QUOTIENTS OF BOOLEAN ALGEBRAS AND REGULAR SUBALGEBRAS

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Abstract. Let $\mathbb{B}, \mathbb{C}$ be Boolean algebras and $e : \mathbb{B} \to \mathbb{C}$ an embedding. We examine the hierarchy of ideals on $\mathbb{C}$ for which $\overline{e} : \mathbb{B} \to \mathbb{C}/\mathcal{I}$ is a regular (i.e. complete) embedding. As an application we deal with the interrelationship between $\mathcal{P}(\omega)/\mathrm{fin}$ in the ground model and in its extension. If M is an extension of V containing a new subset of ω, then in M there is an almost disjoint refinement of the family $([\omega]^\omega)^V$. Moreover, there is, in M, exactly one ideal $\mathcal{I}$ on ω such that $(\mathcal{P}(\omega)/\mathrm{fin})^V$ is a dense subalgebra of $(\mathcal{P}(\omega)/\mathcal{I})^M$ if and only if M does not contain an independent (splitting) real.

We show that for a generic extension $V[G]$, the canonical embedding

$$\mathcal{P}^V(\omega)/\mathrm{fin} \hookrightarrow \mathcal{P}(\omega)/(U(Os)(\mathbb{B}))^G$$

is a regular one, where $U(Os)(\mathbb{B})$ is the Urysohn closure of the zero - convergence structure on $\mathbb{B}$.

1. Introduction

Let V be a model of ZFC and M its extension. Then $(\mathcal{P}(\omega)/\mathrm{fin})^V$ is a subalgebra of the Boolean algebra $\mathcal{P}(\omega)/\mathrm{fin}$ in M. By an extension of a model V we mean a transitive model M of ZFC, that has the same class of ordinal numbers as V and $V \subset M$.

It is natural to ask whether $(\mathcal{P}(\omega)/\mathrm{fin})^V$ is a regular subalgebra of $(\mathcal{P}(\omega)/\mathrm{fin})$.

This question makes sense only in cases when there are new reals in the extension M, otherwise these algebras coincide. Hence in what follows we suppose that M is an arbitrary ZFC extension of the ground model V containing new reals. In this situation the answer is negative, but under certain circumstances it leads to interesting ideals on ω.

L. Soukup posed the following question:

Does the family $([\omega]^\omega)^V$ have an almost disjoint refinement in any generic extension which contains a new real?

It was known that this holds for different types of generic extensions, e.g. adding one Cohen real [Hec78]. Note, that the generic extension is a special type of ZFC extension.

We shall consider a more general situation when we take into account an arbitrary ZFC extension M of V and arbitrary family $\mathcal{S} \subset V$, $S \in M$, consisting of infinite sets. Clearly to have any chance for a refinement, the extension M has to contain a new real, i.e.

$$(\mathcal{P}(\omega))^V \subsetneq (\mathcal{P}(\omega))^M.$$

In this generalised setting we show in paragraph 3 the following theorem. This result was achieved for $\mathcal{P}(\omega)/\mathrm{fin}$ independently by J. Brendle, his proof is rather different and can be found in L. Soukup's paper [Sou08].

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Theorem 1. *Assume that M is an extension of V containing new reals. For any cardinal κ and any set family $\mathcal{S} \subset \mathcal{P}(\kappa) \cap V$ consisting of infinite sets there is an almost disjoint refinement of $\mathcal{S}$ in M.*

Let us recall some basic set theoretical facts and notions used here. In the following $A, B \in [X]^\omega$; $A \subset^* B$ will denote the fact that $A \setminus B$ is finite.

Definition 2. A family $\mathcal{S} \subset [\kappa]^{\geq\omega}$ has an *almost disjoint refinement (ADR)* by countable sets if there is an almost disjoint family $\mathcal{A}$ such that for every $X \in \mathcal{S}$ there is $A \in \mathcal{A}$ such that $A \subset X$.

For systems on ω we have the following proposition.

Proposition 3. *For a family $\mathcal{S} \subset [\omega]^\omega$ the following are equivalent:*

(i) *The family $\mathcal{S}$ has ADR,*
(ii) *There is an almost disjoint family $\{A_X : X \in \mathcal{S}\}$ such that $A_X \in [X]^\omega$ for every $X \in \mathcal{S}$.*
(iii) *There is an almost disjoint family $\mathcal{A}$ such that for any $X \in \mathcal{S}$*
$$|\{A \in \mathcal{A} : |X \cap A| = \omega\}| = 2^\omega.$$

Proof. *(ii) → (i)* This implication is trivial since the almost disjoint family from *(ii)* satisfies also *(i)*.
(i) → (iii) Let $\mathcal{A}$ be an almost disjoint family as in *(i)*. In $[\omega]^\omega$ there is a maximal almost disjoint family $\langle B_i^A : i \in 2^\omega \rangle$ of a size 2^ω below any $A \in \mathcal{A}$. Hence $\langle B_i^A : i \in 2^\omega, \quad A \in \mathcal{A} \rangle$ satisfies *(iii)*.
(iii) → (ii) First enumerate $\mathcal{S} = \{X_\alpha : \alpha \in 2^\omega\}$ and for any $X \in \mathcal{S}$ denote $\mathcal{A}_X = \{A \in \mathcal{A} : |X \cap A| = \omega\}$, $|\mathcal{A}_X| = 2^\omega$. Now proceed by recursion and for each $X_\alpha \in \mathcal{S}$ choose an $A_\alpha \in \mathcal{A}_{X_\alpha} - \bigcup\{A_\beta : \beta < \alpha\}$. The family $\{A_\alpha \cap X_\alpha : \alpha \in 2^\omega\}$ gives an almost disjoint refinement for $\mathcal{S}$. □

Our approach to Theorem 1 strongly benefits from results of [BPS80] or see [BS89]; let us quickly summarise the results we use. For undefined notions concerning Boolean algebras see [Kop89] and for the basic forcing notions see [Jec86].

