Kant, Herbart and Riemann[1]

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Introduction

Johann Friedrich Herbart (1776–1841) succeeded to Kant's chair at Königsberg in 1809. He is known today as a philosopher of education and as a seminal psychologist; but in the 19th century, Herbart's metaphysics and psychology had an extraordinary, still largely undocumented, influence on science and mathematics. Bernhard Riemann, in his famous 1854 *Probevorlesung* on geometry, named as his influences only C. F. Gauss and "some philosophical investigations of Herbart". Hermann Grassmann's groundbreaking *Ausdehnungslehre* (1844) was influenced by Herbart.[2] And Ernst Mach, the Austrian physicist-philosopher, was a devoted Herbartian in his youth, whose theory of space drew on Herbart's *Metaphysik*.[3] In his *Knowledge and Error*, Mach referred to Herbart's metaphysics as a model for a "chemical manifold" of energies (electromagnetic, gravitational and thermal) for a future physics, mature enough to dispense with space:

> Our intuitions of space and time form the most important foundations of our sensory view of the world and as such cannot be eliminated. However this does not prevent us from trying to reduce the manifold of qualities of place-sensations to a physiological-chemical manifold. We might think of a system of mixtures in all proportions of a number of chemical qualities (processes). If such an attempt were one day to succeed, it would lead also to the question whether we might not give a physical sense to the speculations that Herbart, following Leibniz, conducted as regards the construction of intelligible space, so that we might reduce physical space to concepts of quality and magnitude. There is of course much to be objected to in Herbart's metaphysics. His tracking down of contradictions that are in part artificially contrived and his eleatic tendencies are none too attractive, but he will hardly have produced nothing but errors.[4]

What in the metaphysics and psychology of this "minor" philosopher made such an impression on three first-class minds of the nineteenth century?

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[2] Jammer, Max: *Concepts of Space*. Third Edition. New York: Dover 1994, 177.

[3] See my *Ernst Mach's World Elements: A Study in Natural Philosophy*. Dordrecht: Kluwer Academic Publishers 2003.

[4] *Knowledge and Error*. Translated by Thomas McCormack and Paul Foulkes. Dordrecht: D. Reidel 1976, 349.

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One possible answer is that Herbart was a brilliant *clarifier* and interpreter who sharpened several lines of thought in Kant and Fichte, making their ideas more acceptable to scientists. In doing so, Herbart returned to a realistic construal of the Leibnizean dynamic community of substances, which Kant had fought so strenuously to limit to the realm of *appearances*. Herbart also firmly separated spatial representation (*Raumvorstellung*) from the properties of physical space. He proposed a Kantian theory of spatial representation by means of memory and imagination in the *Psychologie als Wissenschaft* (1824–25), and a Leibnizean construction of intelligible space from space-less elements in the *Allgemeine Metaphysik* (1828–29). Both lines of argument had been attempted by Kant (and rejected) but Herbart's distinction became crucial for scientific progress later in the century, especially after Gauss and Riemann, when it could not be maintained that the human psychological capacity to represent space as Euclidean or even three-dimensional indicated *synthetic a priori* knowledge about space's physical properties.

Herbart's Metaphysics

Herbart's metaphysics, of which his *Psychologie* was a chapter, was an *Elementenlehre* of matter and form.[5] The matter of the world was made up of sensations with their qualities and magnitudes considered as primitive properties. The forms of space, time and extended objects indicated abstract orderings of those elements together. Herbart was a realist about forms of combination; his elements always occurred in a causal nexus of some sort, even if it was impossible to know which elements in particular combined with which:

> Are the forms of experience given? Yes indeed they are given, although only as determinations of the manner in which sensations are bound up together. Were they not given, we could not only sunder them from sensation in such a way that the sensed could occur completely isolated, without any connection; rather we could also, at pleasure, see different shapes, hear other time intervals; similarly we could put things together arbitrarily out of properties and change them.[6]

Herbart's doctrine of quality was originally presented in his *Psychologie als Wissenschaft*, where his elements were considered analogous to Newtonian forces. The quality of an element was like the direction of a force. Where two elements of opposing quality met, they pressed against one another by a kind of intensive action and reaction. If the two qualities were independent, they passed through one another harmlessly with no effect, like forces situated at right angles. When two qualities opposed one another, the degree to which the stronger overcame the weaker gave their respective magnitudes.[7]

[5] Herbart, J. F.: *Allgemeine Metaphysik*. Königsberg 1829, Sec. 169.
[6] *Metaphysik*, Sec. 171, 21.
[7] Two assumptions are necessary for that to be true. First a standard quality must be chosen as a unit of intensive magnitude and second those measures must be transitive for all other qualities for which the standard is not in contact.

When Herbart's qualities exactly cancelled one another in magnitude and direction they were in spaceless equilibrium and could group together into stable point-like objects, which he called *Wesen* ("beings," "objects"). A *Wesen* was like a Leibnizean monad in being unextended and simple. *Wesen* were really point-like nodes held in position by the mutually constraining forces acting there and not by their mere existence. In fact, since they were little more than functions of their qualities, Mach saw fit to do away with them and let the qualities, which he called elements, abut upon one another in functional groupings.[8]

Herbart claimed that each *Wesen* conserved itself against the others by "pressing back" through its qualities. In fact he described two sorts of pressures. A *Wesen* could, for example, press outward against others in its environment, in which case Herbart said the *Wesen* was Together (*Zusammen*) with the others. Or its qualities could press inward against each other and away from the outside, in which case Herbart said the *Wesen* was Not-Together (*Nicht-Zusammen*) with the others. Despite the fact that both inward pressures and outward pressures existed, alternating over times, all the qualities in the community pressed one another in general causal relations of action and reaction. Indeed, were the forces in the community not generated in those mutually opposing pairs, the *Wesen*, too, would not exist:

> We could say through a sensory comparison what they [Wesen] do. They *press* one another. For in the world of the senses we find resistance in pressure, where nothing gives way although each is supposed to move. Pressure is rest, through reciprocal endurance against another. But sensory comparison is dangerous here. We are not talking about spatial relations […] Here we are only talking about a change in quality that each should suffer from the other, however against which it conserves itself as that which it is. Disturbance should result; self-conservation cancels the disturbance in such a way that it does not occur at all.[9]

Kant, too, had seen the world of physics, in good Newtonian fashion, as consisting of forces set to work against and resisting one another. When these forces cancelled, Kant remarked, we do not have nothing, as in the A and not-A of logic, but rather a stable phenomenal reality (*realitates phaenomena*).[10]

None of Herbart's metaphysical ideas was unprecedented of course. The idea of an unextended community of qualities and monads beneath the appearance of matter in empty space was long thought to be the true face of the world enjoyed by the divine Intellect, which did not require spatial or temporal representation. Leibniz, and the pre-critical Kant, had adumbrated such a community of unextended substances changing their states qualitatively, either by pre-established harmony emanating from their insides (Leibniz) or by external forces like gravitation acting at a distance (Kant). (Herbart seems to have combined these mechanisms into inner and outer pressures.) Even in the *Critique of Pure Reason*, Kant was still convinced

[8] For this episode in Mach's Herbartian apprenticeship, see Laass, Andreas: Vom Sinne des Machschen Philosophierens. In: Hoffmann/Laitko (Eds.): *Ernst Mach: Studien und Dokumente zu Leben und Werk*. Berlin: Deutscher Verlag der Wissenschaften 1991.

[9] *Metaphysik*, Sec. 234, 103.

[10] See KrV: A 272–273/B 328–329.

that a community of substances undergoing dynamical, reciprocal changes among themselves, would be sufficient for a construction of space within, if only "pure understanding could penetrate to things in themselves, and if they admitted of space and time determinations".[11]

Herbart's instantaneous, space-less sensations were extrapolated from another of Kant's positions: his doctrine of intensity. Possibly drawing from an "intensive" conception of the infinitesimal in the analytic mechanics of his time, Kant had been among the first to emphasize that sensations, and even perhaps instantaneous moments of gravity and other physical forces,[12] exemplified intensive magnitudes in nature:

> Since sensation *an sich* is no objective representation, and in it neither the intuition of space, nor of time, is met with, no extended magnitude is attributed to it, but to be sure a magnitude (and one in which empirical consciousness grows from nothing = 0 to a given measure in apprehending it in a certain time), that is an *intensive magnitude*. Correspondingly intensive magnitude must be attributed to all objects of perception, in so far as they contain sensation, i.e. a degree of influence on the senses.[13]

Kant did not hold that the pure flux of sensations could be consciously experienced, for, as he stated, an *empirical consciousness* had to apprehend a sensation rising or falling in time and his axioms of intuition demanded that *experience* always be of extended magnitudes. That thought applied to every part of experience as well; space consisted of ever smaller spaces, time of ever smaller times. What was non-spatial, for Kant, were sensations *an sich* when regarded as mere qualities with intensity, abstracted from their attachment to an object, which Kant also called *Eindrücke*, impressions. He nearly always qualified such hints by adding that a sensation without a form, not belonging to a spatio-temporal object, was not to be exhibited in intuition and belonged to a preconscious state of sensing before awareness.[14] The aspatiality or atemporality of raw sensations or impressions is a point that has been bandied about by Kant scholars. Kemp Smith classes sensation with unextended magnitudes,[15] while a great many others hold that sensations were subject to the receptive conditions of the transcendental aesthetic, and thus already spa-

[11] KrV: A 266–A 267/B 322–B 323: "[…] wenn der reine Verstand unmittelbar auf Gegenstände bezogen werden könnte, und wenn Raum und Zeit Bestimmungen der Dinge an sich selbst wären."

