited fact that symmetries in quantum theory are given by $*$-automorphisms.

Clearly if α is a symmetry and $\pi(A)$ is a representation of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$, $\pi \circ \alpha(A) = \pi(\alpha(A))$ is also a representation of $\mathfrak{A}$ on $\mathcal{H}$. In this case α will act as a bijective mapping from π to $\pi \circ \alpha$. We call α *unitarily implementable in the representation* π when there is a unitary mapping $U : \mathcal{H} \longrightarrow \mathcal{H}$ such that

$$\pi'(A) = \pi(\alpha(A)) = U\pi(A)U^*. \tag{3}$$

This means the symmetry α is unitarily implementable in π iff π and $\pi \circ \alpha$ are unitarily equivalent representations of $\mathfrak{A}$.

This is where spontaneous symmetry breaking comes in. In general, a state ω breaks the symmetry α only if α is not unitarily implementable in ω's GNS representation.[2] When this occurs, ω's GNS representation and the GNS representation of the symmetry-transformed state $\alpha'(\omega) = \omega \circ \alpha^{-1}$ will be unitarily inequivalent.

In one example, the CAR algebra (so-called because it obeys the canonical anticommutation relations) is the field algebra for a system of interacting spin-1/2 systems. We may use the infinite version of the algebra to represent an infinitely long chain of spins confined to a one-dimensional lattice, as in the Heisenberg model of a ferromagnet. This infinite CAR algebra possesses a non-unitarily implementable automorphism which represents a symmetry of the ferromagnet: namely, a 180-degree rotation which flips all of the spins in the chain. The rotation is therefore a spontaneously broken symmetry. See Ruetsche (2006) for a detailed study of this case; we present only a few general features it shares in common with other examples of SSB.

The lowest-energy states available to an infinite spin chain are ones in which all of the spins align in the same direction. The Heisenberg ferromagnet has two such ground states: ω, in which all of the spins point along $+x$ (where x is the axis of the one-dimensional chain) and ω', in which they all point along $-x$. These states each define a GNS representation (π, π' respectively) on Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$. If α is the automorphism of the CAR algebra representing a 180-degree rotation along an axis perpendicular to x, $\omega' = \alpha'(\omega)$. But since α is spontaneously broken, π must be unitarily inequivalent to π'. This is where things get

[2] On the approach shared by Strocchi and Earman, this is both a necessary and sufficient condition.

confusing.

3 Wigner's theorem and Earman's problem

The expectation values of observables aren't the only important quantities in quantum physics. We should also expect a symmetry to preserve the transition probabilities between (pure) states of any quantum theory. In the Hilbert space formalism these are given by inner products: $\langle \psi, \psi' \rangle$ represents the likelihood of a spontaneous transition from vector state ψ to ψ'.[3]

It seems obvious that no symmetry worth its salt will alter any of the transition probabilities. This led Wigner to conclude that any symmetry worth its salt is given by a unitary operator.[4] For the following can be proven (see Bargmann, 1964):

Wigner's Theorem. *Any bijection from the unit rays (vector states) of a Hilbert space $\mathcal{H}$ to the unit rays of $\mathcal{H}'$ which preserves the inner product is given by a unitary mapping $W : \mathcal{H} \longrightarrow \mathcal{H}'$.*

This is puzzling. A spontaneously broken symmetry should still map bijectively between the vector states of the two representations, and it would be very strange if it failed to preserve transition probabilities. But we also know that broken symmetries are not unitarily implementable. How can such a symmetry be given by a unitary operator in the sense important to Wigner's theorem?

We are not the first to notice this puzzle. As noted in our introduction, Earman (2003) posed it some time ago. But while Earman successfully casts the puzzle in precise mathematical terms, he does so in a peculiar way. Earman resolves his interpretation of the puzzle without addressing what we consider the most natural and interesting conceptual question in the vicinity.

Earman begins by noting the implications of Wigner's theorem: any symmetry α' on states that preserves transition probabilities on a Hilbert space $\mathcal{H}$ is given by a unitary operator U on $\mathcal{H}$. Earman calls this a "Wigner symmetry." Further, U is unitary iff it

[3] Only on collapse interpretations do such transitions actually occur, of course. But we should nevertheless expect other interpretations to retain the statistical predictions codified in transition probabilities.

