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# NOTIONS OF RELATIVE UBIQUITY FOR INVARIANT SETS OF RELATIONAL STRUCTURES

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**Abstract.** Given a finite lexicon $L$ of relational symbols and equality, one may view the collection of all $L$-structures on the set of natural numbers $\omega$ as a space in several different ways. We consider it as: (i) the space of outcomes of certain infinite two-person games; (ii) a compact metric space; and (iii) a probability measure space. For each of these viewpoints, we can give a notion of relative ubiquity, or largeness, for invariant sets of structures on $\omega$. For example, in every sense of relative ubiquity considered here, the set of dense linear orderings on $\omega$ is ubiquitous in the set of linear orderings on $\omega$.

**§0. Introduction.** Herein we investigate various ways in which a class of countable relational structures is ubiquitous, or large, relative to a containing class. For example, the class of dense linear orderings is ubiquitous in the class of linear orderings, in every sense of relative ubiquity considered here. We lend meaning to the notion of ubiquity by employing game-theoretic, topological, and measure-theoretic methods.

For example, consider the following game: Player (I) constructs a finite linear ordering, player (II) extends that ordering to a new finite ordering by adding at least one new element, (I) now properly extends (II)'s play, and so on forever. Player (II) wins just in case the union of the infinite chain of linear orderings produced is a dense ordering. It is not hard to see that (II) has a winning strategy for this game. Consequently, in a game-theoretic sense, "almost every linear ordering is dense." Note that the game just described is a thinly-disguised Banach-Mazur game.

This paper originated in a seminar talk by Peter J. Cameron [4] at Simon Fraser University in November of 1984, which one of us (Bankston) attended. In his talk, Cameron introduced the notions of "absolute ubiquity," "ubiquity in category," and "ubiquity in measure" as applicable to a particular countable relational

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structure (e.g. the ordered set of rational numbers). We presently extend these ideas to apply to classes of countable structures, and consider as well how game-theoretic notions of ubiquity compare with other such notions.

There are ten sections: §1 deals generally with games and probabilities on finitely branching trees of countable height; §2 introduces "evolution" trees of finite models, relating the "canonical" topology on the branch set and the joint embedding property; in §3 definability of classes of countable models and the Borel hierarchy in the canonical topology are explored; §4 talks briefly about P. J. Cameron's notion of "absolute ubiquity"; §5 involves various game-theoretic notions of ubiquity; in §6 we treat companions of universal theories (ubiquitous, model, and forcing); §7 applies the theory so far developed to specific first order examples; §8 introduces probabilistic notions of ubiquity; §9 presents more examples as applications of the probabilistic theory; and §10 is a short note relating probability measures on the branch set and asymptotic relative frequencies.

We are grateful to Professor Cameron for starting us off on this project. We are also grateful to several other people for their stimulating ideas, interest, and help in guiding us to a very rich literature on the uses of game-theoretic, topological, and probabilistic methods in model theory. At the risk of slighting some by inadvertent omission, we thank: Wilfrid Hodges, Matt Kaufmann, Dugald Macpherson, Alan Mekler, Evelyn Nelson, Marion Scheepers, John Simms, Michael Slattery, and Rastislav Telgársky.

**§1. Preliminaries on trees.** As suggested by the game-theoretic example in the Introduction, we are interested in how countably infinite structures "evolve" as chain unions of finite structures. To this end, we need some preliminary results on certain kinds of trees.

Let us define an *evolution tree* to be a partial ordering $(T, <)$ satisfying the following requirements: (i) the predecessors of each element form a finite chain; (ii) each element has a finite nonzero number of immediate successors; and (iii) there is a unique minimal element $\lambda$. For $n = 0, 1, \ldots$, the $n$th level of $T$, a finite set, is denoted by $T_n$; the set of immediate successors of $t \in T$ is $\mathrm{sc}(t)$; and $T \restriction n = \bigcup_{m<n} T_m$. The *rank* $\mathrm{rk}(t)$ of $t$ is the unique $n < \omega$ such that $t \in T_n$. In our applications, nodes of $T$ are finite relational structures over a finite lexicon of relation symbols, and $s < t$ means that $s$ is a proper substructure of $t$. For $t \in T$, the subtree with root node $t$ is $\{s: t \leq s\}$ and is referred to by using the interval notation $[t, \infty)$.

A *branch* of $T$ is a maximal chain in $T$. We use letters $a, b, c, \ldots$ to designate branches, identifying branches with leaf nodes for $T$. Thus, $t < a$ is synonymous with $t \in a$. In keeping with this view, we let $T_\omega$ be the set of all branches of $T$. In our applications, branches correspond to countably infinite structures.

If $a \in T_\omega$ and $n < \omega$, we let $a \restriction n$, the *restriction* of $a$ to $n$, be the unique element $t$ of $a \cap T_n$.

If $a, b \in T_\omega$, we define the distance $\rho(a,b)$ to be $1/(n+1)$ just in case $a \restriction n = b \restriction n$ and $a \restriction (n+1) \neq b \restriction (n+1)$. This defines a non-Archimedean metric on $T_\omega$ with $\rho(a,c) \leq \max\{\rho(a,b), \rho(b,c)\}$. It is easy to see that $\rho$ is complete and totally bounded, hence compact. Typical basic open sets look like $t^{\#} = \{a: t < a\}$; that is, the branches of $[t, \infty)$. Each $t^{\#}$ is clopen. For $F \subseteq T$, let $F^{\#} = \bigcup_{t \in F} t^{\#}$. If $F$ is finite, then

$F^\#$ is a finite union of clopen sets, hence clopen itself. By compactness, every clopen set must be of the form $F^\#$ for some finite $F \subseteq T$. Let $\Omega$ be the collection of open sets, and let $\Gamma$ be the clopen sets of the metric space $(T_\omega, \rho)$. Let $|S|$ denote the cardinality of a set $S$. By well-known results in topology (see, e.g., [24]), $(T_\omega, \rho)$ is homeomorphic to the Cantor discontinuum if and only if there are no isolated points (i.e. the space is self-dense) if and only if for each $t \in T$ there is a $t' > t$ with $|\mathrm{sc}(t')| > 1$.

$\Gamma$ is a Boolean algebra under the usual finitary operations. The $\sigma$-algebra generated by $\Gamma$ is the collection of Baire sets. Since $(T_\omega, \rho)$ has a countable basis of clopen sets, this $\sigma$-algebra coincides with the $\sigma$-algebra generated by $\Omega$; that is the collection of Borel sets. We will use the standard notation (see [23]) for specifying the levels of the Borel hierarchy: $\boldsymbol{\Sigma}^0_0 = \boldsymbol{\Pi}^0_0 = \Gamma$; for $0 < \alpha < \omega_1$, $\boldsymbol{\Sigma}^0_\alpha$ (resp. $\boldsymbol{\Pi}^0_\alpha$) is the set of all countable unions (resp. intersections) of members of $\bigcup_{\beta<\alpha} \boldsymbol{\Pi}^0_\beta$ (resp. $\bigcup_{\beta<\alpha} \boldsymbol{\Sigma}^0_\beta$). Thus, $\boldsymbol{\Sigma}^0_1 = \Omega$, $\boldsymbol{\Pi}^0_2 =$ the set of $G_\delta$ sets, and so on.

Since $(T_\omega, \rho)$ is a complete metric space, the Baire category theorem says that every countable intersection of dense open sets is dense. A set $R \subseteq T_\omega$ is called *residual* if it contains such an intersection; *somewhere residual* if $R \cap t^\#$ is residual in $t^\#$ for some $t \in T$; and *meager* if it is the complement of a residual set. A $\boldsymbol{\Pi}^0_2$-set $U$ is, as a topological subspace of $T_\omega$, completely metrizable [24]. If $T_\omega$ is self-dense and $U$ is dense, then $U$ is self-dense also. Thus, in the event $T_\omega$ is self-dense and $R$ is residual, $|R| = 2^\omega =$ the cardinality of the continuum.

One of our uses of the words "ubiquitous" and "almost" involves residual sets. Such sets form a countably complete filter on $T_\omega$, which is nonprincipal if $T_\omega$ is infinite. From now on we ignore the metric $\rho$ and concentrate on the generated topology, which we term the *canonical* (*tree*) *topology* on $T_\omega$.

Let us now turn to game-theoretic notions of large. Given $T$, let $X \subseteq T_\omega$, and let $R_{\mathrm{I}}$ and $R_{\mathrm{II}}$ be maps from $\{\varnothing\} \cup T$ to nonempty subsets of $T$ such that $R_{\mathrm{I}}(\varnothing) \subseteq T \backslash \{\lambda\}$, and each $t$ is sent to a subset of $[t, \infty) \backslash \{t\}$. We call these maps *regulators*: They spell out the legal moves for the two players of the game $G(T, X, R_{\mathrm{I}}, R_{\mathrm{II}})$, described as follows: Player (I) chooses $t_0 \in R_{\mathrm{I}}(\varnothing)$; player (II) chooses $t'_0 \in R_{\mathrm{II}}(t_0)$; (I) picks $t_1 \in R_{\mathrm{I}}(t'_0)$; and so on. The chain $t_0$, $t'_0$, $t_1$, $t'_1, \ldots$ is called a *legal play* with *outcome* $a = \lim_{n\to\infty} t_n \in T_\omega$. In all games $G(T, X, R_{\mathrm{I}}, R_{\mathrm{II}})$, (II) is trying to force the play into $X$; (I) is trying for the complement $T_\omega \backslash X$. So (II) wins just in case the outcome $a$ is in $X$. $G(T, X, R_{\mathrm{I}}, R_{\mathrm{II}})$ is *unrestricted* for player (I) (resp. (II)) if $R_{\mathrm{I}}(\varnothing) = T \backslash \{\lambda\}$ and $R_{\mathrm{I}}(t) = [t, \infty) \backslash \{t\}$ for $t \in T$ (resp. $R_{\mathrm{II}}(t) = [t, \infty) \backslash \{t\}$ for $t \in T$). We say that (I) (resp. (II)) plays by a *handicap* otherwise. The game that is unrestricted for both players is denoted $G(T, X)$. The game $G(T, X, R_{\mathrm{I}}, R_{\mathrm{II}})$ in which $R_{\mathrm{I}}$ is unrestrictive and $R_{\mathrm{II}}(t) = \{t' > t \colon \mathrm{rk}(t') \leq \mathrm{rk}(t) + m\}$ is denoted $G_m(T, X)$.

A *strategy* is a function $\sigma$ which assigns a value in $T$ to each finite chain of $T$. $\sigma$ is *legal* for (II) if $\sigma$ takes every finite chain $t_0 < t'_0 < \cdots < t_n$ to $R_{\mathrm{II}}(t_n)$. If $t_0 < t'_0 < t_1 < \cdots$ is a legal play, we say (II) plays *according to* $\sigma$ if $t'_n = \sigma(t_0, t'_0, \ldots, t_n)$, $n < \omega$. We say that $\sigma$ is a *winning strategy* for (II) if $\sigma$ is legal for (II) and whenever (II) plays according to $\sigma$, the outcome is in $X$. The corresponding notions for (I) are defined in the obvious way. $X$ is *determined* for $G(T, X, R_{\mathrm{I}}, R_{\mathrm{II}})$ if one of the players has a winning strategy. The strategy $\sigma$ is *forgetful* (called "stationary" by R. Telgársky [22], and a "tactic" by M. Scheepers) if its value depends only on the opponent's immediately preceding move; that is, if it is essentially a map from $\{\varnothing\} \cup T$ to $T$.

$F \subseteq T$ is *cofinal* in $T$ if for each $t \in T$ there is a $t' > t$ with $t' \in F$.

**1.1.** PROPOSITION. *Assume* (II) *has a winning strategy for* $G(T, X, R_{\mathrm{I}}, R_{\mathrm{II}})$, *and that* $R_{\mathrm{I}}(\varnothing)$ *is cofinal in* $T$. *Then* $X$ *is dense in* $T_\omega$.

PROOF. Since (II) has a winning strategy for $G(T, X, R_{\mathrm{I}}, R_{\mathrm{II}})$, we know that $X \cap t^\# \neq \varnothing$ for every $t \in R_{\mathrm{I}}(\varnothing)$. But $R_{\mathrm{I}}(\varnothing)$ is cofinal in $T$; hence $X \cap t^\#$ is nonempty for every $t \in T$. □

**1.2.** PROPOSITION. (i) *Assume* $X$ *is residual in* $T_\omega$ *and that* $R_{\mathrm{II}}(t)$ *is cofinal in* $[t, \infty)$ *for every* $t \in T$. *Then* (II) *has a forgetful winning strategy for* $G(T, X, R_{\mathrm{I}}, R_{\mathrm{II}})$.

(ii) *Assume* $T_\omega \backslash X$ *is somewhere residual,* $R_{\mathrm{I}}(\varnothing)$ *is cofinal in* $T$, *and that* $R_{\mathrm{I}}(t)$ *is cofinal in* $[t, \infty)$ *for every* $t \in T$. *Then* (I) *has a forgetful winning strategy for* $G(T, X, R_{\mathrm{I}}, R_{\mathrm{II}})$.

PROOF. (i) Let $X \supseteq \bigcap_{n<\omega} U_n$, where each $U_n$ is dense open in $T_\omega$. Given $t \in T$, look for the least $n < \omega$ such that $t^\#$ is not a subset of $U_n$. If no such $n$ exists, choose $\varphi(t)$ arbitrarily in $R_{\mathrm{II}}(t)$. Otherwise, let $\varphi(t)$ be any member $t'$ of $R_{\mathrm{II}}(t)$ such that $(t')^\# \subseteq U_n$. Then $\varphi$ describes a forgetful winning strategy for (II).

(ii) Suppose $T_\omega \backslash X$ is residual in $t^\#$. Since $R_{\mathrm{I}}(\varnothing)$ is cofinal in $T$, $t$ can be chosen in $R_{\mathrm{I}}(\varnothing)$. Let this be the opening move. We then have the restricted game $G([t, \infty), t^\# \backslash X, R_{\mathrm{I}}, R_{\mathrm{II}})$ in which (II) is the first player. By (i) above, (I), the new second player, has a forgetful winning strategy for this game. Thus (I) has a forgetful winning strategy for $G(T, X, R_{\mathrm{I}}, R_{\mathrm{II}})$. □

The residuality of $X$ in Proposition 1.2(i) is not necessary, even if $R_{\mathrm{I}}(\varnothing) = T \backslash \{\lambda\}$. We use the following lemma, due to Morton Davis [5], to construct a counterexample.

**1.3.** LEMMA (DAVIS [5]). *Let* $T$ *be the full binary tree, and let* $X \subseteq T_\omega$. *Then* (I) (*resp.* (II)) *has a winning strategy for* $G_1(T, X)$ *if and only if* $T_\omega \backslash X$ *contains a Cantor set* (*resp.* $T_\omega \backslash X$ *is countable*). □

**1.4.** EXAMPLE. A game $G(T, X, R_{\mathrm{I}}, R_{\mathrm{II}})$ in which: (i) (II) has a forgetful winning strategy; (ii) $R_{\mathrm{II}}$ is unrestrictive; (iii) $R_{\mathrm{I}}(\varnothing) = T \backslash \{\lambda\}$; but (iv) $X$ is meager in $T_\omega$.