[12] See KrV: B 210–215.

[13] KrV: A 166/B 208: "Da nun Empfindung an sich gar keine objektive Vorstellung ist, und in ihr weder die Anschauung vom Raum, noch von der Zeit angetroffen wird, so wird ihr zwar keine extensive, aber doch eine Größe (und zwar durch die Apprehension derselben, in welcher das empirische Bewußtsein in einer gewissen Zeit von nichts =0 bis zu ihrem gegebenen Maße erwachsen kann), also eine intensive *Größe* zukommen, welcher correspondirend allen Objecten der Wahrnehmung, so fern diese Empfindung enthält, *intensive Größe*, d.i. ein Grad des Einflusses auf den Sinn, beigelegt werden muß."

[14] See KrV: A 291–292, B 347–348.

[15] Smith, Norman Kemp: *A Commentary on Kant's Critique of Pure Reason*. London: MacMillan, 351.

tial. One cannot thus claim consensus on the aspatiality thesis, but many of Kant's immediate successors held it, including Maimon, Fichte and Herbart.[16]

Herbart did not in general abide by Kant's stricture to stay within the bounds of experience and its preconditions. A naturalized epistemologist avant de la lettre, Herbart believed the structure of human experience was simply a matter for a scientific psychology, whose job it was to analyze away confused apprehension until what remained was the true underlying form and material of things in themselves, i.e. the point-like *Wesen* and their changing states:

> Complexes and fusions, shaded off in inexhaustible variety, interwoven and roused to activity, give our representations in part invented, in part empirical forms. The Mechanics of mind, which not only gets into the representations but also the states of the representing process, displays the possible forms and modes of influence [*Wirkungsarten*] of complexes and fusions. It thus teaches the conditions under which spatial shapes, time intervals, and series of changes are represented. The fulfillment of this condition is nature's affair; we thus posses a knowledge of nature, subject indeed to error and improvement, but of which we cannot be robbed, and which emerges victorious from all difficulties. For in the connections of our ideas, in so far as they are formed by experience anyway, are reflected the connections of things to one another and with us; and this connection between that which is in us and that which is outside of us, becomes clear through psychology in such a way that redounds not insignificantly to the confirmation of the true realistic metaphysics.[17]

Like many post-Kantians, Herbart did speculative metaphysics freely; in fact, he even considered it to be doing science at a higher level of generality. Just as science was forced to admit hypotheses to account for the unobservable, metaphysics was employed to find higher level conceptual grounds for those hypotheses.[18] But Herbart still held that any direct evidence for metaphysical constructions flowed from experience, i.e. sensational data. Metaphysical thought constructions could only be remotely and indirectly "verified" through the success of scientific theories that employed them:

> It is not required that metaphysics be more certain and penetrate deeper than it can in following experience. It rests on this, as on its own proper hypotheses. Should human experience be found too limited, too incomplete, not enough in accord with hopes and wishes on some points to ground a completely satisfactory conviction, the fault cannot be unjustly foisted on metaphysics, which does not light with its own light, but merely renders up what it has received.[19]

As part of his freedom to speculate on the basic metaphysical conceptions underlying science, Herbart permitted himself to consider the pure underlying flux of sensations *an sich*, before considering their confinement in spatial and temporal objects. Under the heading of sensations, Herbart also seems to have included the physical energies and forces that made up matter at a fundamental level.

[16] The locus classicus for this interpretation is Hermann Cohen *Kants Theorie der Erfahrung* and *Das Prinzip der Infinitesimal-Methode und seine Geschichte*. In: *Werke*. Hildesheim/Zürich/New York: Georg Olms Verlag 1987.
[17] *Metaphysik*, Sec. 170. 21.
[18] *Metaphysik*, Sec. 162, 12–13.
[19] *Metaphysik,* 12.

Given his prior Leibnizean allegiances, it is probable that Kant at least, *entertained* the idea that experiences in space and time were constructed by the mind out of a primordial array of spaceless, timeless sensations.[20] For example, in the A edition of the Transcendental Deduction, Kant ventured the hypothesis *conditionally* that if an observer had impressions raw, they would appear in disjoint, strobe-like bursts of intensity. Each state would vanish when the next occurred. If that were so, we could not draw a line or collect up a series of points, because the points just drawn would constantly fall away.

> It is apparent that when I draw a line in thought, or think the time from one noon to another, or merely imagine a certain number, I must necessarily first conceive these manifold representations one after another in thought. Should I leave out and not reproduce the preceding (first parts of the line, the preceding parts of time or the imagined units after one another) in passing to what follows, then never would a whole representation, or the aforementioned thoughts, or even the purest and first fundamental representations of space and time be able to arise.[21]

In the A deduction, at least, Kant seemed to have reasoned that the very possibility of experience in space and time depended on the ability to reproduce past impressions in memory and imagination that were no longer present. In experience, as opposed to passive sensing, the productive *Einbildungskraft* produced and adjoined images of past and future impressions to present impressions and created an imaginary, abiding framework of past, present and future objects out of raw sensations that simply appeared and vanished.

Psychology as a Science

Herbart turned some of these hints into a theory of psychological space in his *Psychologie als Wissenschaft*. Herbart's theory of *Raumvorstellung* was called the "reproduction series", a term that mirrored Kant's "synthesis of reproduction".

In Herbartian psychology, the mind, or "apperceptive mass", was an agglomeration of mutually resisting ideas all fighting to maintain their strength and clearness. In the preconscious flux, every sensation began to vanish immediately after it occurred, but all sensations left behind memory traces or *Vorstellungen*. *Vorstellung*

[20] Wayne Waxman has advanced this interpretation of Kant's transcendental idealism, i.e. that space and time first arise in the imagination, and that raw sensation is aspatial and atemporal. See his "What are Kant's Analogies About?" *Review of Metaphysics* 47, September 1993, 81–82.

[21] KrV: A 101–102: "Nun ist offenbar, daß, wenn ich eine Linie in Gedanken ziehe, oder die Zeit von einem Mittag zum andern denken, oder auch nur eine gewisse Zahl mir vorstellen will, ich erstlich nothwendig eine dieser mannigfaltigen Vorstellungen nach der andern in Gedanken fassen müsse. Würde ich aber die vorhergehende (die erste Theile der Linie, die vorhergehende Theile der Zeit oder die nach einander vorgestellte Einheiten) immer aus den Gedanken verlieren und sie nicht reproduciren, indem ich zu den folgenden fortgehe, so würde niemals eine ganze Vorstellung und keiner aller vorgenannten Gedanken, ja gar nicht eimal die reinste und erste Grundvorstellungen von Raum und Zeit entspringen können."

has the connotation in German of a product of the imagination, that is something that one sets before one's mind rather than something perceived or experienced. These *Vorstellungen*, ideas, then inhibited one other until the stronger ideas had pressed the weaker beneath the limin of consciousness.

From these assumptions, and a few quantitative principles besides, Herbart derived a *Mechanik des Geistes*. One of these primitive mechanisms of this *Mechanik* was to reproduce past impressions after their disappearance. For example, in the case represented in figure 1 below of three qualities A, B and C, if A occurred instantaneously and then disappeared, it would leave behind memory images, gradually sinking in strength, which Herbart denoted with primed letters, A', A'', A''' and so forth; B left behind its images B', B'', B''' and so on. These images were then, in their turn, inhibited by newly appearing impressions and after images.