[4] Or by an antiunitary operator; for present purposes we ignore the difference.

corresponds to an automorphism of $B(\mathcal{H})$ that takes an operator O to UOU^*. So any Wigner symmetry induces an automorphism of $B(\mathcal{H})$, which automorphism must be unitarily implementable–and conversely, any such automorphism must be a Wigner symmetry. Earman asks,

> How then can a symmetry in the guise of an automorphism of a C*-algebra A fail to be an unbroken or Wigner symmetry? The answer is that two different senses of "broken symmetry" are in play. For a broken symmetry in the sense of spontaneous symmetry breaking, the C*-algebra $\mathfrak{A}$ is not isomorphic to $B(\mathcal{H})$; indeed, a representation π of $\mathfrak{A}$ is into rather than onto $B(\mathcal{H})$, and there is no continuous extension of $\pi(A)$ to all of $B(\mathcal{H})$. An automorphism $[\alpha]$ of $\mathfrak{A}$ is broken in the sense of spontaneous symmetry breaking not because it is broken in the Wigner sense in that it fails to preserve probabilities but because it is not an automorphism of $B(\mathcal{H})$. (Earman, 2003, 341)

Earman has correctly solved a puzzle concerning how a broken symmetry α can fail to be unitarily implementable without violating Wigner's theorem–but it is not, we think, the most obvious or pressing puzzle of this sort. He addresses what we will call

Earman's problem: Given that any automorphism of $B(\mathcal{H})$ is unitarily implementable on $(\pi, \mathcal{H})$, how can a symmetry α fail to be unitarily implementable on $(\pi, \mathcal{H})$?

The answer, as Earman notes, is that α is an automorphism of $\mathfrak{A}$, not $B(\mathcal{H})$. If α were defined as an automorphism of $B(\mathcal{H})$, as opposed to $\mathfrak{A}$, it would be unitarily implementable on π for the reasons explained by Earman. To clarify his explanation a bit, the crucial fact is not that $\pi(A)$ does not extend continuously to $B(\mathcal{H})$–even in the absence of broken symmetry, so such extension exists for physically interesting C*-algebras. What's important is that α itself does not extend continuously to an automorphism of $B(\mathcal{H})$, because if such an extension existed it would constitute a unitary operator implementing α on π, and hence intertwining π and π'.

As helpful as it is to understand the resolution of Earman's problem, a further puzzle also challenges our understanding of how Wigner's theorem can coexist with SSB. For Wigner's theorem implies the existence of a unitary operator which looks for all the world like it ought to implement α on π. Whether α is an automorphism of $B(\mathcal{H})$ or not, α' is still a bijection between the states of the representation Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$. Furthermore, as Roberts and Roepstorff (1969, 335) prove, α' must preserve all transition probabilities. By Wigner's theorem, this guarantees the existence of a unitary operator whose action on states is the same as α's, prompting

Our problem: Given that the action of a symmetry α on the states of $\mathcal{H}$ preserves transition probabilities–and is therefore given by a mapping $\phi \to W\phi$ for a unitary operator W–how can W fail to unitarily implement α on $(\pi, \mathcal{H})$?

There is of course a sense in which Earman's solution to his problem implies that there *must* be a solution to our problem. After all, it is impossible for W, or any other unitary operator, to intertwine π and π' on the assumption that α does not extend continuously to an automorphism of $B(\mathcal{H})$. In a sense, this is simply to reiterate that α is a broken symmetry and SSB requires unitary inequivalence. But subsuming W under this general fact is not the same as providing a constructive explanation for why W in particular fails to intertwine the representations, especially since its unique properties seem to imply that it must do so.

