The Conventionality of Parastatistics

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Abstract

Nature seems to be such that we can describe it accurately with quantum theories of bosons and fermions alone, without resort to parastatistics. This has been seen as a deep mystery: paraparticles make perfect physical sense, so why don't we see them in nature? We consider one potential answer: every paraparticle theory is physically equivalent to some theory of bosons or fermions, making the absence of paraparticles in our theories a matter of convention rather than a mysterious empirical discovery. We argue that this equivalence thesis holds in all physically admissible quantum field theories falling under the domain of the rigorous Doplicher-Haag-Roberts approach to superselection rules. Inadmissible parastatistical theories are ruled out by a locality-inspired principle we call Charge Recombination.

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1 Introduction

Our most fundamental theories of matter provide a highly accurate description of subatomic particles and their behavior. They do so, in part, by classifying particles based on the different ways that groups of (intrinsically) identical particles behave under permutation symmetry. Accordingly, all particles in the Standard Model can be classified as either bosons or fermions. The former obey symmetric Bose-Einstein statistics, the latter obey antisymmetric Fermi-Dirac statistics. Given the success of our present theories, it may come as a surprise, even to the physics-literate, to learn that these two families of particles do not exhaust the space of quantum mechanical possibilities. In fact, quantum theory allows for the existence of infinitely many families of *paraparticles*, which obey mixed-symmetry statistics. Their conspicuous absence from nature presents an intriguing foundational puzzle for quantum physics.

The extent to which this puzzle stands in need of resolution is a matter of some debate. Many physicists are content to treat the non-existence of paraparticles as a contingent feature of our world and its laws. Others (often unwittingly) rule out paraparticles by fiat, placing ad hoc restrictions on the representational structure of Hilbert space.[1] In contrast, philosophers, mathematicians, and physicists with a more foundational eye, including many of the fathers

[1] Introductory quantum mechanics texts frequently assert that pure states must be represented by normalized rays (i.e. 1-dimensional subspaces) in a Hilbert space. Sometimes a quick heuristic argument from the unobservability of quantum phases is given to justify this restriction. These arguments, however, apply to any normalized subspace that is invariant under permutation symmetry, regardless of its dimension. These higher dimensional subspaces represent paraparticles. For a more thorough discussion, see §2.

of quantum mechanics, have long viewed nature's abhorrence of parastatistics as a deeper problem. Commenting upon the matter in his 1945 Nobel Prize lecture, Wolfgang Pauli laments: "The impression that the shadow of some incompleteness fell here on the bright light of success of the new quantum mechanics seems to me unavoidable."[2]

While few would go as far as to say that the lack of any observational evidence for paraparticles *falsifies* standard quantum mechanics, it is in apparent tension with the kind of plenitude principles that physicists routinely employ. Beginning with Wigner's seminal work on irreducible representations of the Poincaré group and continuing through Gell-Mann's "Eightfold Way" classification of the meson and baryon octets, physicists have used group representation theory to classify and predict the existence of new particles. According to the standard recipe, roughly, there should be particle types corresponding to every irreducible positive-energy representation of a theory's gauge group. The permutation symmetry that determines a particle's statistical properties has many of the hallmarks of a gauge symmetry. But applying the standard gauge recipe to this symmetry leads inexorably to paraparticles. In light of this, there have been numerous attempts to find natural principles which rule out the possibility of paraparticles in conjunction with standard quantum mechanics. None have been readily forthcoming.[3] Lacking a persuasive no-go result, it was hoped, for a time, that the physics of mixed-symmetry particles would prove to be so complex that they could be safely ignored as physical pathologies. The work of Hartle, Stolt, and Taylor on the classification and physical interpretation of parastatistics largely scuttled this prospect.[4] Paraparticles are indeed strange, but not strange enough to banish outright.

There is another option. It has often been claimed — sometimes offhand, sometimes with considerable supporting argument — that every theory of paraparticles is physically equivalent to some theory of regular bosons or fermions. For short, we'll simply call this the *equivalence thesis.* In this paper we explore the historical contours of the equivalence thesis and formulate and prove a highly general version of the thesis that applies to quantum field theories with local charges. If we are right, any such theory involving paraparticles is simply a notational variant of some theory positing only bosons or fermions.

While none of the existing arguments for the equivalence thesis are fully satisfactory as they stand, there are two prevailing trains of thought that hold considerable promise. The first is that the Doplicher-Roberts reconstruction theorem (which undergirds the mathematically rigorous theory of superselection rules) offers a way of constructing, from any

[2] Pauli (1946).

[3] Steinmann (1966), for instance, attempts to show that paraparticles are inconsistent with the cluster decomposition principle. This argument was later refuted by Hartle and Taylor (1969). For an overview of some other failed attempts, see van Fraassen (1991), Ch. 11.

[4] See Hartle and Taylor (1969) and Stolt and Taylor (1970a).

parastatistical theory, a theory with ordinary statistics. Unfortunately, while the theory thus constructed is sometimes equivalent to the original parastatistical theory, it isn't always. The second sort of argument (due to Druhl *et al.* (1970)) proceeds by forming a taxonomy of the possible quantum field theories for a given set of parastatistical fields, and arguing that the only ones which are physically admissible or "well-behaved" are equivalent to ordinary Bose/Fermi theories. This argument is neither fully fleshed-out nor fully general, but it suggests a well-motivated physical principle with which the argument based on Doplicher-Roberts reconstruction can be made sound. By assuming this principle, we show that within the domain where superselection theory is mathematically well understood (i.e. theories with local charges), the only physically admissible paratheories are equivalent to theories with ordinary statistics.

In addition to resolving our foundational quandary, the equivalence thesis has ramifications for a number of other philosophical debates in which paraparticles have made occasional cameo appearances:

(i) **Interpretations of QM** — Even those who think that the absence of paraparticles does not pose a deep problem for standard quantum mechanics generally agree that a version of quantum theory which successfully predicts the impossibility of paraparticles would possess an explanatory advantage over the basic theory. A number of interpretations including Bohmian mechanics, stochastic mechanics, and modal interpretations have offered no-go theorems for paraparticles.[5] If the equivalence thesis is correct, it would appear to nullify the comparative theoretical advantage these interpretations claim in this arena.

(ii) **Structuralism and Haecceities** — Paraparticles have also featured in philosophical arguments concerning identical particles and structural realism, most recently by Caulton and Butterfield (2012). They claim that the possibility (in principle) of paraparticles undermines the "quantum hole argument" of Stachel (2002), and supports anti-haecceitism about fundamental particles. If it is always possible to rewrite theories with paraparticles in terms of ordinary statistics, the arguments of Caulton and Butterfield must be critically reassessed.

(iii) **History of Particle Physics** For a brief time in the 1960s some theorists speculated that quarks might in fact be paraparticles.[6] Ultimately the paraquark theory was abandoned in favor of a rival quark model which posited a new kind of funda-

[5] See Bacciagaluppi (2003), Dürr *et al.* (2006), Nelson (1985), and Kochen (unpublished).
[6] See Greenberg (2004).

mental charge, *color*, and was more easily incorporated into the framework of local gauge theory. French (1995) analyzes this historical episode from a general philosophy of science angle. Drawing upon early versions of the equivalence thesis, French argues that although the paraquark and color models were empirically equivalent, it was the heuristic fruitfulness of the color model, demonstrated by its extension to quantum chromodynamics (QCD), that militated in its favor amongst the community of physicists.

The plan of the paper is as follows: in the next section we establish some necessary background — the quantum mechanics of mixed-symmetry particles and the connection to quantum field theories obeying "abnormal" commutation relations. §3 contains a discussion of different notions of empirical and theoretical equivalence. Here we provide a sufficient condition for the theoretical equivalence of two *field systems*, the kind of quantum theories within the scope of our main thesis. In §4 we turn to the history of the equivalence thesis focusing on the arguments supplied by Druhl *et al.* (1970) and the Doplicher-Roberts reconstruction theorem. We then proceed to our strengthened version of the equivalence thesis in §5 and demonstrate that it satisfies the criteria for theoretical equivalence established in §3. In the concluding discussion, we explore the limitations of our version of the equivalence thesis and its possible impact on the philosophical debates listed above.

2 Paraparticles in quantum theory

States representing identical particles are required to be invariant under permutation transformations which, intuitively, switch the order of the particles in the state description. The different varieties of quantum statistics are rooted in the different ways that vectors representing these states can transform under permutations. A state vector representing n identical bosons must be completely *symmetric* with respect to permutations of any two particles — the result of such a permutation leaves the state vector unchanged. In contrast, a state vector representing n identical fermions must be completely *antisymmetric* — any permutation of two particles changes the vector by an overall multiplicative factor of -1. The resulting differences between Bose-Einstein and Fermi-Dirac statistics have important physical consequences. For example, collections of identical bosons can be prepared in the same quantum state, a property which gives rise to the physics of lasers and Bose-Einstein condensates. Collections of identical fermions behave quite differently. No two fermions may be prepared in the same quantum state, a result familiar to students of high-school chemistry as the Pauli Exclusion Principle. This statistical property explains important features of the periodic table of elements and facts about the stability of matter.[7]

Paraparticles obey statistics of *mixed-symmetry* type. A state vector representing n paraparticles is neither required to be completely symmetric nor completely antisymmetric. Paraparticles are divided into two general categories, *parabosons* and *parafermions*, and further classified by a natural number $p \geq 1$, their *order*.[8] According to the standard physical interpretation, a state representing n parafermions of order p can be symmetric in up to p particle indices and must be antisymmetric in the remaining indices. (The analogous n-particle paraboson state can be antisymmetric in up to p indices.) Hence, parafermions obey a generalized exclusion principle: up to p parafermions can be prepared in the same quantum state, but any additional members of the ensemble must be in a different state. The spin-statistics theorem also generalizes: particles with integer spin must be parabosons and particles with half-integer spin must be parafermions. Ordinary particles can be thought of as paraparticles of order $p = 1$.

There are two primary methods for constructing quantum theories with parastatistics.[9] The first approach begins with a tensor product of single-particle Hilbert spaces and considers the effects of permutation symmetry on this multi-particle configuration space. (We expect this will be the framework most familiar to philosophers of science.) The configuration space of a one-particle system, H^1, is given by the space of square-integrable wavefunctions of one coordinate, $\psi(k)$.[10] Pure states of the system are in one-to-one correspondence with rays in this space. In order to represent a system of n particles, we take the n-fold tensor product,

$$H^n = H^1 \otimes H^1 \otimes \ldots \otimes H^1 \tag{1}$$

H^n is the space of square-integrable wavefunctions of n coordinates, $\psi(k_1, \ldots, k_n)$. Each

[7] If the state itself is permutation invariant, one might wonder how different transformation properties of the state *vector* (which is simply a mathematical device for representing the state) can have any physical consequences. The key is to remember that the states in question are multiparticle states, and are thus linear functions of single particle states. Consider the 2-particle state vectors $|\Phi_S\rangle = 1/\sqrt{2}(|\phi_1\rangle|\phi_2\rangle + |\phi_2\rangle|\phi_1\rangle)$ and $|\Phi_A\rangle = 1/\sqrt{2}(|\phi_1\rangle|\phi_2\rangle - |\phi_2\rangle|\phi_1\rangle)$, where $|\phi_1\rangle$, $|\phi_2\rangle$ are 1-particle state vectors. $|\Phi_S\rangle$ is symmetric and $|\Phi_A\rangle$ is anti-symmetric under a permutation of the two particles, although the states they represent are invariant since $\langle\Phi_S|A|\Phi_S\rangle = \langle\Phi'_S|A|\Phi'_S\rangle$ and $\langle\Phi_A|A|\Phi_A\rangle = \langle\Phi'_A|A|\Phi'_A\rangle$, for all $A \in \mathfrak{A}$, where $|\Phi'_S\rangle$, $|\Phi'_A\rangle$ are the permuted state vectors. Now, if the two particles are in the same quantum state, either $|\phi_1\rangle = |\phi_2\rangle$ or $|\phi_1\rangle = -|\phi_2\rangle$. Either way, $|\Phi_A\rangle = 0$, whereas $|\Phi_s\rangle \neq 0$. Hence particles characterized by anti-symmetric wavefunctions obey an exclusion principle whereas particles characterized by symmetric wavefunctions do not.

[8] It is also possible to have paraparticles of infinite order whose physical interpretation does not fit cleanly into this scheme. Most of the theorems we discuss only apply to finite parastatistics, and for present purposes we will restrict attention to this case.

[9] In the paraparticle literature these are sometimes referred to as the *first-* and *second-quantized* approaches, respectively.

[10] For ease of exposition, we are suppressing spinor indices. Nothing crucial turns on this omission.

wavefunction has a vector representation, $|\psi\rangle$, in H^n, which is expressible as a sum of products of state vectors lying in the one-particle spaces H^1. These one-particle vectors are indexed by a number $1, \ldots, n$ corresponding to which H^1 they are elements of.