*Construction.* Let $T$ be the full binary tree, $R_{\mathrm{I}}(\varnothing) = T \backslash \{\lambda\}$, and $R_{\mathrm{I}}(t) = \mathrm{sc}(t)$, and let $R_{\mathrm{II}}$ be unrestrictive. For each $t \in T$ let $X_t$ be a Cantor set in $t^\#$ which is nowhere dense in $T_\omega$; let $X = \bigcup_{t \in T} X_t$. Then $X$ is a dense meager subset of $T_\omega$. In order to prove that (II) has a winning strategy for $G(T, X, R_{\mathrm{I}}, R_{\mathrm{II}})$, let (I) play $t \in T$. Then (II) is now the first player in the game $G_1([t, \infty), t^\# \backslash X_t)$. By Lemma 1.3, the first player can win this game. That (II) has a forgetful winning strategy follows from the next result. □

**1.5.** THEOREM (with J. SIMMS [20]). *If a player has a winning strategy for* $G(T, X, R_{\mathrm{I}}, R_{\mathrm{II}})$, *then that player has a forgetful winning strategy.*

PROOF. It suffices to prove the result for player (II).

For each $t \in T$, let $t^-$ be the set of all finite increasing chains of $T$ which terminate with $t$. Order $t^-$ lexicographically by saying $(s_0, \ldots, s_m) \sqsubset (t_0, \ldots, t_n)$ if $s_i < t_i$ for some $i$ and $s_j = t_j$ for all $j < i$. Obviously, $\sqsubset$ well-orders the finite set $t^-$; the maximal element is $(t)$, and the minimal element is the full predecessor chain for $t$.

Let $\sigma$ be a winning strategy for (II) in the game $G(T, X, R_{\mathrm{I}}, R_{\mathrm{II}})$. For each $t \in T$, let $\varphi(t)$ be arbitrary in $R_{\mathrm{II}}(t)$ if there is no legal play where (II) plays according to $\sigma$ and $t$

appears as a move by (I). Otherwise, let $\varphi(t)$ be the value when $\sigma$ is applied to the $\sqsubset$-least chain $(t_0,\ldots,t)$ in $t^-$ such that the chain is a legal initial play according to $\sigma$. Note that $\varphi(t)$ is always in $R_{\mathrm{II}}(t)$.

Let $t_0 < \varphi(t_0) < t_1 < \cdots$ be a legal play according to the forgetful strategy $\varphi$, and let $b = \lim_{n\to\infty} t_n$. To see that $b \in X$, we construct a legal play according to $\sigma$ with $b$ also as its limit. Each $t_n$ has an associated legal initial play $v_n = (u_{n,0}, u'_{n,0}, u_{n,1}, u'_{n,1}, \ldots, u_{n,l(n)})$ in $t_n^-$ which is played according to $\sigma$. Pick the $\sqsubset$-least one. We now show that for each $i < \omega$ there is an $N < \omega$ such that for all $n \geq N$ we have $l(n) \geq i$ and $u_{n,j} = u_{N,j}$, $j \leq i$ (so $\lim_{n\to\infty} l(n) = \infty$ and the sequences $u_{1,i}$, $u_{2,i}, \ldots$ eventually "settle down"). The proof is by induction on $i < \omega$. We have $l(n) \geq 0$. Since $v_n * (\varphi(t_n), t_{n+1})$ is a legal initial play according to $\sigma$, we know it $\sqsubset$-dominates $v_{n+1}$. Thus $u_{n+1,0} \leq u_{n,0}$ for all $n$; hence for $n$ beyond some $N$, $u_{n,0} = u_{N,0}$. For the induction step, assume there is some $N < \omega$ such that for all $n \geq N$ we have $l(n) \geq i-1$ and $u_{n,j} = u_{N,j}$, $j < i$. Now $t_N < \varphi(t_N) < t_{N+1}$, so for $n \geq N+1$ we have $l(n) \geq i$. Also for $n \geq N+1$,

$$v_n = (u_{N,0}, u'_{N,0}, \ldots, u_{N,i-1}, u'_{N,i-1}, u_{n,i}, \ldots, u_{n,l(n)})$$

and $v_{n+1} \sqsubseteq v_n * (\varphi(t_n), t_{n+1})$. Thus the sequence $u_{N+1,i}, u_{N+2,i}, \ldots$ is decreasing, hence eventually constant. This completes the induction.

To finish, let $u_i$ be the limit of the eventually constant decreasing sequence $u_{N,i}, u_{N+1,i}, \ldots$, with $u'_i = \sigma(u_0, u'_0, \ldots, u_i)$. Since each initial segment of $u_0$, $u'_0$, $u_1$, $u'_1, \ldots$ is an initial segment of some $v_n$, we know we have a legal play according to $\sigma$. Moreover, each $u_n$ is dominated by some $t_m$, whence $\lim_{n\to\infty} u_n = b$. This completes the proof. □

A regulator $R$ is *monotone* if $t \leq t'$ implies $R(t) \supseteq R(t')$.

**1.6.** THEOREM (with J. SIMMS [20]). (i) *Assume* (II) *has a winning strategy for* $G(T, X, R_{\mathrm{I}}, R_{\mathrm{II}})$, $R_{\mathrm{I}}(\varnothing)$ *is cofinal in* $T$, *and* $R_{\mathrm{I}}$ *is monotone. Then* $X$ *is residual in* $T_\omega$.

(ii) *Assume* (I) *has a winning strategy for* $G(T, X, R_{\mathrm{I}}, R_{\mathrm{II}})$, $R_{\mathrm{II}}$ *is monotone, and* $R_{\mathrm{II}}(t)$ *is cofinal in* $[t, \infty)$ *for all* $t$. *Then* $T_\omega \backslash X$ *is somewhere residual in* $T_\omega$.

PROOF. We only prove (i); (ii) follows using "restriction," as in the proof of Proposition 1.2(ii). By Theorem 1.5 we can assume (II) has a forgetful winning strategy $\varphi$. For $n < \omega$, define $U_n = \{b \in T_\omega$: there is a legal initial play $t_0 < \varphi(t_0) < \cdots < t_n < \varphi(t_n) < b\}$. Let $b \in U_n$. Then there is a legal initial play ending in some $\varphi(t_n)$ witnessing this. Thus $b \in (\varphi(t_n))^\# \subseteq U_n$, so $U_n$ is open. $U_n$ is dense; for let $t \in T$ and let $t_0 \in R_{\mathrm{I}}(\varnothing)$ dominate $t$. We can then construct a legal initial play beginning with $t_0$; hence $t$ extends to some member of $U_n$. We now claim $X \supseteq \bigcap_{n<\omega} U_n$. Suppose $b \in \bigcap_{n<\omega} U_n$; we construct a legal play $t_0 < \varphi(t_0) < t_1 < \cdots$ with $b$ as limit, by induction on $n$. Since $b \in U_0$, we can get $t_0 < \varphi(t_0)$. Suppose we have built up $t_0 < \varphi(t_0) < \cdots < t_n < \varphi(t_n) < b$. Given $m < \omega$, let $s_0 < \varphi(s_0) < \cdots < s_m < \varphi(s_m) < b$ witness that $b \in U_m$. We can pick $m$ so large that $\varphi(t_n) \leq \varphi(s_{m-1})$. Then $s_m \in R_{\mathrm{I}}(\varphi(s_{m-1})) \subseteq R_{\mathrm{I}}(\varphi(t_n))$, so we let $t_{n+1} = s_m$. This proves $b \in X$. □

**1.7.** REMARKS. (i) When $R_{\mathrm{I}}$ and $R_{\mathrm{II}}$ are unrestrictive, Proposition 1.2 and Theorem 1.6 follow directly from J. C. Oxtoby's work on Banach-Mazur games [18]. Our small improvement uses forgetful winning strategies in, we believe, an essential way, and seems new (see [12] and [22] for more historical details). One corollary of Oxtoby's purely topological characterization of "winnable" sets for the game $G(T, X)$ is Borel determinacy. More generally, if $X \subseteq T_\omega$ has the Baire property (i.e. is in the $\sigma$-algebra generated by $\Omega$ together with the residual sets) then $X$ is determined for $G(T, X)$, since $X$ is residual or $T_\omega \backslash X$ is somewhere residual. Borel determinacy for other games $G(T, X, R_{\mathrm{I}}, R_{\mathrm{II}})$ follows from the deeper analysis of D. Martin [14].

(ii) The following example of a nondetermined game $G(T, X)$ is basically folklore, but we have not encountered it in print. Let $T$ be the full binary tree of finite sequences of zeros and ones, as in Lemma 1.3 and Example 1.4, and let $X$ be a free ultrafilter of subsets of $\omega$, viewed as a set of branches of $T$ by identifying subsets of $\omega$ with their characteristic functions: For $S \subseteq \omega$, $\chi_S(n) = 1$ if and only if $n \in S$. We claim $G(T, X)$ is undetermined. For assume that (II) has a winning strategy. By Theorem 1.5 we may assume this to be a forgetful strategy $\varphi$. Define $\eta\colon \omega \to \omega$ by $\eta(n) = \max\{\mathrm{rk}(\varphi(t))\colon t \in T_n\}$, and let $\psi_i$, $i = 0, 1$, be the forgetful strategy which takes the finite sequence $t \in T$ and appends $\eta(\mathrm{rk}(t))$ $i$'s. It is not hard to show, since $X$ is a filter, that $\psi_1$ is a forgetful winning strategy for (II). However, if (I) plays according to $\psi_0$, then, again using that $X$ is a filter, we obtain an infinite string whose complement is also in $X$. This cannot happen, since $X$ is a proper filter. So (II) has no winning strategy. Similarly we can show that (I) has no winning strategy, since $T_\omega \backslash X$ is isomorphic to the same ultrafilter.

(iii) The characterization in Lemma 1.3, although expressed in topological language, depends upon the structure of the binary tree. Let $T$ be the evolution tree for linear orderings on the ordinals $n = \{m\colon m < n\}$, and let $s \leq t$ mean that $s$ is a subordering of $t$. Let $X$ be the set of all linear orderings on $\omega$ which are isomorphic to the rational order type. Then one easily shows that (II) can win $G_1(T, X)$ (this will be an easy consequence of some general results presented later on). However, $T_\omega \backslash X$ has cardinality continuum.

(iv) Independently, in a recent paper [25], F. Galvin and R. Telgársky prove a general result implying our Theorem 1.5.

Given $T$, $R_{\mathrm{I}}$, and $R_{\mathrm{II}}$, let $W$ be the set $\{X \subseteq T_\omega\colon \text{(II) has a winning strategy for } G(T, X, R_{\mathrm{I}}, R_{\mathrm{II}})\}$. The following application of Theorem 1.5 will be used in §6.

**1.8.** THEOREM. *Assume $R_{\mathrm{I}}$ is unrestrictive. Then $W$ is a countably complete filter of subsets of $T_\omega$.*

PROOF. Clearly $\varnothing \notin W$, $T_\omega \in W$, and $W$ is closed under superset. Assume $X_n \in W$, $n < \omega$, and let $X = \bigcap_{n<\omega} X_n$. For each $n < \omega$, let $\varphi_n$ be a forgetful winning strategy for (II) in the game $G(T, X_n, \mathrm{R}_{\mathrm{I}}, R_{\mathrm{II}})$. We show how (II) can win $G(T, X, R_{\mathrm{I}}, R_{\mathrm{II}})$ as follows. Let $\eta\colon \omega \to \omega$ be a surjection such that the preimage of each $i < \omega$ is infinite. In response to (I)'s move $t_n$, (II) plays $t'_n = \varphi_{\eta(n)}(t_n)$. For each $i < \omega$, choose $n_0 < n_1 < \cdots$ such that, for all $k < \omega$, $\eta(n_k) = i$ and $\varphi_i(t_{n_k}) < t_{n_{k+1}}$. Then each play $t_{n_0} < \varphi_i(t_{n_0}) < t_{n_1} < \varphi_i(t_{n_1}) < \cdots$ is a legal play with outcome in $X_i$. But all of these plays are chains which are cofinal in $t_0 < t'_0 < t_1 < t'_1 < \cdots$; hence, $\lim_{n\to\infty} t_n \in X$. $\square$

We now consider the probabilistic notions of large. To do this, we construct natural probability measures on the Borel sets of $T_\omega$.

For each $t \in T$, the (unbiased) *branching weight* $\mathrm{W_b}(t)$ is defined to be 1 if $t \in T_0$, and to be the product $(\prod\{|\mathrm{sc}(t')|\colon t' < t\})^{-1}$ otherwise. If $F \subseteq T$ is any finite (order) independent set then define $W_b(F) = \sum_{t\in F} W_b(t)$. In order to get a probability measure on $T_\omega$ we use the following well-definedness condition.

**1.9.** LEMMA. *Suppose $F$ and $G$ are finite independent subsets of $T$, with $F^\# \subseteq G^\#$. Then* $W_b(F) \le W_b(G)$.

PROOF. Pick $n$ so that $F \cup G \subseteq T \upharpoonright (n+1)$, and let $F_n = \{t' \in T_n : t \le t'$ for some $t \in F\}$. Likewise define $G_n$. Then it is clear that $F^\# = F_n^\#$, $G^\# = G_n^\#$, and $W_b(F) = W_b(F_n)$, $W_b(G) = W_b(G_n)$. Since $F_n^\# \subseteq G_n^\#$, it is obvious that $F_n \subseteq G_n$; hence $W_b(F) = W_b(F_n) \le W_b(G_n) = W_b(G)$. □

A probability measure $P$ is *positive* if $P(U) > 0$ for all nonempty open sets $U$. $P$ is *continuous* if $P(\{a\}) = 0$ for all $a \in T_\omega$.

**1.10.** PROPOSITION. *There is a probability measure* $P_b$, *called the* ***branching probability*** *and defined on the Borel sets of $T_\omega$, such that $P_b(F^\#) = W_b(F)$ for each finite independent $F \subseteq T$. Moreover, $P_b$ is positive, and is continuous just in case $T_\omega$ is self-dense.*

PROOF. First define $P_b$ on $\Gamma$. If $F^\# \in \Gamma$; find $G \subseteq T$; finite and independent, so that $F^\# = G^\#$. Then define $P_b(F^\#) = W_b(G)$. By Lemma 1.9 this is unambiguous. Since $G$ is independent, we have $0 \le P_b(F^\#) \le 1$. Clearly $P_b$ is finitely additive; for if $F$ and $G$ are finite independent and $F^\# \cap G^\# = \varnothing$, then $F \cup G$ is independent. Thus $P_b(F^\# \cup G^\#) = P_b((F \cup G)^\#) = W_b(F \cup G) = W_b(F) + W_b(G) = P_b(F^\#) + P_b(G^\#)$. One shows that $P_b$ extends uniquely to a Borel probability measure by employing the Carathéodory extension theorem [8]. All we need to check is that if $F_0^\# \subseteq F_1^\# \subseteq \cdots$ and $\bigcup_{n<\omega} F_n^\# = F^\#$ then $P_b(F^\#) = \sup_{n<\omega} P_b(F_n^\#)$. But $T_\omega$ is a compact topological space, each $F_n^\#$ is open, and $F^\#$ is closed. Thus $F^\# = F_n^\#$ for some $n < \omega$.

Now suppose $U \subseteq T_\omega$ is open and nonempty. Then $P_b(U) \ge P_b(t^\#)$ for some $t$. Thus $P_b(U) \ge W_b(t) > 0$.

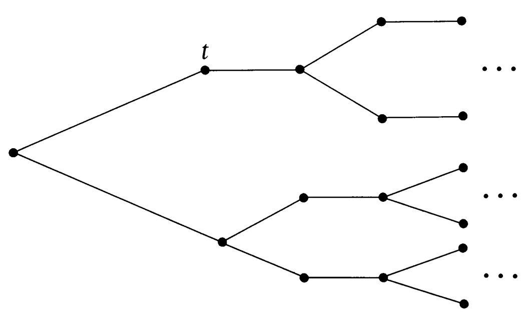
If $a$ is an isolated point of $T_\omega$, then $P_b(\{a\}) = W_b(t) > 0$ for some $t < a$. Otherwise, $P_b(\{a\}) \le 1/2^n$ for every $n < \omega$. □

This brings us to a new notion of large. Define $S \subseteq T_\omega$ to be of *branching measure one* if $P_b(S) = 1$ (where $P_b$ is extended in such a way that any subset of measure zero also has measure zero).