Although W has so far been defined in terms of its action on the state vectors in $\mathcal{H}$, it does of course act on operators as well, taking the operator $\pi(A)$ on $\mathcal{H}$ to $W\pi(A)W^*$. And it is trivial that the expectation value of this transformed operator in the transformed state $W\psi$ is given by $(W\psi, W\pi(A)W * W\psi) = (\psi, \pi(A)\psi)$. This is the same as the expectation value of the operator $\alpha(A)$ in the state $\alpha'(\omega)$, where ω is the abstract state corresponding to ψ. The natural conclusion to draw is that $W\pi(A)W^* = \pi(\alpha(A))$. But this can't be a correct conclusion, since if it were true W would intertwine the representations π and π', which contradicts the assumption that α is broken.

In sum, the problem posed by Earman cannot be fully laid to rest without exploring the properties of the unitary symmetry whose existence is implied by Wigner's theorem.

4 The resolution

Since Wigner's theorem applies to all symmetries, a spontaneously broken symmetry must in some sense give a unitary mapping between the states of unitarily inequivalent representations. So there must be some wiggle room in the definition of unitary equivalence that makes this possible. To solve our problem, we must look again at the definition and find the wiggle room.

In effect, we have two data points to work with. First, as Roberts and Roepstorff prove, Wigner's theorem applies to all algebraic symmetries. This means that any symmetry α', as it acts on states, must be induced by some unitary operator. Since the existence of this operator is guaranteed by Wigner's theorem, we'll call it the *Wigner unitary* W. Since for any state ω, $\alpha'(\omega) = \omega \circ \alpha^{-1}$, Wigner's theorem is telling us that W must take the state vector ψ that represents ω to a vector $W\psi$ that represents $\omega \circ \alpha^{-1}$.

Our second data point is the fact that spontaneous symmetry breaking is possible. This implies that some symmetries are not unitarily implementable in a ground state's GNS representation, in the sense that they fail to satisfy Eq. (3) for any unitary U. For these symmetries, the unitary W must map the vector states of $\mathcal{H}$ to those of $\mathcal{H}'$ without satisfying Eq. (1) for $U = W$. In such a case, the representations are unitarily inequivalent even though their Hilbert spaces are related by a unitary operator. There is nothing strictly contradictory about this, since the existence of a unitary operator implies unitary equivalence only if the operator intertwines the representations. Their respective representations π, π' of the C^*-algebra may nonetheless be preserved by W, as long as W does not map the representation π pointwise to the representation π'. So it may still be that

$$W\pi(\mathfrak{A}) = \pi'(\mathfrak{A})W \tag{4}$$

although there are individual operators $A \in \mathfrak{A}$ for which (1) does not hold.

In fact, an operator meeting these criteria exists whenever the states of two irreducible representations are connected by a symmetry. We will show in steps that in every such case a unitary W exists which satisfies Eq. (4), and which implements the symmetry as it acts on states without intertwining the representations. First, we establish its existence:

Representation Wigner Theorem. *Let $\langle \mathcal{H}, \pi, \Omega \rangle$ be a GNS representation for ω, and let $\langle \mathcal{H}', \pi', \Omega' \rangle$ be a GNS representation for $\omega \circ \alpha^{-1}$. Then there is a unique unitary operator $W = W_{\pi,\pi'} : \mathcal{H} \longrightarrow \mathcal{H}'$ such that $W\Omega = \Omega'$, and $W\pi(\alpha^{-1}(A)) = \pi'(A)W$ for all $A \in \mathfrak{A}$.* (Proof in Appendix 1.)

Since $\alpha^{-1}(\mathfrak{A}) = \mathfrak{A}$ and W intertwines $\pi \circ \alpha^{-1}$ and π', Eq. (4) follows. This means that when π and π' are not unitarily equivalent, W maps between these two representations without mapping π pointwise to π', which is what we expected. In other words, W acts as a bijection between the operators of these representations but does not, in general, intertwine the representations.

We have established that a unitary mapping preserving transition probabilities can exist even between unitarily inequivalent representations. Indeed, such a mapping always exists in cases of SSB. This is not yet enough to establish that the conclusion of Wigner's theorem is true (as it must be). Wigner's theorem ensures, not just that such a mapping exists, but that every mapping which preserves transition probabilities must be unitary. This includes every symmetry (whether spontaneously broken or not) as it acts on states.