If the particles are identical, then intuitively the order in which they appear in the state description is irrelevant. This intuition gives rise to the *Quantum Indistinguishability Postulate*, which asserts that for systems of identical particles, no two state vectors differing by a permutation of the particle indices can be distinguished by measurements at any time. Letting S_n denote the group of permutations of n objects, for any $P \in S_n$ there is a corresponding unitary U_P acting on vectors in H^n. If $|\psi\rangle$ is a simple tensor product of one particle state vectors, then $U_P|\psi\rangle = U_P|\psi\rangle_1|\psi\rangle_2 \ldots |\psi\rangle_n = |\psi\rangle_{P1}|\psi\rangle_{P2} \ldots |\psi\rangle_{Pn}$. The action of U_P on more complex state vectors is fixed by this definition.[11]

The indistinguishability postulate requires that for any state vector and any permutation, $|\psi\rangle$ and $U_P|\psi\rangle$ yield the same expectation values on observables:

$$\langle\psi|A|\psi\rangle = \langle\psi|U_P{}^* A\ U_P|\psi\rangle \quad \forall A \in \mathfrak{A} \tag{2}$$

This entails that all observables must commute with the permutation operators:

$$[A, U_P] = 0 \quad \forall A \in \mathfrak{A},\ \forall P \in S_n \tag{3}$$

In this manner the indistinguishability postulate places a constraint on the algebra of observables. Since states can be completely characterized by their expectation values on $\mathfrak{A}$, the indistinguishability postulate also places a corresponding constraint on how vectors in H^n can represent states. Specifically, it requires that

$$\langle\psi|A|\psi\rangle = \langle\xi|A|\xi\rangle, \tag{4}$$

for all $A \in \mathfrak{A}$, iff $|\psi\rangle$ and $|\xi\rangle$ represent the same state. The standard way to implement constraint (4) is to posit that each state of the system corresponds to a unique ray in H^n. This posit is sometimes known as the *Symmetrization Postulate*. If $|\psi\rangle$ and $U_P|\psi\rangle$ are required to lie in the same ray, they must be proportional, and so only vectors that are completely symmetric or completely antisymmetric can represent distinct states of the

[11] As an explicit example, consider two spin-1/2 particles in a singlet state. Let $P = (12)$. The action of U_P is given by:

$$U_P|\psi\rangle = U_P(1/\sqrt{2})(|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2) = (1/\sqrt{2})(|\uparrow\rangle_2|\downarrow\rangle_1 - |\downarrow\rangle_2|\uparrow\rangle_1)$$

system. In other words, every state must either have bosonic or fermionic statistics.

The symmetrization postulate effectively rules out parastatistics by fiat. Rays in H^n correspond to 1-dimensional irreducible representations of S_n (either the completely symmetric or completely antisymmetric representations), but the indistinguishability postulate itself does not force this dichotomy. Typically there will be additional subspaces of H^n corresponding to higher-dimensional irreducible representations of S_n (known as *generalized rays*). Vectors lying in such subspaces will automatically satisfy (4), therefore in principle any such subspace can represent a distinct state of the system. States represented by higher-dimensional subspaces of H^n will exhibit mixed-symmetry, parastatistical properties. Extending the symmetrization postulate to include these subspaces naturally incorporates paraparticles into the theory.

Unfortunately this approach only works for non-relativistic quantum mechanics. A second, more general technique that applies equally well to both non-relativistic and relativistic theories, begins from the standpoint of second-quantization and proceeds by introducing creation/annihilation operators satisfying generalized trilinear commutation relations (as opposed to the standard bilinear ones). To streamline our presentation, we will present the relativistic version here.[12] For each distinct type of particle in the theory, we introduce field operators $\phi(x)$ satisfying the Wightman axioms. For present purposes we will treat the fields as localized at spacetime points, ignoring the issue of smearing with test functions without loss of generality. These operators can be expanded in terms of positive-frequency components in the following fashion,

$$\phi(x) = \sum_k a(+k)\phi_+(x) + a(-k)^*\phi_-(x) \tag{5}$$

where $\phi_\pm(x)$ represent a complete set of orthonormal functions with only positive and negative frequency components and $a(k)^*, a(k)$ represent creation and annihilation operators respectively. These carry the physical interpretation of either creating or annihilating a particle of the relevant type with momentum k (and positive energy). The vacuum state $|\Omega\rangle$ is defined to be the state of lowest energy:

$$a(k)|\Omega\rangle = 0 \tag{6}$$

In their relativistic form, the Heisenberg equations of motion require that

$$\partial_\mu \phi(x) = i[P_\mu, \phi(x)] \ , \tag{7}$$

[12]See Stolt and Taylor (1970b) for a treatment of the non-relativistic case.

where P_μ is the relativistic 4-momentum. The 4-momentum in turn can be expressed in terms of the creation/annihilation operators as

$$(a) \quad P^\mu = \sum_k p_k^\mu [a(k)^*, a(k)] \qquad (b) \quad P^\mu = \sum_k p_k^\mu \{a(k)^*, a(k)\} \ , \tag{8}$$

where (8.a) holds for particles of half-integer spin and (8.b) holds for particles of integer spin. Ordinarily, one posits that the these operators satisfy bilinear commutation relations:

$$\begin{aligned} (a) \quad & \{a(k), a(l)\} = 0 & (b) \quad & [a(k), a(l)] = 0 \\ & \{a(k)^*, a(l)\} = \frac{1}{2}\delta_{kl} & & [a(k)^*, a(l)] = \frac{1}{2}\delta_{kl} \end{aligned} \tag{9}$$

where again (9.a) holds for particles of half-integer spin and (9.b) holds for particles of integer spin. Given (8), these commutation relations ensure that the equations of motion have the proper form (7). The corresponding field operators satisfy similar bilinear commutation relations, and the particles they describe must either obey fermionic or bosonic statistics.[13]

Like the symmetrization postulate in the first approach, the restriction to commutation relations of the form (9) amounts to a stipulation that paraparticles do not exist. In principle, the constraints imposed by the equations of motion allow for more general, multi-linear commutation relations. A necessary condition for (7) and (8) to be satisfied is that the creation/annihilation operators obey trilinear relations

$$\begin{aligned} (a) \quad & [a(k), [a(l), a(m)]] = 0 & (b) \quad & [a(k), \{a(l), a(m)\}] = 0 \\ & [a(k), [a(l)^*, a(m)]] = \frac{1}{2}p \ \delta_{kl} a(m) & & [a(k), \{a(l)^*, a(m)\}] = \frac{1}{2}p \ \delta_{kl} a(m) \end{aligned} \tag{10}$$

(This follows from the requirement that (7) and (8) be invariant under unitary transformations of the creation/annihilation operators.[14]) The corresponding field operators satisfy similar trilinear commutation relations, and describe either (10.a) para-Fermi or (10.b) para-Bose particles of order p.

[13]Or so the standard story goes. The relationship between field commutation relations and particle statistics is somewhat subtle. Using Haag-Ruelle scattering theory, one can show that in a generic interacting theory, if the interacting fields obey standard commutation relations, then they continue to do so asymptotically. This is taken as an explication of the notion of particle statistics by Streater and Wightman (161). As we will go on to see in §6, it is possible to have ordinary fields in a superselection sector whose statistical dimension is greater than one and therefore a *parasector*. By Streater and Wightman's lights this would appear to indicate that it is possible to have ordinary particles in a parasector; however, Dopplicher and Roberts (1972) argue that a particle inherits its statistics from the sector making it impossible to have ordinary particles in a parasector (although it remains possible to have ordinary fields).

[14]Bialynicki-Birula (1963)

Fortunately, in non-relativistic quantum theories this second approach is fully equivalent to the first. Stolt and Taylor (1970b) provide a scheme for translating between the two frameworks that preserves all important physical content.[15] In this context the two formalisms are effectively interchangeable. Since the second-quantized approach is needed in order to treat paraparticles in quantum field theory, however, the literature on the equivalence thesis predominantly employs this method. We shall follow suit here.[16]

3 Theoretical equivalence

Before evaluating the arguments for the equivalence thesis, it is worth noting that the term 'physical equivalence' is used (especially in the physics literature) to denote two distinct notions. One of these has been extensively studied in the philosophy of science, where it goes by the name of "empirical equivalence." Theories are said to be empirically equivalent when they make the same observable predictions. The other notion might aptly be named "theoretical equivalence." Although it is harder to define in theory-neutral terms, the idea is that two theories are theoretically equivalent if they postulate the same unobservable reality, in addition to being empirically equivalent. In the present context, a proof of the full theoretical equivalence of paraparticle theories with certain theories of bosons and fermions would be holy grail from the standpoint of the scientific realist. There would be no mystery

[15]That such a translation exists is not obvious. Two difficulties immediately present themselves. The first concerns the fact that states in the first-quantized theory are represented by higher dimensional subspaces of H^n whereas the standard connection between states and rays holds in the second-quantized theory. This means that a correspondence between states cannot be established directly by a one-to-one correspondence between vectors in the relevant Hilbert spaces. The second problem concerns the fact that there is no clear analogue in the second-quantized approach to particle label permutation operators U_P that were so central in the first-quantized approach. The natural choice would be an operator that acts on vectors generated from the vacuum by permuting the creation operators

$$a(k_1)^* \dots a(k_n)^* |\Omega\rangle \to a(k_{P1})^* \dots a(k_{Pn})^* |\Omega\rangle \ , \tag{11}$$

but it can be demonstrated that no such operator can be implemented unitarily. Stolt and Taylor circumvent these problems by exploiting the fact that the Hamiltonian commutes with permutation operators, and so within the first-quantized formalism each generalized ray in H^n can be faithfully represented by one of it's basis vectors. This effectively reestablishes the link between states and rays in a reduced Hilbert space that is provably isomorphic to the Hilbert space of the second-quantized formalism.

[16]There is a third approach to parastatistics developed by Doplicher, Haag, and Roberts that seeks to characterize the statistical properties of fields entirely in terms of the superselection structure of the algebra of observables, independent of any particular Hilbert space representation. It is not directly apparent how this approach is related to the two more traditional methods. We will postpone discussion of this issue until §6.

as to why the latter appear adequate by themselves to describe nature.[17]

As we will show, every parafield theory satisfying certain physical requirements can be translated into an ordinary field theory. This translation preserves a significant amount of theoretical structure that does more than just save the phenomena. Is it a full blown theoretical equivalence? Do the field systems in question describe the same unobservable reality?

The obvious method to tackle this question would be to proceed as follows: determine the complete unobservable structure of theory T_1, do the same for theory T_2, compare. An equally obvious objection looms: until we have solved the measurement problem, we won't know what the fundamental structure of reality looks like according to either theory.

For several reasons, we believe an interesting notion of theoretical equivalence can nonetheless be developed using only the resources at hand. To begin with, as Ruetsche (2011) has noted, an interpretation of a physical theory need not be complete, in the sense of identifying every metaphysical consequence of that theory. Thus we may think of quantum theory, even in the absence of a solution to the measurement problem, as *partially* interpreted.[18] While the various interpretations of QM tend to disagree about the fundamental ontology of the quantum world, they tend to agree with each other — as well as with the standard formalism — regarding which models of the theory are physically equivalent. For example, the Bohmian, Everettian, and collapse theorist will all agree that the Heisenberg and Schrödinger models of the simple harmonic oscillator are theoretically equivalent, while the theory of three free particles is not equivalent to the theory of two free particles. Absent evidence to the contrary, this agreement can be expected to carry over to field theories as well. Even if it turns out that different interpretations disagree about the status of parastatistics, at least some interpretations (such as Everett/many worlds) possess the exact same theoretical structure as standard QM. Hence our claims here should carry over straightforwardly at least to those interpretations. Lastly, a proof of theoretical equivalence in the partially interpreted framework could place interesting constraints on the interpretive project itself. The task of interpretation does not start from unassailable foundational axioms, but rather begins *in medias res*. Questions of theoretical equivalence are not just an afterthought, but rather provide crucial pieces of data to be brought into reflective equilibrium with the to-

[17] Of course a proof of empirical or partial theoretical equivalence would still go some way towards explaining why we have never *observed* paraparticles in nature.

[18] For example, all extant interpretations agree that the process of decoherence occurs in systems correlated with a large environment, although the physical details are characterized differently. In Bohmian mechanics, decoherence brings about the effective collapse of the wave function, while in the contemporary Everett/many-worlds interpretation, decoherence is what explains the branching structure of the multiverse of outcomes.

tality of interpretive evidence. For instance, the Stone-von Neumann theorem in standard QM and the existence of the Legendre transformation interpolating between Hamiltonian and Lagrangian models of classical mechanics have proven to be crucial fixed points for interpreters of those theories.

Tabling these broader methodological issues, we now turn to the details of the case at hand. First we introduce the mathematical structure of field systems in algebraic quantum field theory (AQFT). Then we argue that the technical notion of *quasiequivalence* provides a sufficient condition for theoretical equivalence of field systems.

3.1 Field systems in AQFT

In many ways, the basic formalism of AQFT is a simple generalization of the familiar Hilbert space formalism of ordinary quantum mechanics. Because our central question concerns, in part, the status of unobservable fields, we must explain how such entities are represented in AQFT, which will complicate matters somewhat.