Note that an algebra $\mathbb{B}$ is $(\kappa, \cdot, 2)$ distributive if and only if any κ-many partitions of unity have a common refinement, or equivalently if the intersection $\cap_{\alpha<\kappa} D_\alpha$ of κ-many open dense sets is dense.

The cardinal invariant $\mathfrak{h}$ (*non-distributivity number*) is characterised through distributivity properties of the algebra $\mathcal{P}(\omega)/\text{fin}$ as follows:

Definition 4.
$$\mathfrak{h} = \min\{\kappa : \mathcal{P}(\omega)/\text{fin is not } (\kappa, \cdot, 2) \text{ distributive}\}.$$

In the proof of Theorem 1 we use the base tree technique. Base tree is a special kind of dense subset of $\mathcal{P}(\omega)/\text{fin}$; see e.g. [BS89].

Theorem 5. *There is a* base tree $(T, \supset^*)$ *for $[\omega]^\omega$, i.e.*

(i) *$(T, \supseteq^*) \subset [\omega]^\omega$ is a tree,*
(ii) *if $B \in T$ then the family of immediate successors of B in T is a maximal almost disjoint family below B of full (2^ω) size,*
(iii) *for each $A \in [\omega]^\omega$ there is $B \in T$ such that $B \subset A$,*
(iv) *the height of T is $\mathfrak{h}$.*

It is well known that if a new real is added, then $(\mathcal{P}(\omega)/\mathrm{fin})^V$ is not a regular subalgebra of $(\mathcal{P}(\omega)/\mathrm{fin})^M$. There is a natural question whether there is an ideal $\mathcal{I}$ containing fin such that the canonical embedding

$$(\mathcal{P}(\omega)/\mathrm{fin})^V \hookrightarrow \mathcal{P}(\omega)^M/\mathcal{I}$$

becomes regular. We show in paragraph 2 the following more general theorem:

Theorem 6. *Let $\mathbb{B}$ be a subalgebra of a Boolean algebra $\mathbb{C}$. There is an ideal $\mathcal{I}$ on $\mathbb{C}$ such that the canonical homomorphism*

$$\begin{aligned} i : \mathbb{B} &\longrightarrow \mathbb{C}/\mathcal{I} \\ b &\longmapsto [b]_{\mathcal{I}} \end{aligned}$$

is a regular embedding of $\mathbb{B}$ into $\mathbb{C}/\mathcal{I}$.

Finally in paragraphs 4 and 6 we compute the minimal *regularisation* ideal $\mathcal{I}_{\min} \supset$ fin for embeddings $(\mathcal{P}(\omega)/\mathrm{fin})^V \hookrightarrow \mathcal{P}(\omega)^M/\mathcal{I}_{\min}$ and $\mathbb{B} \hookrightarrow \mathbb{B}^\omega/Fin$. Both of these regularisation ideals are closely connected with the order sequential topology on Boolean algebras, which we briefly introduce in The Topological Intermezzo.

2. Regularisation ideals

We start with Theorem 6. First, let us recall the definition of a regular subalgebra $\mathbb{B}$ of a Boolean algebra $\mathbb{C}$ and its equivalents.

A subalgebra $\mathbb{B}$ of a Boolean algebra $\mathbb{C}$ is called *regular* if any $X \subset \mathbb{B}$ which has a supremum $\bigvee^{\mathbb{B}} X$ in $\mathbb{B}$, has the same element as a supremum in $\mathbb{C}$, i.e. $\bigvee^{\mathbb{B}} X = \bigvee^{\mathbb{C}} X$. An embedding $i : \mathbb{B} \to \mathbb{C}$ is *regular* if the image $i[\mathbb{B}]$ is a regular subalgebra of the algebra $\mathbb{C}$.

Proposition 7. *For a subalgebra $\mathbb{B} \subset \mathbb{C}$ the following are equivalent.*

(i) *$\mathbb{B}$ is a regular subalgebra of $\mathbb{C}$,*
(ii) *every maximal pairwise disjoint family in $\mathbb{B}$ is maximal in $\mathbb{C}$,*
(iii) *for each $c \in \mathbb{C}^+$ there is a* 'pseudoprojection' *$b_c \in \mathbb{B}^+$; i.e. for every $a \leq b_c$, $a \in \mathbb{B}^+$*

$$a \wedge c \neq \mathbf{0},$$

(iv) *for every generic filter F on $\mathbb{C}$, $F \cap \mathbb{B}$ is a generic filter on $\mathbb{B}$.*

Proof. The proofs of the implications *(i)* $\leftrightarrow$ *(ii)* $\leftrightarrow$ *(iii)* $\leftrightarrow$ *(v)* and *(vi)*$\rightarrow$ *(ii)* are straight forward.

To show that *(ii)*$\rightarrow$ *(vi)* let $c \in \mathbb{C}^+$. Take an arbitrary maximal pairwise disjoint family $B_c \subset \{b \in \mathbb{B} \ : \ b \wedge c = \mathbf{0}\}$. From *(ii)* it follows that B_c is not maximal in $\mathbb{B}$, hence there is some b_c disjoint with B_c and we are done. □

Let $\mathbb{B}, \mathbb{C}$ be Boolean algebras and $e : \mathbb{B} \to \mathbb{C}$ an embedding. We are looking for ideals on $\mathbb{C}$ for which the factor embedding $\overline{e}$ is regular. We call such an ideal a *r*egularisation ideal for the embedding e. If the corresponding embedding is clear from context we omit it.