So we must also show that a spontaneously broken symmetry can be given by a unitary operator in the sense just discussed, without intertwining π and π' — that is, without satisfying Eq. (3). To establish this, we will show that W itself induces the symmetry as it acts on states.

Keep in mind that any representation $\pi : \mathfrak{A} \longrightarrow B(\mathcal{H})$ of a C^*-algebra gives rise to a map T_π of unit vectors of $\mathcal{H}$ into the state space of $\mathfrak{A}$. In particular,

$$T_\pi(x)(A) = \langle x, \pi(A)x \rangle, \qquad (A \in \mathfrak{A}).$$

We now use this map to show that W induces the symmetry α' as it acts between the states of representations π and π'.

Corollary 1. *Let $(\mathcal{H}, \pi, \Omega)$ be a GNS representation for ω. Then the Wigner unitary W for α implements the action of α on vectors in $\mathcal{H}$. That is, $T_{\pi'}(Wx) = T_\pi(x) \circ \alpha^{-1}$ for any unit vector x in $\mathcal{H}$.* (Proof in Appendix 1.)

In other words, when we apply W to the state vector $x \in \mathcal{H}$ which represents the algebraic state $T_\pi(x)$ in the GNS representation π, the result is the vector $Wx \in \mathcal{H}'$ which represents the state $\alpha'(T_\pi(x)) = T_\pi(x) \circ \alpha^{-1}$ in the representation π'. This is just what it means for W to implement the symmetry as it acts on states.

Finally, we confirm that W does not in general intertwine the representations π and π'. In fact, we can show that W intertwines these representations only if it is trivial:

Corollary 2. *If the Wigner unitary $W : \mathcal{H} \longrightarrow \mathcal{H}'$ also induces a unitary equivalence between π and π', then $\alpha' \circ T_\pi = T_\pi$.* (Proof in Appendix 1.)

That is, every vector state in π's Hilbert space is invariant under the symmetry α', and hence left unchanged by W. This means W must be the identity.

The astute reader may be puzzled by some of the properties we ascribe to the Wigner unitary. In particular, we've shown that the Wigner unitary, which implements a symmetry as it acts on states, never implements that same symmetry unitarily on an irreducible representation π unless it is the trivial identity operator. How, then, can a non-trivial symmetry be unitarily implemented on π (and hence unbroken)? This sort of puzzle is best resolved by looking at concrete examples of Wigner unitaries in the case of both broken and unbroken symmetries, which examples we provide in Appendix 2.

We've shown that whenever two states are related by a symmetry, a unitary mapping exists between the Hilbert spaces of their GNS representations and has the properties we would expect. This is the Wigner unitary. Its existence vindicates Wigner's theorem, in that it shows how the theorem can be true even when spontaneous symmetry breaking prevents a symmetry from being unitarily implemented between irreducible representations.

5 Foundational significance

Besides the dissolution of a confusing puzzle, are there foundational implications of this result? We believe so. To underscore the foundational importance of our problem and its solution, let's briefly explore how it bears on one vexed question in the philosophy of quantum field theory. What are the necessary conditions for physical equivalence between field-theoretic states? In AQFT, the representations of the field algebra separate the states into natural "families:" the folia of states given by density operators in each representation. We may therefore ask what conditions must be met for two such families of states – two folia – to represent the same set of physical possibilities.

For two folia to be physically equivalent, they must at least be empirically equivalent. The mistaken line of reasoning that led to Earman's problem suggests that unitary equivalence is necessary if we want to preserve transition probabilities. Since a quantum theory's transition probabilities are part of its empirical content, it would seem to follow that the folia of unitarily inequivalent representations cannot predict the same empirical consequences – making them physically inequivalent by the above reasoning.