A	B	C
	A'	B'
		A''

Figure 1

But suppose, Herbart asked, that an impression similar enough to A should recur at a later point in time and was recognized as a repetition of the same, by comparing it with the memory trace left behind by the original A. Herbart said that, at this point, the new sensation would enliven the memory images of the original A. A's strongest lingering memory trace (A') was first inhibited by B so as A' pushed its way back up above the limin of consciousness B's image was pushed up ahead of it. A's second strongest image was inhibited by C, so C was the next image reproduced, and so on. The reproduction series was so called because it exactly repeated the original order of inhibition, and, with it, a facsimile of the original series A, B, C.

Because vanished impressions were still imagined as present after their disappearance, the overall psychological effect of the reproductive imagination was that a sequence of disjoint impressions A, B, C, appeared to be part of an extended stretch from A to C, in which A shaded off continuously as B and C arose. Herbart's teacher, Fichte, had interpreted the Kantian *Einbildungskraft* in similar fashion to account for extended intervals of time between impressions. For example, when the impression A vanished and B instantaneously took its place, Fichte thought that the productive imagination provided the interval between them:

> Suppose that at the point X at time A, light should occur and in the immediately following time, B, darkness. So the light and the darkness are separated sharply from one another as it should be. However the moments A and B bound each other immediately and there is no gap between them. Imagine the sharp boundary between them =Z. What is in Z? Not light; for that is in moment A, and Z is not =A; and just as little darkness for that is in moment B. Therefore neither of the two. But I can just as well say both of them, since if there is no gap between them, there is no gap between light and darkness, therefore they touch each other in Z immediately. It could be said that in the latter reasonings I extend the Z, which should only be a boundary,

by means of the imagination into a moment; and so it is indeed. The moments A and B themselves are extended in no other way than by means of the imagination.[22]

He added pregnantly: "only for the imagination is there any time."[23] Fichte had also employed the Kantian *Einbildungskraft* to bridge the gap between universals and particulars, in what he called a *Schweben*. The present impression was apprehended as a particular and completely determinate in its properties. The missing parts of a perception, such as its past and future stages, were merely imagined and not completely determinate as to their properties or descriptions. Thus the combination of the two in the schema could have the properties of both particulars and universals, which is precisely what Kant had demanded of the transcendental schema.

In his *Psychologie als Wissenschaft*, Zweyter Teil, I. 3., Herbart applied his reproductive series to the psychological problems of spatial and temporal representation, such as the hearing of a melody, or the generating of a monocular field of vision from the retina. When a melody was played, only one note at a time was heard. Therefore the manifestation of the whole was an object of the imagination, except for the parts that happened to be present. Of course the *locus classicus* for the idea that objects and times are a manifestation of memory or imagination is St. Augustine, *Confessions* XI. 35.–38.

Similarly, in interpreting a mosaic of disjoint light sensations on the retina as a two dimensional extended manifold, Herbart believed that each retinal point a, b, c had to possess its own separate reproduction series. The sum of all of them, extending radially from the center, made up a spatial field, reproduced constantly in time as the eye moved about. As Mach described, while still a Herbartian, in his 1863 "Lectures on Psychophysics":

> If the eye has run through the series a,b,c several times and in the opposite direction, the stimulation of one member of the series will have the consequence of stimulating the running through of the series in both directions from this member. From this member on, the members to each side of it will appear simultaneously, with descending clarity. If we consider that from an *a*, the eye can run through not just one but infinitely many series in different directions, and, if we apply the law of reproduction in series, we attain something at least very similar to spatial perception.[24]

As Mach remarked, in any reproduction series where spatial extension was to be intuited, Herbart believed it was necessary to assume multiple series running at once along independent directions. One series, for example that of a melody, at most ordered the impressions temporally but not spatially. The extension was not directly intuited but simply imagined. For Herbart a truly intuitable linear extension required *two* overlapping and independent reproduction series, *durchgekreuzt* or "crossed over" one another.[25] A thin wire seen head-on, for example, looks like a

[22] J. G. Fichte: *Grundlage der gesamten Wissenschaftslehre (1794)*. Hamburg: Felix Meiner Verlag 1997, 127.
[23] *Wissenschaftslehre,* 136.
[24] Mach: Vorträge über Psychophysik. In: *Oesterreiche Zeitschrift für praktische Heilkunde* 9, 1863, 338.
[25] *Psychologie als Wissenschaft,* §§ 109–116.

point. Only when it is turned through an independent, dissociating, direction does it actually appear as a line. Thus by interposing a linear series independent of the first, to be run through as the original series is run through, the original series will actually be *seen* tracing-out an extended line.

Intelligible Space

With his theory of psychological space completed by 1825, Herbart turned to his construction of "intelligible space" in the *Allgemeine Metaphysik*. This was the construction of space out of spaceless elements and the community of *Wesen* that seems to have influenced Riemann, Grassmann and Mach. Here, too, there were several other Kantian *Denkwege* that Herbart followed up, especially concerning spaceless sensations.

Kant had held that physical intensities were logically prior in some sense to extension, since in one and the same instant of time or space, the real could communicate itself to any intensive degree:

> Since every reality has a degree, which can diminish to nothing (the void) through infinite gradations without in any way altering the extensive magnitude of the appearance, there must be infinitely many degrees in which space and time may be filled. Intensive magnitude can in different appearances be smaller or greater, though the extensive magnitude of the intuition remain the same.[26]

Conversely, if the magnitudes of intensities depended on their extension then ten times the force would always require ten times longer, or ten times the space, to communicate; a greater quantity of matter would fill a greater volume since each unit quantity would have to be set aside in a separate space.[27]

On the aspatiality thesis, outlined above, Kant had implied that the real was intensively communicated in a sensation, in which space or time were not to be met with. In a famous passage, in the schematism of the pure principles of the understanding, Kant appeared to be describing how extended temporal intervals were obtained by schematizing instantaneous magnitudes:

> Reality is that in a pure concept of the understanding which corresponds to a sensation, that, therefore, whose concept in itself points to a being (in time); negation is that whose concept represents non-being (in time). The opposition between the two happens in the distinguishing of the same time as a filled or empty time. Just as time is only the form of intuition, therefore of objects as appearances, so is that which corresponds to sensation in them the transcendental material of all objects as things in themselves (thing-hood, reality).

[26] KrV: A 171/B 214: "[…] weil jede Realität ihren Grad hat, der bei unveränderter extensiven Größe der Erscheinung bis zum Nichts (dem Leeren) durch unendliche Stufen abnehmen kann, unendlich verschiedene Grade, mit welchen Raum oder Zeit erfüllt sei, geben und die intensive Größe in verschiedenen Erscheinungen kleiner oder größer sein können, obschon die extensive Größe der Anschauung gleich ist."

[27] See KrV: A 173/B 215–A 175/B 217.

Now every sensation has a degree or magnitude, through which it fills the same time, in respect to the self-same representation of an object, to a greater or lesser degree, until it ends in nothing (=0 negatio). Hence there is a relation and connection, or rather a transition [*Übergang*] from reality to negation, which makes every reality representable as a quantum; and the schema of a reality as the quantity of something in so far as it fills time is just its continuous and uniform production in time, in that one goes, in time, from a sensation with a certain degree downwards to its disappearance, or gradually climbs up from negation to the same magnitude.[28]

The passage begins with Kant's assumption that all sensations that appeared in a single instant, manifested a ratio, or, as he says, an opposition [*Entgegensetzung*] between their intensive reality (a maximum of intensity) and negation (the zero). When Kant spoke of sensations "filling" time in this passage, he seemed to be speaking of two things:

1. the filling of an *instantaneous* moment by means of a constant reality to negation ratio, or unit measure such as R+N=1.
2. The filling of an temporally *extended* series of moments by means of a schema, i.e. a stringing together of the many instantaneous moments of 1. into a time over which an apprehended sensation could be represented as rising or falling.