We define now a second probability measure on $T_\omega$. Its definition proceeds much the same as that above, though a bit more problematically. It basically coincides with the probability measure Cameron used in [4]. Given $t \in T$ and $n < \omega$, define $F_n(t) = |\{t' \in T_n : t \le t'\}|/|T_n|$, the relative frequency of extensions of $t$ at level $n$. The *frequency weight* $W_f(t)$ is then $\lim_{n\to\infty} F_n(t)$, if it exists.

**1.11.** EXAMPLE. A tree in which $W_f$ is not defined.

*Construction.* Let $T$ be the tree



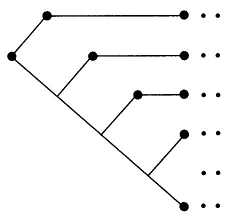
Then, for $n \geq 1$,

$$F_n(t) = \begin{cases} 1/2 & \text{if } n \text{ is odd,} \\ 1/3 & \text{if } n \text{ is even.} \end{cases} \quad \square$$

Define $T$ to be *frequency stable* if $\mathrm{W_f}(t)$ is always defined. If $T$ is indeed frequency stable, then one may proceed to extend $\mathrm{W_f}$ to a Borel probability measure exactly as before. This gives rise to the *frequency probability* $\mathrm{P_f}$. We then define $S \subseteq T_\omega$ to be of *frequency measure one* if there is a Borel set $B \subseteq S$ with $\mathrm{P_f}(B) = 1$.

**1.12.** Example. A tree in which $\mathrm{P_b}$ and $\mathrm{P_f}$ do not coincide. Moreover $\mathrm{P_f}$ is not positive, even on dense open sets; nor is it continuous.

*Construction.* Let $T$ be the tree



Let $t \in T$. Then for sufficiently large $n < \omega$,

$$F_n(t) = \begin{cases} 1/(n+1) & \text{if } |\mathrm{sc}(t)| = 1, \\ (n+1-\mathrm{rk}(t))/(n+1) & \text{if } |\mathrm{sc}(t)| > 1. \end{cases}$$

So let $S \subseteq T_\omega$ consist of all but the bottommost branch $a$. Then $S$ is dense open and $\mathrm{P_f}(S) = 0$. Also we see that $\mathrm{P_f}(\{a\}) = 1$, so $\mathrm{P_f}$ fails to be continuous. On the other hand, $\mathrm{P_b}(S) = \frac{1}{2} + \frac{1}{4} + \cdots = 1$. $\square$

Define a tree $T$ to be *balanced* if, whenever $t_1$ and $t_2$ have the same rank, $|\mathrm{sc}(t_1)| = |\mathrm{sc}(t_2)|$.

**1.13.** Proposition. *If $T$ is a balanced tree, then $T$ is frequency stable; in fact,* $\mathrm{W_f}(t) = \mathrm{W_b}(t) = |T_{\mathrm{rk}(t)}|^{-1}$.

Proof. Let $T$ be balanced and let $s\colon \omega \to \omega$ be such that $|\mathrm{sc}(t)| = s(\mathrm{rk}(t))$ for $t \in T$. Then, for each $n < \omega$, $|T_{n+1}| = s(n) \cdot |T_n|$; so, for $n = 1, 2, \ldots, |T_n| = s(n-1) \cdots s(0)$. Let $t \in T_n$ and let $k \geq 1$. Then

$$F_{n+k}(t) = \frac{s(n) \cdot s(n+1) \cdots s(n+k-1)}{s(0) \cdot s(1) \cdots s(n) \cdot s(n+1) \cdots s(n+k-1)}$$

$$= \frac{1}{s(0) \cdot s(1) \cdots s(n-1)} = \frac{1}{|T_n|}.$$

Thus $\mathrm{W_f}(t) = 1/|T_n|$. It is a trivial computation to show that $\mathrm{W_b}(t) = 1/|T_n|$ as well. $\square$

**§2. Invariant sets and their trees.** Let $L$ be a finite lexicon of finitary relation symbols. We treat equality as a logical predicate in the various languages associated with $L$, and define functions (including constants) via axioms in the first order language $L_{\omega\omega}$. The evolution tree associated with $L$ will be denoted $T$: $T_n$ is the set of all $L$-structures with universe $n = \{m: m < n\}$; $\lambda \in T_0$ is the empty structure; $s < t$ means that $s$ is a proper substructure of $t$; and $T_\omega$ is identified with the set of $L$-structures with universe $\omega$, the *canonical* $L$-structures.

$T$ is clearly a balanced tree: Given $L = \{R_1, \ldots, R_k\}$, where each $R_i$ is $n_i$-ary, we may compute $|\mathrm{sc}(t)|$, where $t \in T_m$, as

$$\prod_{i=1}^{k} 2^{((m+1)^{n_i} - m^{n_i})}.$$

Thus, by Proposition 1.13, branching probability and frequency probability agree on $T_\omega$.

For $X \subseteq T_\omega$, we define $\bar{X}$ to be the closure of $X$ in the canonical tree topology on $T_\omega$. A new ingredient here is the idea of isomorphism of structures, denoted $a \cong b$. We define $X^+$ to be $\{a \in T_\omega: a \cong b \text{ for some } b \in X\}$, and say that $K \subseteq T_\omega$ is *invariant* if $K = K^+$.

**2.1.** REMARK. The terminology "invariant set" follows R. Vaught [23]. $X^+$ is referred to there as the "outer invariantization" or "saturation" of $X$. Vaught considers a topology on $T_\omega$ by taking a countable Tichonov power of the two-point discrete space 2 (e.g. if $L = \{R\}$, $R$ binary, then each canonical $L$-structure can be identified with a subset of $\omega \times \omega$; hence the space of canonical $L$-structures is $2^{\omega \times \omega}$). Vaught's topology and the tree topology are identical because $L$ is finite.

Let $X \subseteq T_\omega$. For each $n < \omega$, define $T_n(X)$ to be $\{t \in T_n: t \text{ extends to a member of } X\}$. If $X$ is invariant, $t$ extends to a member of $X$ if and only if $t$ embeds in a member of $X$. The evolution tree associated with $X$ is defined to be $T(X) = \bigcup_{n<\omega} T_n(X)$, a (usually unbalanced) subtree of $T$. $T_\omega(X)$ is the set of branches of $T(X)$. Clearly, an $L$-structure $A$, on any countable set, is isomorphic to some member of $T(K) \cup T_\omega(K)$, $K$ invariant, if and only if every finite substructure of $A$ embeds in a member of $K$.

A class of $L$-structures has the *joint embedding property* (JEP) if any two members of the class embed in a third. The following proposition lists some elementary facts about the trees $T(K)$.

**2.2.** PROPOSITION. (i) *If $X_1 \subseteq X_2 \subseteq T_\omega$, then $T_\omega(X_1)$ is a closed metric subspace of $T_\omega(X_2)$.*

(ii) *$X$ is dense in $T_\omega(X)$; hence, $\bar{X} = T_\omega(X)$.*

(iii) *If $X_1 \subseteq X_2 \subseteq T_\omega(X_1)$, then $T_\omega(X_2) = T_\omega(X_1)$.*

*Let $K$ be invariant. Then:*

(iv) *$T_\omega(K)$ is an invariant set.*

(v) *If $U$ is open in $T_\omega(K)$, then so is $U^+$.*

(vi) *If $T(K)$ has the JEP and $U$ is nonempty and open in $T_\omega(K)$, then $U^+$ is dense and open in $T_\omega(K)$.*

(vii) *If $T_\omega(K) \neq T_\omega$, then $T_\omega(K)$ is nowhere dense in $T_\omega$.*

PROOF. (i) Clearly the metric on $T_\omega(X_1)$ is inherited from the metric on $T_\omega(X_2)$. $T_\omega(X_1)$ is compact, and is therefore closed in $T_\omega(X_2)$.

(ii) If $t \in T(X)$, then $t < a$ for some $a \in X$. Hence $t^\# \cap X \neq \varnothing$. This says $X$ is dense in $T_\omega(X)$. By (i), we have $\bar{X} = T_\omega(X)$.

(iii) $X_1 \subseteq X_2 \subseteq \bar{X}_1$, so $T_\omega(X_2) = \bar{X}_2 = \bar{X}_1 = T_\omega(X_1)$.

(iv) Obvious.

(v) Let $U \subseteq T_\omega(K)$ be open and let $a \in U^+$. Pick $b \cong a$ with $b \in U$, and let $t < b$ be chosen so that $b \in t^\# \subseteq U$. Then $a \in (t^\#)^+ \subseteq U^+$. Since $t$ embeds in $a$, there is an $n < \omega$ such that $t$ embeds in $a \restriction n$. If $c \in (a \restriction n)^\#$ then $t$ embeds in $c$, so we can find a $d \cong c$ so that $t < d$; hence $c \in (t^\#)^+$. This tells us that $a \in (a \restriction n)^\# \subseteq (t^\#)^+ \subseteq U^+$, so $U^+$ is open.

(vi) Let $t \in T(K)$. It suffices to show $t$ embeds in some member of $U$. This will prove $t^\# \cap U^+ \neq \emptyset$, establishing density. Suppose $t_1^\# \subseteq U$. Using the JEP, we find $t_2 \in T(K)$ so that both $t$ and $t_1$ embed in $t_2$. Since $K$ is invariant, we can arrange matters so that $t_1 \leq t_2$, whence $t$ embeds in some member of $t_1^\# \subseteq U$.

(vii) $T_\omega(K)$ is a closed subspace of $T_\omega$ by (i). Suppose $T_\omega(K)$ contains $t^\#$ for some $t \in T$. Then, since $T$ obviously enjoys the JEP, $(t^\#)^+$ is dense in $T_\omega$ by (vi). Thus $T_\omega(K)$ is dense as well as closed in $T_\omega$; so $T_\omega(K) = T_\omega$. □

**2.3.** Remark. That $C$ is closed in $T_\omega(K)$ need not imply that $C^+$ is closed in $T_\omega(K)$. Indeed, if $T(K)$ has the JEP and $t \in T(K)$ then $(t^\#)^+$ is dense open in $T_\omega(K)$ by Proposition 2.2(vi). But $t^\#$ is also closed; if $(t^\#)^+$ were closed as well, then it would be $T_\omega(K)$ itself. It is easy to find counterexamples to this: Let $t \in T$, $t \neq \lambda$, and $L \neq \emptyset$. Then $(t^\#)^+$ is never $T_\omega$.

**§3. Definable subsets of $T_\omega$.** In §1 we introduced the levels $\boldsymbol{\Pi}^0_\alpha$ and $\boldsymbol{\Sigma}^0_\alpha$, $\alpha < \omega_1$, of the Borel hierarchy for $T_\omega$ with the canonical topology. In this section we explore briefly the relationship between these levels and analogous levels of definability. Let $\mathscr{L}$ be any lexicon, possibly infinite. The first order language (with equality) over $\mathscr{L}$ is denoted $\mathscr{L}_{\omega\omega}$. The infinitary language $\mathscr{L}_{\omega_1\omega}$ is constructed in like manner, except that disjunctions are allowed over those countable sets of formulas in which only finitely many different free variables appear. As usual, we drop the subscripts in the case of first order languages, there being small likelihood of ambiguity.

The hierarchies of formulas of $\mathscr{L}$ and $\mathscr{L}_{\omega_1\omega}$ are defined analogously. We first define the finite levels $\boldsymbol{\Pi}^0_n$ and $\boldsymbol{\Sigma}^0_n$ for $\mathscr{L}$ inductively: $\boldsymbol{\Pi}^0_0 = \boldsymbol{\Sigma}^0_0 =$ the quantifier-free formulas; for $n \geq 1$, the $\boldsymbol{\Pi}^0_n$-formulas (resp. $\boldsymbol{\Sigma}^0_n$-formulas) are those of the form $\forall x_1 \cdots x_m \varphi$ (resp. $\exists x_1 \cdots x_m \varphi$), where $\varphi$ is a $\boldsymbol{\Sigma}^0_{n-1}$-formula (resp. $\boldsymbol{\Pi}^0_{n-1}$-formula). In the infinitary case, we define the countable levels $\boldsymbol{\Pi}'^0_\alpha$ and $\boldsymbol{\Sigma}'^0_\alpha$, also by induction: $\boldsymbol{\Pi}'^0_0 = \boldsymbol{\Sigma}'^0_0 = \boldsymbol{\Pi}^0_0$; for $\alpha \geq 1$, the $\boldsymbol{\Pi}'^0_\alpha$-formulas (resp. $\boldsymbol{\Sigma}'^0_\alpha$-formulas) are those of the form $\bigwedge_{n<\omega} \forall x_1 \cdots x_n \varphi_n$ (resp. $\bigvee_{n<\omega} \exists x_1 \cdots x_n \varphi_n$), where each $\varphi_n$ is a $\boldsymbol{\Sigma}'^0_{\beta_n}$-formula (resp. $\boldsymbol{\Pi}'^0_{\beta_n}$-formula) for some $\beta_n < \alpha$. Clearly, every $\boldsymbol{\Pi}^0_n$-formula (resp. $\boldsymbol{\Sigma}^0_n$-formula) is a $\boldsymbol{\Pi}'^0_n$-formula (resp. $\boldsymbol{\Sigma}'^0_n$-formula).

Let us return now to the finite lexicon $L$ with relation symbols only. Let $\alpha \leq \omega$. By adjoining a constant for each $n < \alpha$, we obtain the expanded lexicon $L(\alpha)$. In any interpretation, the constant $n$ shall denote itself. Given a sentence $\sigma$ of $(L(\omega))_{\omega_1\omega}$, we denote by $[\![\sigma]\!]$ the set of canonical models of $\sigma$. For a set $\Sigma$ of sentences, $[\![\Sigma]\!] = \bigcap_{\sigma \in \Sigma} [\![\sigma]\!]$.

Let $\mathscr{L}$ be either $L$ or $L(\omega)$, and let $\mathscr{L}^*$ be either first order or infinitary logic over $\mathscr{L}$. A set $X \subseteq T_\omega$ is *definable in* $\mathscr{L}^*$ if $X = [\![\Sigma]\!]$ for some countable subset $\Sigma$ of $\mathscr{L}^*$. $X$ is *basically definable in* $\mathscr{L}^*$ if $\Sigma$ can be chosen to be finite. The meanings of such utterances as "$X$ is $\boldsymbol{\Pi}'^0_3$-definable in $(L(\omega))_{\omega_1\omega}$" should now be obvious; clearly any set which is $\boldsymbol{\Pi}^0_n$-definable is basically $\boldsymbol{\Pi}'^0_n$-definable (over $L$ or $L(\omega)$).

The following assertion is well known; its proof is an easy induction on levels $\alpha < \omega_1$.

**3.1.** PROPOSITION. *Let $X \subseteq T_\omega$. If $X$ is basically $\boldsymbol{\Pi}'^0_\alpha$-definable (resp. basically $\boldsymbol{\Sigma}'^0_\alpha$-definable) over $L(\omega)$, then $X$ is a $\boldsymbol{\Pi}^0_\alpha$-set (resp. $\boldsymbol{\Sigma}^0_\alpha$-set) in the Borel hierarchy.*

PROOF. Assume $\sigma$ is quantifier free, and let $k_1, \ldots, k_n$ be the constants occurring in $\sigma$. If $a \in [\![\sigma]\!]$, we find $t < a$ large enough to contain each $k_i$. Then $t \models \sigma$, whence $a' \models \sigma$ for each $a' \in t^\#$. This says $X = [\![\sigma]\!]$ is open. Similarly, the complement $[\![\neg\sigma]\!]$ is open. Thus $[\![\sigma]\!]$ is closed as well.