This is why Arageorgis, while attempting "to clarify the connection between 'intertranslatability' [a necessary condition for physical equivalence] and 'unitary equivalence,'" writes in his seminal dissertation,

> Intertranslatability requires a mapping between theoretical descriptions that preserves the reports of empirical findings. These are couched in terms of probabilities in quantum theory. And as Wigner has taught us, the preservation of probabilities in the Hilbert space formulation implies the existence of a unitary (or antiunitary)[5] operator. (Arageorgis, 1995, 302 fn 111)

He takes this point to establish that folia must belong to unitarily equivalent representations if they are to count as physically equivalent. But his argument includes a false premise: the assumption that the existence of a unitary operator connecting the folia of two representations implies a unitary equivalence between those representations. As we have shown, though, there is no such implication if the unitary operator is what we've called a Wigner unitary. Arageorgis's argument is unsound.

This means that at least one significant part of the empirical content of a quantum theory – its transition probabilities – can be preserved by a mapping between the folia of two inequivalent representations. If we further assume (as conventional wisdom dictates) that a quantum theory's symmetries preserve all empirical content, then the folia of at least some pairs of inequivalent representations must be empirically equivalent if spontaneous symmetry breaking is possible. Indeed, the existence of a symmetry is often taken to imply physical equivalence in the fullest sense (see Baker, 2011).[6] The notion that unitary equivalence is a necessary condition for physical equivalence should now appear quite suspect. Insofar as the so-called "Hilbert space conservative" interpretation of quantum field theory identifies physical equivalence with unitary equivalence (see Ruetsche, 2002), that interpretation must come into question as well.

Appendix 1: Representation Wigner Theorem

Here we prove the results mentioned in the main text.

[5] Recall that for purposes of this paper we ignore the distinction between unitary and antiunitary.

[6] Our view should be carefully distinguished from the notion that any automorphism between representations of $\mathfrak{A}$ implies their physical or empirical equivalence. In our opinion, this holds only for automorphisms which are also symmetries (i.e., which commute with the theory's dynamics).

Definition. Let ω be a state on a C^*-algebra $\mathfrak{A}$, let $\mathcal{H}$ be a Hilbert space with Ω a unit vector in $\mathcal{H}$, and $\pi : \mathfrak{A} \longrightarrow B(\mathcal{H})$ a representation of $\mathfrak{A}$. We say that the triple $\langle \mathcal{H}, \pi, \Omega \rangle$ is a *GNS representation* for ω just in case:

1. $\langle \Omega, \pi(A)\Omega \rangle = \omega(A)$, for all $A \in \mathfrak{A}$, and
2. $\{\pi(A)\Omega : A \in \mathfrak{A}\}$ is dense in the Hilbert space $\mathcal{H}$.

The GNS theorem shows that for each state ω, there is a GNS representation; and that any two GNS representations of ω are unitarily equivalent.

Let $\mathfrak{A}$ be a C^*-algebra, let ω be a state of $\mathfrak{A}$, and let α be a $*$-automorphism of $\mathcal{A}$. Let $(\mathcal{H}, \pi, \Omega)$ be the GNS triple of $\mathfrak{A}$ induced by ω, and let $(\mathcal{H}', \pi', \Omega')$ be the GNS triple of $\mathfrak{A}$ induced by $\omega \circ \alpha^{-1}$. For brevity, we sometimes just use π and π' to denote the corresponding triples.

Representation Wigner Theorem. *Let $\langle \mathcal{H}, \pi, \Omega \rangle$ be a GNS representation for ω, and let $\langle \mathcal{H}', \pi', \Omega' \rangle$ be a GNS representation for $\omega \circ \alpha^{-1}$. Then there is a unique unitary operator $W = W_{\pi,\pi'} : \mathcal{H} \longrightarrow \mathcal{H}'$ such that $W\Omega = \Omega'$, and $W\pi(\alpha^{-1}(A)) = \pi'(A)W$ for all $A \in \mathfrak{A}$.*

In fact this is a special case of Roberts and Roepstorff (1969), Proposition 6.2, but we provide a more elementary proof that does not appeal to Wigner's theorem as a premise.