The job of the schematism of the real (i.e., the imagination) seems to have been to pass from 1. to 2., that is, "from intensity to extension", from a timeless, instantaneous ratio to the extended experience of sensations increasing or decreasing (Figure 2):

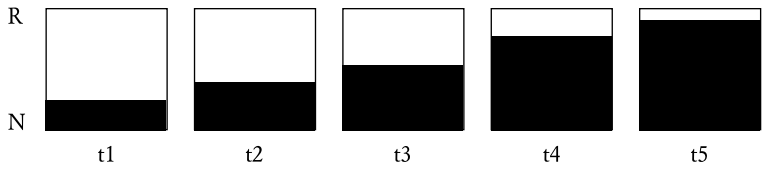

Figure 2

[28] KrV: A 142–143/B 182–183: "Realität ist im reinen Verstandesbegriffe das, was einer Empfindung überhaupt correspondirt, dasjenige also, dessen Begriff an sich selbst ein Sein (in der Zeit) anzeigt; Negation, dessen Begriff ein Nichtsein (in der Zeit) vorstellt. Die Entgegensetzung beider geschieht also in dem Unterschiede derselben Zeit, als einer erfüllten oder leeren Zeit. Da die Zeit nur die Form der Anschauung, mithin der Gegenstände als Erscheinungen ist, so ist das, was an diesen der Empfindung entspricht, die transscendentale Materie aller Gegenstände als Dinge an sich (die Sachheit, Realität). Nun hat jede Empfindung einen Grad oder Größe, wodurch sie dieselbe Zeit, d.i. den innren Sinn, in Ansehung derselben Vorstellung eines Gegenstandes mehr oder weniger erfüllen kann, bis sie in Nichts (=*0=negatio*) aufhört. Daher ist ein Verhältniß und Zusammenhang, oder vielmehr ein Übergang von Realität zur Negation, welcher jede Realität als ein Quantum vorstellig macht; und das Schema einer Realität als der Quantität von Etwas, so fern es die Zeit erfüllt, ist eben diese continuirliche und gleichförmige Erzeugung derselben in der Zeit, indem man von der Empfindung, die einen gewissen Grad hat, in der Zeit bis zum Verschwinden derselben hinabgeht, oder von der Negation zu der Größe derselben allmählig aufsteigt."

Kant did guarantee that every quality could be anticipated to fall somewhere between a certain maximum, R, and a minimum of intensity, N, which not only set the scale for comparing all intensities, but also manifested a constant quantum, R+N in which *any* intensive magnitude was contained. This would happen if the reality and the negation content always added to a fixed quantity or to zero. Kant did not say how he could be so sure that all qualities could be compared along the same scale.

Once the instants had been marked off as intensive quanta, the schematization of timeless moments into stretches of time could have proceeded with the help of the imagination (as represented by figure 2). Past appearances of a quality could be imagined with their intensive ratios in a former instant, and the imagination could then string the instants together by picturing the quality rising or falling over the remembered stretch of time.

Within a community of qualities, such as that considered by Herbart, assuming that the R/N ratios of all the sensations were mutually comparable, that is that whatever some A gains, some B loses, the set of present and remembered reality-to-negation ratios, (N+R), (N'+R'), (N''+R'') … ($N^{(n)}$+$R^{(n)}$), provide a kind of procedure for arranging all intensive magnitudes into the moments of a universal time order.

Some discussion is necessary here, because Kant's schematism of the real is nearly always read in the opposite sense, i.e., that intensities are already temporal magnitudes climbing or falling *in time* as they pass from reality to negation. No doubt that is how things stand *after* schematization, but not before. Paul Guyer, for example, states that intensities must be interpreted as "filling time" prior to being schematized, which leads him to remark that:

> There seems to be no other connection between the structure of time and the fact that what fills it, namely sensation, comes in different degrees than there is between, say, the structure of wine bottles and the fact what fills them comes in different vintages.[29]

Recalling, however, that Kant believed extended intervals of time or space could be "filled" to any intensive degree, it is certainly impossible that we should measure intensity by time or space, and Kant was not suggesting it. In fact, Guyer, whose main subject in *Kant and the Claims of Knowledge* is the rules of time schematization, concludes that "Kant's principle of intensive magnitude […] does not have a clear place in his transcendental theory of experience." On the contrary, Kant's whole theory of time schematism begins with intensive magnitudes.

Such an interpretation seemed clear enough to Herbart, who took it for granted that the schematizing of rising and falling intensities provided the essentials of time determination. His chapter of the *Metaphysik* "Von der Zeit" began with a discussion of the various types of intensive magnitude and the ways in which they

[29] Guyer, Paul: *Kant and the Claims of Knowledge*. Cambridge University Press 1987, 202.

might fall, rise or remain the same. Herbart called special attention to the annihilation of past impressions and the appearance of new ones, claiming that these positings and cancellations were what was numbered by time.[30]

Overall, Fichte's and Herbart's guiding interpretation of Kant seems to have been that time schematization completed present sensations into *ein Bild*, by adding to them the images of non-present past sensations and future ones, linking them together with the present impression. The result is an ideal, permanently extended *time experience*, the smallest stretch of which always contained a past, a present and a future, again a property of time identified by St. Augustine. If so, the real, the sensations *an sich,* were rightly considered timeless, as well as spaceless, and did not even form objects of experience for Kant.[31]

Herbart took it upon himself to correct what he saw as Kant's omissions. For example, Kant had assigned every quality its instantaneous magnitude of R+N, which could always be assumed to be a fixed quantum. This was not justified unless the relations of the qualities of the world to one another were always such that whatever one quality gained another would lose, i.e., that "action and reaction" was be observed in their changes. For Herbart, every quality was matched against an opposite, so that the reality content of one was the negation content of the other and vice versa. What was for Kant a ratio of reality to nothing became for Herbart a ratio of reals to *other* reals. That would certainly have fit with Kant's view of the world (*qua realitatis phaenomena*) as consisting of an equilibrium of opposing forces.

Action and reaction also provided a sufficient condition for the qualities to form into *a* community, the existence of which Herbart deduced immediately from the fact that qualities depended upon and pressed against one another.[32] In fact, once he had assumed the community of reciprocally determining qualities, Herbart stated that the opposition of qualities to one another contained the needed time determination in the numbering of their changes by a constant quantum. Here he may have been thinking of a matching procedure, like that described above. It could easily be imagined that as the qualities pressed against one another, they changed one another's magnitudes. They filled time through their action, which could be represented instantaneously by ratios, or over imaginary stretches of time and space. The actions were all in step with one another, in a *Wechselwirkung*:

[30] *Metaphysik*, Sec. 287.

[31] The mathematician W. K. Clifford (On the Nature of Things in Themselves, *Mind* 3, 1878, 57–67.) had the aperçue that "Kant threw out a suggestion that the Ding an sich might be of the nature of mind." If so, that is if the sensations an sich are direct unschematized "experiences" of the real, they are still limited by Kant to a transcendental pre-conscious before the experiences of self and objects.

[32] It is not a sufficient requirement to determine the objects of the community uniquely, for the action and reaction pairings could still be done in any number of possible ways.

The togetherness and separateness of substances is subject to an alternation [*Wechsel*], which encloses immediately a time-determination. One easily sees that motion and space are also assumed within; [...] Provisionally we call that space, which we unavoidably think in addition with the coming and going of substance, intelligible space.[33]

Herbart did, in fact, give a measure of the time instant deduced from the mutual action of his force-like qualities. If kown intensities are set to work against one another, the instantaneous ratio they fall into gives a measure of their relative intensities.[34] This ratio can then be extended to other oppositions by transitivity. In fact the quantum can be any standard ratio one chooses to define t or dt, without involving extended time intervals.

In his construction of intelligible space, however, Herbart was less concerned with working out the variety of relations possible at the level of qualities and more concerned with the waves of *action* that undulated through the community and traced out lines, surfaces and solids in time. Of course, the *Wechsel* of qualities still drove the whole construction since the *Wesen* were practically no more than patterns formed by the play of qualities beneath.

Natural Number: Starre Linien

Herbart applied his ideas to a construction of the natural number line, which he imagined to be constructed by a generating process that could progress only in two directions, left and right. To exhibit the structure of the natural number line, Herbart felt that the construction of a *Nacheinander* of points had to create both points *and* gaps setting one point apart from another over time. It was essential for him that the separateness of points be established, as well as their togetherness.

Herbart accounted for these properties of points and gaps by building them into the generating process that traced out the line. The points were generated by the meeting of two *Wesen* together (*Zusammen*) and the gaps by their not being together (*Nicht-Zusammen*). He understood two *Wesen* to be together, for example, when their qualities were in direct mutual influence, without a causal intermediary. He understood two *Wesen* formerly together to separate when one suffered a change in its qualities and the other remained in equilibrium[35]. When this happened, an intermediary *Wesen*, or rather *Bild*, represented by primed letters A' A'' A''' took the place of the *Wesen* that had since changed and represented the continuing presence of this past *Wesen*, as the images of past impressions had done in Herbart's psychological reproduction series (Figure 3):

[33] *Metaphysik*, Sec. 244, 118.

[34] *Psychologie als Wissenschaft*, Erster, synthetischer Teil, § 44.

[35] To be completely clear, Herbart says we are to abstract A and B from the underlying play of qualities. This is because the qualities and the *Wesen* are always unique, whereas for the sake of the construction, one has to consider the abstract patterns A and B as movable and reidentificable as they more through the community. (See loc. cit.).