For the inductive steps, existential quantifiers and infinitary disjunctions (resp. universal quantifiers and infinitary conjunctions) get converted to countable unions (resp. countable intersections). $\square$

**3.2.** REMARKS. (i) $\boldsymbol{\Sigma}^0_1$-definable sets need not be open. For if we let $L = \{R\}$, $R$ unary, and let $\Sigma$ be a set of $\boldsymbol{\Sigma}^0_1$-sentences which asserts that the interpretation of $R$ must be an infinite set, then every $t \in T$ extends to some $a \in T_\omega$ in which $R$ is interpreted as finite. Thus $T_\omega \backslash [\![\Sigma]\!]$ is dense, so $[\![\Sigma]\!]$ cannot be open.

(ii) An immediate corollary of Proposition 3.1 is that $\boldsymbol{\Pi}'^0_2$-definable sets are residual in their closures. Thus, by Proposition 1.2(i), if $X \subseteq T_\omega$ is $\boldsymbol{\Pi}'^0_2$-definable, then player (II) has a forgetful winning strategy for any game $G(T(X), X, R_{\mathrm{I}}, R_{\mathrm{II}})$ in which $R_{\mathrm{II}}(t)$ is cofinal for all $t$. We will take up this theme again in §5.

(iii) In [23], Vaught proves a converse to Proposition 3.1: If $K$ is an invariant set which is also a $\boldsymbol{\Pi}^0_\alpha$-set (resp. $\boldsymbol{\Sigma}^0_\alpha$-set) in the Borel hierarchy when the product topology is used on $T_\omega$ (see Remark 2.1), then $K$ is basically $\boldsymbol{\Pi}'^0_\alpha$-definable (resp. basically $\boldsymbol{\Sigma}'^0_\alpha$-definable) over $L$. A key lemma in the proof is that the topological group $\omega!$ of permutations on $\omega$, viewed as a subspace of the product space $\omega^\omega$, acts continuously on the space $T_\omega$. That is, the obvious group action $\omega! \times T_\omega \to T_\omega$ is continuous in both variables separately. When the canonical tree topology is put on $T_\omega$, the same analysis works.

(iv) Although $L$ contains no function or constant symbols, such symbols may, of course, be simulated using relation symbols. One may thus view a group as an $L$-structure in which $L$ consists of a ternary, a binary, and a unary symbol; in this view, the invariant set of canonical groups is basically $\boldsymbol{\Pi}^0_2$-definable.

One consequence of the finiteness of $L$ is that finite structures can be completely characterized in a first order manner. Given $t \in T_n$, $n > 0$, let $\delta_t(x_0, \ldots, x_{n-1})$ be the complete open description of $t$, i.e. the conjunction of all atomic and negated atomic formulas, in variables among $\{x_0, \ldots, x_{n-1}\}$, which hold for $t$ when $i$ is substituted for $x_i$, $i < n$. Let $\sigma_t$ be the sentence $\exists x_0 \cdots x_{n-1}\, \delta_t$. Then, for any $a \in T_\omega$, $a \models \sigma_t$ if and only if $t$ embeds in $a$. Thus $[\![\sigma_t]\!] = (t^\#)^+$, the smallest invariant set containing $t^\#$.

**3.3.** PROPOSITION. *Let $K \subseteq T_\omega$ be an invariant set.*

(i) *If $K$ is closed, then $K$ is $\boldsymbol{\Pi}^0_1$-definable over $L$.*

(ii) *If $K$ is open, then $K$ is basically $\boldsymbol{\Sigma}'^0_1$-definable over $L$.*

(iii) *If $K$ is open and definable in $L$, then $K$ is basically $\boldsymbol{\Sigma}^0_1$-definable over $L$.*

(iv) *If $K$ is a $\boldsymbol{\Pi}^0_2$-set and definable in $L$, then $K$ is $\boldsymbol{\Pi}^0_2$-definable over $L$.*

PROOF. (i) Let $K$ be invariant, and let $\Pi = \{\neg\sigma_t : t \notin T(K)\}$. Then $[\![\Pi]\!]$ is easily seen to be $T_\omega(K)$. So if $K$ is also closed, we have $K = [\![\Pi]\!]$, a $\boldsymbol{\Pi}^0_1$-definable set.

(ii) If $K$ is open invariant, let $T_\omega \backslash K = [\![\Pi]\!]$ as in (i) above. Then $K = [\![\neg \bigwedge \Pi]\!]$, a basically $\boldsymbol{\Sigma}'^0_1$-definable set.

(iii) If $K$ is open and definable in $L$, say $K = [\![\Sigma]\!]$, let $T_\omega \backslash K = [\![\Pi]\!]$, as in (i) above. By the compactness theorem of first order logic, there is a $\boldsymbol{\Pi}_1^0$-sentence $\pi$ in $L$ with $[\![\pi]\!] = [\![\Pi]\!]$. Thus $\neg\pi$ is a $\boldsymbol{\Sigma}_1^0$-definition of $K$.

(iv) This part requires Vaught's analysis in [23] (see Remark 3.2(iii)). Use the fact $K$ is an invariant $\boldsymbol{\Pi}_2^0$-set to infer that $K$ is $\boldsymbol{\Pi}_2'^0$-definable over $L$, hence closed under direct limits of chains of embeddings. Then use the full hypothesis, plus the Chang-Łoś-Suszko theorem (see, e.g., [13]), to infer that $K$ is $\boldsymbol{\Pi}_2^0$-definable over $L$. $\square$

3.4. COROLLARY. *There are only countably many open subsets of $T_\omega$ which are definable in $L$.* $\square$

3.5. COROLLARY. *Given any invariant set $K \subseteq T_\omega$, $T_\omega(K)$ is defined by all the $\boldsymbol{\Pi}_1^0$-sentences in $L$ which hold in each member of $K$.* $\square$

3.6. REMARK. Proposition 3.3 does not extend to higher levels of the Borel hierarchy, by Keisler's finite approximation interpolation theorem [26].

**§4. Absolute ubiquity.** When we speak of an invariant set $K$ as being ubiquitous in a larger invariant set $M$, we have in mind that $K \subseteq M \subseteq T_\omega(K)$ and $K$ is somehow large in its closure. Thus we are, in a sense, justified in saying "almost every structure in $M$ is a structure in $K$." This will be the underlying theme throughout the remainder of the paper.

We begin with a notion of ubiquity which was introduced by P. J. Cameron [4] and explored to a great extent by Cameron, I. M. Hodkinson, and H. D. Macpherson [10], [15].

A structure $a \in T_\omega$ is *absolutely ubiquitous* (a.u.) if, whenever $b \in T_\omega$ is such that $t$ embeds in $b$ if and only if $t$ embeds in $a$ for every $t \in T$, we have that $b \cong a$.

4.1. REMARKS. (i) For $a \in T_\omega$, let $a^+$ be the isomorphism type $\{a\}^+$ of $a$. The structure $a$ is a.u. just in case for no $b \in T_\omega(a^+)\backslash a^+$ is it true that $b^+$ is dense in $T_\omega(a^+)$; i.e., if $b \in T_\omega(a^+)$ and $T(b^+) = T(a^+)$, then $b \in a^+$. For any invariant set $K$, let $\Pi_K = \{\neg\sigma_t : t \notin T(K)\}$, as in Proposition 3.3, so $[\![\Pi_K]\!] = T_\omega(K)$. Let $\Sigma_K = \{\sigma_t : t \in T(K)\}$. We write $\Pi_a$ (resp. $\Sigma_a$) for $\Pi_{a^+}$ (resp. $\Sigma_{a^+}$). Then $a$ is a.u. if and only if $a^+ = [\![\Pi_a \cup \Sigma_a]\!]$; whence $a^+$ is $\boldsymbol{\Pi}_2^0$-definable over $L$ for any a.u. structure $a \in T_\omega$.

(ii) Macpherson [15] gave a complete characterization of a.u. undirected loop free graphs, and in later work, he and Hodkinson were able to extend that result to the general situation: $b \in T_\omega$ is a.u. if and only if there is a partition of $\omega$ into finitely many equivalence classes $S_1, \ldots, S_n$ such that whenever $\pi \in \omega!$ takes each $S_i$ to itself, $\pi$ is an automorphism on $b$ [10].

(iii) In special cases, we can apply this theorem to give simple characterizations of the a.u. structures.

(a) Given a graph $g$, define the binary relation on $g$ which pairs two vertices just in case the sets of vertices they are connected to are the same. This is an equivalence relation on $\omega$, and $g$ is a.u. if and only if there are only finitely many equivalence classes (conjectured and partially proved earlier by Cameron (see [15])).

(b) An equivalence relation is a.u. if and only if it has only finitely many equivalence classes which have more than one member.

(c) A partial ordering $p$ is a.u. if and only if $p$ can be partitioned into finitely many antichains $A_1, \ldots, A_n$ such that for $1 \le i, j \le n$, if some member of $A_i$ is less than some member of $A_j$, then every member of $A_i$ is less than every member of $A_j$. In particular, no canonical linear ordering is a.u.

Absolute ubiquity, a very strong property, is defined combinatorially in terms of finite structures and embeddings. This definition is quite impredicative: One defines absolute ubiquity of an invariant set $K$ by referring to the larger invariant set $T_\omega(K)$. The power of the Hodkinson-Macpherson theorem is to "internalize" the notion.

**§5. Ubiquity and games.** In this section we concern ourselves with games $G_\alpha(T(K), M)$, where $K, M \subseteq T_\omega$ are invariant sets, $M \subseteq T_\omega(K)$, and $1 \leq \alpha \leq \omega$ (see §1). $G_\omega(T(K), M)$ is a completely unrestricted Banach-Mazur game, and shall be denoted $G(T(K), M)$. We define $M$ to be *$\alpha$-winnable* in $T_\omega(K)$ if player (II) has a winning strategy for $G_\alpha(T(K), M)$. The properties $\alpha$-winnable clearly become weaker with increasing $\alpha$; by results of §1, $\omega$-winnable is synonymous with residual. If $K \subseteq M \subseteq T_\omega(K)$ and $K$ is $\alpha$-winnable in $T_\omega(K)$, we can say, from a strategic point of view, that "$K$ is large in $M$," or "almost every structure in $M$ is in $K$."

**5.1.** LEMMA. *Let $K \subseteq T_\omega$ be an invariant set such that $T(K)$ satisfies the JEP, and assume $M \subseteq T_\omega(K)$ is a somewhere residual invariant set. Then $M$ is residual in $T_\omega(K)$.*

PROOF. If $t \in T(K)$, then $(t^\#)^+$ is open in $T_\omega(K)$ by Proposition 2.2(v). Suppose $M \subseteq T_\omega(K)$ is a somewhere residual invariant set. Then we can find $u \in T(K)$ and open sets $U_n$, $n < \omega$, such that each $u^\# \cap U_n$ is dense in $u^\#$, and $u^\# \cap \left(\bigcap_{n<\omega} U_n\right) \subseteq M$. Suppose $u$ embeds in $s$. Then there is $t \geq u$ such that $s \cong t$, each $t^\# \cap U_n$ is dense in $t^\#$, and $t^\# \cap \left(\bigcap_{n<\omega} U_n\right) \subseteq M$. Then there is a permutation $\pi \in \omega!$ which fixes each $n \geq \mathrm{rk}(t) = \mathrm{rk}(s)$ and which takes $t$ onto $s$. Let $\bar{\pi}$ be the induced bijection on $T_\omega(K)$. Then $\bar{\pi}$ is a homeomorphism such that $\bar{\pi}(a) \cong a$ for each $a \in T_\omega(K)$. Since $M$ is invariant we have $\bar{\pi}[M] = M$; and the images $\bar{\pi}[U_n]$, $n < \omega$, witness that $M$ is residual in $s^\#$ as well as in $t^\#$. From this it is easy to see that $M$ is residual in $(u^\#)^+$. But $T(K)$ satisfies the JEP, so, by Proposition 2.2(vi), $(u^\#)^+$ is dense in $T_\omega(K)$. This implies that $M$ is residual in $T_\omega(K)$. □

Coupling Lemma 5.1 with Remark 1.7(i) on Borel determinacy, we immediately obtain

**5.2.** THEOREM. *Let $K \subseteq T_\omega$ be an invariant set such that $T(K)$ satisfies the JEP, and assume $M \subseteq T_\omega(K)$ is an invariant Borel set. Then either $M$ is residual in $T_\omega(K)$ or $T_\omega(K)\backslash M$ is residual in $T_\omega(K)$.* □

We use Theorem 5.2 in §6 when we talk about the completeness of certain theories and game-theoretic zero-one laws.

From Proposition 3.1 we know that if $X \subseteq T_\omega$ is $\Pi_2'^0$-definable over $L(\omega)$, then $X$ is residual, and hence $\omega$-winnable, in $T_\omega(X)$. If we strengthen the hypothesis slightly, we may also strengthen the conclusion:

**5.3.** THEOREM. *Let $0 < m < \omega$, and assume $X \subseteq T_\omega$ has a $\Pi_2'^0$-definition over $L(\omega)$ in which only finitely many constants occur and no block of existential quantifiers has length exceeding $m$. Then $X$ is $m$-winnable in $T_\omega(X)$.*

PROOF. Let $\sigma$ be a $\Pi_2'^0$-definition of $X$ over $L(r)$, $r < \omega$, of the form

$$\bigwedge_{i<\omega} \forall x_1 \cdots x_i \bigvee_{j<\omega} \exists y_1^{ij} \cdots y_m^{ij} \varphi_{ij}(x_1, \ldots, x_i, y_1^{ij}, \ldots, y_m^{ij}).$$

For each $i < \omega$ and $\mathbf{k}_i = (k_1, \ldots, k_i) \in \omega^i$, let

$$U_{i,\mathbf{k}_i} = \left\{ a \in T_\omega(X) \colon a \models \bigvee_{j<\omega} \exists \mathbf{y}^{ij} \varphi_{ij}[\mathbf{k}_i](\mathbf{y}^{ij}) \right\}.$$

Then $U_{i,\mathbf{k}_i}$ is dense open in $T_\omega(X)$, and $X = \bigcap\{U_{i,\mathbf{k}_i}: i < \omega, \mathbf{k}_i \in \omega^i\}$. The strategy for (II), stated informally, is the following: Play arbitrarily until (I) plays a $t$ whose domain contains $\mathbf{k}_i$ for some $i < \omega$, as well as all the constants occurring in $\sigma$. Since $U_{i,\mathbf{k}_i}$ is dense in $T_\omega(X)$, some $a \in U_{i,\mathbf{k}_i}$ extends $t$; so let $j < \omega$ and $\mathbf{l} = (l_1, \ldots, l_m) \in \omega^m$ be such that $a \models \varphi_{ij}[\mathbf{k}_i, \mathbf{l}]$. Now permute the natural numbers in such a way that the members of $t$ are fixed, and each member of $\mathbf{l}$ gets sent below $\mathrm{rk}(t) + m$. This gives rise to an isomorphic copy $a' \cong a$ such that $t < a'$ and $\bigvee_{j<\omega} \exists \mathbf{y}^{ij} \varphi_{ij}[\mathbf{k}_i](\mathbf{y}^{ij})$ is satisfied in $a'$ by stage $\mathrm{rk}(t) + m$. (II) should now play $t' = a' \restriction (\mathrm{rk}(t) + m)$. No matter what future plays are made, the outcome will lie in $U_{i,\mathbf{k}_i}$. Player (II) is ready now to take care of the other sets $U_{i,\mathbf{k}_i}$ in turn. This ensures a win for (II) for $G_m(T(X), X)$. □

One can readily deduce from Morton Davis's Lemma 1.3 that not all residual sets are 1-winnable. The following example establishes that the properties $\alpha$-winnable, $1 \le \alpha \le \omega$, are all distinct.