Proof. Let $(\mathcal{H}, \pi, \Omega)$ be a GNS representation of $\mathfrak{A}$ for the state ω, and let $(\mathcal{H}', \pi', \Omega')$ be a GNS representation of $\mathfrak{A}$ for the state $\omega \circ \alpha^{-1}$. Define $W : \mathcal{H} \longrightarrow \mathcal{H}'$ by setting

$$W\pi(A)\Omega = \pi'(\alpha(A))\Omega', \qquad \forall A \in \mathfrak{A}.$$

Since $\alpha(I) = I$, it follows that $W\Omega = \Omega'$. Since

$$\|\pi'(\alpha(A))\Omega'\|^2 = \langle \Omega', \pi'(\alpha(A^*A))\Omega' \rangle = \omega(\alpha^{-1}(\alpha(A^*A))) = \omega(A^*A) = \|\pi(A)\Omega\|^2,$$

it follows that W is well defined and extends uniquely to a unitary operator from $\mathcal{H}$ to $\mathcal{H}'$. Note that since $\pi(A)\Omega = W^*W\pi(A)\Omega = W^*\pi'(\alpha(A))\Omega'$, it follows that $W^*\pi'(B)\Omega' = \pi(\alpha^{-1}(B))\Omega$ for all $B \in \mathfrak{A}$. Therefore,

$$W^*\pi'(A)W\pi(B)\Omega = W^*\pi'(A\alpha(B))\Omega' = \pi(\alpha^{-1}(A)B)\Omega = \pi(\alpha^{-1}(A))\pi(B)\Omega,$$

for all $A, B \in \mathfrak{A}$. Since the vectors $\pi(B)\Omega$, for $B \in \mathfrak{A}$, are dense in $\mathcal{H}$, it follows that $W^*\pi'(A)W = \pi(\alpha^{-1}(A))$ for all $A \in \mathfrak{A}$. That is, W implements a unitary equivalence from $\pi \circ \alpha^{-1}$ to π'.

To show the uniqueness of W, it suffices to note that Ω is a cyclic vector for $\pi \circ \alpha$, Ω' is a cyclic vector for π', and $W\Omega = \Omega'$. Thus, there is at most one unitary intertwiner from $\pi \circ \alpha^{-1}$ to π' that maps Ω to Ω'. □

Note that if $\alpha = \iota$ is the identity automorphism, and if we take $\langle \mathcal{H}', \pi', \Omega' \rangle = \langle \mathcal{H}, \pi, \Omega \rangle$, then I satisfies the conditions of the theorem, hence by uniqueness $W_{\pi,\pi'} = I$.

For the following corollary, recall that any representation $\pi : \mathfrak{A} \longrightarrow B(\mathcal{H})$ of a C^*-algebra gives rise to a map T_π of unit vectors of $\mathcal{H}$ into the state space of $\mathfrak{A}$. In particular,

$$T_\pi(x)(A) = \langle x, \pi(A)x \rangle, \qquad (A \in \mathfrak{A}).$$

Corollary 1. *Let $(\mathcal{H}, \pi, \Omega)$ be a GNS representation for ω. Then the Wigner unitary W for α implements the action of α on vectors in $\mathcal{H}$. That is, $T_{\pi'}(Wx) = T_\pi(x) \circ \alpha^{-1}$ for any unit vector x in $\mathcal{H}$.*

Proof. By the Theorem, a Wigner unitary W intertwines $\pi \circ \alpha^{-1}$ and π', that is

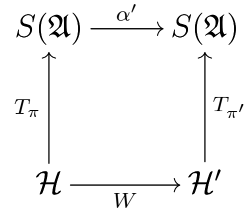
$$\pi(\alpha^{-1}(A)) = W^*\pi'(A)W,$$

for all $A \in \mathfrak{A}$. Hence

$$T_{\pi'}(Wx)(A) = \langle Wx, \pi'(A)Wx \rangle = \langle x, W^*\pi'(A)Wx \rangle = \langle x, \pi(\alpha^{-1}(A))x \rangle = T_\pi(x)(\alpha^{-1}(A)),$$

for all $A \in \mathfrak{A}$. □

The preceding corollary can be conveniently pictured via a commuting diagram:



where $S(\mathfrak{A})$ is the state space of $\mathfrak{A}$, and $\alpha' : S(\mathfrak{A}) \longrightarrow S(\mathfrak{A})$ is the symmetry $\omega \mapsto \omega \circ \alpha^{-1}$.