The most basic notions are those of an algebra of observables and a state on that algebra. A theory's algebra of observables $\mathfrak{A}$ is a C*-algebra composed of a collection of operators denoting physical quantities, whose self-adjoint elements stand for real-valued quantities which are (in principle) measurable. States then correspond to probabilistic predictions about which values these quantities will take on. Formally, we may represent such a state by a (normed, linear) functional $\omega : \mathfrak{A} \to \mathbb{C}$, which assigns a (complex-valued) expectation value to every operator in the algebra.

These two elements, algebras and states, will be our basic tools. The first complication we must introduce is relativity. For our AQFT to conform to the laws of special relativity, quantities must be assigned to regions of Minkowski spacetime. Thus we must assign a subalgebra of $\mathfrak{A}$ to every region. For convenience, we will restrict ourselves to double cones, the "diamond-shaped" open regions O given by the intersection of a future-directed light cone and a past-directed light cone. For every such region, we define a subalgebra $\mathfrak{A}(O) \subseteq \mathfrak{A}$ of operators localized within O. The collection of all these local algebras $\mathfrak{A}(O)$ is called a *net* of algebras. In relativistic field theory, we equip the net with a natural representation of the Poincaré group to ensure that the theory obeys the symmetries of special relativity. The observables also satisfy the axiom of microcausality, which requires that all observables in $\mathfrak{A}(O)$ commute with all observables in $\mathfrak{A}(O')$ if O and O' are spacelike separated. These posits, along with some more technical ones that will not concern us here, form the Haag-Kastler axioms for AQFT.

The resources of the more well-known Hilbert space formalism of quantum theory will also be useful for some purposes. Fortunately, the connection between the algebraic and Hilbert space formalisms is well understood. A *representation* is a Hilbert space with a distinguished algebra of operators that mirror the algebraic structure of $\mathfrak{A}$. Mathematically, a representation of $\mathfrak{A}$ is a pair (H, π) consisting of a Hilbert space H and a *-homomorphism π from $\mathfrak{A}$ into $\mathbf{B}(H)$ (the bounded operators on H). The density operators on H will then correspond to a subset of the states on $\mathfrak{A}$. Furthermore, every state ω is associated with a unique "home" representation, its GNS representation, where it is represented by a cyclic vector.

To represent fields with parastatistical (and regular Fermi) commutation relations, we'll need a way of defining unobservable field operators. A suitably general method is to make the observable algebra $\mathfrak{A}$ a subalgebra of a *field algebra* $\mathfrak{F}$ which contains the unobservable fields along with the observables. In particular, we may define a group G of internal symmetries of the fields in $\mathfrak{F}$, in such a way that $\mathfrak{A}$ is the subalgebra invariant under these symmetries. Such a group is called a *gauge group* (although the symmetries are global rather than local).

The formal structure needed is a *field system with gauge symmetry:*

Field System. Let $\mathfrak{A}$ be a net of C^*-algebras satisfying the Haag-Kastler axioms, let ω_0 be a vacuum state, and let (H_0, π_0) be the GNS representation of $\mathfrak{A}$ induced by ω_0. A 4-tuple $(\mathfrak{F}, H, \pi, G)$ is a *field system with gauge symmetry* for $\mathfrak{A}$ and ω_0 just in case:

1. π is a representation of $\mathfrak{A}$ on H containing (H_0, π_0) as a subrepresentation;
2. G is a compact group of unitary operators acting on H and leaving H_0 pointwise invariant;
3. $O \mapsto \mathfrak{F}(O)$ is a net of algebras acting on H (not necessarily satisfying microcausality!), and the field algebra $\mathfrak{F}$ acts irreducibly on H;
4. the $g \in G$ act as automorphisms of each $\mathfrak{F}(O)$, with $\mathfrak{A}(O)$ as the fixed points;
5. for each double cone O, H_0 is cyclic for $\mathfrak{F}(O)$ (i.e. any sector can be reached from the vacuum sector);
6. if O_1 is spacelike to O_2 then $[\pi(\mathfrak{A}(O_1)), \mathfrak{F}_2(O_2)] = \{0\}$, i.e. observables commute with spacelike separated field operators.

The motto behind this formalism is: the observables are the gauge-invariant fields.

Example. Consider the free Fermi field. In this case, the field algebra $\mathfrak{F}$ is the algebra of the canonical anticommutation relations (CAR algebra). Since the basic field operators anticommute at spacelike distances, if they were observables they would violate the microcausality axiom of AQFT, which states that spacelike-separated observables must commute.

But there is an automorphism γ of $\mathfrak{F}$ that acts on its generators (the annihilation operators $a(f)$) by

$$\gamma(a(f)) = -a(f).$$

The algebra $\mathfrak{A}$ of fixed points under γ is called the even CAR algebra. This is the algebra of observables for the free Fermi theory. Since there is a single non-trivial symmetry that we "mod out" to get the observables, the theory's gauge group (consisting of that symmetry plus the identity) is isomorphic to $\mathbb{Z}_2$, the group of integers modulo two.

In addition to accommodating Fermi and parastatistical fields, field systems are also useful in the theory of superselection rules, which will be relevant to some aspects of our study. A superselection rule is a law of nature that forbids the superposition of states with different values of some physical quantity (the superselected quantity or "charge"). For example, in nature we never find physical systems in superpositions of different values of electric charge. Thus we say that each value of charge corresponds to a distinct superselection sector.

The gauge group G of a field system gives rise to its characteristic superselection structure. The Hilbert space H decomposes under the action of G into a direct sum of superselection sectors which transform irreducibly under G and hence correspond to irreducible representations of the observable algebra. These correspond to the sectors that can be reached from the vacuum sector by the action of local unobservable fields in $\mathfrak{F}$. Hence the action of the field operators on states serves to move states between different superselection sectors.

The mathematically rigorous theory of superselection sectors is, at present, only well understood within a limited domain. This is relevant for our purposes because the domain of superselection theory will also limit the applicability of our argument for the equivalence thesis. The canonical method is that of Doplicher *et al.* (1969), often called the DHR approach. DHR theory proceeds by identifying a privileged vacuum state, and designating its home (GNS) representation as the neutral or zero-charge superselection sector. The physically admissible states are required to satisfy the

DHR selection criterion: Let (H_0, π_0) be the GNS representation induced by the privileged vacuum state ω_0 of $\mathfrak{A}$. A representation (H, π) of $\mathfrak{A}$ is *DHR* iff (1) for each Minkowksi double cone O, the representations $\pi_0|_{\mathfrak{A}(O')}$ and $\pi|_{\mathfrak{A}(O')}$ are unitarily equivalent; and (2) (H, π) possesses finite statistics, that is, a finite-dimensional representation of the permutation group. Here O' is the spacelike complement of O, $\pi|_{\mathfrak{A}(O')}$ is the restriction of the representation π to the subalgebra $\mathfrak{A}(O')$, and $\mathfrak{A}(O')$ is the C^*-algebra generated by $\mathfrak{A}(O_1)$ with O_1 a double cone spacelike separated from O. A state is DHR iff it is representable by a density operator in a DHR representation.

Intuitively, a DHR state initially differs from the vacuum (in its expectation values for quantities in $\mathfrak{A}$) only within some finite region, within which the state's charge is considered localized. The DHR condition restricts the theory's domain to charges whose effects propagate only finitely far. In particular, charged states in electromagnetism — whose effects can be observed at any arbitrary distance — do not meet the DHR condition.

There is considerable promise for extending a DHR-type theory of superselection even to electrodynamics (Buchholz *et al.* (in preparation)), and it has already been extended to a more general class of theories with charges localized within a single spacelike cone (Buchholz and Fredenhagen, 1982). But for now, since our argument will depend crucially on results from superselection theory, it will apply only to those theories of short-ranged forces whose states meet the DHR condition.

3.2 Equivalence of field systems

In order to determine whether two field systems are theoretically equivalent, we must look for a mapping (morphism) between the two systems that preserves both their empirical predictions and (physically significant) theoretical structure. Viewed in the abstract, quantum theories consist of a collection of physical quantities, a collection of kinematically possible states, and a specification of dynamically possible histories. Equivalent field systems must agree on which quantities, states, and histories are physically possible. In addition each of these collections is not a bare set; there are important relations within and between each category of theoretical structure that must be preserved as well.

According to the standard credo of AQFT, physical quantities are determined by the structure of the algebra of observables. Thus theoretical equivalence at a minimum requires a $*$-isomorphism of observable algebras.[19] What about operators in $\mathfrak{F}$ that are not elements of $\mathfrak{A}$? For purposes of adjudicating this question, the term standardly used to refer to these quantities is unfortunate. "Unobservable fields" sound like exactly the sort of thing a scientific realist should be a realist about. But when their theoretical role is taken into account, it becomes far from clear that anyone should accept realism about operators living in $\mathfrak{F}$ but not $\mathfrak{A}$.

To begin with: the unobservable fields are not invariant under gauge symmetries. The gauge symmetries of an AQFT field system are global internal symmetries, analogous to the phase symmetry of basic QM. So as with phase, one would expect that the physically signifi-

[19]It is possible that the physical quantities form a proper subalgebra of $\mathfrak{A}$, or that not all algebraic relations between observables are physically relevant. Since we are concerned with a sufficient condition for theoretical equivalence, this will not be an issue.

cant quantities should be left unchanged by these symmetries. This notion is highly plausible in light of a broad consensus that symmetries are analogous to coordinate transformations, in that applying them to a state changes nothing physical about the system described by that state.[20]

This is not to say that $\mathfrak{F}$ is purely a notational convenience. As discussed in the previous section, important physical information about the superselection structure of local charges is encoded in $(\mathfrak{F}, H, \pi, G)$. Unobservable elements of the field algebra map states between different superselection sectors. In effect, they express relationships between states from different sectors, without corresponding to quantities whose values are manifested by those states. Thus superselection theory provides a plausible story according to which the unobservable fields possess significant theoretical utility, but do not denote physical quantities.

Again, the free Fermi field serves as a useful example. In this case, a representative example of an unobservable field operator is the fermion creation operator $a^\dagger(f)$. Given a state ψ, $a^\dagger(f)\psi$ gives the same state, but with an additional particle whose wavefunction is f. So $a^\dagger(f)$ expresses a relationship between states whose particle numbers differ by one.

As in any AQFT, the free Fermi theory's superselection sectors correspond to the irreducible representations of its gauge group. Since $\mathbb{Z}_2$ has two irreducible representations, there are two of these — the even subspace, containing all states of even particle number (including the zero-particle vacuum), and the odd subspace, which contains states of odd particle number. Obviously $a^\dagger(f)$ will always take a state from one of these sectors to another. So it also represents a relationship between states in different superselection sectors. These relations between states expressed by the creation operator exhaust its theoretical role. In particular, no one would expect the expectation value of the creation operator to be of any physical interest. It does not stand for a physically significant quantity, the way the operators for spin, mass and energy do.

Despite this, there are operators outside the algebra of observables which appear to represent physically significant quantities. These are the so-called "parochial observables" belonging to the weak closure $\pi(\mathfrak{A})^-$ of $\mathfrak{A}$ in a given Hilbert space representation (H, π) of $\mathfrak{A}$; they cannot be defined independent of a representation. Examples include temperature, the stress-energy tensor and (in Fock space theories) global particle number. The existence of physically significant parochial observables has been cited as an important reason to deny that all physically meaningful quantities are given by operators in $\mathfrak{A}$ (Ruetsche, 2002).

There is more to a theory than its specification of quantities. For example, the Klein-Gordon theory of the free Bose field and the ϕ^4 theory of a self-interacting Bose field share

[20]Belot (forthcoming) has, however, raised concerning objections to this orthodoxy.

the same algebra of observables, but no one in their right mind would claim that these two theories are theoretically equivalent. One is a free (linear) theory, while the other describes a non-linear interaction. The difference between these two theories manifests not in the algebra of observables, but in which states are counted as physically possible (as well as in the set of parochial observables).

What does it take for two field systems to agree about the parochial observables and physically admissible states? Two field systems will agree on their expectation values for all observables, parochial and otherwise, if they are *quasiequivalent*.[21] Quasiequivalence essentially requires that the field systems' weak closure algebras (the algebras containing the parochial observables) are isomorphic, and that the isomorphism preserves the structure of the observable algebra $\mathfrak{A}$. Precisely:

Quasiequivalence: Representations π and π' of $\mathfrak{A}$ are *quasiequivalent* iff there is a *-isomorphism α from $\pi(A)^-$ onto $\pi'(A)^-$ and $\alpha(\pi(A)) = \pi'(A)$ for all $A \in \mathfrak{A}$. We will call two field systems for $\mathfrak{A}$ quasiequivalent if their representations of $\mathfrak{A}$ are quasiequivalent.

By this definition, quasiequivalent representations always share the same roster of parochial observables, with the same functional relations among them. Moreover, quasiequivalent representations always share the same folium of states (i.e. the same states of $\mathfrak{A}$ are representable by density operators on their Hilbert spaces). So the states of quasiequivalent representations agree about the expectation values of all observables, including parochial ones. As a consequence, quasiequivalent representations also possess the same superselection structure.