**5.4.** Example. A closed invariant set $K \subseteq T_\omega$ and invariant subsets $K_1 \supseteq K_2 \supseteq \cdots \supseteq K_\omega$ of $K$ such that, for each $1 \le \alpha \le \omega$, $K_\alpha$ is $\alpha$-winnable in $K$, but not $m$-winnable in $K$ for $1 \le m < \alpha$.

*Construction.* Let $L = \{R\}$, where $R$ is binary, and let $K = \{b \in T_\omega$: the interpretation of $R$ in $b$ is a partial injection$\}$. Then $K = [\![\sigma]\!]$, where $\sigma$ is the $\boldsymbol{\Pi}^0_1$-sentence

$$\forall xyz((Rxy \wedge Rxz \to y = z) \wedge (Rxz \wedge Ryz \to x = y)).$$

For each $1 \le \alpha \le \omega$, let $K_\alpha = \{b \in K$: the interpretation of $R$ in $b$ is a total bijection, and there is at least one orbit of each finite positive length $\le \alpha\}$.

One can easily check that $K_\alpha$ has a $\boldsymbol{\Pi}^0_2$-definition in which at most $\alpha$ variables are existentially quantified. Thus, by Theorem 5.3, $K_\alpha$ is $\alpha$-winnable in $T_\omega(K_\alpha) = K$. Now let $1 \le m < \alpha$. We claim that (I) has a winning strategy for $G_m(T(K), K_\alpha)$: (I) plays $t_0$, a single orbit of length $m + 2$. No matter what $t'_0$ is now played by (II), there can be no orbit of length $m + 1$. (I) plays $t_1 > t'_0$ in such a way that any incomplete orbits in $t'_0$ are completed into orbits of length $m + 2$. Again, (II) cannot establish an orbit of length $m + 1$. This pattern is repeated with the outcome $\lim_{n\to\infty} t_n \in K \backslash K_\alpha$. □

We have seen how the JEP influences the winning of games $G(T(K), M)$ by player (II). We will now explore the role of the (usually stronger) amalgamation property in this connection; namely in (II)'s being able to win the handicap games $G_m(T(K), M)$.

Recall that a class of structures satisfies the *amalgamation property* (AP) if whenever $A_0$, $A_1$, $A_2$ are members of that class and $\eta_i$: $A_0 \to A_i$ is an embedding for $i = 1, 2$, then there is a fourth member $A$ of the class and embeddings $\mu_i$: $A_i \to A$, $i = 1, 2$, such that the resulting mapping square is commutative: $\mu_1\eta_1 = \mu_2\eta_2$. Note that if we allow the empty structure in our class, then the AP implies the JEP. This will be the case when the class in question is some $T(K)$.

Let $L$ be given, and let $t, t' \in T$ with $\mathrm{rk}(t) = n$ and $t' \in \mathrm{sc}(t)$. We define the formula $\varepsilon_{t,t'}(x_0, \ldots, x_n)$ to be the implication

$$\delta_t(x_0, \ldots, x_{n-1}) \to \delta_{t'}(x_0, \ldots, x_n),$$

where $\delta_t$ is the complete open description of $t$, and we define $\sigma_{t,t'}$ to be the sentence $\forall x_0 \cdots x_{n-1} \exists x_n \varepsilon_{t,t'}$. Note that an $L$-structure $A$ satisfies $\sigma_{t,t'}$ if and only if every embedded copy of $t$ in $A$ extends to an embedded copy of $t'$ in $A$. These sentences were used originally by H. Gaifman [7], who credited their invention to M. Rabin and D. Scott. Consequently, we shall refer to the sentences $\sigma_{t,t'}$ and $\sigma_t$ of Proposition 3.3 as *Rabin-Scott sentences*. (To complicate matters, J. F. Lynch [12] attributes the invention of these sentences to unpublished work of S. Jaśkowski.)

Let $K \subseteq T_\omega$ be an invariant set. As we saw before in §§3 and 4, $\Pi_K = \{\neg\sigma_t\colon t \notin T(K)\}$ is a $\boldsymbol{\Pi}_1^0$-axiomatization of $T_\omega(K)$. Letting $\Sigma_K = \{\sigma_t\colon t \in T(K)\}$ as in Remark 4.1(i), we see that $[\![\Pi_K \cup \Sigma_K]\!]$ consists of those members of $T_\omega(K)$ in which every member of $T(K)$ embeds; that is, the members of $T_\omega(K)$ that are *universal models* for $T(K)$. Given these remarks, the following is easy to prove:

**5.5.** PROPOSITION. *The following are equivalent for an invariant set $K \subseteq T_\omega$:*

(i) *$\Pi_K \cup \Sigma_K$ is a consistent $\boldsymbol{\Pi}_2^0$-theory.*

(ii) $[\![\Pi_K \cup \Sigma_K]\!] \neq \varnothing$.

(iii) *$[\![\Pi_K \cup \Sigma_K]\!]$ is a dense $\boldsymbol{\Pi}_2^0$-subset of $T_\omega(K)$.*

(iv) *$T(K)$ has a universal model.*

(v) *$T(K)$ satisfies the JEP.* $\square$

When we add the remaining Rabin-Scott sentences, we get an analogous result involving the AP. Given $K \subseteq T_\omega$, let $\Gamma_K = \{\sigma_{t,t'}\colon t, t' \in T(K) \text{ and } t' \in \mathrm{sc}(t)\}$.

**5.6.** REMARKS. (i) If $b \in [\![\Pi_K \cup \Sigma_K \cup \Gamma_K]\!]$, then $b$ is not only universal for $T(K)$, but also homogeneous (in the sense of R. Fraïssé): Given finite substructures $A$, $B \subseteq b$ and an isomorphism $\eta\colon A \to B$, $\eta$ can be extended to an automorphism on $b$ (by a back-and-forth argument). Conversely, let $b \in T_\omega(K)$ be universal for $T(K)$ and homogeneous as well. Suppose $\eta\colon t \to b$ is an embedding and $t' \in \mathrm{sc}(t)$. Since $b$ is universal, there is an embedding $\varepsilon\colon t' \to b$. By homogeneity, there is an automorphism which takes $\varepsilon[t]$ to $\eta[t]$. This tells us that $b \models \sigma_{t,t'}$. So $b \models \Pi_K \cup \Sigma_K \cup \Gamma_K$.

(ii) Any two countable models of $\Pi_K \cup \Sigma_K \cup \Gamma_K$ are isomorphic: The Rabin-Scott sentences are designed so that one can carry out a classic back-and-forth argument.

**5.7.** THEOREM. *The following are equivalent for an invariant set $K \subseteq T_\omega$:*

(i) *$\Pi_K \cup \Sigma_K \cup \Gamma_K$ is a consistent $\aleph_0$-categorical $\boldsymbol{\Pi}_2^0$-theory.*

(ii) $[\![\Pi_K \cup \Sigma_K \cup \Gamma_K]\!] \neq \varnothing$.

(iii) *$[\![\Pi_K \cup \Sigma_K \cup \Gamma_K]\!]$ is a $\boldsymbol{\Pi}_2^0$-subset of $T_\omega(K)$ which is also* 1*-winnable in $T_\omega(K)$.*

(iv) *$T(K)$ has a homogeneous universal model.*

(v) *$T(K)$ satisfies the AP.*

*Moreover, $\Pi_K \cup \Sigma_K \cup \Gamma_K$ is model complete, if consistent.*

PROOF. The equivalence of (i), (ii), and (iv) was established in Remark 5.6; (iii) trivially implies (ii). We first prove that (ii) implies (iii).

Assuming (ii), let $b \models \Pi_K \cup \Sigma_K \cup \Gamma_K$. Since $b$ is universal for $T(K)$, we know $[\![\Pi_K \cup \Sigma_K \cup \Gamma_K]\!]$ is dense in $T_\omega(K)$. Let $\Sigma = \{\sigma_t \in \Sigma_K\colon \mathrm{rk}(t) = 1\}$. Then it is easy to show that $\Pi_K \cup \Sigma \cup \Gamma_K$ axiomatizes $\Pi_K \cup \Sigma_K \cup \Gamma_K$. Since $\Pi_K \cup \Sigma \cup \Gamma_K$ is a $\boldsymbol{\Pi}_2^0$-set of sentences in which only one variable appears in any block of existential quantifiers, we infer from Theorem 5.3 that $[\![\Pi_K \cup \Sigma_K \cup \Gamma_K]\!]$ is 1-winnable in $T_\omega(K)$. Thus (iii) holds.

It remains to prove the equivalence of (ii) and (v).

Assume (ii) and an amalgamation situation $\eta_i\colon t_0 \to t_i$, $i = 1,2$. Let $b \in [\![\Pi_K \cup \Sigma_K \cup \Gamma_K]\!]$. For simplicity we can arrange matters so that both $\eta_1$ and $\eta_2$ are inclusions. Since $b \models \sigma_{t_0}$, we can find $A_0 \cong t_0$, $A_0 \subseteq b$. Let $t_0 \subseteq t_0' \subseteq \cdots \subseteq t_1$ be a list of all intermediate steps between $t_0$ and $t_1$ in the tree $T(K)$. Then we can use the appropriate sentences $\sigma_{t,t'}$ of $\Gamma_K$ to extend $A_0$ to a copy $A_1$ of $t_1$ in $b$. Similarly extend $A_0$ to a copy $A_2$ of $t_2$ in $b$. The amalgamation we want is thus isomorphic to $A_1 \cup A_2$, so (v) holds.

Now assume (v). We wish to prove that there is a countable $b \models \Pi_K \cup \Sigma_K \cup \Gamma_K$. Let $(A_n, B_n)$, $n < \omega$, be an enumeration of all pairs of finite structures such that: (1) $A_n \subseteq B_n$ and $B_n$ is a one-point extension of $A_n$; (2) the domain of each $B_n$ is a subset of $\omega$; and (3) the $A_n$'s and $B_n$'s are members of $T(K)$. Note that we allow the empty structure to appear among the $A_n$'s. We construct a sequence of finite structures $M_0 \subseteq M_1 \subseteq \cdots$ whose union is a model of $[\![\Pi_K \cup \Sigma_K \cup \Gamma_K]\!]$. Let $M_0 = \varnothing$. Assume $M_i \in T(K)$ to be constructed. There exists a smallest $k$ such that $A_k \subseteq M_i$, and whenever $B \cong B_k$, then $B$ does not embed in $M_i$. Using the AP, we can find an extension $M_{i+1} \in T(K)$ of $M_i$ and an embedding $\eta\colon B_k \to M_{i+1}$ which is an inclusion when restricted to $A_k$. Clearly, $M = \bigcup_{i<\omega} M_i$ is an element of $T_\omega(K)$, as it is obtained as a proper chain union of countably many copies of members of $T(K)$.

To say that a Rabin-Scott sentence $\sigma_{t,t'}$ fails in $M$ is to allow the existence of some smallest $m$ such that $A_m \subseteq M$; but whenever $B \cong B_m$ and $A_m \subseteq B$, then $B$ is not a substructure of $M$. Let $i$ be the least such that $A_m \subseteq M_i$. Then there must be $B \cong B_m$ such that $A_m \subseteq B \subseteq M_{i+m} \subseteq M$. This gives a contradiction, so every Rabin-Scott sentence $\sigma_{t,t'}$ holds in $M$. If $t \in T_1(K)$ then $M \models \sigma_t$, since the above argument holds even if $A_m$ is empty. Since $\Pi_K \cup \Sigma_K \cup \Gamma_K$ can be axiomatized by $\Pi_K \cup \Sigma \cup \Gamma_K$, we have $M \models \Pi_K \cup \Sigma_K \cup \Gamma_K$. This completes the proof of the equivalence. The model completeness then follows from (i) and Lindström's theorem. $\square$

**5.8.** Remark. Most of Theorem 5.7 is already known (see, e.g., [27]); Cameron [4] stated that the isomorphism type of a homogeneous structure is residual in its closure. The connection with handicap games is new.

The following example shows that no converse to Theorem 5.3 is possible.

**5.9.** Example. An invariant set $K \subseteq T_\omega$ that is basically $\boldsymbol{\Sigma}^0_3$-definable, 1-winnable in $T_\omega(K)$, but not $\boldsymbol{\Pi}'^0_2$-definable.

*Construction.* Let $L = \{R\}$, where $R$ is binary, and let

$$K = [\![\forall x(\neg Rxx) \wedge \forall xy(Rxy \to Ryx) \wedge \exists x \forall y \exists z(Rxz \wedge Ryz)]\!].$$

Each $g \in K$ is a graph with a "center" that is connected to every vertex via an edge path of length 2. Clearly $\Pi_K$ is the theory of graphs, and player (II) can win $G_1(T(K), K)$ simply by adding a vertex at each turn and connecting it to each previously played vertex. To see that $K$ is not $\boldsymbol{\Pi}'^0_2$-definable, we show that $K$ is not closed under chain unions. Let $A_0$ be a countably infinite graph with no edges. Assume we have constructed $A_n$; let $A_{n+1}$ consist of $A_n$, together with three new vertices $v_0, v_1, v_2$, edges joining $v_0$ to $v_1$ and $v_1$ to $v_2$, and edges joining $v_0$ to each vertex of $A_n$. Clearly $v_0$ and $v_1$ are "centers" for $A_{n+1}$, but not for $A_{n+2}$; thus the union of the chain $A_1 \subseteq A_2 \subseteq \cdots$ has no "center". $\square$

**5.10.** Remark. The notion of absolute ubiquity, defined by Cameron for isomorphism types, has an obvious generalization to arbitrary invariant sets: $K \subseteq T_\omega$ is *absolutely ubiquitous* if $K = [\![\Pi_K \cup \Sigma_K]\!]$. In light of Proposition 5.5, the following statements may be made:

(i) $K$ is a.u. if and only if $K$ consists of those structures which are universal for $T(K)$.

(ii) Let $K$ be a.u. Then $K$ is nonempty just in case $T(K)$ satisfies the JEP.

(iii) Let $K$ be a.u. Then $K$ is $\omega$-winnable in $T_\omega(K)$. (Cameron in [4] stated that the isomorphism type of an a.u. structure is residual in its closure.)

(iv) Given any invariant $K$ in $T_\omega$, let $M = [\![\Pi_K \cup \Sigma_K]\!]$. Then one easily shows that either $M$ is empty or $T(M) = T(K)$. In any event, $M = [\![\Pi_M \cup \Sigma_M]\!]$; hence $M$ is a.u. This tells us exactly which invariant sets can be a.u., and indicates that the property of absolute ubiquity becomes much weaker when we move away from isomorphism types.

**§6. Ubiquitous companions.** Let $K \subseteq T_\omega$ be an invariant set. Then, by Corollary 3.5, $\Pi_K$ axiomatizes the set of $\boldsymbol{\Pi}^0_1$-sentences over $L$ that are true for all members of $K$; so $[\![\Pi_K]\!] = T_\omega(K)$, the closure of $K$ in $T_\omega$. For each $1 \leq \alpha \leq \omega$, define the *αth ubiquitous companion* $\Pi_K^{u,\alpha}$ of $\Pi_K$ to be the set of sentences $\varphi$ of $L$ such that $[\![\Pi_K \cup \{\varphi\}]\!]$ is $\alpha$-winnable in $T_\omega(K)$. Clearly $\Pi_K \subseteq \Pi_K^{u,1} \subseteq \Pi_K^{u,2} \subseteq \cdots \subseteq \Pi_K^{u,\omega}$; consequently $[\![\Pi_K]\!] \supseteq [\![\Pi_K^{u,1}]\!] \supseteq [\![\Pi_K^{u,2}]\!] \supseteq \cdots \supseteq [\![\Pi_K^{u,\omega}]\!]$. We let $\Pi_K^u$ denote $\Pi_K^{u,\omega}$, the *ubiquitous companion* of $\Pi_K$.