Theorem 6. *Let $\mathbb{B}$ be a subalgebra of a Boolean algebra $\mathbb{C}$. There is a minimal ideal $\mathcal{I}_{min}$ on $\mathbb{C}$ such that the canonical homomorphism*

$$\begin{aligned} i : \mathbb{B} &\longrightarrow \mathbb{C}/\mathcal{I}_{min} \\ b &\longmapsto [b]_{\mathcal{I}_{min}}, \end{aligned}$$

is a regular embedding of $\mathbb{B}$ into $\mathbb{C}/\mathcal{I}_{min}$.

Proof. Let

$\mathcal{I} = \{u \in \mathbb{C} : \exists$ maximal pairwise disjoint family $X \subset \mathbb{B}$ such that $u \wedge x = \mathbf{0}$ for any $x \in X\}$.

We check that $\mathcal{I}$ is an ideal. The set $\mathcal{I}$ is downward closed. Let $u, v \in \mathcal{I}$. Take maximal pairwise disjoint families X and Y that guarantee that u respectively v belongs to $\mathcal{I}$. Then $Z = \{x \wedge y \neq \mathbf{0} : x \in X \ \& \ y \in Y\}$ is a maximal pairwise disjoint family of elements of $\mathbb{B}$ and $u \vee v$ is disjoint with every element of Z. Therefore $u \vee v \in \mathcal{I}$, hence $\mathcal{I}$ is an ideal.

No $b \in \mathbb{B}^+$ belongs to $\mathcal{I}$, so the mapping $i : \mathbb{B} \to \mathbb{C}/\mathcal{I}$ is an embedding. We show that i is a regular embedding. Let $\{c_i : i \in I\}$ be a maximal pairwise disjoint family in $\mathbb{B}$, the family $\{[c_i] : i \in I\}$ is a pairwise disjoint family in $\mathbb{C}/\mathcal{I}$. Assume that there is $[u]$, disjoint with every $[c_i]$ in $\mathbb{C}/\mathcal{I}$, i.e. $c_i \wedge u \in \mathcal{I}$, hence there is a maximal pairwise disjoint set $X_i \subset \mathbb{B} \upharpoonright c_i$ such that u is disjoint from every element of X_i. The set $\bigcup\{X_i : i \in I\}$ is maximal in $\mathbb{B}$ and so $u \in \mathcal{I}$, i.e. $[u] = \mathbf{0} \in \mathbb{C}/\mathcal{I}$.

The ideal $\mathcal{I}$ obtained in this way is minimal and we denote it $\mathcal{I}_{min}$. □

The following fact was proved by M. Rubin for other purposes [Rub83], cf. [Kop89].

Proposition 8. *Let $\mathbb{B}$ be a subalgebra of a Boolean algebra $\mathbb{C}$ and let $\mathcal{J} \subset \mathbb{C}$ be a maximal ideal such that $\mathbb{B} \cap \mathcal{J} = \{\mathbf{0}\}$. Then the canonical embedding*

$$i : \mathbb{B} \longrightarrow \mathbb{C}/\mathcal{J},$$

is regular. In this case $i[\mathbb{B}]$ is even dense in $\mathbb{C}/\mathcal{J}$.

Proof. Suppose that $i[\mathbb{B}]$ is not dense in $\mathbb{C}/\mathcal{J}$. Then there is a $c \in \mathbb{C}$, $c \notin \mathcal{J}$ such that for any $b \in \mathbb{B}^+$ $b \not\leq_{\mathcal{J}} c$. Since $\mathcal{J}$ is maximal and $c \notin \mathcal{J}$ there is a $j \in \mathcal{J}$ such that there is a $b \in \mathbb{B}^+$ so that $b \leq c \vee j$ i.e. $b \leq_{\mathcal{J}} c$, a contradiction. □

Corollary 9. Let $\mathbb{B}$ be a subalgebra of a Boolean algebra $\mathbb{C}$ and let $\mathcal{J} \subset \mathbb{C}$ be a maximal regularising ideal. Then

(i) if $\mathbb{B}$ is complete, then $\mathbb{B} \simeq \mathbb{C}/\mathcal{J}$;
(ii) if $\mathbb{C}$ is complete, then the completion of $\mathbb{B}$ is isomorphic with $\mathbb{C}/\mathcal{J}$; $\mathrm{RO}(\mathbb{B}) \simeq \mathbb{C}/\mathcal{J}$;
(iii) for any ideal $\mathcal{I}$ on $\mathbb{C}$ such that $\mathcal{I} \cap \mathbb{B}^+ = \emptyset$ there is a regularisation ideal $\mathcal{J} \supset \mathcal{I}$.

Proposition 10. *Let $\mathbb{B}$ be a subalgebra of a Boolean algebra $\mathbb{C}$ and let*

$$\mathcal{K} = \{\mathcal{J} \ : \ \mathcal{J} \text{ is an ideal on } \mathbb{C} \text{ maximal with respect to } \mathcal{J} \cap \mathbb{B}^+ = \emptyset\}$$

then

(i) $\bigcap \mathcal{K} = \mathcal{I}_{min}$ *and*
(ii) $\bigcup \mathcal{K} = \{c \in \mathbb{C} \ : \ \neg(\exists b \in \mathbb{B}^+) \ b \leq c\}$.
(iii) *If $\mathcal{I}, \mathcal{J}$ are regularisation ideals, then $\mathcal{I} \cap \mathcal{J}$ is a regularisation ideal.*