There are, of course, other physically significant features of a quantum field theory, but in AQFT these depend on the states and observables. Inner products of pure algebraic states, for example, can be defined as a function of those states' expectation values (Roberts and Roepstorff, 1969). And because AQFT is, effectively, formulated within the Heisenberg picture, field systems which agree (everywhere in spacetime) on the observables and on the space of states must have the same dynamics. So long as we stick to basic quantum theory, with unitary dynamics and no hidden variables, quasiequivalence offers everything we require of a sufficient condition for physical equivalence.[22]

[21] More precisely: if their respective representations of $\mathfrak{A}$ are quasiequivalent.

[22] For irreducible representations, quasiequivalence reduces to unitary equivalence. This can be seen by applying the following (equivalent) definition: Two representations are quasiequivalent iff every subrepresentation of one is unitarily equivalent to some subrepresentation of the other.

4 A brief history of the equivalence thesis

Given a parafield theory $(\mathfrak{F}_P, H_P, \pi_P, G_P)$ for $(\mathfrak{A}, \omega_0)$ whose field operators obey generalized trilinear commutation relations as discussed in §2, a series of results (primarily due to H.S. Green and Huzihiro Araki) show that it is always possible to find a corresponding theory $(\mathfrak{F}, H, \pi, G)$ with ordinary commutation relations and the same observable algebra $\mathfrak{A}$. The ordinary field theory is generated from the parafield theory by applying a series of non-local *Klein transformations* to the underlying field algebra $\mathfrak{F}_P$. Drühl, Haag, and Roberts (1970) extend these results, demonstrating that for certain gauge groups, G_P, the Klein transformation preserves not only the observable algebra $\mathfrak{A}$, but also the field system's superselection structure. Other natural choices of G_P are ruled out as unphysical. However, their conclusion is not fully general since only a handful of possible field systems are examined. The Doplicher-Roberts reconstruction theorem (Doplicher and Roberts (1990)) offers a completely different approach that avoids the explicit construction of Klein transformations. A direct corollary of the theorem asserts that any parafield system whose superselection structure satisfies certain completeness conditions will be quasiequivalent to some complete ordinary field system. Again, the result fails to be fully general since the corollary breaks down when the field systems in question are not complete. As we shall prove in §5, however, there is a general physical principle that can be drawn from the critique of Drühl *et al.*, which entails that any physically reasonable field system must be complete, thus tying these two historical strands together and establishing a general version of the equivalence thesis for field systems.

4.1 The Green decomposition

The construction of $(\mathfrak{F}, H, \pi, G)$ from $(\mathfrak{F}_P, H_P, \pi_P, G_P)$ proceeds in two stages. The first stage draws upon pioneering work done by H.S. Green in the early 1950s on parafield theories. (It was Green who first explored the connection between paraparticles and generalizations of second-quantization discussed in §2.) Green showed that any parafield operator, $\phi(x)$, obeying trilinear commutation relations can be uniquely decomposed into a sum of component fields obeying ordinary bilinear commutation relations:

$$\phi(x) = \sum_{i=1}^{p} \phi_i(x), \tag{12}$$

The number of Green components is equal to the order of the parafield, and although the $\phi_i(x)$ satisfy ordinary commutation relations, they exhibit the wrong connection between spin and statistics. According to the spin statistics theorem, if the fields ϕ_1, ϕ_2 have integer spin and ψ_1, ψ_2 have half-integer spin, then

$$\begin{aligned}[\phi_1(x), \phi_2(y)] &= 0 \\ \{\psi_1(x), \psi_2(y)\} &= 0 \\ [\phi(x), \psi(y)] &= 0 \quad , \end{aligned} \tag{13}$$

for $(x-y)^2 < 0$ (spacelike separation). This represents the "right" connection between spin and statistics. Bosons commute with each other and with fermions, and fermions anticommute with each other. In contrast, Green components have abnormal commutation relations. Components with different indices anticommute when they represent fields with integer spin and commute when they represent fields with half-integer spin:

$$\begin{aligned}\{\phi_i(x), \phi_j(y)\} &= 0 \\ [\psi_i(x), \psi_j(y)] &= 0 \end{aligned} \tag{14}$$

Because of their abnormal spin-statistics properties, the individual Green components are physically pathological, lacking even the attenuated interpretive significance of the field operators in $\mathfrak{F}_P$ or $\mathfrak{F}$. They are normally viewed as a convenient way to rewrite the parafields in $\mathfrak{F}_P$, nothing more. The field algebra generated by the Green components, $\mathfrak{F}_G$, properly contains $\mathfrak{F}_P$.[23] Consequently, the Hilbert space $H_G = \overline{\mathfrak{F}_G\Omega}$ is larger than $H_P = \overline{\mathfrak{F}_P\Omega}$. In order to compensate, the gauge group of the Green field system must be enlarged too. Only particular combinations of fields in $\mathfrak{F}_G$ carry any theoretical weight. For instance, the vacuum state in H_G is defined by the condition that $(\phi_i(x) + \ldots + \phi_j(x))|\Omega\rangle = 0$.

4.2 Klein transformations

Exploiting earlier work by Klein, Araki (1961) provided a general proof indicating how the Green field algebra, $\mathfrak{F}_G$ could be reparametrized in order to restore the right connection between spin and statistics for the basic fields while preserving the structure of the observable algebra. The basic idea is as follows: recall that observables are constructed out of algebraic combinations of field operators. Let $M \in \mathfrak{A}$ be a monomial of the basic fields $\phi_i(x)$. M's commutation relations underdetermine the commutation relations of the basic fields. Indeed, the commutation relations of M with respect to all other observables only depend on the

[23]Formally, $\mathfrak{F}_G$ is the net generated by $\mathfrak{F}_G(O) = \mathfrak{F}_1(O) \otimes \ldots \otimes \mathfrak{F}_n(O)$, where $\mathfrak{F}_i(O)$ is the Clifford algebra generated by all ith-indexed Green components with support in O.

parity of the powers of the basic fields comprising M (i.e. whether each basic field occurs an even or odd number of times in the monomial). The commutation relations for arbitrary polynomials of the basic fields are in turn determined by the relations for their constituent monomials. Thus, an observable can obey normal bilinear commutation relations with other observables even if its constituent fields obey abnormal bilinear commutation relations. A Klein transformation swaps out abnormal fields in favor of normal ones, but does so in a way that leaves the commutation relations between observables unaffected.

Araki's proof proceeds by showing that any theory with parafields possesses certain symmetries called *even-oddness conservation laws*. A theory $(\mathfrak{F}, H, \pi, G)$ possesses an even-oddness conservation law for a set of fields $\alpha \subseteq \mathfrak{F}$ if every vacuum expectation value containing an odd number of fields from α vanishes.[24] Let H_e and H_o denote the spaces generated from the vacuum by the action of operators which are polynomials, even/odd respectively in fields from α. Because of the even-oddness law for α, these spaces are orthogonal. Moreover, both spaces are invariant under the action of the (proper orthochronous) Poincaré group, $\mathcal{P}_+^\uparrow$. This entails that the self-adjoint, unitary operator $\mathbf{p}(\alpha)$, which is equal to 1 on vectors in H_e and -1 for vectors in H_o, is well defined and commutes with the unitary representation $U(\mathcal{P}_+^\uparrow)$ on H. For every additional set $\beta \subseteq \mathfrak{F}$, there exists a Klein transformation relative to this conservation law. We define the Klein transformation $K_\beta^\alpha(\phi) = \tilde{\phi}$ as follows:

$$\begin{aligned} \tilde{\phi} &= \mathbf{p}(\alpha)\phi \ , \qquad && \text{if } \phi \in \beta \\ \tilde{\phi} &= \phi \ , && \text{if } \phi \notin \beta \end{aligned} \tag{15}$$

for all $\phi \in \mathfrak{F}$. The unitary $\mathbf{p}(\alpha)$ anticommutes with ϕ if $\phi \in \alpha$, and commutes with ϕ if $\phi \notin \alpha$. Hence the Klein transformation has the following effect on commutation relations: for any two fields $\phi_1, \phi_2 \in \mathfrak{F}$, if one is in α and one is in β, and at least one is not in both α and β, then their commutation relations are altered (i.e. $[\phi_1, \phi_2] \to \{\tilde{\phi}_1, \tilde{\phi}_2\}$, and vice versa), otherwise they are unchanged. The set β is a free parameter that can be chosen to change the abnormal commutation relations into normal ones.

As an explicit example, we consider a field system consisting of a single para-Fermi field of order $p = 3$. The field operator $\psi(x)$ can be decomposed into three Green components satisfying abnormal commutation relations; components with the same Green index obey ordinary anticommutation rules,

$$\{\psi_i(x), \psi_i(y)^*\} = \delta^3(x-y), \quad \{\psi_i(x), \psi_i(y)\} = 0 \quad \forall\ i = 1,2,3\ , \tag{16}$$

[24] Here it is necessary to assume that the vacuum state is cyclic with respect to all fields in $\mathfrak{F}$.

but components with different indices commute

$$[\psi_i, \psi_j^*] = 0, \quad [\psi_i, \psi_j] = 0 \quad \forall\, i,j = 1,2,3 \; . \tag{17}$$

Since observables of the theory will be polynomials of the parafield and (its adjoint), gauge invariance requires that only polynomials containing all three Green components can be candidates for observables. Thus for any two components, there is an even-oddness conservation law. Let α be the set consisting of ψ_2, ψ_3. We choose β to be the set consisting of ψ_1, ψ_2. The associated Klein transformation yields:

$$\tilde{\psi}_1 = \mathbf{p}(\alpha)\psi_1 \;, \quad \tilde{\psi}_2 = \mathbf{p}(\alpha)\psi_2 \;, \quad \tilde{\psi}_3 = \psi_3 \tag{18}$$

(An identical transformation is applied to the adjoint of each Green component.) The new Klein transformed fields satisfy ordinary anticommutation relations,

$$\{\tilde{\psi}_i, \tilde{\psi}_j^*\} = 0, \quad \{\tilde{\psi}_i, \tilde{\psi}_j\} = 0 \quad \forall\, i,j = 1,2,3 \; , \tag{19}$$

restoring the proper connection between spin and statistics.[25]

The construction method illustrated by the example is completely general. Araki proved that any parafield system satisfying microcausality and cluster decomposition will have enough even-oddness conservation laws to generate the requisite sequence of Klein transformations.[26] Thus, given a physically realistic parafield system, we can always translate it into an ordinary field system with the same observable algebra.[27] Moreover, one can show that the Wightman functions of the two theories differ by at most a sign, and hence their S-matrix elements are the same. This ensures that the two theories will agree on the outcomes of all scattering experiments.[28] Since the individual Klein operators K^α_β are unitary, there is an inverse Klein transformation; starting with an ordinary field system possessing an appropriate non-abelian gauge group, the inverse Klein transformation returns an equivalent parafield system.

[25] Note that Klein transformed fields with the same green index continue to satisfy relations analogous to equation (16).

[26] For an excellent exposition of the details of the proof, see Streater and Wightman (1989), §4.4.

[27] The restriction of a Klein transformation to $\mathfrak{A}$ is just the identity operator.

[28] Araki (1961).

4.3 The argument of Drühl, Haag, and Roberts

While Araki's result is strong — demonstrating the existence of a (reversible) translation preserving physical quantities (the observable algebra $\mathfrak{A}$), possible states (understood as linear functionals on $\mathfrak{A}$), dynamics (understood as privileged automorphisms of the net of observables), and experimental outcomes — it does not address the status of parochial observables or the superselection structure of the two field systems. In their 1970 paper Drühl, Haag, and Roberts attempt to fill in this gap. Their claim is that, if we impose the DHR condition (discussed in §3.1), every physically admissible parastatistical QFT is physically equivalent to some non-parastatistical theory.[29]

They begin by considering the case of an order-2 para-Fermi field algebra, $\mathfrak{F}_p$. For any physically permissible para-field system, the local observable algebras associated with a given spacetime region O_i must be subalgebras of $\mathfrak{F}_p(O_i)$ and satisfy microcausality. Subject to these constraints, Drühl *et al.* demonstrate that the net of observables must be a subnet of $\mathfrak{A}_0$, the net consisting of local algebras generated by even combinations of field components from $\mathfrak{F}_p$. These correspond to combinations that are invariant under the gauge transformation sending $\psi(x) \to -\psi(x)$. The relevant gauge group is $\mathbb{Z}_2$.

In addition to $\mathfrak{A}_0$, they consider two other natural choices for the algebra of observables, $\mathfrak{A}_1$, and $\mathfrak{A}_2$, the subnets of $\mathfrak{A}_0$ generated by commutators and chargeless combinations of the field components, respectively.[30] This gives rise to two additional parafield systems with gauge groups G_1 and G_2. Using the Green decomposition theorem and Klein transformations, they translate all three of these theories into ordinary field systems:

$$\begin{aligned}
(\mathfrak{F}_P, \mathfrak{A}_0, \mathbb{Z}_2) &\to (\mathfrak{F}, \mathfrak{A}_0, SO(2)) \\
(\mathfrak{F}_P, \mathfrak{A}_1, G_1) &\to (\mathfrak{F}, \mathfrak{A}_1, O(2)) \\
(\mathfrak{F}_P, \mathfrak{A}_2, G_2) &\to (\mathfrak{F}, \mathfrak{A}_2, U(2))
\end{aligned} \tag{20}$$

In addition to sharing the same observables, Drühl *et al.* show that the $\mathfrak{A}_2$ para-theory has the same superselection structure as the ordinary $\mathfrak{A}_2$ Fermi theory. That is, the two theories share the same folium of states, which breaks down into the same family of superselection sectors. This is not true for the $\mathfrak{A}_0$ and $\mathfrak{A}_1$ para-theories, which possess different superselection structure than their corresponding Fermi theories. In both these cases, the

[29] At the time, the full account of DHR superselection theory had yet to be published. As a result, their argument is presented somewhat opaquely. In our exposition we will take advantage of these subsequent developments to present a significantly streamlined version of their main argument.