**6.1.** PROPOSITION. *For each* $1 \leq \alpha \leq \omega$, $\Pi_K^{u,\alpha}$ *is* $\alpha$*-winnable.*

PROOF. This is immediate, by Theorem 1.8, since $\Pi_K^{u,\alpha}$ is countable. □

**6.2.** EXAMPLE. An invariant set $K$ such that, for each $1 \leq m < \omega$, there is a $\boldsymbol{\Pi}^0_2$-sentence in $\Pi_K^{u,m+1}$ that is not in $\Pi_K^{u,m}$. Hence, the invariant sets $[\![\Pi_K^{u,\alpha}]\!]$ are all distinct, $1 \leq \alpha \leq \omega$.

*Construction.* Just use the construction in Example 5.4. □

We wish to view the theories $\Pi_K^{u,\alpha}$, $1 \leq \alpha \leq \omega$, as companions of $\Pi_K$ in the tradition of A. Robinson (see [13]). However, Example 6.2 points to problems when $\alpha < \omega$: If $\sigma$ is a $\boldsymbol{\Pi}^0_2$-sentence such that $[\![\sigma]\!]$ is dense in $T_\omega(K)$, $\sigma$ need not be in $\Pi_K^{u,\alpha}$.

To prove the next result, let $A$ be any $\mathscr{L}$-structure. Diag($A$) is the set of all atomic and negated atomic $\mathscr{L}$-sentences, with constants from $A$, which hold in $A$. Thus $B \models \text{Diag}(A)$ just in case $A$ embeds in $B$.

**6.3.** PROPOSITION. *Let $A$ be a model of $\Pi_K$. Then $A$ embeds in some model $B$ of $\Pi_K^u$. If $A$ is canonical then $B$ can be chosen to be canonical also.*

PROOF. We need to show that $\text{Diag}(A) \cup \Pi_K^u$ is consistent. Let $\Delta \subseteq \text{Diag}(A)$ be finite. Then there is a finite $A_0 \subseteq A$ satisfying $\Delta$. Let $t_0 \in T(K)$ be isomorphic with $A_0$. By Proposition 6.1, $[\![\Pi_K^u]\!]$ is residual in $T_\omega(K)$. Thus $t_0^\# \cap [\![\Pi_K^u]\!]$ is nonempty; hence $\Delta \cup \Pi_K^u$ is consistent. By the compactness theorem, $\text{Diag}(A) \cup \Pi_K^u$ is consistent; hence $A$ embeds in a model $B$ of $\Pi_K^u$. If $A$ is infinite then $B$ can be chosen to be of the same cardinality, by the downward Löwenheim-Skolem theorem. □

Recall the definition of the *model companion* $\Sigma^*$ of a set $\Sigma$ of $\mathscr{L}$-sentences (see [13]): Every model of $\Sigma^*$ embeds in a model of $\Sigma$, and vice versa; and $\Sigma^*$ is model complete. By work of A. Robinson, $\Sigma^*$ is essentially unique when it exists; in this case we call $\Sigma$ *companionable.*

**6.4.** PROPOSITION. *Let $K \subseteq T_\omega$ be an invariant set, and assume $\Pi_K$ is companionable. Then $\Pi_K^* = \Pi_K^u$.*

PROOF. In light of Proposition 6.3, the fact that $\Pi_K \subseteq \Pi_K^u$, and Robinson's uniqueness theorem, all we need to show is that $\Pi_K^u$ is model complete whenever $\Pi_K$ is companionable. By the definition of model companion, $[\![\Pi_K^*]\!]$ is dense in $[\![\Pi_K]\!] = T_\omega(K)$. Since $\Pi_K^*$ is model complete, ascending chains of models of $\Pi_K^*$

become elementary chains, so by the Chang-Łoś-Suszko theorem we know that $\Pi_K^*$ is $\boldsymbol{\Pi}_2^0$-axiomatizable. By Proposition 3.1, $[\![\Pi_K^*]\!]$ is a dense $\boldsymbol{\Pi}_2^0$-set in $T_\omega(K)$, hence winnable. Thus $\Pi_K^* \subseteq \Pi_K^{\mathrm{u}}$. Now let $A$ and $B$ be models of $\Pi_K^{\mathrm{u}}$, with $A \subseteq B$. Then $A$ and $B$ are models of $\Pi_K^*$, a model complete theory. This says $A$ is an elementary submodel of $B$; whence $\Pi_K^{\mathrm{u}}$ is model complete. □

When $\Pi_K$ is companionable, the theories $\Pi_K^{\mathrm{u},m}$, $1 \le m < \omega$, approximate $\Pi_K^{\mathrm{u}} = \Pi_K^*$.

**6.5.** Proposition. *Let $K \subseteq T_\omega$ be an invariant set, and assume $\Pi_K$ is companionable. Then $\Pi_K^{\mathrm{u}} = \bigcup_{m=1}^{\infty} \Pi_K^{\mathrm{u},m}$.*

Proof. By the proof of Proposition 6.4, $\Pi_K^{\mathrm{u}} = \Pi_K^*$ is $\boldsymbol{\Pi}_2^0$-axiomatizable. Let $\sigma \in \Pi_K^{\mathrm{u}}$, say $\{\varphi_1, \ldots, \varphi_n\} \vdash \sigma$, where each $\varphi_i \in \Pi_K^{\mathrm{u}}$ is a $\boldsymbol{\Pi}_2^0$-sentence. $[\![\sigma]\!]$ is thus easily seen to be $m$-winnable for some $m < \omega$, by Theorem 5.3. □

**6.6.** *Question.* Does Proposition 6.5 hold even without the companionability assumption?

In Example 6.2, the theories $\Pi_K^{\mathrm{u},\alpha}$ are all distinct, and it is easy to show that $T(K)$ satisfies the JEP in this case. The story is entirely different when the AP holds, however.

**6.7.** Proposition. *Let $K \subseteq T_\omega$ be an invariant set such that $T(K)$ satisfies the AP. Then $\Pi_K$ is companionable, and $\Pi_K^{\mathrm{u},1} = \Pi_K^*$. Moreover, $\Pi_K^*$ is complete and $\aleph_0$-categorical.*

Proof. This follows immediately from Theorem 5.7 using the Łoś-Vaught test. □

**6.8.** *Question.* Does completeness of $\Pi_K^{\mathrm{u},1}$ imply the AP for $T(K)$?

**6.9.** Remarks. (i) If $b \in T_\omega$ is a.u., then $b^+ = [\![\Pi_b \cup \Sigma_b]\!]$; hence, $\Pi_b \cup \Sigma_b$ is an $\aleph_0$-categorical $\boldsymbol{\Pi}_2^0$-theory and, therefore, model complete by Lindström's theorem. Thus $\Pi_b \cup \Sigma_b$ is complete and the model companion of $\Pi_b$.

(ii) Asserting that $\Pi_K^{\mathrm{u},\alpha}$ is complete is a way of stating a strategic zero-one law: Given a first order sentence $\varphi$, player (II) can win either $G_\alpha(T(K), [\![\varphi]\!])$ or $G_\alpha(T(K), [\![\neg\varphi]\!])$. This is a stronger statement than saying that one of these games is determined, since it is linked with the JEP, as we presently show.

**6.10.** Proposition. *Let $K \subseteq T_\omega$ be an invariant set. Then $\Pi_K^{\mathrm{u}}$ is complete if and only if $T(K)$ satisfies the JEP.*

Proof. Suppose $\Pi_K^{\mathrm{u}}$ is complete, and let $b \models \Pi_K^{\mathrm{u}}$. For each $t \in T(K)$, $[\![\sigma_t]\!]$ is a nonempty open subset of $T_\omega(K)$, hence somewhere residual. Thus (I) can win $G(T(K), [\![\neg\sigma_t]\!])$. By completeness of $\Pi_K^{\mathrm{u}}$, (II) can win $G(T(K), [\![\sigma_t]\!])$; hence $\sigma_t \in \Pi_K^{\mathrm{u}}$, and we have $b \models \sigma_t$. Therefore $t$ embeds in $b$, and $b$ is universal for $T(K)$. $T(K)$ satisfies the JEP by Proposition 5.5.

Conversely, if $T(K)$ satisfies the JEP, we can invoke Theorem 5.2 for the completeness of $\Pi_K^{\mathrm{u}}$. □

A theorem similar to Proposition 6.10 was proved by A. Robinson for the finite forcing companion $\Pi_K^{\mathrm{f}}$ (see [1]). The reader may well have guessed that this is no coincidence: $\Pi_K^{\mathrm{u}}$ and $\Pi_K^{\mathrm{f}}$ are the same. To see this, we refer the reader to [11] for background on model-theoretic forcing. In this instance, conditions are finite sets of atomic and negated atomic sentences of $L(\omega)$ which are satisfied in some $t \in T(K)$ or, equivalently, in some $a \in K$. Let $P = P_K$ be the set of all conditions, ordered by set-theoretic inclusion. For $Q \subseteq P$, $q \in Q$, and $\varphi$ a sentence of $L(\omega)$, define the forcing relation $q \Vdash_Q \varphi$ by induction on the complexity of $\varphi$ as follows:

$$q \Vdash_Q \varphi \quad \text{if } \varphi \in q, \text{ for atomic } \varphi;$$
$$q \Vdash_Q \varphi \vee \psi \quad \text{if } q \Vdash_Q \varphi \text{ or } q \Vdash_Q \psi;$$
$$q \Vdash_Q \neg\varphi \quad \text{if } r \nVdash_Q \varphi \text{ for all } r \geq q,\ r \in Q; \text{ and}$$
$$q \Vdash_Q \exists x\, \varphi(x) \quad \text{if } q \Vdash_Q \varphi[n] \text{ for some } n < \omega.$$

The *finite forcing companion* $\Pi_K^{\mathrm{f}}$ of $\Pi_K$ is just $\{\varphi: \varphi$ is an $L$-sentence and $\varnothing \Vdash_P \neg\neg\varphi\}$. Note that $q \Vdash_Q \neg\neg\varphi$ if and only if for all $r \in Q$, $r \geq q$, there is an $s \in Q$, $s \geq r$, with $s \Vdash_Q \varphi$. Also note that if $q \Vdash_Q \varphi$ and $q \leq r \in Q$ then $r \Vdash_Q \varphi$.

**6.11.** LEMMA. (i) *Let $Q \subseteq P$ be cofinal in P. Then, for all $q \in Q$, $q \Vdash_Q \varphi$ if and only if $q \Vdash_P \varphi$.*

(ii) *Let $Q = \{\mathrm{Diag}(t): t \in T(K)\}$, for some invariant set $K \subseteq T_\omega$. Then Q is cofinal in $P_K$, and, for all $t \in T(K)$ and $\varphi \in L(\omega)$, $\mathrm{Diag}(t) \Vdash_Q \neg\neg\varphi$ if and only if $[\![\varphi]\!]$ is residual in $t^\#$.*

PROOF. (i) An easy induction on the complexity of $\varphi$, the least trivial step being negation. Obviously, if $q \Vdash_P \neg\varphi$, then $r \nVdash_P \varphi$ for all $q \leq r \in Q$; hence by induction $r \nVdash_Q \varphi$ for all $q \leq r \in Q$. Thus $q \Vdash_Q \neg\varphi$. Conversely, assume $q \Vdash_Q \neg\varphi$. For all $q \leq r \in Q$, we have $r \nVdash_Q \varphi$. Let $q \leq s \in P$. By cofinality there is some $s \leq r \in Q$; so $r \nVdash_Q \varphi$. By the inductive hypothesis, $r \nVdash_P \varphi$. Thus $s \nVdash_P \varphi$; so $q \Vdash_P \neg\varphi$.

(ii) $Q$ is clearly cofinal in $P$. We prove by induction on the complexity of $\varphi$ in $L(\omega)$ that $\mathrm{Diag}(t) \Vdash_Q \neg\neg\varphi$ if and only if $[\![\varphi]\!]$ is residual in $t^\#$. The proof is straightforward; we check two of the induction steps.

(1) $\mathrm{Diag}(t) \Vdash_Q \neg\neg\exists x\varphi(x)$ iff there is a cofinal subset $S \subseteq [t, \infty)$ such that for all $s \in S$ there is an $n < \omega$ such that $\mathrm{Diag}(s) \Vdash_Q \varphi[n]$ iff there is a cofinal subset $S \subseteq [t, \infty)$ such that for all $s \in S$, $[\![\exists x\varphi(x)]\!]$, which is $\bigcup_{n<\omega}[\![\varphi[n]]\!]$, is residual in $s^\#$ iff $[\![\exists x\varphi(x)]\!]$ is residual in $t^\#$.

(2) $\mathrm{Diag}(t) \Vdash_Q \neg\neg(\neg\varphi)$ iff $\mathrm{Diag}(t) \Vdash_Q \neg\varphi$ iff $[\![\varphi]\!]$ is nowhere residual in $t^\#$ iff $[\![\neg\varphi]\!]$ is residual in $t^\#$ (using Borel determinacy). □

As an immediate consequence of Lemma 6.11, we have:

**6.12.** PROPOSITION. *Let $K \subseteq T_\omega$ be an invariant set. Then $\Pi_K^{\mathrm{u}} = \Pi_K^{\mathrm{f}}$.* □

**6.13.** REMARK. Robinson's theorem states that if $\Sigma$ is any set of sentences, then $\Sigma^{\mathrm{f}}$ is complete if and only if the class of models of $\Sigma$ has the JEP. In Proposition 6.10 we consider the JEP only for the class of finite models of $\Pi_K$. However it is easy to show, using diagrams and compactness, that for any universal theory $\Pi$, the JEP holds for the finite models of $\Pi$ just in case it holds for all models of $\Pi$.

Given an invariant set $K \subseteq T_\omega$, one can form $E_K \subseteq T_\omega(K)$, the invariant set of structures existentially closed in $T_\omega(K)$, defined as follows (see [13]): $a \in E_K$ just in case whenever $A \models \Pi_K$, $\varphi$ is a $\Sigma_1^0$-sentence from $L(\omega)$, and $a \subseteq A \models \varphi$ (constants are interpreted standardly, as always), then $a \models \varphi$. In general there is no relation between $E_K$ and $[\![\Pi_K^{\mathrm{u}}]\!]$, unless $\Pi_K$ is companionable, in which case equality holds. A theorem of P. Eklof and G. Sabbagh (see [13]) states that $\Pi_K$ is companionable if and only if $E_K$ is definable (in $L$). One can easily show $E_K$ is residual in $T_\omega(K)$; in fact, more is true: The smaller invariant set $F_K$ of generic models is residual. We see this as follows.

Let $K \subseteq T_\omega$ be fixed, and let $P$ be the set of finite conditions as before. We identify $T(K) \subseteq P$ in the obvious way. A set $G$ of atomic and negated atomic sentences from $L(\omega)$ is *generic* if: (i) each finite $p \subseteq G$ is an element of $P$; and (ii) for each $L(\omega)$-sentence $\varphi$, one can find a condition $p \subseteq G$ such that either $p \Vdash_P \varphi$ or $p \Vdash_P \neg\varphi$. A standard fact is that each $p \in P$ is contained in some generic $G$. (See [11]. This is also an immediate consequence of Proposition 6.14 below.) Moreover, for each generic $G$, there is a unique (up to isomorphism) $a(G) \in T_\omega(K)$ such that, for each $\varphi$ of $L(\omega)$, $a(G) \models \varphi$ if and only if $p \Vdash_P \varphi$ for some $p \subseteq G$. Let $F_K$ be the invariant set of all *generic models* $a(G)$. Then [11] $F_K \subseteq E_K$. In general, $\Pi^{\mathrm{f}}_K = \mathrm{Th}(F_K)$. If $\Pi_K$ is companionable, then $F_K = E_K = \llbracket \Pi^*_K \rrbracket$.