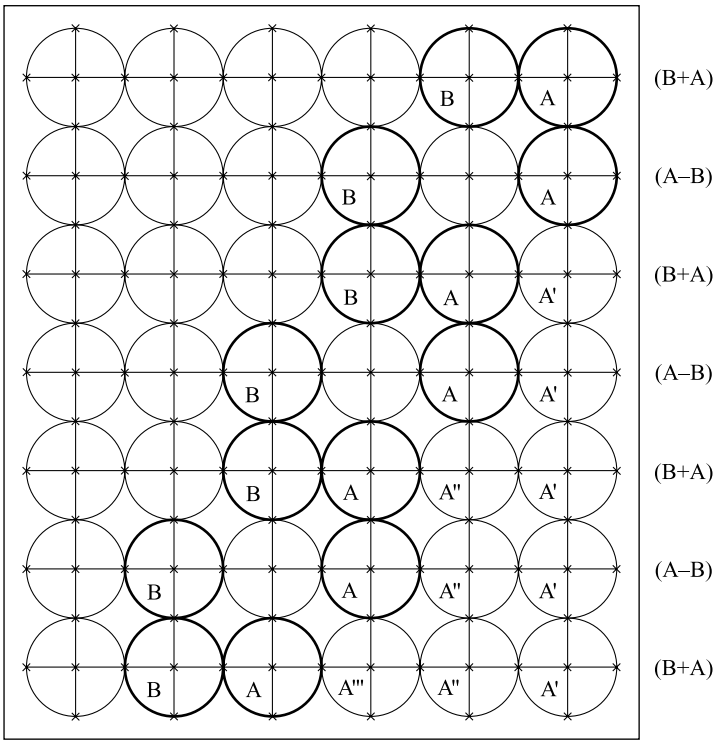

Figure 3

A simple notation involving the symbols {**A, B, +, –**} can be employed to show all of the possible directed paths Herbart could generate with such a process. Let **A** and **B** in figure 3 be two *Wesen*. When they are together with each other, their qualities are in complete dependence, which means that when a quality of one rises in intensity, a quality in the other falls. The ratio (**R+N**) is upheld as the time instant demanded. Suppose, however, that the qualities of **B** suffer a change and that the qualities of **A** remain of the same magnitude. In that case, as Herbart represented in words, the situation is written (**A–B**), writing the fixed *Wesen* first and the changing *Wesen* second. When **A** changes and **B** remains the same, we have (**B–A**). If, on the other hand, **A** and **B** are two formerly independent *Wesen* just passing into dependence on one another, they are represented as Together. (**A+B**) indicates that an independent **B** has passed into dependence on a fixed **A**, and (**B+A**) indicates that an independent **A** has passed into dependence on a fixed **B**. Taking all of the possibilities in hand, the process of producing a set of points in either direction (in Figure 3) is an alternation that could be represented by "right" and "left" sequences:

Right: **(A–B),(B+A),(A–B),(B+A) …**

Left: **(A+B),(B–A),(A+B),(B–A) …**

Synoptically represented in Figure 4:

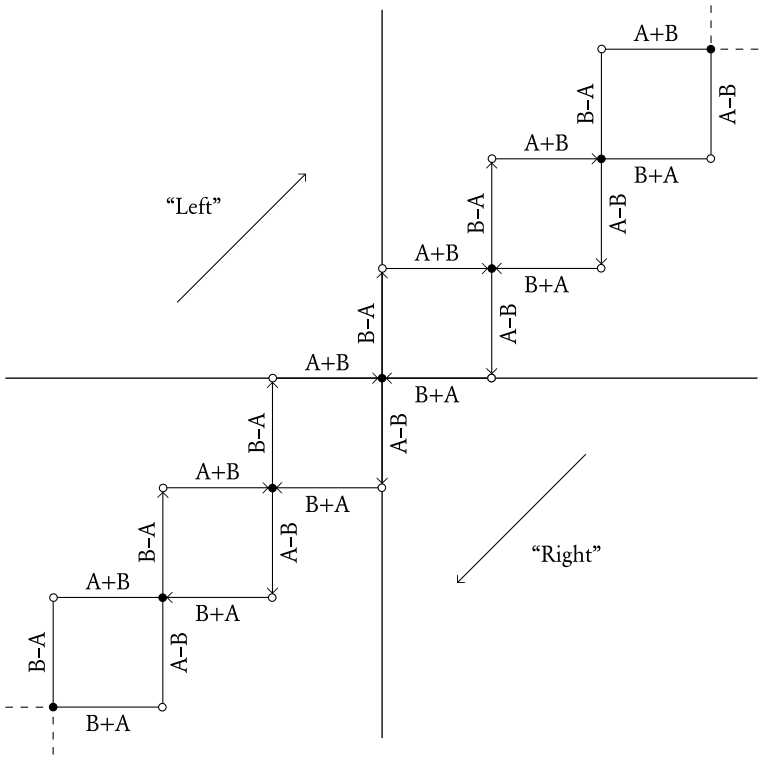

Figure 4

Herbart and the Problem of Change

We can credit Herbart with the knowledge that extension is a complex concept in that it includes the sub-concepts both of association and *dis*sociation, something he made clear in his construction of psychological extended series, with a causally dissociating series *durchgekreuzt* with a given reproduction series.

Herbart claimed that a time sequence, natural number line or series of degrees climbing and falling was not a series of points where the qualities rigidly coincide everywhere, but a constructed set of points *and* dissociating gaps. In the gaps the

qualities of *Wesen* that begin together suddenly became independent of one another (*Nicht-Zusammen*), until equilibrated at the next instant, this time perhaps with a different *Wesen* as partner.

This process is worth reconstructing in some detail, employing the same notation, but this time applied to the qualities **R**, **N** within the *Wesen*.

Let figure 5 below represent a rising intensity on Herbart's scheme. That means the reality content (**R**) of the quality increases at the expense of the negation content (**N**). At first **R** and **N** are rigidly connected to one another. Then they are allowed to slip out of dependence as a gap occurs. These changes can be instantaneous "in-between instants". In the dissociation of the two qualities **R** and **N**, they are free to change independently of each other and come into dependence instead on two further qualities **N'** and **R'** since every quality's magnitude always depends on some other. Let **N**, in fact, depend on some **R'** that is greater than **R**, so that the intensity of **N** is now suddenly lower. Meanwhile let **R** drop out of connection with **N**, and into dependence on an **N'** whose magnitude is less than **N**, meaning that **R**'s inten-

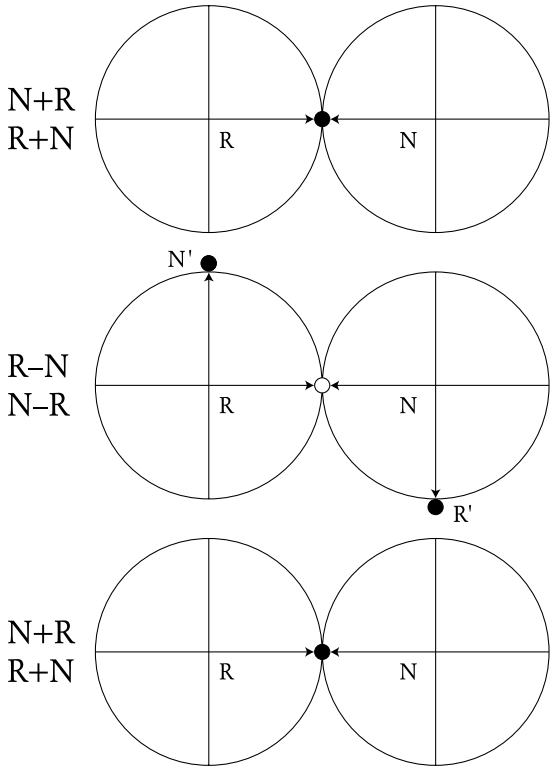

Figure 5

sity is now greater. After these switches have been made, and assuming they still add to the same fixed quantum, we can join **R** and **N** together again at their new intensities. After joining them again, **R** has a higher intensity and **N** a lower.

Starre and stetige Linien

Herbart claimed that his construction in Figure 3 gave every "thinkable distancing" that could occur between **A** and **B** in both directions.[36] Every directed path was representable. For example, a process could move forwards any number of spaces and then backwards retracing these spaces, or it could move infinitely in either direction, or infinitely back and forth over the same three points.