[30] Formally, these generators are characterized most easily by their Green decompositions. The generator of the net $\mathfrak{A}_1$ is $r(x,y) = \psi_1(x)\psi_1(y) + \psi_2(x)\psi_2(y)$. The generator of $\mathfrak{A}_2$ is $\varrho(x,y) = \psi_1(x)^*\psi_1(y) + \psi_2(x)^*\psi_2(y)$.

parastatistical states occupy only a proper subset of the superselection sectors occupied in the corresponding ordinary theory. But Drühl *et al.* argue that the $\mathfrak{A}_0$ and $\mathfrak{A}_1$ para-theories can be ruled out on physical grounds, for exactly this reason: these theories' folia of states are incomplete in a physically problematic way.

Although they go on to generalize their argument to include para-Fermi fields of arbitrary order, they do not consider the case of para-Bose theories. In addition they offer no argument that the three nets considered are the only permissible algebras of observables for parafield theories. Indeed, there are many other possible algebras. But their argument that the $\mathfrak{A}_0$ and $\mathfrak{A}_1$ theories are incomplete is highly suggestive. Considering the $\mathfrak{A}_0$ para-theory, they note that it possesses only two superselection sectors, the charge 0 and charge 1 sectors, even though a localized field operator which generates negative charge in a region can be defined:

> Furthermore only a small part of the physically relevant states are described by the vectors in H_P. We have for example no states with total charge -1 in H_P, although we do have states in which the charge of a certain region is -1. The true physical content of the theory in this case is completely equivalent to that of a theory with Fermi statistics; the parastatistics are here only simulated by an artificial and physically inadmissible restriction on the manifold of states which are considered. (Druhl *et al.*, 1970, 215)[31]

This restriction does seem peculiar from the standpoint of physical intuition. If the laws allow a region (what's more, any compact region of any finite size) to contain total charge -1, what could motivate us to conclude that the same laws forbid a total charge of -1 for the global state (that is, for the whole universe)? This is especially peculiar given the nature of the localized charges described by theories meeting the DHR condition. Such charges are localized in a very strong sense, in that they make no difference to observables located at spacelike distances from the finite region where they are localized. In §5 we will distill this intuition into a physical principle forming the core of a much more general argument in favor of the equivalence thesis. The generality that is lacking in the arguments of Drühl *et al.* will come from a different source: the Doplicher-Roberts reconstruction theorem.

4.4 The Doplicher-Roberts reconstruction theorem

The Doplicher-Roberts reconstruction theorem is the capstone to the DHR analysis of superselection structure in AQFT. The theorem demonstrates that imposing the DHR selection

[31] Although they do not analyze the case in detail, Drühl *et al.* conclude that similar reasoning applies in the $\mathfrak{A}_1$ case: "the parafield description is again inadequate in this case because H_P does not contain all the relevant states over $\mathfrak{A}_1$" (216).

criterion on states of a given observable algebra, $\mathfrak{A}$, suffices to generate a unique, privileged field system for $\mathfrak{A}$:

DR Reconstruction Theorem. Let $\mathfrak{A}$ be an algebra of observables satisfying the axioms of AQFT and ω_0 a vacuum state on $\mathfrak{A}$. Then there exists a unique, complete field system $(\mathfrak{F}, H, \pi, G)$ of $\mathfrak{A}$ with normal commutation relations.

It is important to flag three features of the theorem. First, the privileged field system is required to satisfy ordinary statistics, not parastatistics. Second, it must be *complete*, meaning that the states of its folium include all (and only) DHR states of $\mathfrak{A}$. Third, the claim of uniqueness is made up to *DR-equivalence*, a much stronger notion of equivalence than quasiequivalence. DR-equivalence requires an isomorphism between the field algebra and the gauge groups in question, not just between the representations of the observable algebra.[32]

The task of superselection theory is then, in essence, to explain why the superselection sectors of this privileged field system correspond one-to-one with the irreducible representations of the gauge group G. This is accomplished by noticing that in AQFT, unitarily inequivalent irreducible representations of the algebra of observables function as superselection sectors. Since our field system is complete, its irreducible, unitarily inequivalent subrepresentations of $\mathfrak{A}$ are the DHR representations of $\mathfrak{A}$. And it can be shown that the DHR representations of $\mathfrak{A}$ correspond one-to-one with the irreducible representations of the gauge group assigned to $\mathfrak{A}$ by DR reconstruction (Baker and Halvorson, 2010). Thus complete field systems contain all of the superselection structure implied by DHR theory.[33]

When applied to an algebra of observables $\mathfrak{A}$, the DR reconstruction gives us a complete field system for $\mathfrak{A}$ with ordinary commutation relations. This offers another possible route for establishing the equivalence thesis that avoids the explicit construction of Klein transformations and would hold for para-Fermi and para-Bose systems of arbitrary order. It has occasionally been suggested that the reconstruction theorem by itself is sufficient to establish the equivalence of para- and ordinary statistics. For example, in a review article Schroer (2001) writes,

[32]Specifically, DR-equivalence requires that there is a unitary operator $W : H_1 \to H_2$ such that (i) $W\pi_1(A) = \pi_2(A)W$, $\forall A \in \mathfrak{A}$, (ii) $WU(G_1) = U(G_2)W$, and (iii) $W\mathfrak{F}_1(O) = \mathfrak{F}_2(O)W$, for each double cone O. Conditions (ii) and (iii) are not guaranteed by quasiequivalence, but neither are they necessary for physical equivalence, by our argument of §3.2.

[33]More precisely, a theory's superselection sectors will be isomorphic to the category of DHR representations of its algebra of observables. Thus a field system $(\mathfrak{F}, H, \pi, G)$ with algebra of observables $\mathfrak{A}$ is complete iff the representation π of $\mathfrak{A}$ contains (as subrepresentations) copies of all representations of $\mathfrak{A}$ meeting the DHR condition.

> [T]he general statement, that it is always possible to convert parastatistics into Fermi/Bose statistics (plus multiplicities for an internal symmetry group to act on), is one of the most nontrivial theorems in particle physics. For its proof one needs the full power of the superselection theory in local quantum physics as well as some more recent group theoretical tools.

He ends this passage by citing Doplicher and Roberts (1990), the implication being that Doplicher and Roberts proved the equivalence thesis. In fact, the DR theorem implies this only in a special class of cases — namely, for parafield theories corresponding to *complete* field systems. This is a direct corollary of the fact that any two complete field systems with the same algebra of observables are quasiequivalent:

Fact. *If* $(\mathfrak{F}_1, H_1, \pi_1, G_1)$ *and* $(\mathfrak{F}_2, H_2, \pi_2, G_2)$ *are complete field systems for* $(\mathfrak{A}, \omega_0)$ *then* (H_1, π_1) *and* (H_2, π_2) *are quasiequivalent representations of* $\mathfrak{A}$*.*

Proof. To begin, it is known that if $(\mathfrak{F}, H, \pi, G)$ is a field system for $(\mathfrak{A}, \omega_0)$, then π is a direct sum of irreducible representations, each of which is a DHR representation of $\mathfrak{A}$. (For a proof of this fact, see p. 809ff of Halvorson and Müger (2006).)

A field system $(\mathfrak{F}, H, \pi, G)$ is complete just in case each DHR representation of $\mathfrak{A}$ is a subrepresentation of π. Thus, if a field system is complete then: (a) every DHR representation of $\mathfrak{A}$ occurs as a subrepresentation of π, and (b) every subrepresentation of π is a DHR representation. It follows then that if $(\mathfrak{F}, H, \pi, G)$ is complete, then the folium of π consists of all and only DHR states. It is well-known that the following two conditions are equivalent:

(1) $\pi_1(\mathfrak{A})$ and $\pi_2(\mathfrak{A})$ are quasiequivalent.

(2) The folium of π_1 is equal to the folium of π_2.

(For a proof, see Bratteli and Robinson (1981), Theorem 2.4.26 and Kadison and Ringrose (1997), Proposition 10.3.13.) Therefore, if $(\mathfrak{F}_1, H_1, \pi_1, G_1)$ and $(\mathfrak{F}_2, H_2, \pi_2, G_2)$ are complete field systems for $(\mathfrak{A}, \omega_0)$, then π_1 and π_2 are quasiequivalent representations of $\mathfrak{A}$.

□

For parafield systems that are complete, the DR reconstruction theorem thus provides a strong, general case for the equivalence thesis. Given such a system, the theorem guarantees the existence of a complete ordinary field system that agrees on which states are possible, and on their expectation values for quantities in the algebra of observables as well as for all *parochial* observables. The only potential grounds for denying full theoretical equivalence in such a case would be if we counted unobservable fields as physically significant quantities. As we've seen (§3.2), this position is difficult to support.

For incomplete parafield systems, however, the picture is quite different. Since the ordinary field system supplied by the DR reconstruction theorem is complete, the two theories will disagree about which states are physically possible. The parafield theory's states will be a proper subset of the ordinary field system's, so the para-theory recognizes fewer physical possibilities even though the two theories share the same algebra of observables. The argument sketched here is silent about the possible existence of an *incomplete* ordinary field system with the same superselection structure as the parafield theory. An additional argument is needed either to fill in this gap or to rule out incomplete field systems altogether. It is to this task that we now turn.

5 Sharpening the thesis

The prospects for a general theorem translating incomplete parafield systems into equivalent ordinary field systems look dim. The category theoretic basis for the DR reconstruction theorem crucially relies on the completeness restriction, so there's no obvious analogue to the argument from §4.4 in sight. Additionally, as Drühl *et al.* showed, the Klein transformation of an incomplete field system is not guaranteed to preserve the superselection structure of the theory. Indeed, this is precisely what goes wrong in the $\mathfrak{A}_0$ and $\mathfrak{A}_1$ case. The corresponding parafield theories are not complete: their Hilbert spaces do not contain all of the states deemed physical by the lights of the DHR criterion. Their Klein transformed counterparts do have the missing sectors, however, rendering them physically inequivalent.[34]

If the equivalence thesis is to have any legs, the second strategy seems to be the best option. Is there a natural physical principle that would rule out incomplete field systems? For field systems with localized charges, we believe the answer is yes.

The seed for this principle lies within the arguments supplied by Drühl *et al.* Recall their rejection of the $\mathfrak{A}_0$ para-Fermi theory. The problem was that this theory does not permit states with a certain global charge, even though states with the very same local charge are allowed. They are implicitly appealing to a principle about what we should look for in physically admissible quantum field theories. Given that Q is a physically possible value

[34]One might try the following procedure: take an incomplete parafield system and find its Klein transformed counterpart. Then identify the superfluous sectors in the Klein transformed theory and toss them out. The problem here is that in order to maintain the standard connection between the gauge group and superselection structure, the gauge group must be appropriately enlarged. There's no guarantee that there will be a natural choice of a (compact) gauge group for the ordinary field system that rules out all and only the offending sectors.

for the charge of a region, we should prefer (as better motivated, or less *ad hoc*) theories according to which it is physically possible for Q to be the total charge of the whole universe. Let's call this criterion

Charge Recombination. If it is physically possible for any arbitrary finite region to possess charge Q, then it must be physically possible for the global charge to be Q.

Why prefer theories satisfying Charge Recombination? First, out of a sense of methodological conservatism: all of our classical theories with conserved charge quantities, like electrodynamics, satisfy the principle. Second, because theories that don't satisfy Charge Recombination exhibit a weird sort of non-locality that conflicts with the very notion of localized charge. For, if Recombination is violated, the laws must witness not just the local (force-mediated) interactions between charges, but also the total amount of charge in the entire universe, however far separated these charges may be. So if Q is a charge that could not possibly be the total charge of the universe, if any region possesses charge Q, the laws ensure that somewhere out there — no matter how far off — there must be another nonzero charge. And this must hold even if these two charges never once interact.