**6.14.** PROPOSITION. *$F_K$ is residual in $T_\omega(K)$.*

PROOF. Let $\varphi_0, \varphi_1, \dots$ be a list of all sentences of $L(\omega)$. Define the "Markov" strategy $\mu$: $T(K) \times \omega \to T(K)$ (terminology from [22]) using that for any $t \in T(K)$ and sentence $\varphi$, either $\llbracket\varphi\rrbracket$ is residual in $t^\#$ or $\llbracket\neg\varphi\rrbracket$ is somewhere residual in $t^\#$. Thus:

$$\mu(t,n) = \begin{cases} \text{some } s > t \text{ with } s \Vdash_{T(K)} \varphi_n & \text{if } \llbracket\varphi_n\rrbracket \text{ is residual in } t^\#, \\ \text{some } s > t \text{ with } s \Vdash_{T(K)} \neg\varphi_n & \text{otherwise.} \end{cases}$$

Let $a = \bigcup_{n<\omega} t_n$ be the result of a game in which (II) uses the strategy $t'_n = \mu(t_n, n)$. Let $G = \mathrm{Diag}(a)$. We show that $G$ is generic and $a = a(G)$. Suppose $\varphi$ is $\varphi_n$. If $\llbracket\varphi_n\rrbracket$ is residual in $(t'_n)^\#$ then, by Lemma 6.11, $t'_n \Vdash_P \varphi_n$. If $\llbracket\neg\varphi_n\rrbracket$ is somewhere residual in $(t'_n)^\#$, then $t'_n \Vdash_P \neg\varphi_n$. To show that $a = a(G)$, we induct on complexity of sentences. Note that every finite $p \subseteq G$ extends to some $t_n$ (i.e. to some $\mathrm{Diag}(t_n)$). Thus, given $\varphi$, $p \Vdash_P \varphi$ for some $p$ if and only if $t_n \Vdash_P \varphi$ for some $n$. Let $\varphi$ be atomic. $a \models \varphi$ if and only if $\varphi \in \mathrm{Diag}(a)$ if and only if $\varphi \in \mathrm{Diag}(t_n)$ for some $n$ if and only if $t_n \Vdash_P \varphi$ for some $n$. The least trivial inductive step is negation: $a \models \neg\varphi$ if and only if $a \not\models \varphi$ if and only if $t_n \not\Vdash_P \varphi$ for all $n$. If $t_m \Vdash \neg\varphi$ for some $m$, then, for all $n \geq m$, $t_n \not\Vdash_P \varphi$. But this implies, since the $t_n$'s form a chain, that $t_n \not\Vdash_P \varphi$ for all $n$. Conversely, if $t_n \not\Vdash_P \neg\varphi$ for all $n$, then for each $n$ there is some $m \geq n$ with $t_m \Vdash_P \varphi$. Thus it is not the case that $t_n \not\Vdash_P \varphi$ for all $n$. From this we see that $a \models \neg\varphi$ if and only if $t_n \Vdash_P \neg\varphi$ for some $n$. Thus $a$ is a generic model. □

**6.15.** REMARKS. (i) If $K$ is a $\boldsymbol{\Pi}^0_2$-definable invariant set, then $F_K \subseteq K$ (see [11]). Thus, in the topological sense, "almost every model in $K$ is generic."

(ii) Although Propositions 6.12 and 6.13 are essentially known (see [9]), our proofs and viewpoints are somewhat different from what has gone before.

For any invariant set $K \subseteq T_\omega$, we have an infinite descending chain of residual invariant subsets of $T_\omega(K)$: $\llbracket \Pi^{\mathrm{u},1}_K \rrbracket \supseteq \llbracket \Pi^{\mathrm{u},2}_K \rrbracket \supseteq \cdots \supseteq \llbracket \Pi^{\mathrm{u}}_K \rrbracket \supseteq F_K$. The question naturally arises as to whether or not there exists a minimal residual invariant subset $M_K$ of $T_\omega(K)$, necessarily unique if it exists. We collect what we know in the next result. We are grateful to Wilfrid Hodges for suggestions on how to get the nonexistence of $M_K$ in the presence of the JEP.

**6.16.** PROPOSITION. *Let $K \subseteq T_\omega$ be an invariant set.*

(i) *If $T(K)$ has the JEP, then $M_K$ exists if and only if $M_K = a^+$ for some $a \in T_\omega(K)$.*

(ii) *If $T(K) = T(a^+)$ for some absolutely ubiquitous $a \in T_\omega$, then $M_K = a^+$.*

(iii) *If $T(K)$ has the AP, then $M_K = \llbracket \Pi_K \cup \Sigma_K \cup \Gamma_K \rrbracket$.*

(iv) *If $T(K)$ has a universal model $a$ with a $\Pi_2'^0$-Scott sentence, then $M_K = a^+$. This can happen in the absence of the hypotheses in* (ii) *or* (iii).

(v) *The JEP for $T(K)$ and the existence of $M_K$ are independent.*

(vi) *The companionability of $\Pi_K$ does not entail the existence of $M_K$.*

PROOF. (i) By Proposition 3.1 and the existence of Scott sentences of $L_{\omega_1\omega}$, which define the isomorphism type of a countable structure, each $a^+$ is a Borel subset of $T_\omega(K)$ whenever $a \in T_\omega(K)$. By Theorem 5.2, then, $a^+$ is either residual or meager in $T_\omega(K)$. If some $a^+$ is residual, then $M_K = a^+$; otherwise each $a^+$ is meager and the intersection of all invariant residual subsets of $T_\omega(K)$ is empty.

(ii) Use Remark 6.9(i).

(iii) Use Theorem 5.7.

(iv) Use Proposition 3.1 for the first assertion. For the second assertion, use Example 5.4. In that example, the universal models for $T(K)$ must have infinitely many orbits of each finite positive length. The additional properties of being a total bijection with no infinite orbits completely describe, with a $\Pi_2'^0$-sentence of $L_{\omega_1\omega}$, a universal model $a$. This structure is easily seen to be nonhomogeneous; hence the AP fails. To see that $T(K)$ cannot be $T(b^+)$ for any a.u. structure $b$, note that such a $b$ would have to be isomorphic to $a$, by (ii) above. But $a$ is not a.u.; by a simple ultrapower argument, plus the Löwenheim-Skolem theorem, $a$ is elementarily equivalent to a canonical structure with infinite orbits. Hence $a$ is not even $\aleph_0$-categorical.

(v) *The JEP can fail for $T(K)$, but $M_K$ can still exist.* Let $L = \{R\}$, $R$ unary, and let $K = [\![\forall x Rx \vee \forall x \neg Rx]\!]$. Then the JEP clearly fails, but $M_K = K$. A more interesting example is detailed in Example 7.13.

*The JEP can hold for $T(K)$, but $M_K$ can fail to exist.* In a private communication, W. Hodges pointed out to us the relevant information necessary to construct the following example: Let $L = \{\cdot, (\ )^{-1}, 1\}$ be the lexicon of groups, where we view an $n$-ary function symbol as an $(n+1)$-ary relation symbol (see Remark 3.2(iv)). Let $K$ be the invariant set of groups; then $K$ is $\Pi_2^0$-definable and hence residual in $T_\omega(K)$. Clearly, $T(K)$ satisfies the JEP; the free product of all finitely generated groups is universal for $T(K)$. The nonexistence of $M_K$ is an immediate consequence of the following two facts, both due to A. Macintyre, and proved in [9]. Fact (1): Every existentially closed group has a finitely generated subgroup with unsolvable word problem (Corollary 3.3.8 in [9]). Fact (2): If $g \in K$ is finitely generated with unsolvable word problem, then $\{h \in T_\omega(K)\colon g \text{ fails to embed in } h\}$ is residual in $T_\omega(K)$. Fact (2) actually follows from the proof of the apparently weaker Theorem 3.4.6 in [9].

(vi) Use a slight variation on Example 5.4. Let $K$ be the closed invariant set of canonical partial injections, subject to the condition that for each odd whole number $n$, if there is an orbit of length $n$, there can be no orbit of length $n+1$. It is easy to see that for any $a \in T_\omega(K)$ and $t \in T(K)$, there is a $t' \in T(K)$ such that $t \leq t'$ and $t'$ does not embed in $a$. Thus the JEP fails very strongly; in fact player (II) has an easy winning strategy for $G(T(K), T_\omega(K)\backslash a^+)$ for all $a \in T_\omega(K)$. Thus each $a^+$ is meager in $T_\omega(K)$, and $M_K$ fails to exist. However, $\Pi_K$ is companionable; $[\![\Pi_K^*]\!] = \{b \in T_\omega(K)\colon b$ is a total bijection such that for each odd $n$, $b$ has infinitely many orbits of length $l$ for some $l \in \{n, n+1\}\}$. □

**6.17.** *Question.* If $\Pi_K$ has a complete model companion, does $M_K$ exist?

**6.18.** REMARK. Note that in the proof of Proposition 6.16(vi), $\Pi_K^*$ is not complete. For by Proposition 6.4, $\Pi_K^* = \Pi_K^{\mathrm{u}}$; and by Proposition 6.10, completeness fails because the JEP fails for $T(K)$.

**§7. Examples and remarks.** In this section we present some examples that apply the techniques developed so far, as illustrations of the theme: "Almost every structure in $K$ is in $M$," where $M \subseteq T_\omega(K)$.

**7.1.** EXAMPLE. Let $L$ be arbitrary, and let $K = T_\omega$. Then $T(K)$ satisfies the AP, $\Pi_K = \varnothing$, and $\Sigma_K \cup \Gamma_K$ is the set of Rabin-Scott sentences studied by H. Gaifman [7] and R. Fagin [6], among others. Denote the isomorphism type $[\![\Sigma_K \cup \Gamma_K]\!]$ by $\iota_0(L)$. Then $\iota_0(L)$ is 1-winnable, and we can say "almost every $L$-structure is isomorphic to a model in $\iota_0(L)$" in the strongest strategic sense. We also add that $\Pi_K^{\mathrm{u},1} = \Pi_K^{\mathrm{u},2} = \cdots = \Pi_K^{\mathrm{u}} = \Pi_K^*$; these theories are all complete, thus all strategic 0-1 laws hold.

**7.2.** EXAMPLE. Let $L$ consist of one binary relation, and let $K$ be the canonical linear orderings. Then $K$ is closed, $T(K)$ satisfies the AP, and $[\![\Pi_K \cup \Sigma_K \cup \Gamma_K]\!] = \eta_0$, where $\eta_0$ is the order type of the rational line. Interestingly, every $r \in K$ is universal for $T(K)$; only the members of $\eta_0$ are also homogeneous. The remaining comments in Example 7.1 apply here as well.

**7.3.** EXAMPLE. Let $L$ consist of one binary relation, and let $K$ be the canonical loop-free undirected graphs with no multiple edges. Then $K$ is closed, $T(K)$ satisfies the AP, and $[\![\Pi_K \cup \Sigma_K \cup \Gamma_K]\!] = \rho_0$, where $\rho_0$ is the isomorphism type of R. Rado's random graph (studied also by P. Erdös and A. Rényi, viz. [3]). The isomorphism type $\rho_0$ is characterized by the following property of a canonical graph $g$: For each disjoint pair of finite sets of vertices, there is a single vertex which is edge-joined to each vertex in one of the sets and to none in the other. In particular, $g$ is a connected graph, each pair of disjoint vertices being connectable via an edge-path of length 2. All the remaining comments from Examples 7.1 and 7.2 apply here, except that not every canonical graph is universal for $T(K)$, though many besides those in $\rho_0$ are.

**7.4.** EXAMPLE. Let $L$ consist of one binary relation, and let $K$ be the canonical equivalence relations. Then $T(K)$ satisfies the AP, and $[\![\Pi_K \cup \Sigma_K \cup \Gamma_K]\!] = \varepsilon_0$, where $\varepsilon_0$ is the isomorphism type of those canonical equivalence relations which consist of infinitely many infinite equivalence classes, the "totally infinite" equivalence relations. All the remaining comments from Examples 7.1 and 7.3 apply here as well.

**7.5.** EXAMPLE. Let $L$ consist of one binary relation, and let $K$ be the canonical partial injections, as in Example 5.4. Then $T(K)$ satisfies the JEP, but not the AP. Let $b \in T_\omega(K)$ be a total bijection in which there are infinitely many orbits of each finite positive length and no infinite orbits. Then one may readily verify that $b$ is universal for $T(K)$, and $b^+$ has a $\boldsymbol{\Pi}_2'^0$-definition (Scott sentence). Thus $b^+$ is a dense $\boldsymbol{\Pi}_2^0$-set, hence residual in $T_\omega(K)$. Note that the AP fails, since $b$ is easily seen to be nonhomogeneous. Moreover, by Example 5.4, $b^+$ is not $m$-winnable in $T_\omega(K)$ for any $1 \leq m < \omega$. Thus, "almost every partial injection on $\omega$ is isomorphic to $b$" is true in only the weakest game-theoretic sense.

**7.6.** EXAMPLE. Let $L$ be arbitrary. An $L$-structure $A$ is a *partial algebra* if the interpretation in $A$ of an $(n + 1)$-ary relation $R \in L$ is a partial $n$-ary operation. Let $K \subseteq T_\omega$ be the invariant set of canonical partial algebras which are total (that is, the

$n$-ary operations apply to all $n$-tuples). Then $K$ is $\Pi_2^0$-definable and $T_\omega(K)$ is the set of all canonical partial algebras. Moreover, since $\Pi_2^0$-sentences which define totality satisfy the hypotheses of Theorem 5.3, we can conclude that "almost every partial algebra is total" in the strongest strategic sense: $K$ is 1-winnable in $T_\omega(K)$. But the JEP does not hold if there are two or more unary predicates (partial constants).

**7.7.** REMARK. In the examples that follow, we consider lexicons $L$ that include function symbols. It is clear how to replace an $n$-ary function symbol $f$ by an $(n+1)$-ary relation symbol $\bar{f}$ and, likewise, how to convert $L$-structures with function symbols $f$ to $\bar{L}$-structures with corresponding relation symbols $\bar{f}$. We must add axioms of the form

$$\forall x_1 \cdots x_n y_1 y_2 (\bar{f} x_1 \cdots x_n y_1 \wedge \bar{f} x_1 \cdots x_n y_2 \rightarrow y_1 = y_2)$$

and

$$\forall x_1 \cdots x_n \exists y \bar{f} x_1 \cdots x_n y$$

to express that the predicates $\bar{f}$ represent functions. Our intention is to use Proposition 3.1 and Theorem 5.3 in the algebraic setting as well. The above axioms contain only blocks of one existential quantifier, and therefore present no difficulty when we apply Theorem 5.3. However, we must pay attention to how $L$-sentences become converted to $\bar{L}$-sentences.