Proof. Suppose that $\mathcal{I}_{\min} \setminus \mathcal{J} \neq \emptyset$ and $a \in \mathcal{I}_{\min} \setminus \mathcal{J}$. Since $\mathcal{J}$ is maximal then there is a $j \in \mathcal{J}$ for which there is a $b \in \mathbb{B}^+$ such that $b \leq j \vee a$. Since $a \in \mathcal{I}_{\min}$, there is a maximal antichain M in $\mathbb{B}$ such that $m \wedge a = \mathbf{0}$, for each $m \in M$. Every $b \in \mathbb{B}$ has to intersect some $m \in M$, so $\mathbf{0} \neq m \wedge b \leq j \vee a$, but the m and a are disjoint hence $m \wedge b \leq j$, which is a contradiction with the assumption that $\mathcal{J}$ does not contain any element from $\mathbb{B}^+$. Hence, $\bigcap \mathcal{K} \supset \mathcal{I}_{\min}$.

Take an arbitrary $c \in \mathbb{C}^+ \setminus \mathcal{I}_{\min}$, the set $X = \{b \in \mathbb{B}^+ \ : \ b \leq -c\}$ is not dense in $\mathbb{B}$ as $c \notin \mathcal{I}_{\min}$. This means that there is a $b_0 \in \mathbb{B}^+$ such that

$$(\forall b \in X) \ \ b - b_0 \neq \mathbf{0}.$$

If $b_0 \le c$ then c does not belong to any regularisation ideal, otherwise if $b_0 - c \notin \mathbb{B}^+$ one can take a maximal regularisation ideal $\mathcal{J}$ extending $\mathbb{C} \restriction (b_0 - c)$. This shows that $c \notin \mathcal{J}$; and we are done.

(*ii*) and (*iii*) are easy. □

3. Almost disjoint refinement of ground model reals

Let M be a ZFC extension of V. We ask about the existence of an almost disjoint refinement of $[\omega]^\omega \cap V$ in M. Clearly, to have any chance for a refinement, the extension M has to contain a new real, i.e.

$$(\mathcal{P}(\omega))^V \subsetneq (\mathcal{P}(\omega))^M.$$

Hence, from now on we will assume, that the extension M contains new reals. In fact we ask about the existence (of course in M) of a mapping

$$\varphi : ([\omega]^\omega)^V \to [\omega]^\omega$$

such that for each $x \neq y,\ x, y \in ([\omega]^\omega)^V$

(i) $\varphi(x) \subset x$ and
(ii) $\varphi(x) \cap \varphi(y) =^* \emptyset$.

First we show the well known fact that the subalgebra $((\mathcal{P}(\omega)/\mathrm{fin})^V \subsetneq (\mathcal{P}(\omega)/\mathrm{fin})^M$ is not regular.

Lemma 11. *There is $\sigma \subset \omega$, $\sigma \in M$ such that for each $X \in [\omega]^\omega \cap V$ there is a $Y \in [X]^\omega \cap V$ such that $Y \cap \sigma = \emptyset$.*

Proof. Instead of ω one can consider the countable set

$$A = \bigcup \{\ {}^n\{0,1\}\ :\ n \in \omega\}.$$

Let χ be the characteristic function of a new real. Define $\sigma = \{\chi \restriction n : n \in \omega\}$, note that σ is set of compatible functions. Then σ has the desired properties:

Let $X \subset A$, $X \in V$ be infinite. Since A with inverse inclusion is a tree it follows that X either contains an infinite subset Y of compatible functions or it contains an infinite subset Y of pairwise disjoint functions. In the latter case $|Y \cap \sigma| \le 1$. Now suppose that Y is a set of compatible functions and $Y \cap \sigma$ is infinite. Then $\bigcup Y = \chi$, but $\bigcup Y \in V$ and $\chi \notin V$, a contradiction. Hence $Y \cap \sigma =^* \emptyset$ and we are done. □

This yields a list of straightforward corollaries. Note that if there is a $H \subset [\omega]^\omega$ dense in $(\mathcal{P}(\omega)/\mathrm{fin})^M$ such that $H \subset V$. Then $\mathcal{P}^M(\omega) = \mathcal{P}^V(\omega)$.

Corollary 12. Assume that V is ZFC model and M its extension containing new reals. Then

(i) there is $\sigma \subset \omega$, $\sigma \in M$ such that σ does not contain an infinite ground model set; i.e. $(\mathcal{P}(\omega)/\mathrm{fin})^V$ is not a regular subalgebra of the Boolean algebra $\mathcal{P}(\omega)/\mathrm{fin}$ in M,
(ii) there is a MAD family in $[\omega]^\omega \cap V$ which is no longer MAD in M; cf. Proposition 7.

Theorem 1. *In any ZFC extension M of V containing a new real there is an almost disjoint refinement of $([\omega]^\omega)^V$.*

Proof. From Corollary 12 we know, that there is a destructible MAD family $\mathcal{A}$ in $[\omega]^\omega \cap V$, with its 'destructor' $\sigma \in [\omega]^\omega$, $\sigma \in M$, i.e. $\sigma \cap A =^* \emptyset$ for all $A \in \mathcal{A}$.

Let $(T, \supseteq^*) \subset [\omega]^\omega$ be a base tree for $[\omega]^\omega$, in the ground model V. Our aim is to construct another base tree $T^* \in V$ and for each $a \in T^*$ we find $\sigma_a \in [\omega]^\omega$ such that $\{\sigma_a : a \in T^*\}$ will be an almost disjoint refinement for the base tree T^*, hence for $[\omega]^\omega \cap V$.

We denote by T_α the α-level of the tree T. By recursion we construct a base tree $T^* \in V$ for $[\omega]^\omega \cap V$.