This is a bit too quick, though. We can imagine theories that violate Charge Recombination in a way that admits of local explanation. For example, consider a toy theory in which the force laws prevent any positively-charged particle from straying more than one nanometer from a corresponding negative charge. The global state will then, of course, be neutral, but any finite region could still possess total charge $+1$, since a positive charge could be located less than one nanometer from the border of the region with its corresponding negative charge just outside the region.While this theory violates Recombination, the violation is explained by the local force laws (and indeed, it must be posited if those laws hold universally).[35]

[35]While this example is inspired by the phenomenon of quark confinement, a local dynamical explanation need not have this form. For instance, a theory in which charged particles are always created in $+1/-1$ pairs from local interactions could explain the failure of Charge Recombination without confinement-like behavior. Whether or not QCD itself satisfies Charge Recombination is an interesting open question. Kijowski and Rudolph (2005) carry out a DHR inspired analysis of the charge structure of lattice QCD, finding that there are three sectors compatible with the theory's global conservation laws: a $+1$ charged color sector, a -1 charged anti-color sector, and a 0 charged colorless sector. Prima facie one would expect that quark confinement rules out the ± 1 sectors, ensuring that the global state is colorless, although the exact explanation is somewhat muddled since there is no rigorous proof of confinement in 4d QCD. Roughly, the spacetime trajectories of a quark-antiquark pair created from the vacuum are represented by a gauge invariant observable called the Wilson loop. Ordinarily, the Wilson loop action is proportional to the loop perimeter, but in theories exhibiting confinement (e.g. 4d QCD, 2d and 3d compact Abelian gauge theories, the Schwinger model), it is proportional to the area, and hence to the separation between the quark-antiquark pair. This suppresses the production of free quarks. For more details see Greensite (2011).

Fair enough. Rather than requiring Charge Recombination hold as an a priori posit, we suggest the following principle of theory choice:

Charge Recombination Principle. If we face a choice between two theories with the same dynamical laws, one obeying Charge Recombination and one which violates it, the theory which obeys Charge Recombination should be preferred.

This is another (more precise) way of requiring that any violation of Charge Recombination should be explained by a theory's force laws.

Several supporting arguments can be marshaled in favor of this principle. First, as we've noted, locality is itself a plausible principle of theory choice. Second, the non-locality that appears when Charge Recombination is violated (without a dynamical explanation) amounts to an odd sort of unobservable law. A law that sets limits on the global charge without placing any limits on the charge within finite regions obviously can't be confirmed or disconfirmed by observations confined to any finite region, however large. Finally, theories violating charge recombination place unhelpful limits on our ability to accurately idealize. For example, it is often helpful in quantum physics to idealize a spatially finite physical system by pretending it is spatially infinite, using a state over all of spacetime to represent the state of a region. But if a global state cannot possess charge Q while local states can, our options for representing finite systems using global states will be correspondingly limited. Although some readers may demur here, we maintain that the availability of useful idealizations is an important virtue for physical theories, and thus theories which make idealization easier are preferable.

Not only does Charge Recombination stand as a general theoretical principle undergirding the arguments of Drühl *et al.*, it provides the missing link needed to shore up the argument for equivalence from the reconstruction theorem. If we restrict our attention to theories of local charges (i.e. those within the domain of DHR theory), it can be shown that the only field systems satisfying Charge Recombination are complete field systems:

Proposition. *Let* $(\mathfrak{F}, H, \pi, G)$ *be a field system satisfying the requirements of DHR theory. The system satisfies Charge Recombination iff it is complete.*

There is an easy, physically intuitive way to see why incomplete field systems must violate charge recombination.[36] (The converse direction follows trivially from the fact that in a complete theory, for any local charge Q, there is a sector with global charge Q.) Suppose

[36]It is important to emphasize that we are not arguing that the DHR selection criteria is a necessary and sufficient condition for physically possible *states*. Rather, we are only arguing that it is a necessary and sufficient condition for physically possible *sectors*. One is free to impose additional physicality conditions on states as long as every charge sector is occupied by some possible state.

a hypothetical incomplete field system forbids states in the DHR sector with charge Q. (For every incomplete field system, there will be some Q meeting this condition, or else that field system would be complete.) In DHR theory, every charge has a conjugate, or opposite, which we may call $\overline{Q}$ (Baker and Halvorson, 2010). It is physically possible for a state to have charge Q and still live in the vacuum sector, provided that the opposite charge $\overline{Q}$ is located in a different region. So every value of charge (for any DHR sector) can be located in a finite region of some state in the vacuum sector. Thus, it is possible for any finite region to have charge Q, even according to this hypothetical incomplete field system. Since Q is not a possible global charge, the hypothetical field system violates Recombination.

It's simple to make this intuitive argument rigorous in the special case where $\mathfrak{A}$'s natural gauge group privileged by DR reconstruction is abelian. It is useful to represent the DHR representations and their states in terms of charge-creating *localized, transportable endomorphisms* ρ, a species of automorphisms of $\mathfrak{A}$ which "shuffle around" the operators in $\mathfrak{A}$. Each ρ is localized in the sense that it acts trivially outside some given compact spacetime region (for simplicity, a double-cone). These morphisms correspond one-to-one with the DHR states, and their unitary equivalence classes $\hat{\rho}$ correspond one-to-one with $\mathfrak{A}$'s DHR representations, so that we may think of each morphism as acting on states (or on sectors) to generate some fixed quantity of charge within the region where it is localized. In general, $\omega_0 \circ \rho$ is a state containing charge localized within the region where ρ is non-trivial, while $\rho \circ \pi$ gives the DHR representation with total charge equal to $\hat{\rho}$, the amount of charge ρ generates.[37]

For any incomplete field system, there will be some DHR representations whose states are not among the states of that field system — the "empty" superselection sectors, if you will. Suppose that ρ is a charge-creating morphism that maps the vacuum to one of these sectors. This means the state $\omega_0 \circ \rho$ is not one of the physical states of our incomplete field system. We will say that Q is the charge quantum number corresponding to ρ's superselection sector in a complete field system for $\mathfrak{A}$. Thus Q is not a possible total charge for a state of our incomplete field system.

Like all charge-creating morphisms in DHR theory, ρ has a conjugate $\overline{\rho}$, corresponding to the opposite charge. Since the natural gauge group for $\mathfrak{A}$ is abelian, $\rho \circ \overline{\rho}$ is the identity morphism. Therefore, $\omega_0 \circ \rho \circ \overline{\rho}$ is the vacuum, which is of course a state of our incomplete field system. Now, suppose ρ is localized in the Minkowski double cone O (which may be of any finite size), so that $\omega_0 \circ \rho$ is a state with charge localized in O. We may, if we like, transport ρ to another region O', spacelike separated from O, without changing its superselection sector.

[37] For a more detailed explication, see Halvorson and Müger (2006, 785-803).

Call the resulting morphism ρ'. Then $\omega_0 \circ \rho' \circ \overline{\rho}$ is a state with positive charge Q localized in O and negative charge $\overline{Q}$ localized in O'. This state is located in the vacuum sector, and hence is a state of our incomplete field system. But since the state contains charge Q in region O, and O is just an arbitrary finite region, our incomplete field system violates Charge Recombination (as Q cannot be the global charge by hypothesis).

Things are more complicated when the natural gauge group is nonabelian, since in that case $\rho \circ \overline{\rho}$ is not the identity morphism, and $\omega_0 \circ \rho \circ \overline{\rho}$ is not the vacuum (nor even a state in the vacuum sector). So to be completely general, we must proceed in a more roundabout way. First, we must establish the following lemma:

Lemma. *For any double cone O and DHR morphism ρ localized in O, there is a state σ in the vacuum sector such that $\sigma|_{\mathfrak{A}(O)} = (\omega_0 \circ \rho)|_{\mathfrak{A}(O)}$.*

Proof. Let (K, ϕ) be the GNS representation of $\mathfrak{A}$ induced by the state $\omega_0 \circ \rho$. Then (K, ϕ) is locally quasiequivalent to the vacuum representation (see Halvorson and Müger). In other words, the restriction of ϕ to any region O, $\phi|_{\mathfrak{A}(O)}$, is always quasiequivalent to $\pi_0|_{\mathfrak{A}(O)}$. As we have already seen, two representations are quasiequivalent iff they have the same folium of states. Thus the folia of $\phi|_{\mathfrak{A}(O)}$ and $\pi_0|_{\mathfrak{A}(O)}$ coincide. Obviously $(\omega_0 \circ \rho)|_{\mathfrak{A}(O)}$ is in the folium of $\phi|_{\mathfrak{A}(O)}$, hence it is also in the folium of $\pi_0|_{\mathfrak{A}(O)}$.

To complete the argument, we need only show that every state (of $\mathfrak{A}(O)$) in the folium of $\pi_0|_{\mathfrak{A}(O)}$ extends to a state (of $\mathfrak{A}$) in the folium of π_0. In fact, this can be shown with complete generality. Suppose that $\mathfrak{B} \subseteq \mathfrak{A}$ is an inclusion of C^*-algebras, and that (H, π) is a representation of $\mathfrak{A}$. A state ρ of $\mathfrak{B}$ is in the folium of $\pi|_{\mathfrak{B}}$ just in case there is a density operator D on H such that $\rho(B) = \mathrm{Tr}(D\pi(B))$, for all $B \in \mathfrak{B}$. Define the state σ of $\mathfrak{A}$ by setting $\sigma(A) = \mathrm{Tr}(D\pi(A))$ for all $A \in \mathfrak{A}$. Then σ is in the folium of π and $\sigma|_{\mathfrak{B}} = \rho$. Therefore every state in the folium of $\pi|_{\mathfrak{B}}$ is the restriction of a state in the folium of π.

□

This lemma suffices to establish the main proposition. Suppose that $(\mathfrak{F}, H, \pi, G)$ is an *incomplete* field system for $(\mathfrak{A}, \omega_0)$, so that there is a DHR morphism ρ localized in O such that $\omega_0 \circ \rho$ is *not* in the folium of π. By the lemma, there is a state σ in the vacuum sector that is equal on $\mathfrak{A}(O)$ to $\omega_0 \circ \rho$. Since every field system's folium includes the vacuum sector, every field system deems the state σ "physically possible." So every field system claims that it is possible for the region O to have charge $\hat{\rho}$. But it is impossible, according to the field system $(\mathfrak{F}, H, \pi, G)$, for the global charge to be $\hat{\rho}$, since this field system doesn't admit any of the states in the sector of $\omega_0 \circ \rho$.

This result is not quite enough, by itself, to rule out incomplete theories by our principle of theory choice. Recall that our principle only rules out theories violating Charge Recombination if there is an alternative theory with the same dynamical laws that obeys Charge Recombination. So to solidify our case for the equivalence thesis, we must show that any incomplete field system has the same dynamics as the complete theory with the same algebra of observables. Fortunately a good argument to this effect is available — establishing, in effect, that there can be no dynamical explanation for violations of charge recombination in the DHR framework.

Plausibly, for any algebra of observables $\mathfrak{A}$, two representations of $\mathfrak{A}$ exhibit the same dynamical laws if they are unitarily equivalent. This follows, for example, from the platitude that a quantum theory's dynamics is always given by a group of unitary time-evolution operators. Moreover, since a field system is the going definition of a full quantum theory on the DHR approach, all the sectors of a field system share the same dynamics. If π is an incomplete field system for $\mathfrak{A}$ and π' is a complete one, every sector of π is unitarily equivalent to some sector of π'. Thus every sector of π has the same dynamics as a sector of π'. Since all the sectors of π share the same dynamics, and the same goes for all sectors of π', we conclude that the two field systems have the same dynamical laws.

This conclusion is quite strong! In fact, it appears to rule out any possibility of providing a local dynamical explanation for a violation of Charge Recombination within the DHR framework. Since any such violation entails that the theory in question is incomplete, if the dynamics are inconsistent with Charge Recombination, they are incompatible with at least one charge sector. But this is impossible if two unitarily equivalent representations share the same dynamics, because as we have seen above, any incomplete field system shares its dynamics with some complete field system (where the latter's existence is ensured by the DR reconstruction theorem). Thus the dynamics could not really be inconsistent with Charge Recombination.[38]

Here, then, is where we stand. Consider any parastatistical QFT satisfying the DHR condition. It is either a complete field system or an incomplete one. By the DR reconstruction theorem, there is a complete field system with ordinary statistics that shares the same

[38]This may seem puzzling, since the DHR selection criterion is not intuitively incompatible with force laws violating Charge Recombination. Perhaps dynamical explanations of Charge Recombination failure exist, but they require moving outside the framework of DHR superselection theory. Or perhaps dynamics can only rule out sectors in conjunction with auxiliary assumptions about initial or boundary conditions. A more detailed investigation of confinement phenomena and similar types of explanations is needed to resolve this issue. At present, we take the above argument as a strong point in favor of Charge Recombination as a plausible criterion of theory choice (at least for theories satisfying the DHR requirements). If it turns out that local dynamical explanations for its failure are impossible in DHR, then so much the better.

algebra of observables. By our argument above, this has the same dynamics as our para-theory. If the para-theory is incomplete, then it violates Charge Recombination, and is therefore unacceptable on physical grounds. If it is complete, then it is quasiequivalent to the ordinary field system whose existence is guaranteed by DR reconstruction. And as we've seen (in §3), quasiequivalence is sufficient for theoretical equivalence. The interpretive upshot: any para-theory which is not ruled out by the physically reasonable principle of Charge Recombination is theoretically equivalent to a theory with ordinary statistics.