Corollary 2. *If the Wigner unitary $W : \mathcal{H} \longrightarrow \mathcal{H}'$ also induces a unitary equivalence between π and π', then $\alpha' \circ T_\pi = T_\pi$.*

Recall that $\alpha' : S(\mathfrak{A}) \longrightarrow S(\mathfrak{A})$ is defined by $\alpha'(\omega) = \omega \circ \alpha^{-1}$.

Proof. If W induces a unitary equivalence between π and π' then

$$\pi(A)W^* = W^*\pi'(A) = \pi(\alpha^{-1}(A))W^*,$$

for all $A \in \mathfrak{A}$. Canceling the unitary operator W^* on the right gives $\pi(A) = \pi(\alpha^{-1}(A))$ for all $A \in \mathfrak{A}$, that is $\pi = \pi \circ \alpha^{-1}$. From the latter equation it clearly follows that $T_\pi = \alpha' \circ T_\pi$. □

Appendix 2: Properties of the Wigner unitary operator

We now give a special case of the Representation Wigner Theorem which will illustrate some properties of the Wigner unitary. But first we need a lemma.

Lemma. *If $\langle \mathcal{H}, \pi, \Omega \rangle$ is a GNS representation for the state ω then $\langle \mathcal{H}, \pi \circ \alpha^{-1}, \Omega \rangle$ is a GNS representation for the state $\omega \circ \alpha^{-1}$.*

Proof. Since $\langle \Omega, \pi(\alpha^{-1}(A))\Omega \rangle = \omega(\alpha^{-1}(A))$ and since Ω is cyclic under $\{\pi(\alpha^{-1}(A)) : A \in \mathfrak{A}\}$, it follows that $\langle \mathcal{H}, \pi \circ \alpha^{-1}, \Omega \rangle$ is a GNS representation for $\omega \circ \alpha^{-1}$. □

We can now apply the Representation Wigner Theorem to the representations $\langle \mathcal{H}, \pi, \Omega \rangle$ and $\langle \mathcal{H}, \pi \circ \alpha^{-1}, \Omega \rangle$

Specialized Representation Wigner Theorem. *If $W : \mathcal{H} \longrightarrow \mathcal{H}$ is the Wigner operator for the representations $\langle \mathcal{H}, \pi, \Omega \rangle$ and $\langle \mathcal{H}, \pi \circ \alpha^{-1}, \Omega \rangle$, then $W = I$.*

Proof. By RWT, $W\Omega = \Omega$ and $W\pi(\alpha^{-1}(A)) = \pi(\alpha^{-1}(A))W$ for all $A \in \mathfrak{A}$. Since α is an automorphism, $W\pi(A) = \pi(A)W$ for all $A \in \mathfrak{A}$. Hence

$$W\pi(A)\Omega = \pi(A)W\Omega = \pi(A)\Omega,$$

for all $A \in \mathfrak{A}$. Since the set $\{\pi(A)\Omega : A \in \mathfrak{A}\}$ of vectors is dense in $\mathcal{H}$, it follows that $Wx = x$ for every vector in $\mathcal{H}$; that is, $W = I$. □

We now apply SRWT to the case of symmetries in elementary quantum mechanics. Let $\mathcal{H}$ be a finite-dimensional Hilbert space, and let $U : \mathcal{H} \longrightarrow \mathcal{H}$ be a unitary operator that induces a symmetry. Then we have the following transformations:

$$\begin{array}{ll} \varphi \longmapsto U\varphi & \text{transformed state} \\ A \longmapsto UAU^* & \text{transformed observable} \end{array}$$

Of course, $B(\mathcal{H})$ is a C^*-algebra, and each (pure) state of $B(\mathcal{H})$ is represented uniquely by a ray in $\mathcal{H}$. The unitary U induces the automorphism $\alpha(A) = UAU^*$ of $B(\mathcal{H})$, as well as the corresponding state mapping.