We start with the level T_0, which is a MAD family on ω. Each $t \in T_0$ is an infinite subset of ω, take an arbitrary bijection $b_t : t \to \omega$ in V. So $b_t^{-1}[\mathcal{A}]$ is a destructible MAD family on t with a destructor $b_t^{-1}(\sigma) \in M$. Put $T_0^* = \bigcup\{b_t^{-1}[\mathcal{A}] : t \in T_0\}$.

If all T_β^*'s are known for $\beta < \alpha$, denote by T'_α a common almost disjoint refinement of $\{T_\beta^* : \beta < \alpha\}$ and T_α. Every $t \in T'_\alpha$ is an infinite subset of ω, so as in the initial step, take an arbitrary bijection $b_t : t \to \omega$ in V and put $T_\alpha^* = \bigcup\{b_t^{-1}[\mathcal{A}] : t \in T'_\alpha\}$.

The tree $T^* \in V$ is clearly a base tree for $([\omega]^\omega) \cap V$. Moreover, for each $t \in T^*$ we found a subset $b_t^{-1}(\sigma) \in M$. Note that each $b_t^{-1}(\sigma)$ is almost disjoint with every $s \in T_\beta^*$ for each $\beta > \alpha$. Hence, for each $t \neq s$, $b_s^{-1}(\sigma)$ is almost disjoint from $b_t^{-1}(\sigma)$ and

$$\{b_t^{-1}(\sigma) \ : \ t \in T^*\}$$

is an almost disjoint refinement of $([\omega]^\omega)^V$, which completes the proof. □

Corollary 13. Assume that M is an extension of V containing new reals. For any cardinal κ and any set family $\mathcal{S} \subset \mathcal{P}(\kappa) \cap V$ consisting of infinite sets there is an almost disjoint refinement of $\mathcal{S}$ in M.

Proof. Let $\mathcal{A}$ be a MAD family on κ in V consisting of countable sets. For each $A \in \mathcal{A}$ apply previous theorem. For each $A \in \mathcal{A}$ we have a refinement by countable sets R_A. $\{\bigcup R_A : A \in \mathcal{A}\}$ is the desired refinement: let X be infinite subset of κ, then there is $A \in \mathcal{A}$ such that $X \cap A$ is infinite hence there is some $r \in R_A$, $r \subset X$. □

4. The Regularisation Ideal for $\mathcal{P}(\omega)/\mathrm{fin}$

From the previous paragraphs we know, that for an arbitrary ZFC extension M, there is a minimal ideal such that the embedding $(\mathcal{P}(\omega)/\mathit{fin}) \hookrightarrow (\mathcal{P}(\omega)/\mathit{fin})^M/\mathcal{I}$ is regular. Since $\mathcal{I}$ contains fin, we can simplify the notation and write $(\mathcal{P}(\omega)^M/\mathcal{I}$. We are able to describe the minimal regularisation ideal only in the case of a generic extension rather then an arbitrary one. i.e. the minimal ideal $\mathcal{I}_{\min}$ such that the embedding

$$(\mathcal{P}(\omega)/\mathit{fin})^V \hookrightarrow (\mathcal{P}(\omega)^{V(\mathbb{B})}/\mathcal{I}_{\min}$$

is regular. To describe $\mathcal{I}_{\min}$ we introduce *the order sequential topology* on Boolean algebras.

Topological Intermezzo. In order to equip a Boolean algebra with a topological structure that agrees with the Boolean operations we start with a *convergence structure.* It is enough to determine which sequences converge to $\mathbf{0}$ because using the symmetrical difference operation we can move convergent sequences to an arbitrary element $a \in \mathbb{B}$; i.e. $\lim a_n = a$ if and only if $\lim a_n \vartriangle a = 0$. It is natural to use the following notion of a limit; i.e. $\lim a_n = \mathbf{0}$ if and only if

$$\limsup a_n = \bigwedge_n \bigvee_{k \geq n} a_k = \mathbf{0} = \bigvee_n \bigwedge_{k \geq n} a_k = \liminf a_n.$$

It is clear that the right-hand side of the previous formula is redundant and one can define *the order convergence structure* on Boolean algebra $\mathbb{B}$ as the following ideal

$$Os(\mathbb{B}) = \{f \in \mathbb{B}^\omega \ : \ \limsup f = \mathbf{0}\}.$$

Note that it follows directly from the definition that $f \in Os(\mathbb{B})$ if and only if there is $g \in \mathbb{B}^\omega$ so that $g \searrow \mathbf{0}$ and $f \leq g$.

The order convergence structure $Os(\mathbb{B})$ determines *the order sequential topology* τ_s on the Boolean algebra: The set $A \subset \mathbb{B}$ is τ_s*-closed* if and only if

$$(\forall f \in A^\omega)\ (f \text{ is convergent sequence} \longrightarrow \lim f \in A).$$

$(\mathbb{B}, \tau_s)$ is generally a T_1 topological space. The τ_s topology allows us to define *The Urysohn closure* of $Os(\mathbb{B})$; i.e. an ideal

$$U(Os(\mathbb{B})) = \{f \in \mathbb{B}^\omega : f \xrightarrow{\tau_s} \mathbf{0}\}.$$

There is a well known relation between algebraic and topological convergence.

Proposition 14. *A sequence $\langle x_n \rangle$ converges to x in the topology τ_s, $x_n \xrightarrow{\tau_s} \mathbf{0}$, if and only if any subsequence of $\langle x_n \rangle$ has a subsequence that converges to $\mathbf{0}$ algebraically.*

The definition of the topological structure sketched here works well only in case the Boolean algebra in question is σ-complete (we use it here on complete Boolean algebras). In general, the assumption of σ-completeness of $\mathbb{B}$ is not necessary. We give the general definition here. The definitions coincide whenever $\mathbb{B}$ is σ-complete; for more details see [Vla69], [BFH99], [BJP05] or [Paz07].