6 Discussion

The equivalence thesis has been shown to hold for any theory satisfying the constraints of DHR superselection theory. Our version of the thesis is both more general and has a more explicit foundational motivation (the principle of Charge Recombination) than previous versions advanced in the physics literature. We see no obvious obstacle that would prevent our argument from generalizing to other quantum field theories, as well, once a rigorous formalism for superselection theory is applied to these theories. Thus we see good reason to conclude that the difference between parastatistics and ordinary statistics is a matter of convention rather than physical reality. Nonetheless, it is important to understand the limitations of the thesis in its current form. We view the present paper as an important stepping-off point for future work on statistics in AQFT rather than an endpoint.

First there is the restriction to theories with local charges satisfying the requirements of DHR theory. This covers a broad range of QFTs including massive non-abelian gauge theories like QCD, but it ignores theories like quantum electrodynamics (since the DHR selection criteria rules out any states with non-zero electric charge) as well as massive non-abelian theories with topological charges. In the latter case, Buchholz and Fredenhagen (1982) have developed a rigorous superselection theory accommodating topological charges. The BF selection criteria relaxes the DHR requirement that physical states must differ from the vacuum state only within a bounded spacetime region and instead allows physical states that differ from the vacuum in some spacelike cone.[39] The more restrictive DHR setting is technically simpler and still serves as a useful test-case for studying superselection rules in more general theories satisfying the BF condition. Indeed we expect that our equivalence result will carry over to the BF setting more-or-less unchanged. Since the folium of a BF-complete field system contains all and only BF states and any two BF sectors are locally

[39] Formally, a BF representation (H, π) is one such that $\pi_0|_{\mathfrak{A}(C')}$ and $\pi|_{\mathfrak{A}(C')}$ are unitarily equivalent. Here C' is the causal compliment of an arbitrary space-like cone C. This condition can be substituted mutatis mutandis into the DHR selection criterion stated in §3.1 to yield the BF selection criteria.

quasiequivalent, any two BF-complete systems will be quasiequivalent. What is still needed is a physical principle like Charge Recombination that rules out BF-incomplete field systems.

Extending our argument to theories with massless particles like QED is much more difficult and will likely have to wait until new mathematical tools are developed for treating superelection principles in such theories.[40] Nonetheless we view the existence of a Klein transformation preserving the structure of the observable algebra as a good indicator that the equivalence thesis will continue to hold in this case. Extant analyses of superselection rules operate by placing restrictions on states defined as linear functionals over the observable algebra. Both the DHR and BF analyses culminate in a reconstruction theorem demonstrating that complete field systems can be reconstructed (up to physical equivalence) from the observable algebra, vacuum state, and selection criteria as inputs. Thus, unless our understanding of superselection theory stands to be radically altered by QFTs with massless particles, the existence of a Klein transformation should be enough to secure the equivalence thesis for suitably complete field systems.

A second limitation of our argument concerns its extension to interacting QFTs. The problem here comes from two different directions. One is the notorious difficulty of modeling interacting fields within (4-dimensional) AQFT. The other is our current lack of understanding of interacting parafield theories, even within the framework of standard Lagrangian field theory. We have good reasons to think that the tools of DHR theory, including the reconstruction theorem crucial for our argument, will continue to apply in interacting theories — these include their important role in lower dimensional interacting AQFTs as well as the explanations they provide of central phenomena from standard field theory like the spin-statistics theorem and the PCT theorem. Still, the devil is in the details. On the parafield side, the mathematical complexities of trilinear commutation relations have hindered progress at classifying the Fock-like representations of interacting parafield theories. While the Green decomposition, which is a crucial component of the free parafield classification scheme discussed in §2, is known to exist in the interacting case, its uniqueness has yet to be determined. Although there have been promising recent developments on both fronts,[41] wading into this complex domain is beyond the scope of the present paper. Despite the lack of general results, a few individual cases of interest have been investigated in con-

[40]Recent work by Buchholz, Doplicher, and Roberts (Buchholz *et al.* (in preparation)) has made important progress in this direction, however. The chief technical difficulty stems from the infrared divergences of QED. This yields the possibility of infraparticles, charged particles surrounded by an infinite cloud of soft photons. These charged states cannot be compactly localized and are not Lorentz covariant. This in turn creates difficulties for the standard Wigner classification of particles as well as the DHR analysis of superselection sector structure.

[41]For the former, see Summers (2012); for the latter, see Kanakoglou (2012).

nection with paraquark theories, and here the equivalence thesis shows no sign of breaking down. In §6.3 we will discuss some of the complications that arise in this case.

Apart from these two issues of scope, there is a lingering problem regarding the physical interpretation of the equivalence theorem. In addition to the two methods for treating parastatistics discussed in §2, there is a third method supplied by DHR superselection theory itself. According to this approach, given a field system $(\mathfrak{F}, H, \pi, G)$ for $(\mathfrak{A}, \omega)$, the statistical properties of a state ψ are determined by the *statistical dimension* of the superselection sector ψ occupies. As noted in §5, states satisfying the DHR condition are in one-to-one correspondence with localized, transportable endomorphisms of the observable algebra. Recall that ρ_i denotes the endomorphism that creates charge Q in spacetime region O_i. The categorical structure of these endomorphisms allows for the construction of tensor products, $\rho \otimes \rho$, and gives rise to a natural representation of S_n on the endomorphisms of $\rho_1 \otimes \ldots \otimes \rho_n$.[42] The categorical dimension of ρ corresponds to the Hilbert space dimension of the superselection sector associated with ρ, and is said to be the *statistical dimension* of the sector. If the dimension of the sector is $d = 1$, its states obey ordinary statistics — that is, if ψ_1 and ψ_2 are two state vectors lying in that sector, the composite state vector $\psi_1 \times \psi_2$ transforms according to a 1-dimensional representation of S_2 (either the trivial or alternating representation depending on wether the sector is bosonic or fermionic).[43] If the dimension of the sector is $d > 1$, its states obey parastatistics — the composite vector will transform according to a higher dimensional representation of S_2.[44]

The trouble is that a field system will possess superselection sectors with statistical dimension $d > 1$ iff G is a non-abelian (global) gauge group.[45] But the equivalence thesis asserts that any parafield system is equivalent to an ordinary field system with an additional global non-abelian symmetry. Indeed a quick survey of the field systems canvassed by Drühl et al. shows that in general the Klein transformed partner of a parafield system will have a non-abelian gauge group. But according to the DHR analysis of statistics, these field systems will still have *parasectors*, and hence state vectors transforming under higher-dimensional representations of S_n. What's going on here?

The apparent paradox is resolved, at least formally, by noting that while the field algebra commutation relations and gauge group determine the sector statistics, the converse is not true. The sector statistics only determine the commutation relations up to a possible Klein

[42]In particular, the category of localized transportable endomorphisms is a symmetric, tensor $*$-category.

[43]Note, the composite state vector is not the tensor product of ψ_1 and ψ_2 on H. Rather $\psi_1 \times \psi_2 = F_1 F_2 \psi_0$ where F_i is an operator on H such that $F_i \psi_0$ implements the state $\omega_0 \circ \rho_i$ in the representation $(H_0, \pi_0 \circ \rho_i)$.

[44]For a full discussion of the DHR treatment of statistics see Halvorson and Müger (2006), especially §11.4 and §8.5. Also see Baker (in preparation).

[45]For a proof see Halvorson and Müger (2006), §11.4.4.

transformation.[46] Therefore it is possible to have a theory with parasectors whose underlying field algebra obeys ordinary commutation relations. But if such a theory has states that transform under higher-dimensional representations of S_n in what sense can we claim that it is a theory with ordinary particles and ordinary statistics at all? What physical information do the commutation relations of the field algebra encode anyways? After all, we have seen that the field operators themselves are of dubious physical significance. Suddenly, the equivalence theorem threatens to vanish before our eyes.

This pessimistic conclusion is too hasty. It is crucial to note that while the unobservable field operators do not denote physical quantities, relations between the operators do carry important information. For example, in standard versions of the spin-statistics theorem requiring that the fields obey either commutation or anti-commutation relations ensures the appropriate statistics for the theory in question. The commutation relations for the creation/annihilation operators determine the structure of the corresponding Fock space representation which underwrites our talk of particle states in QFT. In the present case, the fact that the parafield creation/anihilation operators obey trilinear commutation relations gives rise to a Fock space representing particle states characterized by mass and spin, which have intrinsic parastatistics. In contrast, their Klein transformed counterparts which obey ordinary bilinear commutation relations generate a Fock space describing particle states, characterized by mass, spin, *and* some additional new quantity (characterized by invariance under the new expanded non-abelian gauge group), which have intrinsic Bose/Fermi statistics. As an example, consider an order $p = 2$ para-Fermi system with gauge group G_2. Any wavefunction representing $n > 2$ particles must be antisymmetrized. The system will allow symmetric one- and two-particle wavefunctions, however, due to the intrinsic statistical properties of the parafermions described by the theory. The equivalence theorem asserts that this theory is equivalent to an ordinary field system with gauge group U_2. This theory will also allow for symmetric one- and two-particle wavefunctions, but here it is because these wavefunctions contain an extra degree of freedom, a global quantum number invariant under U_2 whose value can be changed to yield appropriately symmetrized or antisymmetrized wavefunctions for $n \leq 2$.

The difference is one of physical interpretation — are the symmetry properties of certain multi-particle wavefunctions due to the statistics of the particles or to other hidden quantum numbers? If this all sounds like a minor semantic quibble, that's precisely *the point* of the equivalence thesis. After all the physical states of both theories have the same symmetry properties, its just a matter of where we draw the line between intrinsic statistical properties

[46] Doplicher and Roberts (1972)

of particles and their other properties. Our argument from quasiequivalence suggests that this line is in fact an arbitrary convention.

We will not pretend that the issue has been decisively settled; the relationship between the DHR picture of statistics and the more standard approaches of §2 deserves to be looked at in much greater detail. The presence of parasectors in theories of ordinary particles with global non-abelian gauge groups indicate that these theories retain some parastatistical flair. The question remains open whether or not any of our current physical theories contain this residual parastatistical element. After all, the equivalence thesis is asymmetric. Any paraparticle theory subject to the requirements of Charge Recombination can be translated into a theory of ordinary particles, but the converse is not true. Only ordinary theories with certain global non-abelian gauge groups can be reinterpreted as paraparticle theories.[47] Both QCD and electro-weak theory involve non-abelian gauge symmetries, but these are local, not global symmetries, and as we shall see shortly in the case of paraquark theories, this complicates the matter considerably. It may be that the Standard Model *cannot* be written in parastatistical notation, or it may be that we have simply chosen, as a matter of convention, to express it in terms of ordinary bosons and fermions. If the latter is true, there is no mystery to be explained. If the former is true, then our original question appears to linger on in a new ghostly form — not "why does nature abhor paraparticles," but "why does nature abhor certain global nonabelian gauge symmetries?" In this case, however, the equivalence thesis has deflated much of the punch of the original problem. Whereas Wigner-style plenitude arguments appear to make the absence of paraparticles in nature a striking mystery, there are no analogous principles suggesting that the fundamental laws exhibit all possible symmetries. They just are what they are.

Despite these unresolved matters, the equivalence thesis is a fascinating result that warrants serious attention from philosophers of physics. We wrap up our discussion by considering how the thesis in its current form impacts the three philosophical debates highlighted in the introduction.

6.1 Interpretations of QM

Several interpretations of quantum mechanics have claimed to explain why only bosons and fermions are possible. Bacciagaluppi (2003) has argued that parastatistics are impossible in Bohmian mechanics, if one assumes (as is natural) that corpuscles cannot coincide in space.

[47] For instance the theory and $(\mathfrak{F}, \mathfrak{A}_1, O(2))$ discussed in §3.4 possess non-abelian global symmetry, hence parasectors. Yet it does not possess an equivalent parafield interpretation. A similar conclusion holds for $(\mathfrak{F}, \mathfrak{A}_0, SO(N))$, for $N > 2$.

Kochen (unpublished), meanwhile, has argued that his version of the modal interpretation predicts the physical impossibility of paraparticles; Nelson (1985) offers the same prediction as an advantage of his (now abandoned) stochastic mechanics.

The conventionality of parastatistics undermines these arguments. For example, consider the no-go theorems supplied by Bohmian mechanics. One might have thought that this provides an explanatory advantage over ordinary quantum theory. Or, to compare apples to apples, over the Everett/many-worlds interpretation, which does not modify the mathematics of the bare Hilbert space formalism as Bohm's theory does. It may therefore seem that the Bohmian can explain the absence of paraparticles in nature better than the Everettian can. Other things being equal, we would then (from a scientific realist point of view) have some reason to accept Bohm's interpretation over Everett's.