An infinite set of possible directed paths evolving over time was not yet a natural number line, however. As Herbart seems to have been aware, a natural number line was the set of all possible combinations of directed paths that satisfied certain additional assumptions. Among these was the property that tracing three spaces forwards can be exactly reversed by another process that traces those spaces in the opposite order. For example, 3,2,1 is the opposite of 1,2,3. Herbart thus postulated that his generating process remained similar to itself in both directions forwards and backwards. The process **A+B** (**B** coming to a fixed **A**) is the opposite in sense of **B+A** (**A** coming to a fixed **B**). Likewise **A–B** is the reverse of **B–A**.[37] We can treat these as cases where the process revisits a position it has previously occupied (which we know already to be something of an idealization). The property of isotropy or indifference to direction is thereby established.

By adding the property that tracing three spaces is the same process no matter *where* it occurs on the line, (i.e. that the spaces 1,2,3 are generated by exactly the same process as spaces 17,18,19, and that translating the place of the process makes no difference to the spaces produced), the property of homogeneity, or indifference to place, is established.[38] The more properties, or more accurately *indifferences* to

[36] *Metaphysik*, Sec. 253. 136: "From the determinate beginning point A on, we find on the line AB two-fold every conceivable distancing that allows itself to be thought precisely and without contradiction. Namely right and left to indicate the opposing processes with well-known names."

[37] Hermann Grassmann (while keeping the idea that natural number lines are traced out by travelling processes of "joining and separation") significantly clarified Herbart's ideas in the *Ausdehnungslehre* by introducing binary symbols $\cup$ for joining (+) and $\cap$ for separation (–). The properties of communitivity, associativity, distribution, identity can then be added or taken away, in so far as the constructing process shows or fails to show them. *A New Branch of Mathematics: The Ausdehnungslehre of 1844 and Other Works*. Translated by Lloyd C. Kannenberg. LaSalle, Illinois: Open Court 1995, Introduction and Chapter One to Sec. 15.

[38] In Grassmann's treatment for example, the generating process is considered elementary when the elements produced are always equal, that is if b is produced from a by the identical process by which c is produced from b, (homogeneity or indifference to position), and if the reverse of the process produces equal elements indifferent to direction (isotropy or indifference to direction). Secs. 13–14.

properties, are added to Herbart's directed paths, the more a travelling process of joining and separation appears to be a moving point, revisiting the same positions on an enduring line in space.

Compared to the vanishing exchanges of qualities underneath, Herbart's line was, in the last analysis, an idealization, a space of intelligibilia. As he himself pointed out, the images A', A'', A''', $A^{(n)}$ and B', B'', B''', $B^{(n)}$, used in the actual collecting up of previously reproduced points were only abstractions [*leere Bilder*]. They did not subsist after the constructing process moved on, because all previous places touched upon changed in the interim; – thus "intelligible space" depended in the last analysis on whether one remembered, or chose to remember, revisited points as part of "the same" line. This is the abstraction that supports both basic properties of isotropy and homogeneity. For Herbart, if there were certain remembered regularities in the causal structure of the community, the tracing process could be imagined to return the way it had come, revisiting, more or less, the same places. The regularity of these causal relations makes possible the apparent tracing out of uniform lines in one space over and above the particulars in it.

Herbart went on to describe the tracing of surfaces and spherically propagating waves by adding independent *Wesen* **C**, **D** that could undergo processes of Together and Not-Together with **A** and **B**. For the four processes **A+B, B+A, A–B, B–A** there were sixteen further combinations the process can undergo with a *Wesen* **C**.

Herbart also called explicit attention to the fact that there was no possible correspondence between a simple *Nacheinander* and a continuous process such as the tracing of a hypotenuse or a circular arc (*Kreislinie*). Herbart regarded these magnitudes as produced by functions in which the independent variables could be simple *Nacheinander*, in step with one another, without the dependent variables being so representable. Thus, the only way he could represent a continuous magnitude was as "a flowing of points into one another" to any degree of density.[39] Herbart was aware of the problems presented by continuity, but could not resolve them except by positing a different kind of line altogether, the *stetige Linie*, which he placed in a different logical category, the set of functions.

Going back to the community on which the entire construction was based, for the qualities of a community of *Wesen*, there was still a universal dependence of all on all, without a particular dependence that always held between any two individual qualities, or *Wesen*. These were permitted to slip in and out of dependence, supporting the waves of action that traced out stretches of space. Such general dependence without particular dependence was what allowed alternating, independent gaps to subsist and, with them, the concept of extended magnitudes.

[39] *Metaphysik*, Sec. 258.

Riemann's 1854 Probevorlesung

As we shall see now, Herbart seems to have influenced Riemann on some fundamental matters relating to the mathematician's concept of a manifold extension. Riemann's acknowledgement in the famous 1854 *Probevorlesung* on geometry reads:

> While I now attempt to solve the [...] problem of the development of the concept of multiply extended manifolds, I think myself the more entitled to ask considerate judgment inasmuch as I have had little practice in such matters of a philosophical nature, where the difficulty lies more in the concepts than in the construction, and because I have not been able to make use of any preliminary studies whatever, aside from some very brief hints which privy Counsellor Gauss has given on the subject in his second essay on biquadratic residues and in his Jubilee booklet, and some philosophical investigations of Herbart.[40]

Riemann may have come into contact with Herbart's ideas in Berlin and Göttingen where he studied philosophical subjects, especially *Naturphilosophie*. Herbart of course had a great influence in both cities at the time and had finished his career in Göttingen. It is therefore not surprising that Riemann's published *Wissenschaftlicher Nachlass* contains a number of notes about Herbart's metaphysics and psychology, including the following remarks:

> My major work concerns a new conception of the known natural laws – the expression of them by means of other fundamental concepts – through which it becomes possible to use experimental data concerning the reciprocal action between heat, light, magnetism and electricity for the exploration of their interconnection. I was led to it in the main by the study of the works of Newton, Euler and, from another side, Herbart. As far as the last, I adhere almost completely to the earliest investigations of Herbart, the results of which are expressed in his Promotions and Habilitations-theses (22 and 23, 1804), but must diverge from the course of his later speculations in one essential point, which is determined by a difference in relation to his *Naturphilosophie* and those laws of psychology which connect it to the *Naturphilosophie*.
>
> The author is a Herbartian in Psychology and in the theory of knowledge (Methodology and Eidolologie), but for the most part he cannot own himself a follower of Herbart's natural philosophy and the metaphysical disciplines related to it (Ontology and Synechologie).[41]

Ontology and Synechologie are the sections in Herbart's *Metaphysik* devoted to his *Wesen* and to intelligible space, and a number of authors, including Bertrand Russell, have used this dismissal to conclude that it was Herbartian psychology, in the reproduction series, not his metaphysics particular that made an impression on Riemann:

> Herbart's actual views on Geometry which are to be found chiefly in his Synechologie, are not of any great value, and have borne no great fruit in the development of the subject. But his psychological theory of space, his construction of extension out of series of points, his comparison of space with the tone and color series, his general preference for the discrete over the continu-

[40] On the Hypotheses which lie at the Foundations of Geometry. In: D. W. Smith (Ed.): *A Source Book in Mathematics*. New York: Dover 1929, 412.

[41] Riemann, Bernhard: *Gesammelte Werke und Wissenschaftlicher Nachlass*. Herausgegeben von Heinrich Weber und Richard Dedekind. 2nd Auflage. New York: Dover 1953, 507–508.

ous, and finally his belief in the great importance of classifying space with other forms of series (*Reihenformen*) gave rise to many of Riemann's epoch-making speculations and encouraged the attempt to explain the nature of space by its analytical and quantitative aspect alone.[42]

Russell's opinion is echoed in the literature,[43] but some authors have tried harder to trace Riemann's philosophical roots as an early Herbartian. Erhard Scholz, who published several extracts from Riemann's *Nachlass*, demonstrates a clear connection between Herbart's qualitative continua produced by the opposition of two qualities, two directions, and Riemann's notion of manifold extension.[44] Another writer, Luciano Boi, who considers the relationship between Riemann's geometry and *Naturphilosophie* writes that Riemann "considers relations of extension by comparison with a continuum of qualitatively different properties (quantitatively undifferentiable), characterized by a kind of "natural topology".[45]

In the 1854 *Probevorlesung*, Riemann developed the concept of multiply extended magnitude and its curvilinear coordinate system in the first part. Then, in the second part, he laid down the requirements for relations of measure on this manifold, for example the preservation of a standard length like a meter stick through transformations of position. What concerns us here however is only the very beginning of Riemann's essay, the definition of multiple extension. There were two stages of this construction: 1. The conceptual definition of 'multiply extended quantity' (*mehrfach ausgedehnte Größe*) (a philosophical task) and 2. The determination of a curvilinear system of coordinate functions within this extension (an empirical task known as geodesy). The second was due to Gauss but the first was probably due to Herbart.