Definition 15. Let $\mathbb{B}$ be an arbitrary Boolean algebra,

$$Os(\mathbb{B}) \ = \ \{f \in \mathbb{B}^\omega : \exists \mathcal{A} \subset \mathbb{B} \text{ a maximal at most countable antichain such that } f \perp \mathcal{A}\},$$

where $f \perp \mathcal{A}$ means that the set $\{n \in \omega : f(n) \wedge a \neq \mathbf{0}\}$ is finite for every $a \in \mathcal{A}$.

If there are no maximal infinite countable antichains in $\mathbb{B}$ it is clear that $Os(\mathbb{B}) = \mathrm{Fin} = \{f \in \mathbb{B}^\omega : \{n : f(n) \neq \mathbf{0}\} < \omega\}$.

The structure $\mathbb{B}^\omega$ with coordinate-wise Boolean operation is again a Boolean algebra; one can also look at $\mathbb{B}^\omega$ as a set of $\mathbb{B}$-names for subsets of ω in the forcing extension by $\mathbb{B}$. From this point of view, the ideal $Os(\mathbb{B})$ consist of names for finite subsets of ω.

Proposition 16. *Let $\mathbb{B}$ be a complete Boolean algebra. Then for any generic G on $\mathbb{B}$*

$$Os^G(\mathbb{B}) \ = \ \{f_G \ : \ f \in Os(\mathbb{B})\} \ = \ \mathit{fin} \ = \ [\omega]^{<\omega},$$

where $f_G = \{n \in \omega : f(n) \in G\}$.

Proof. Let $f \in Os$ and suppose to the contrary that f_G is an infinite set for some generic G. Since $f \in Os$, there exists $g \searrow \mathbf{0}$ such that $f \leq g$. Clearly if $f(n) \in G$ then $g(n) \in G$. Since g is monotone and f_G is infinite, we have $g(n) \in G$ for every $n \in \omega$. This is a contradiction since $\mathbf{0} = \bigwedge\{g(n) : n \in \omega\} \in G$.

On the other hand, suppose that $f \notin Os$ and let $d = \overline{\lim} f > \mathbf{0}$. Choose a generic filter G such that $d \in G$. Clearly, $\forall k \in \omega\ d \leq \bigvee\{f(n) : n > k\}$, which means that $\forall k \in \omega\ \exists m > k$ $f(m) \in G$; hence the set f_G is infinite. $\square$

Computing the minimal regularisation ideal for $\mathcal{P}(\omega)/\mathrm{fin}$. Now we are ready to show that the minimal regularisation ideal $\mathcal{I}_{\min}$ for the canonical embedding of the Boolean algebra $(\mathcal{P}(\omega)/\mathit{fin})^V$ into $(\mathcal{P}(\omega)/\mathit{fin})^{V[G]}$ is given by the evaluation of names from $U(Os(\mathbb{B}))$.

Theorem 17. *Let* $\mathbb{B}$ *be a complete Boolean algebra and let* G *be a generic filter in* $\mathbb{B}$ *over* V. *Then*

$$\mathcal{I}_{\min} \;=\; U(Os)^G.$$

Proof. Let $f \in {}^{\omega}\mathbb{B} \cap V$ be such that $f_G = \rho \subset \omega$ destroys a MAD $\mathcal{A} \in V$. Find a name $g \in U(Os)$ for the set ρ. Suppose $f \notin U(Os)$; i.e. there is $X \subset \omega$ infinite such that $f \restriction Y \notin Os$ for each $Y \in [X]^{\omega}$. Let

$$\mathfrak{X} \;=\; \{\; X \in [\omega]^{\omega} \;:\; \forall Y \in [X]^{\omega}\; f \restriction Y \notin Os \;\}.$$

For $X \in \mathfrak{X}$ there is an $A \in \mathcal{A}$ such that $X \cap A$ is infinite; denote this infinite intersection by $Y_X = X \cap A$. Since $X \in \mathfrak{X}$, $f \restriction Y_X \notin Os$; i.e. $\overline{\lim}_{Y_X} f \notin G$. Otherwise if

$$\bigwedge_{k\in\omega}\; \bigvee_{k\le n\in Y_X} f(n) \;\in\; G,$$

then $\bigvee_{k\le n\in Y_X} f(n) \;\in\; G$ for each $k \in \omega$ and the set $f_G \cap (A \cap X)$ would be infinite, which would contradict the fact that f_G destroys $\mathcal{A}$. Now, put

$$c \;=\; \bigvee_{X\in\mathfrak{X}} \overline{\lim}_{n\in Y_X}\; f(n) \;\notin G,$$

and $g(n) = f(n) - c$; clearly $g_G = f_G = \rho$ and $g \in U(Os)$.

Let $f \in U(Os) \setminus Os$ i.e. for every infinite X there is a $Y_X \in [X]^{\omega}$ such that $f \restriction Y_X \in Os$. Then the family

$$\mathcal{F} \;=\; \{\; Y_X \;:\; X \in [\omega]^{\omega} \;\}$$

is dense in $\mathcal{P}(\omega)/\mathit{fin}$. Now, pick an arbitrary MAD family $\mathcal{A} \subset \mathcal{F}$. Clearly, f_G is an infinite set ($f \notin Os$) and destroys the MAD family $\mathcal{A}$. □

This result together with Corollary 12 yields the following equivalence. This equivalence was proved independently by M. S. Kurilić and A. Pavlović.