Alternatively, suppose that parastatistics is empirically equivalent to ordinary statistics, but not theoretically equivalent. One aspect of theoretical simplicity is the elimination of undetectable surplus structure. For example, the nonexistence of a preferred reference frame in special relativity is widely regarded as an advantage of that theory over Lorentz's ether theory, which posits an unobservable preferred frame. Where the Lorentz theory allows infinitely many undetectably different possibilities, each with a different preferred frame, special relativity eliminates this extra ontology — and importantly, it does so without *ad hoc* stipulation. If Bacciagaluppi is right, the Bohmian can therefore claim that the Everett interpretation includes surplus structure that is not present in Bohm's theory.[48]

But given the conventionality of parastatistics, neither argument applies. Any hypothetical Everettian quantum field theory with parastatistics is simply a notational variant of some theory with ordinary statistics, which will presumably have a Bohmian analogue as well. So there is no surplus structure in Everettian quantum theory. Rather, by using the canonical formalism, the Everettian is able to formulate certain quantum theories in "parastatistical notation" rather than using ordinary Bose and/or Fermi fields. But whichever way the theory is written, the posited physical reality remains the same. Moreover, no explanation is needed for the absence of paraparticles in our present best quantum theories. By the same reasoning, the arguments of Kochen and Nelson offer no evidential advantage to their

[48]The analogy with special relativity is not perfect, however. According to the Lorentz ether theory, all of the undetectably different worlds with different preferred frames are physically possible. The Everettian would presumably not say the same about a parastatistical theory and its ordinary Bose/Fermi counterpart. To illustrate, suppose that the Everettian accepts the present-day Standard Model as exactly true, while the Bohmian accepts a hidden-variable theory with the same empirical predictions. But while there is no well-formulated Bohmian theory including paraparticles, there is a paraparticle theory which the Everettian counts as a legitimate quantum theory and which reproduces the empirical predictions of the Standard Model. Whether this is enough to warrant a preference for Bohmian theories on Occamist grounds is surely a matter for further study.

respective interpretations.

It remains possible that one (or more) of these interpretations might actually disallow the ordinary field systems which standard quantum theory equates with parafield systems. This would put a whole new spin on the corresponding no-go theorem. In effect it would demonstrate that certain non-abelian symmetry groups are inappropriate choices for a global gauge symmetry. This would represent a very different kind of physicality argument, an intriguing possibility that should be studied in greater detail. Since gauge symmetries of the fundamental laws are rarely viewed as requiring an explanation, it is not obvious that this would reintroduce an explanatory advantage for the interpretation in question, however.

6.2 Structuralism and haecceities

In recent debates regarding structuralism and haecceities, Caulton and Butterfield (2012) employ paraparticles to find a way around Oliver Pooley's criticism of Stachel (2002). Stachel has argued that the general relativistic hole argument possesses an analogue in quantum theory where permutation invariance plays the role of diffeomorphism invariance. In both theories, the respective "hole argument" gives reason, according to Stachel, to deny haecceitism about the fundamental entities appearing in the theory. In GR, the fundamental laws are invariant under a smooth reshuffling of spacetime points. They don't care which spacetime points are which. In QM, the laws are invariant under particle permutations. They don't care which particles are which.

Pooley (2006) objects to the analogy, on the following grounds: in GR, while the Einstein field equations are invariant under diffeomorphisms of the spacetime manifold, particular solutions of the field equations are not. While any diffeomorphism preserves the set of solutions to the field equations, it will not leave individual solutions fixed, but rather map (mathematically) distinct solutions onto each other. In this context, adopting anti-haecceitism about spacetime points serves to eliminate excess structure from the theory. Identifying diffeomorphism-linked solutions with one another, the theory views them as different descriptions of physically equivalent spacetime geometries. In contrast, while in QM the Schrödinger equation is similarly invariant under permutation symmetry, the action of S_n on Hilbert space *does* fix individual solutions (assuming that we identify states with rays as the usual formalism dictates), hence there is no surplus structure to get rid of, no mathematically distinct representations to identify as physically equivalent. So adopting anti-haecceitism in this context does not carry the occamist advantages it did in the case of GR.

In response, Caulton and Butterfield note that Pooley's argument tacitly relies on the truth of the symmetrization postulate. When paraparticles are allowed, permutations no longer leave every ray invariant. A ray representing multiple paraparticles will get mapped to a (mathematically) distinct ray in the Hilbert space. We now have the option to reduce excess structure by identifying these permutation linked rays with each other. Accordingly, paraparticle states are represented by higher dimensional invariant subspaces of H instead of rays. So in quantum theory with parastatistics, there is a direct analogue of the hole argument, supporting anti-haecceitism about quantum particles.

Caulton and Butterfield stress that their argument does not presuppose the existence of paraparticles in nature; their quantum hole argument "will only need the possibility of paraparticles." (Caulton and Butterfield, 2012, 235) The equivalence thesis appears to directly threaten this conclusion. If any quantum theory with paraparticles can be written in a theoretically equivalent notation that omits them, this erases the analogy that Caulton and Butterfield have redrawn with the hole argument.

There's a catch, however. The ordinary field theories that we have argued are equivalent to parafield theories do not have a one-to-one relationship between states and rays either. Take the $(\mathfrak{F}, \mathfrak{A}_2, U(2))$ theory that Drühl *et al.* analyze as an example. The theory possesses an infinite number of superselection sectors classified by two quantum numbers I and B (which can be interpreted as isospin and baryon number in some models), where $B \in \mathbb{Z}$, $2I \in \mathbb{Z}_+$, and the sum $2I + B$ must be even. For each value of I, there is a $2I+1$ dimensional subspace of vectors in H which agree on all expectation values for observables in the weak closure algebra $\pi(\mathfrak{A}_2)^-$. It is natural to treat these vectors as degenerate representations of the same physical state. So even if the existence of paraparticles is a matter of convention, theories that posit them and equivalent theories that don't both contain a certain amount of excess descriptive structure. The possibility of a reformulated quantum hole argument remains.

We doubt that this fact is sufficient to reestablish the link with the hole argument originally proposed by Stachel, however. The role of permutation invariance has been replaced by invariance under a particular gauge group. Unlike particle permutations or diffeomorphisms which characterize symmetries of all models of QM and GR respectively, the gauge group in question is specific to a particular field system. Moreover its action doesn't have a clear interpretation as a reshuffling of the identities of individual particles, thus the connection with particle haecceities is lost. Even if a new version of the quantum hole argument could be formulated, it would be the existence of quantum theories with gauge symmetries, not the possibility of paraparticles doing the theoretical work.

6.3 Paraquark theories

Following the introduction of quarks by Gell-Mann and Zweig in 1964, particle physics faced a dilemma. According the quark model, baryons, which have odd half-integer spin, are composed of three quarks, so quarks should be spin-1/2 particles. It follows from the spin-statistics theorem that the single-particle quark wavefunction will be antisymmetric under permutation symmetry. But this implies that the baryon wavefunction will also be antisymmetric under any permutation of two quarks, a prediction contradicted by experimental results including light baryon spectroscopy and measurements of the magnetic moment ratio of protons and neutrons.[49]

In order to ensure the appropriate symmetrization of the baryon wavefunction, Greenberg (1964) proposed a model that treated quarks as parafermions of order $p = 3$. The intrinsic parastatistics of the quarks ensures that there are three-quark bound states which are antisymmetric overall, but which are symmetric under the permutation of two quarks. The model fit the experimental predictions beautifully. Concurrently, Han and Nambu (1965) proposed a quark model that employed ordinary statistics but introduced a new degree of freedom into the quark wavefunction — the *color* quantum number. It was this model, not Greenberg's paraquark theory, that would eventually be developed into QCD, a local non-abelian Yang-Mills theory describing the strong force.

In his philosophical analysis of this episode, French (1995) seeks to answer why, when the two theories were observationally equivalent, physicists chose to develop the color model and not its paraquark rival. He argues against the view that their choice simply reflected theoretical prejudice in favor of a more familiar mathematical formalism, and contends that "it is the objective structural characteristics of the models concerned that contribute to their heuristic fruitfulness."[50] Ultimately it was the color model's ability to be unified within the framework of local Yang-Mills theory that lent it greater theoretical weight: "What is important from our point of view is that the color model was able to be gauged whereas the parastatistics theory was not."[51]

While we will not take issue with French's historical reading, we do wish to clarify the (retroactive) role played by the equivalence thesis in this episode. Throughout his paper, French remains somewhat vague about the scope and nature of the equivalence between color and paraquark models. It is most often treated as a kind of empirical equivalence,[52] although towards the end of the paper a quote from Greenberg implies a possibly stronger

[49] See Greenberg (2004)
[50] French (1995), 87-88.
[51] ibid, 103.
[52] French (1995), 91.

form of theoretical equivalence: "the two theories are equivalent quantum mechanically, but they are apparently not equivalent from the standpoint of quantum field theory."[53]

Drawing upon our analysis of the equivalence thesis, we can say something more precise: the two theories are equivalent in QFT as well, until local gauge symmetry is introduced. The color model can be represented by an ordinary field system with global $SU(3)$ symmetry. This system can be Klein transformed into its paraquark counterpart, which matches Greenberg's theory. Moreover, both field systems are DHR complete, if one assumes that the color degree of freedom is truly hidden, i.e. that only colorless combinations of quarks are physically possible. Hence the two models are not just empirically equivalent, but quasiequivalent, and therefore fully physically equivalent by our standards. This, we propose, is the sense in which the two theories are "equivalent quantum mechanically" (field theoretically as well, in fact).

Does this mean that the quarks we see in nature can actually be reinterpreted as paraparticles? Not necessarily. The original color model was eventually replaced by QCD, which has *local* rather than global $SU(3)$ gauge symmetry. In QCD the color degree of freedom plays a dynamical role; it is the local charge that couples to the strong force. Unlike the original color model, it is not clear that QCD has an equivalent parafield formulation. We would expect such a theory to be the Klein transform of an ordinary field system with local $SU(3)$ gauge symmetry and particle multiplets described by the QCD Yang-Mills Lagrangian. Govorkov (1982) demonstrated that due to the presence of the trilinear Yukawa term in the QCD Lagrangian, one can only find such a model for the smaller symmetry group $SO(3)$. The local $SO(3)$ theory contains fewer varieties of gluons than QCD and as a result makes different empirical predictions from QCD (e.g. regarding decay cross sections and the structure of gluon jets). Essentially the problem is that the parafield Hilbert space is too small to model all eight components of the QCD color current. In fact, the problem is much more general: for order $p = N$ parastatistics, neither the parafield commutator nor anticommutator contains enough components to couple to either an $SU(N)$ or an $SO(N)$ Yang-Mills gauge field, except in the special case of $SO(3)$.[54] This is the sense in which the equivalence between color and parastatistics breaks down "from the standpoint of quantum field theory." The obstacle, however, arises not from a gulf between QM and QFT, but rather from a difference between field systems with global and local gauge symmetries.[55]

It is important to note that even if there is no parafield theory equivalent to full QCD,

[53]ibid, 103. (Here French quotes personal correspondence with Greenberg.)

[54]See Freund (1976).

[55]This conclusion is supported by Greenberg's comments elsewhere. In Greenberg (2004), he claims that the two models are equivalent from the standpoint of a "classification symmetry," but not once the color degree of freedom is treated as part of the dynamics, as in QCD.

this does not impugn the equivalence thesis. *Modulo* the open questions regarding the nature of interacting QFTs discussed previously, it appears that all local paraquark theories that satisfy Charge Recombination will have physically equivalent counterparts with ordinary statistics. The examples that have been studied give us no reason to think otherwise. The no-go result mentioned above does suggest, however, that these counterparts will not include any $SU(N)$ Yang-Mills theories. In this case, we have a historical example of two equivalent theories, the color model and the paraquark model, with physically (and empirically) inequivalent extensions to local interacting QFTs.

This may not be the end of the story, though, as there have been subsequent attempts to construct locally gauged parafield theories that circumvent the size limitations of the standard parafield Hilbert space. One approach, that of Günaydin and Gürsey (1974), proceeds by constructing a parafield theory on an enlarged Hilbert space defined over the octonions rather than the complex numbers. Large swaths of the Hilbert space must be declared unphysical on this approach. Greenberg and Macrae (1983) pursue an alternate strategy. Rather than beginning with the canonical Green decomposition of the parafields, then Klein transforming, they propose an alternate decomposition which expresses the parafields in terms of basis elements generating a complex Clifford algebra.[56] They claim that the resulting parafield theory is fully equivalent to $SU(3)$ Yang-Mills. The interpretive status of these constructions and their relationship to the equivalence thesis remain intriguing open questions.[57]

The equivalence thesis therefore opens up a number of avenues for philosophical investigation in the foundations of physics, naturalized metaphysics, and general philosophy of science. Until now, philosophical debates involving particle statistics have been dominated by considerations of physical equivalence and conventionality raised by permutation symmetry — while our formalism cares about particle labels, nature, apparently, does not. If we are right, there are additional, subtle equivalence relations within our formalism at play in these debates. Not only are individual particle labels conventional, but our ascription of *statistics* to a system of identical particles is also, to some extent, a matter of convention.

[56]This condition holds for the $SU(N)$ case. In the $SO(N)$ case the basis elements generate a real Clifford algebra.

[57]Even if the parafield theory constructed by Greenberg and Macrae is equivalent to an $SU(3)$ Yang-Mills theory, there are additional technical subtleties to address in determining whether or not the theory is equivalent to full QCD. For example, Govorkov (1991) argues that the resulting parafield theory will not contain any charge-symmetric initial states, so only models with an initial imbalance of particles and antiparticles are physically possible (unlike in QCD). Independent of the equivalence issue, this feature could, of course, provide an interesting explanation of our matter dominated universe.

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