In order to apply SRWT, we need to find representations. The first representation is $\langle \mathcal{H}, \iota, \varphi \rangle$, where $\iota : B(\mathcal{H}) \longrightarrow B(\mathcal{H})$ is the identity, and φ is an arbitrarily chosen unit vector. The second representation is $\langle \mathcal{H}, \iota \circ \alpha^{-1}, \varphi \rangle$. By GWT, there is a Wigner unitary $W : \mathcal{H} \longrightarrow \mathcal{H}$, and by SRWT, $W = I$.

So, in what sense does W induce the symmetry U on states? Should we not have $W = U$? No, because a vector φ in $\mathcal{H}$ names different states on $B(\mathcal{H})$ according to which representation we consider, either ι or α^{-1}. Relative to the first, φ represents the state $A \mapsto \langle \varphi, A\varphi \rangle$, and relative to the second, φ represents the state $A \mapsto \langle \varphi, \alpha^{-1}(A)\varphi \rangle$.

What W does is to map a vector representing some state ω relative to π to a vector representing the state $\omega \circ \alpha^{-1}$ relative to $\pi \circ \alpha^{-1}$. In the way we have set things up, $W = 1_H$, which just means that if φ represents ω relative to π, then φ represents $\omega \circ \alpha^{-1}$ relative to $\pi' = \pi \circ \alpha^{-1}$. So, indeed, the identity map implements the symmetry $\omega \mapsto \omega \circ \alpha^{-1}$ of states!

Let us look now, more generally, at the case of an *unbroken* symmetry. By hypothesis, the symmetry α is unbroken just in case the representations $(\mathcal{H}, \pi, \Omega)$ and $(\mathcal{H}, \pi \circ \alpha^{-1}, \Omega)$ are unitarily equivalent. That is, there is a unitary operator $V : \mathcal{H} \longrightarrow \mathcal{H}$ such that $V\pi(\alpha^{-1}(A)) = \pi(A)V$. In fact, in the most interesting case where ω is a pure state, V can be chosen such that $V = \pi(U)$ for some unitary operator $U \in \mathfrak{A}$, hence

$$\pi(\alpha(A)) \;=\; V\pi(A)V^* \;=\; \pi(UAU^*),$$

for all $A \in \mathfrak{A}$. (To verify the existence of such a $U \in \mathfrak{A}$, see Kadison and Ringrose (1997, 730).)

Of course, we are still guaranteed the existence of the Wigner Unitary $W : \mathcal{H} \longrightarrow \mathcal{H}$.

(In fact, we know that $W = I$; but ignore that fact for now.) Which operator, W or V, implements the symmetry α on states? The answer is that they *both* do, but in different senses.

Compare the following two diagrams:

$$\begin{array}{ccc} S(\mathfrak{A}) & \xrightarrow{\alpha'} & S(\mathfrak{A}) \\ {\scriptstyle T_\pi}\uparrow & & \uparrow{\scriptstyle T_{\pi'}} \\ \mathcal{H} & \xrightarrow[W]{} & \mathcal{H} \end{array} \qquad \begin{array}{ccc} S(\mathfrak{A}) & \xrightarrow{\alpha'} & S(\mathfrak{A}) \\ {\scriptstyle T_\pi}\uparrow & & \uparrow{\scriptstyle T_\pi} \\ \mathcal{H} & \xrightarrow[V]{} & \mathcal{H} \end{array}$$

The square on the left shows the action of the Wigner unitary W for the special case of the GNS representation $(\mathcal{H}, \pi \circ \alpha^{-1}, \Omega)$ for $\omega \circ \alpha^{-1}$. The square on the right shows that action of the unitary V that implements the equivalence between π and $\pi \circ \alpha^{-1}$. The key difference, of course, is that V implements the symmetry in such a way that the correspondence between vectors and states can be held invariant (the vertical arrows are the same), whereas W's implementation requires a change of correspondence (T_π versus $T_{\pi'}$). But a state is a way to map observables to numbers, so changing the correspondence between vectors and states is equivalent to leaving this correspondence fixed and instead changing the labels of observables. In equation form:

$$T_{\pi'} = \alpha' \circ T_\pi,$$

i.e. the correspondence $T_{\pi'}$ matches vectors with an observable A in exactly the way that the correspondence T_π matches vectors with the observable $\alpha^{-1}(A)$.

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