Corollary 18. [KP07] For a complete Boolean algebra $\mathbb{B}$ the following are equivalent

(i) $U(Os(\mathbb{B})) \;=\; Os(\mathbb{B})$,
(ii) there is no $V^{\mathbb{B}}$-destructible MAD family on ω in V,
(iii) the algebra $\mathbb{B}$ as a forcing notion does not add new reals.

In the special case when there are no independent reals in the extension M there is even a unique *largest* regularisation ideal (cf. Proposition 8) with a simple and straightforward description. We say that $A \subset \mathcal{P}^M(\omega)$ is an independent real if for every $X \in [\omega]^{\omega} \cap V$ both $A \cap X$ and $X \setminus A$ are infinite.

Definition 19. Let H be the family of subsets of ω that do not contain infinite sets from the ground model

$$H = \{\sigma \in M : \sigma \subset \omega \quad \& \quad \neg\, \exists\, a \in ([\omega]^{\omega})^V \; a \subset \sigma\}.$$

Proposition 20. *The following holds in* M.

(i) H *is an open dense subset of* $([\omega]^{\omega}, \subseteq)$ *if and only if* M *contains new reals.*
(ii) H *is an ideal if and only if* M *does not contain independent reals.*

Proof. First note that if M contains a new real $\chi \subset \omega$, $\chi \notin V$, then H contains an infinite set. It is easy to see that σ given by lemma 11 is an infinite set belonging to H.

To prove *(i)*, let $A \in ([\omega]^\omega)^V$. Then there is a bijection f in V between ω and A and by Lemma 11 there is a subset $\sigma \subset \omega$ in M which does not contain an infinite ground model set, so $f[\sigma] \in H$ is a subset of A. Generally, if $A \in [\omega]^\omega$ then $A \in H$ or there is an $A' \in ([\omega]^\omega)^V$, $A' \subset A$ and we can use the same reasoning.

(ii) Suppose that M contains an independent real σ. Clearly $\sigma \in H$ and $-\sigma \in H$, hence H is not an ideal.

On the other hand if H is not an ideal, then there are $a, b \in H$ and there is an $X \in ([\omega]^\omega)^V$ such that $X \subset a \cup b$. Again, we can identify X and ω in ground model and then $X \cap a$ is an independent real in M. □

It is clear that whenever H is an ideal, then it is the unique regularisation ideal; cf. Proposition 10.

Proposition 21. *Let M be a ZFC extension of V containing new reals. Then M does not contain independent reals if and only if there is a unique ideal H such that the canonical embedding $(\mathcal{P}(\omega)/\mathrm{fin})^V \hookrightarrow \mathcal{P}(\omega)/H$ is regular.*

Proof. This is a direct consequence of Propositions 8 and 10. □

5. Semiselective ideal

Definition 22. Let $\mathcal{I}$ be an ideal on ω containing $[\omega]^{<\omega}$, we say that the coideal $\mathcal{K} = \mathcal{P}(\omega) - \mathcal{I}$ is *semiselective* (cf. I. Farah [Far98]) if

(i) for every countable collection $\{D_n \subset \mathcal{K} : D_n \text{ opendensesetin } (\mathcal{K}, \subset^*)\}$ the intersection $\bigcap_{n\in\omega} D_n$ is dense in $(\mathcal{K}, \subset^*)$ and

(ii) for each $A \in \mathcal{K}$ and for every decomposition R of A into finite sets, there is selector $X \in \mathcal{K}$; i.e. for every $r \in R$ $|r \cap X| \leq 1$.

Typical examples of semiselective coideals are $[\omega]^\omega$, a selective ultrafilter $\mathcal{F}$, or $\mathcal{K}(\mathcal{A})$, where $\mathcal{A} \subset [\omega]^\omega$ is a maximal almost disjoint family and $\mathcal{K}(\mathcal{A}) = \{X \subset \omega : \{A \in \mathcal{A} : |X \cap A| = \omega\} \text{ isinfinite }\}$.

In fact those coideals are even selective (happy families), c.f. [Mat77]. A coideal is selective if we strengthen the condition (i) to the fact that the preordering $(\mathcal{K}, \subset^*)$ is σ-closed. Selective and semiselective coideals play an important role in Ramsey theory see [Mat77], [Far98].

We ask when the coideal $[\omega]^\omega$ in V generates a semiselective coideal in some extension $M \supset V$; i.e. when the structure

$$\mathcal{H} = \{A \subset \omega : (\exists a \in [\omega]^\omega \cap V)\ a \subset A\}$$

is semiselective coideal in M.

Theorem 23. *Let M be a ZFC extension of V. $[\omega]^\omega \cap V$ generates a semiselective coideal in M if and only if*

(i) *M does not contain an independent reals and*

(ii) *$\omega^\omega \cap V$ is a dominating family in M and*

(iii) $\mathrm{cf}^M \mathfrak{h} \neq \omega$.

Proof. The family $\mathcal{H} = \{A \subset \omega : (\exists a \in [\omega]^\omega \cap V)\ a \subset A\}$ is $\mathcal{P}(\omega)^M - H$, where the ideal H is defined in 19. Hence $\mathcal{H}$ is a coideal if and only if H is an ideal if and only if M does not contain independent reals, cf. Proposition 20.

Condition (ii) is for coideal $\mathcal{H}$ equivalent to (ii) in Definition 22 of semiselective coideal. Using a base tree in V, condition (iii) of Theorem is equivalent to (i) in Definition of a semiselective coideal. □

Remark 24. (i) A typical example of M satisfying $(i)-(iii)$ is when M is a generic extension over Sacks forcing. More consistent examples of *ccc*, ${}^{\omega}\omega$ bounding forcings producing suitable extension can be found in [BJP05]

(ii) Generic extension via Rational perfect set forcing [Mil84] satisfies conditions (i) and (iii) and does not satisfy condition (ii). The extension via Measure algebra satisfies condition (ii) and (iii) but does not satisfy condition (i).