In his "Plan der Untersuchung" Riemann said that "space and the first fundamental notions for constructions in space" could not be assumed as given *a priori*, as Euclid and other geometers had done.[46] For Riemann, the common notions of geometry had lain in darkness for so long because "the general concept of multiply extended magnitudes, in which spatial magnitudes are comprehended, has not been elaborated at all". This is already a tell-tale sign that Riemann, philosopher here as well as mathematician, did not intend to begin with space but with simpler concepts, involving him in the investigation of the meaning of extendedness as such.

Riemann then offers his definition of extension by saying that a concept of quantity must first admit of two modes of determination, [*Bestimmungsweise*]

[42] Russell, Bertrand: *An Essay on the Foundations of Geometry*. London: Routledge 1996 orig. 1897, 69.

[43] Torretti, Robert: *The Philosophy of Geometry from Riemann to Poincaré*. Dordrecht: D. Reidel 1978, 107–108; Nowack, Gregory: Riemann's Habilitationsvortrag and the Synthetic a Priori Status of Geometry. In: David Rowe and John McCleary (Eds.): *The History of Modern Mathematics*. Vol 1. Boston: Academic Press 1989.

[44] Scholz, Erhard: Herbart's Influence on Bernhard Riemann. In: *Historia Mathematica* 9, 1982, 413–440.

[45] Boi, Luciano: Die Beziehungen zwischen Raum, Kontinuum und Materie im Denken Riemanns. In: *Philosophia Naturalis* 31(2) 1994, 174.

[46] Riemann: On the Hypotheses …, 411.

and that as there existed a discrete or a continuous transition [*Übergang*] from one mode to the other, there would then exist a manifold of either points or elements:

> Notions of quantity are possible only where there exists already a general concept which allows of various modes of determination. According as there is or is not found among these modes of determination a continuous transition from one to the other, they form a continuous or a discrete manifold; the individual modes are called in the first case points, in the second elements of the manifold.[47]

Riemann claimed that while discrete manifolds were more abundant and exemplified by words in natural language, "in common life … the positions of objects of sense and the colors are probably the only simple notions whose modes of determination form a multiply extended manifold."[48] By mentioning color so prominently alongside continuous position in space, and by using the word "transition" between modes of determination, Riemann seems to suggest a Herbartian construction of extended magnitude by means of a continuous transition from one quality to another opposing quality.

In his *Psychologie*, Herbart had considered qualitative continua of color or tone in which the intermediate places could be determined by proportions of two or more basic qualities. If we look at the position of qualities in a *Wesen* these "transitions," result from one quality's overcoming another by force. If we take a direct analogy with spatial continua, the opposed qualities become like the directions of left and right. A position is located at the junction point of the two and a motion results as the stronger overwhelms the weaker and pushes the point of junction in its own direction. Although only one state is manifested at a time, previous and future states can be substituted in imagination so that the impression of a progression through a series is retained.

As sensory manifolds similar to space, Herbart gave examples of a "tone-space" "a straight line in which intervals can be measured with mathematical precision" and a color manifold in the shape of a triangle with red, blue and yellow at the corners and all interior points represented as mixtures of these extremes. In a particular meeting of two opposed color ideas, if one quality is more forceful and gradually overcomes the other, the process will appear as a colored point "moving" through the continuum as its proportions of the stronger and weaker component exchange. Herbart emphasized that the tone manifold was one-dimensional because there was only a single transition possible between two exchanging qualities of "high" and "low". Thus between any two qualities and any *Wesen*, we would expect to find this extended quality continuum instantiated.

Riemann also believed it was necessary to have *two* modes to determine an extension, not one, even in the case of discrete manifolds. Hypothetically, to locate a point on a line one must be able to locate the same point when travelling from the

[47] Ibid., 412.
[48] Ibid.

left and from the right. Hence, these two "modes" must be mutually dependent so that the left direction yields the same series of points reached in reverse order by travel in the right direction. This seemingly obvious fact is less than obvious if the directions are replaced by colors or tones, since their systematic dependence is a question of fact and empirical opposition. Thus, Riemann's modes of determination seem to be qualities that determine points when they coincide. A point or element of a manifold first arises by means of the transition between two *qualitative* means of fixing an individual, as a position on the tone-row can be determined both from low to high or from high to low. In a Nachlass fragment published by Scholz, Riemann is more explicit that the modes are determinations of a changing object and are not points or numbers:

> If a continuous transition is possible from one mode of determination of a changeable object to every other mode, so do all of the modes of determination (if a continuous transition is possible from every one of a set of different modes of determination to every other, so the totality of these forms a) continuous extended manifold; every individual of this manifold is called a point.[49]

Riemann's musings on the origins of points in "modes" have not attracted many admirers. For Russell, the requirement of two qualitative modes for determinations of quantity was utterly baffling; he asks: "What is meant, to begin with property by a general conception capable of various determinations? Does not this belong to all conceptions?" Surely it does, and surely the fact that we possess independent means of determining objects (color, figure, size, species) helps in isolating an individual from many others. If, for example, we use the properties of color and size as independent "modes" and classify each individual by assigning it a colour and a size, individuals of the same size can be differentiated by colour and individuals of the same colour can be differentiated by size. We can now count these individuals as qualitatively dissociated items, arranged into points by two associated and dissociated modes of determination. But if those adjectives were dependent in meaning, or so ambiguous that two of them picked out the same individuals, then even discrete manifolds would not be possible.

If modes of determination do have a geometrical analogy it is to directions and not points. As in Herbart's one dimensional *Tonlinie*, Riemann points out that the essential mark of a simple manifold is to contain only two directions of forward and back:

> In a concept whose various modes of determination form a continuous manifold, if one passes in a definite way from one mode of determination to another, the modes of determination traversed constitute a simply extended manifold and its essential mark is this, that in it progress is possible from any point only in two directions forward or backward.[50]

The color manifold Riemann had in mind in his remarks was probably one in which the hues were arranged in transitions like right and left or high and low along

[49] Erhard Scholz: Riemanns frühe Notizen zum Mannigfaltigkeitsbegriff und zu den Grundlagen der Geometrie. In: *Archive for History of Exact Sciences* 27, 1982, 213–232, 223.
[50] Riemann: On the Hypotheses, 413f.

the tone series. But it is impossible to tell from the example exactly which colors would pair off as "modes of determination". Nor does it really matter.

A color-coordinate system could be developed analogous to Riemann's Gaussian coordinate system by finding functions in which there is variation along the red and green dimension, say, while the others remain the same, and so forth for the other directions. What we have in the case of the colors then is a manifold in which there are systematically identified places and multiple extension but no conception of a distance from one place to another. As Riemann says, that is a separate question and requires a metric property that is independent of place and capable of moving freely through the manifold without alteration. A rigid body such as a metre stick retains its shape and length through motion from one coordinate position to another, as do analogous shapes such as spherical and concave triangles in spaces of constant curvature. Color coordinates, like city blocks, can be laid down without the need for a metric and thus here the equivalent of a rigid body or meter stick must be sought "outside the manifold".

Thus Russell's criticism of Riemann's analogy of colors with space, viz., that he had given no means of comparing one color with another by motion or superposition, is quite unfair. The metric property, if there is one, need not be the same property by which the coordinates are determined. In fact, it is better to pull apart the stages of manifold determination and metric considerations as Riemann does. Russell – who apparently did not have sense-physiological cases in mind – believed it was "impossible to set up a coordinate system in a manifold in which free mobility or the preservation of distance from one position to another did not also hold." But this is only true if the coordinate system is laid down with the help of spatial distance measurement, and then a property of distance surreptitiously derived by means of those coordinates (then indeed Riemannian geometry would be guilty of question begging, as Russell and others accused). But the whole *point* of comparisons with the color manifold, or with visual space, is to show that the properties of locality and direction are prior to distance. Russell also does not see that his *own* main desideratum of a philosophy of space, namely that spatial properties be abstracted from more fundamental qualitative determinations, is the very thing that Riemann also strove to accomplish.

If there is a criticism to make from a philosophical point of view, it is that Riemann's conception of constructing point or element manifolds from two associated modes of determination does not seem to include Herbart's fundamental idea that these modes, or directions, should both associate and dissociate from one another in the tracing out of extended magnitudes, unless this is somehow logically included in Riemann's notion that concepts of quantity require *two* modes of determination. Consider for example the passage, or change, from a certain qualitative coordinate (B,Y) to a certain other with a different proportion of blue to yellow, (B',Y') close to it but different. Start with B and Y dependent on one another:

$$\{B,Y\}.$$

Now imagine that B and Y dissociate and change their values to B' and Y' as they come into dependence on two qualities X' and Z' from the background:

{{B,Y}, {{B',X'}, {Y',Z'}}}.