Question. Note that if $M = V[G]$, where G is generic over some proper forcing, then item (iii) from theorem can be omitted. We do not know any extension containing new reals, satisfying conditions (i) a (ii) and collapse the cofinality of $\mathfrak{h}$ to ω; i.e. whether the condition (iii) can be omitted in general.

6. The Regularisation Ideal for $\mathbb{B}^{\omega}/Fin$

In this final part we assume that Boolean algebras are at least σ-complete. This assumption is necessary but since our motivation comes from forcing it is not too restrictive.

The canonical embedding

$$\begin{aligned} e : \mathbb{B} &\longrightarrow \mathbb{B}^{\omega} \\ b &\longmapsto \langle b : n \in \omega \rangle \end{aligned}$$

is obviously regular. The more interesting situation is the derived embedding $\hat{e} : \mathbb{B} \hookrightarrow \mathbb{B}^{\omega}/\mathrm{Fin}$, where $\mathrm{Fin} = \{f \in \mathbb{B}^{\omega} : |\{n : f(n) \neq \mathbf{0}\}| < \omega\}$. This embedding is not regular since the image of a maximal countable antichain $\langle a_n : n \in \omega \rangle \subset \mathbb{B}$ is not maximal in $\mathbb{B}^{\omega}/\mathrm{Fin}$. It is enough to put $f = \langle a_n : n \in \omega \rangle \in (\mathbb{B}^{\omega} \setminus \mathrm{Fin})$ and we get $f \wedge e(a_n) \in \mathrm{Fin}$ for every $n \in \omega$. Note that by our assumption that $\mathbb{B}$ is σ-complete, there are countable maximal antichains in $\mathbb{B}$.

It is natural to ask what is the minimal regularisation ideal $\mathcal{I}_{\min}$ for this situation and how does the algebra $\mathbb{B}^{\omega}/\mathcal{I}_{\min}$ behave from the forcing point of view.

Proposition 25. *The canonical embedding of σ-complete Boolean algebra $\mathbb{B}$ into $\mathbb{B}^{\omega}/Os(\mathbb{B})$ is regular. Moreover, whenever the canonical embedding $\mathbb{B} \hookrightarrow \mathbb{B}^{\omega}/\mathcal{I}$ is regular for some ideal $\mathcal{I} \supset \mathrm{Fin}$, then $Os(\mathbb{B}) \subset \mathcal{I}$.*

Proof. Let $f \in \mathbb{B}^{\omega} - Os$ then $d = \overline{\lim} f > \mathbf{0}$ is the required pseudoprojection witnessing the fact that the embedding $\mathbb{B} \hookrightarrow \mathbb{B}^{\omega}/Os$ is regular.

Computing $\mathcal{I}_{min}$ using Theorem 6 we obtain that

$$\mathcal{I}_{min} = \{f \in \mathbb{B}^{\omega} : \exists \text{ max.antichain } \mathcal{A} \text{in} \mathbb{B} \text{ suchthat } f \perp \mathcal{A}\}.$$

It is clear from the definition that $Os \subset \mathcal{I}_{min}$, which completes the proof. □

We conclude with the forcing description of algebra $\mathbb{B}^{\omega}/\mathcal{I}$, where $\mathcal{I}$ is a regularisation ideal.

Theorem 26. *Let $\mathbb{B}$ be a complete Boolean algebra and $\mathrm{Fin} \subset \mathcal{I} \subset \mathbb{B}^{\omega}$ an ideal for which the canonical embedding $\mathbb{B} \hookrightarrow \mathbb{B}^{\omega}/\mathcal{I}$ is regular, then $\mathbb{B}^{\omega}/\mathcal{I}$ is isomorphic with an iteration of $\mathbb{B}$ and $P(\omega)/\mathcal{I}^G$, where G is the generic filter on $\mathbb{B}$; i.e.*

$$\mathbb{B}^{\omega}/\mathcal{I} \quad \cong \quad \mathbb{B} \star (\mathcal{P}(\omega)^{V[G]}/\mathcal{I}^G).$$

Proof. We define

$$\begin{array}{rcl}\varphi : \mathbb{B} \star \mathcal{P}(\omega)/\mathcal{I}^G & \longrightarrow & \mathbb{B}^\omega/\mathcal{I} \\ (b, f) & \longmapsto & e(b) \wedge f,\end{array}$$

where f is a $\mathbb{B}$-name for a subset of ω. Let us recall the ordering

$$(b, f) \leq (c, g) \text{ if and only if } b \leq c \ \&\ b \Vdash \text{“}[f]_\mathcal{I} \leq [g]_\mathcal{I}\text{”},$$

where $b \Vdash$ “$[f]_\mathcal{I} \leq [g]_\mathcal{I}$” means that $e(b) \wedge f \leq_\mathcal{I} e(b) \wedge g$.

It is a routine check to verify that φ preserves ordering, the disjointness relation and that $\varphi[\mathbb{B} \star \mathcal{P}(\omega)/\mathcal{I}^G]$ is dense in $\mathbb{B}^\omega/\mathcal{I}$. □

The following result was originally proved by A. Kamburelis.

Corollary 27. If $\mathbb{B}$ is a complete Boolean algebra, then

$$\mathbb{B}^\omega/Os(\mathbb{B}) \ \cong\ \mathbb{B} \star (\mathcal{P}(\omega)^{V(\mathbb{B})}/\mathrm{Fin}).$$

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