Let us then say that the new values B' and Y' are in a new proportion – whereby whatever B' has lost or gained from B, Y' has gained or lost from Y – and thus can join back together at those new values, B' and Y':

{{{B,Y}, {{B',X'}, {Y',Z'}}}, {B',Y'}}.

That completes the raw extension from the coordinates (B,Y) to (B',Y'). This would then be the simplest concept of quantity or extension, a progressive sequencing of associations and dissociations of two different modes B and Y with the help of certain background qualities (X, Z).

The Naturphilosophie Fragment

Riemann's aforementioned *Naturphilosophie* fragment, which has attracted notice in recent years,[51] was to be, in his own words:

> A complete and self-contained mathematical theory [...] which proceeds from the elementary laws holding at individual points to the processes in the continuously filled space that is actually given to us, without differentiating between the treatment of gravity, electricity or heat.[52]

Riemann again credits Herbart with influencing his thinking, along with Newton and Euler,[53] and, just as Riemann emphasized that space should be analyzed into its very small inner states in the geometry paper, here, too, he says his ambitious aim is "to go beyond the foundations of astronomy and physics laid down by Galilei and Newton and to penetrate to the inside of nature". These inner states of matter are quite similar to the Herbartian force-like sensations and ideas pressing one another in the psyche, and, in opening the discussion, Riemann even argues for his physical hypothesis in psychological terms:

> The foundation of the fundamental laws of motion for ponderables, which are collected at the beginning of Newton's *Principia*, lie in their inner states. Let us try to infer them by analogy with our own inner perception. New complexes of representations are constantly appearing and vanishing from our consciousness. We observe a constant activity of our psyche. Every activity depends upon something permanent, which is noticed as such on particular occasions (through memory) without exerting an enduring influence on phenomena. Thus, something permanent enters our psyche continually (with every act of thought) which however exerts no influence on the world of phenomena. Every act of our psyche thus depends on something permanent, which enters with this act, but which in the same moment vanishes completely from the world of phenomena. Guided by this fact, I make the hypothesis that the universe is filled with a material, which constantly flows through the ponderable atoms and from there vanishes from the phenomenal world (the corporeal world). Both hypotheses can be replaced with one, that in all ponderable atoms permanent material enters the psychical world from the corporeal

[51] In addition to Luciano Boi's study already cited below, see Umberto Bottazzini and Rossana Tazzioli: Naturphilosophie and its Role in Riemann's Mathematics. In: *Revue d'histoire des mathématiques* 1, 1995, 3–38.

[52] *Riemanns Gesammelte Werke*, 507.

[53] Ibid.

world. The reason why the material disappears there is to be sought in the psychical substance formed there, and the ponderable bodies are accordingly the place where the psychical and corporeal worlds meet [...] In every ponderable atom, at every moment, a certain quantity of material proportional to the force of gravity enters and vanishes there.[54]

This strange explanation of the meeting between the psychical and the material worlds in a substance that flows through both and then vanishes, is the basis for Riemann's natural philosophy. He goes on to say that that this material is to be treated as an incompressible homogeneous fluid (constant density), which flows through the atoms in a quantity (*Menge*) proportional to their masses, and with a velocity of flow related to the pressure the atoms experience.

The differential quantities and velocities of fluid flow produce pressures in the space around bodies, apparently induced, as in fluid mechanics, by the variable density of flow lines. The flow reacts like an elastic medium to resist and propagate these disturbances in the form of waves, and Riemann sets out to derive waveforms similar to light on the one hand and gravitation on the other; the propagation of light is explained as the resistance of a linear element of the medium to a change of length and the propagation of gravity is explained as the resistance of a volume element to a distorting stress. (Bottazzini and Tazzioli give a particularly convincing example of Riemann's thinking about stress and resistance from his theory of electricity, which invoked a body's resistance to its own electric state.[55])

Compared with other ether theories, Riemann's is atypical in postulating a fluid flux that vanishes out of the world completely, instead of a conserved space-ether or a simple block of homogeneous elastic material. Why did he introduce the more complicated hypothesis?

Going back to his psychological argument, Riemann points out that the representation of a *Geistesmasse*, or a complex of ideas all fighting against one another for clearness, requires a permanent background against which these changes of force and stress can be represented. In order to represent a stress element as changing, one has to assume a fixed background against which this change can be represented. This background is something permanent, something which cannot come from within in the states themselves and must therefore come from without. In fact, Riemann's fluid medium is an embodiment of the familiar "flow of time", and it has the properties of an embodied time: something which is always identical to itself and ideally reidentifiable, but always continually disappearing, requiring us to represent vanished time, as he says, by an "act of memory". Non-present times exert no physical influence on the present, and so too, the fluid flow is physically inert when it vanishes from the atoms of the material world into the psychical world.

Note also that Riemann infers from the inside out, from the interior states of atoms or egos to the representation of the flow of the permanent. This permanent fluid spead out over space is certainly mathematically fundamental to Riemann's

[54] *Werke*, 528–529.
[55] Bottazzini and Tazzioli, 19–21.

unified treatment of natural forces by potential functions, and there is a temptation to read the inner states of stress and resistance as mere epiphenomena of the fluid flow, but it would make sense on closer reading that the fluid is schematized around the changes of the interior states and not vice versa. Riemann never says anything about where the fluid flow originates outside, whereas inner states are fundamental in the explanatory structure. And except for its role in providing a schematic foundation for objects, the fluid is a physically inert being, most of which ends up being extended in a *Geisteswelt* of memory. The inner states, whether related in *Geistesmassen* or atoms, are the data for Riemann's analogical argument. It is these elements that must change in such a way that the flow of a permanent time ether can be schematized around them, with the properties that Riemann attributes to it, i.e., different quantities and velocities of flow.

In their paper, Bottazzini and Tazzioli also reject the notion that Riemann held a typical ether theory, because, they say, the notion of a physical stress or strain in a medium could be replaced simply by altering the geometry of the space instead, as is done for the force of gravity in general relativity.[56] The changed geometry manifests itself as a pseudoforce and thus does not require stresses or strains in a medium or anywhere else. They add: "This suggests a physical model of the space independent of assumptions about the existence of the ether", and of course makes Riemann a precursor of Einstein.

But there is another choice which seems more in line with the philosophy of Riemann, i.e., of reducing extended space to its their inner states without assuming spatial notions at the outset, a stated goal of his geometry. Space could not be assumed as a simple concept for making constructions in space and this applies to ether as well. Instead, one could regard the extended space, along with its geometry, as a construction out of physical stress elements. These inner states could also ground geometry changes. The difference is that changes in the geometry would be induced by interior changes in the way the elements associate and dissociate in tracing out the intrinsic structure of the extension. These properties then show up in the behavior of the line element traced out in the space, i.e., the fact that one cannot draw Euclidean triangles in the space no matter what coordinate system is applied.

This reading places Riemann in a more direct philosophical line from Herbart and his philosophical project of reducing spatial notions to the behavior of inner states in the small, although an exact interpretation is admittedly difficult. The Herbartian processes generating lines and spaces appear in Riemann's work only in the veiled descriptions of a point moving by transitioning from one mode of determination to the other and in the requirement that there be at least two of these alternating (dependent and independent) modes, logically prior to extension.

But why is the treatment so oblique if that is what Riemann really meant? There is one place where he speaks clearly about the "continuous transition from one mode of determination of a concept to another" and why this is impossible to picture con-

[56] Bottazzini and Tazzioli, 32.

cretely (*unvorstellbar*). This appears in a section of his philosophical *Nachlass* appropriately titled "Antinomies".[57] Here he compares systems of positive concepts with their antitheses, or systems of concepts we cannot put images to. Riemann says that, like most good mathematicians, he is a follower of Newton's method of limits, in that a continuous transition can be represented concretely by a series of discrete steps that grow infinitely small and infinite in number in approaching a limit. However he says that in so doing, we picture continuous transitions by means of "negative predicates", as that to which a limiting process approaches. Actual continuity however lies on the side of things we cannot represent concretely, and thus may be very different from the representation of a limit.

Hence, Riemann may have thought it impossible to give a positive account of the inner states and their transitions. His oblique discussions of these basic concepts are thus the result of a consciously pursued *via negativa* and an indirect way of speaking. And yet any interpretation of Riemann's achievement must deal with those passages, as it must deal with the influence of the philosopher Herbart on the mathematician, which I imagine is just one of many examples of the productive trade in ideas that took place in nineteenth century Germany *bei Nacht und Nebel* betweeen working scientists and speculative metaphysicians.

[57] *Werke*, 519–520.