Suppose that $L$ includes function symbols $f$ of nonzero arity and constants $c$. For each quantifier-free $L$-formula $\varphi$ we associate $\bar{L}$-formulas $\varphi_A(\mathbf{x}) \equiv [\varphi]_A$ and $\varphi_E(\mathbf{x}) \equiv [\varphi]_E$, by induction on complexity, where $\mathbf{x} = x_1 \cdots x_m$ are new variables:

$$[x = y]_A \equiv [x = y]_E \equiv (x = y);$$

$$[x = c]_A \equiv [x = c]_E \equiv \bar{c}x;$$

$$[x = f(\tau_1, \dots, \tau_n)]_A \equiv \left( \bigwedge_{1 \le i \le n} [x_i = \tau_i]_E \right) \rightarrow \bar{f} x_1 \cdots x_n x,$$

where the variables $x_1, \dots, x_n$ are new;

$$[x = f(\tau_1, \dots, \tau_n)]_E \equiv \left( \bigwedge_{1 \le i \le n} [x_i = \tau_i]_E \right) \wedge \bar{f} x_1 \cdots x_n x,$$

where the variables $x_1, \dots, x_n$ are new;

$$[\sigma = \tau]_A \equiv [x = \sigma]_E \rightarrow [x = \tau]_A;$$

$$[\sigma = \tau]_E \equiv [x = \sigma]_E \wedge [x = \tau]_E;$$

$$[\varphi \wedge \psi]_A \equiv [\varphi]_A \wedge [\psi]_A;$$

$$[\varphi \wedge \psi]_E \equiv [\varphi]_E \wedge [\psi]_E;$$

$$[\neg \varphi]_A \equiv \neg [\varphi]_E;$$

$$[\neg \varphi]_E \equiv \neg [\varphi]_A.$$

Each quantifier-free formula $\varphi$ of $L$ now may be associated with either $\forall \mathbf{x} \varphi_A(\mathbf{x})$ or $\exists \mathbf{x} \varphi_E(\mathbf{x})$. This provides us with two ways to translate any formula of $L$. Assume $\psi(\mathbf{x})$ is a quantifier-free formula from $L$. Then we may choose $[\forall \mathbf{x} \psi(\mathbf{x})]^* \equiv \forall \mathbf{x}, \mathbf{y} \psi_A(\mathbf{x}, \mathbf{y})$, where the variables $\mathbf{y}$ are new variables added according to the recipe above. Now assume $\psi(\mathbf{x},\mathbf{y})$ is a quantifier-free formula from $L$ in which at least one of the variables $\mathbf{y}$ occur. Then we may choose

$$[\forall\mathbf{x}\exists\mathbf{y}\psi(\mathbf{x},\mathbf{y})]^* \equiv \forall\mathbf{x}\exists\mathbf{y}\exists\mathbf{z}\psi_E(\mathbf{x},\mathbf{y},\mathbf{z}),$$

where the variables $\mathbf{z}$ are added according to the above. It is not hard to show that if $A$ is any $L$-structure and $\varphi$ is a $\Pi_n^0$-sentence (resp. $\Sigma_n^0$-sentence) then $\varphi^*$ may be chosen such that it is a $\Pi_n^0$-sentence (resp. $\Sigma_n^0$-sentence) as above, and $A \models \varphi$ if and only if $\bar{A} \models \varphi^*$. From a game-theoretic point of view the bad news is that some new variables get existentially quantified; the good news is that we can predict how many.

**7.8.** Example. Let $L = \{\vee, \wedge, \bot, \top, \neg\}$ be the lexicon of Boolean algebras, and let $K$ be the canonical Boolean algebras. Then $\bar{K}$ is $\Pi_2^0$-definable over $\bar{L}$ via a sentence in which only one variable is existentially quantified; hence $K$ is 1-winnable in $T_\omega(K)$, by Theorem 5.3. Let $\alpha_0 \subseteq K$ be the atomless algebras. Then $\alpha_0$ is a single isomorphism type which is dense in $K$ since every Boolean algebra embeds in an atomless one. The isomorphism type $\alpha_0$ is defined by a $\Pi_2^0$-sentence of $L$ in which only one variable is existentially quantified, and it is a straightforward computation to check that the translation of this sentence involves no new variables. Thus $\alpha_0$ is 1-winnable in $K$; hence in $T_\omega(K)$. Consequently, "almost every partial Boolean algebra is total and atomless," in the strongest strategic sense.

**7.9.** Example. Let $L = \{\cdot\}$ be the lexicon of semigroups, and let $K$ be the canonical semigroups. Then $K$ is 1-winnable in $T_\omega(K)$. Let $M \subseteq K$ be the (von Neumann) regular semigroups, i.e. the semigroups satisfying $\forall x\exists y(x \cdot y \cdot x = x)$. When we informally translate $(x \cdot y = z)$ to $\cdot xyz$, we get the translation $\forall x\exists yz(\cdot zxx \leftrightarrow \cdot xyz)$. $M$ is dense in $K$ because every semigroup embeds, by Cayley's theorem, into the regular semigroup of self-maps on a set. Thus, $M$ is 2-winnable in $K$; hence in $T_\omega(K)$. So "almost every partial semigroup is total and regular," in the strongest strategic sense, but one. We do not know whether $M$ is 1-winnable in $K$, but suspect not.

**7.10.** Example. Let $L = \{\cdot, 1\}$ be the lexicon of monoids, and let $K$ be the canonical monoids. Then $K$ is 1-winnable in $T_\omega(K)$. Let $M \subseteq K$ be the groups, i.e. the monoids satisfying $\forall x\exists y(x \cdot y = 1)$. The translation of this sentence, according to the recipe in Remark 7.7, is $\forall x\exists yz(\bar{1}z \wedge \cdot xyz)$. But this new sentence is needlessly complicated, and clearly equivalent, for monoids, with $\forall xz\exists y(\bar{1}z \rightarrow \cdot xyz)$. Thus $M$ is 1-winnable in $T_\omega(M)$. Unfortunately, not every monoid embeds in a group. Thus "almost every group-embeddable partial monoid is a total group," in the strongest game-theoretic sense.

**7.11.** Example. Let $L = \{+, -, 0\}$ be the lexicon of abelian groups, and let $K$ be the canonical abelian groups. Then $K$ is 1-winnable in $T_\omega(K)$. Let $M \subseteq K$ be the divisible groups. Then $M$ is the set of canonical models of the set of sentences $\{\forall x\exists y(ny = x)\colon n = 2, 3, \ldots\}$. However, the translates of these sentences, while still $\Pi_2^0$-sentences of $\bar{L}$, involve the introduction of an unbounded number of new existentially quantified variables. Thus, Theorem 5.3 is of no use here, and it seems that the best that may be said is that $M$ is residual in $T_\omega(M)$. Now, every abelian group has a divisible hull; hence, "almost every partial abelian group is total and divisible," in the weakest game-theoretic sense.

**7.12.** Example. Let $L = \{+, \cdot, -, 0, 1\}$ be the lexicon of unital rings, and let $K$ be the canonical commutative integral domains. Then $K$ is 1-winnable in $T_\omega(K)$. Let $M \subseteq K$ be the fields. As in Example 7.10, the translate of the sentence that says every nonzero element has an inverse is, for unital rings, equivalent to $\forall xz \exists y(\bar{1}z \rightarrow \cdot xyz)$, a $\boldsymbol{\Pi}_2^0$-sentence which has only one existentially quantified variable. Since every commutative integral domain embeds in its field of fractions, we conclude that "almost every partial commutative integral domain is a total field," in the strongest game-theoretic sense.

Example 7.12 can be taken quite a bit further and, as shown in the next example, is relevant to Proposition 6.16(v).

**7.13.** Example. The invariant set $K$ of commutative integral domains in Example 7.12 provides the following properties: (i) $T(K)$ does not satisfy the JEP; (ii) $\Pi_K$ is companionable; and (iii) $T_\omega(K)$ has a smallest residual subset $M_K$, which is not definable in $L$ and which contains a countably infinite number of isomorphism types.

Let $\Sigma_{\mathrm{AC}}$ be the $\boldsymbol{\Pi}_2^0$-theory of algebraically closed fields. Because of the unbounded size of terms in $\Sigma_{\mathrm{AC}}$, we do not have much hope of winning handicap games. By well-known results, $\Sigma_{\mathrm{AC}}$ is the model companion of the theory of commutative integral domains; hence of $\Pi_K$. We get $M_K$ from $[\![\Sigma_{\mathrm{AC}}]\!]$ as follows. For each prime number $p$, there is a $\boldsymbol{\Sigma}_1^0$-sentence stating that $p$ is zero. Let $\varphi$ be the countable disjunction of these sentences. Then $\varphi$ is clearly a $\boldsymbol{\Pi}_2'^0$-sentence; hence $[\![\varphi]\!]$ is residual in its closure ($\varphi$, of course, expresses of a field that the characteristic is prime). To see that $[\![\varphi]\!]$ is dense in $[\![\Sigma_{\mathrm{AC}}]\!]$, let $t \in T_n(K)$ embed in some $f \models \Sigma_{\mathrm{AC}}$ of characteristic 0. Let $R$ be the subring of $f$ generated by the image of $t$. Then $R$ is isomorphic to an integral domain $\mathbf{Z}[\mathbf{x}] = \mathbf{Z}[X_1, \ldots, X_n]/I$, where the elements $\mathbf{x} = x_1, \ldots, x_n$ correspond to the image of $t$. For each pair $1 \leq i < j \leq n$, introduce a new variable $Y_{ij}$. Consider $S = \mathbf{Z}[\mathbf{X}, \mathbf{Y}]$, where $\mathbf{Y} = Y_{1,2}, \ldots, Y_{n-1,n}$, and let $J$ be the ideal of $S$ generated by $I$ and the expressions $(Y_{ij}(X_i - X_j) - 1)$. Then $J$ is contained in a maximal ideal $M \subseteq S$. The relations $(Y_{ij}(X_i - X_j) - 1)$ prevent the $X_i$ from collapsing; thus $t$ embeds in the field $S/M$. Since $S$ is finitely generated over $\mathbf{Z}$, the field $S/M$ must have prime characteristic. So $t$ also embeds in an algebraically closed field of prime characteristic. Thus, "almost every partial commutative integral domain is an algebraically closed field of prime characteristic." Let $p$ be prime and let $\psi_p(x)$ express $(x = 0) \vee (x = 1) \vee \cdots \vee (x = p - 1)$. Then the formula $\varphi_p(x) \equiv (p = 0) \wedge \psi_p(x)$ expresses that $x$ is in the prime subfield $\mathbf{F}_p$. For each $p$ and $n$, let $\psi_{p,n}(x)$ be

$$\exists y_1 \cdots y_n(\varphi_p(y_1) \wedge \cdots \wedge \varphi_p(y_n) \wedge (x^n + y_1 x^{n-1} + \cdots + y_n = 0)),$$

and let $\lambda$ be $\forall x \bigvee_{p,n} \psi_{p,n}(x)$. For a field, the $\boldsymbol{\Pi}_2'^0$-sentence $\lambda$ expresses that the characteristic is prime and that the field is algebraic over its prime subfield. By Hilbert's Nullstellensatz, each finite substructure of a field of characteristic $p$ can be embedded into the algebraic closure of $\mathbf{F}_p$, so $M_K = [\![\Sigma_{\mathrm{AC}} \cup \{\lambda\}]\!]$ is dense in $[\![\Sigma_{\mathrm{AC}}]\!]$. Clearly $M_K$ is the smallest residual invariant subset of $T_\omega(K)$; and "almost every partial commutative integral domain is the algebraic closure of some finite field," in the weakest strategic sense.

**7.14.** Remark. A second theme of this paper, one which is related to the "almost every $K$ is an $M$" theme, is that of "zero-one law." For example, given the invariant set $K \subseteq T_\omega$, we define $K$ to satisfy the *$\alpha$-strategic zero-one law*, $1 \leq \alpha \leq \omega$, if, for every sentence $\sigma$ from $L$, player (II) has a winning strategy for one of the two games $G_\alpha(T(K), [\![\sigma]\!])$ and $G_\alpha(T(K), [\![\neg\sigma]\!])$; equivalently, if $\Pi_K^{u,\alpha}$ is complete. In Examples 7.1–7.4 above, the AP holds, and consequently all strategic zero-one laws hold. In the situations where the JEP fails, we have no strategic zero-one laws, because of Proposition 6.10. These include Examples 7.6 and 7.12 above. (The JEP fails for Example 7.6 only when $L$ has at least two unary predicates.) In the case of Boolean algebras (7.8), the complete theory of atomless algebras is contained in $\Pi_K^{u,1}$; hence all strategic zero-one laws hold. In Examples 7.9–7.11, the JEP holds; so at least the $\omega$-strategic zero-one law holds. As for the other zero-one laws, we have no idea yet. Finally, in the case of Example 7.5, the situation is simple enough so we know that only the $\omega$-strategic zero-one law holds. To see this, we first note that the JEP is true and use Proposition 6.10. Now, for each $1 \leq m < \omega$, let $\sigma$ be the sentence which says there is an orbit of length $m + 1$. Then, as we saw in Example 5.4, (II) cannot win $G_m(T(K), [\![\sigma]\!])$. On the other hand, (I) can win $G_m(T(K), [\![\neg\sigma]\!])$ on the first move; consequently (II) cannot win that game either. Thus neither $\sigma$ nor $\neg\sigma$ is a theorem of $\Pi_K^{u,m}$.

**§8. Ubiquity and probability.** We now switch from games and determinacy to probability and chance. The themes remain the same; only their interpretations differ.

Let $L$ and $K \subseteq T_\omega$, $K$ an invariant set, be given, and let P be a Borel probability measure on $T_\omega(K)$. We define $\Pi_K^{\mathrm{P}}$ to be $\{\varphi\colon \varphi$ is a sentence from $L$ such that $P([\![\Pi_K \cup \{\varphi\}]\!]) = 1$. Note that $[\![\varphi]\!]$ is always a Borel set, so this definition makes sense. We refer to $\Pi_K^{\mathrm{P}}$ as the P-*companion* of $\Pi_K$, and we write $\mathrm{P}(\Sigma)$ in lieu of $\mathrm{P}([\![\Sigma]\!])$. The P-companion need not bear the faintest resemblance to a companion in the sense of A. Robinson.

**8.1.** Proposition. $\mathrm{P}(\Pi_K^{\mathrm{P}}) = 1$.

Proof. This is immediate, since P is a true measure (hence countably additive) and $\Pi_K$ is countable. □

Recall that P is *positive* if $\mathrm{P}(U) > 0$ for each nonempty open set $U \subseteq T_\omega(K)$, and *continuous* if $\mathrm{P}(\{a\}) = 0$ for all $a \in T_\omega(K)$. We saw in Proposition 1.10 that the branching probability $\mathrm{P}_\mathrm{b}$ is always positive, and continuous when $T_\omega(K)$ is self-dense; and in Example 1.12 that the frequency probability $\mathrm{P}_\mathrm{f}$ need not have either property. Of course, by Proposition 1.13, the two probabilities agree when $T(K)$ is balanced.

**8.2.** Proposition. *Let* P *be a positive probability measure on* $T_\omega(K)$, *and let* $A \models \Pi_K$. *Then $A$ embeds in some model $B$ of* $\Pi_K^{\mathrm{P}}$. *If $A$ is canonical, then $B$ can be chosen to be canonical also.*

Proof. Mimic the proof of Proposition 6.3: Replace $\Pi_K^{\mathrm{u}}$ by $\Pi_K^{\mathrm{P}}$, and "residual" with "measure one." The positivity of P ensures that measure one sets are dense. □

We would like to set down general conditions on $K$ and P so that an analogue of Proposition 6.4 would go through. However, we do not know, except in very special cases, that dense $\Pi_2^0$-sentences are in $\Pi_K^{\mathrm{P}}$. The positivity of P is definitely necessary; $\Pi_K^{\mathrm{P_f}}$ and $\Pi_K^{\mathrm{u}}$ can be in wild disagreement (see Example 9.4). In light of this state of affairs, the following result is rather surprising.

**8.3.** THEOREM. *Let* P *be a positive probability measure on* $T_\omega(K)$, *and assume* $\Pi_K^{\mathrm{P}}$ *is complete. Then* $T(K)$ *satisfies the JEP, and, consequently,* $\Pi_K^{\mathrm{u}}$ *is also complete. Thus the zero-one law for* P *implies the* $\omega$*-strategic zero-one law.*

PROOF. Assume $\Pi_K^{\mathrm{P}}$ is complete, and let $a \in [\![\Pi_K^{\mathrm{P}}]\!]$. For any $t \in T(K)$, $[\![\sigma_t]\!]$ is a nonempty open set. Thus, $\mathrm{P}(\sigma_t) > 0$; hence, $\mathrm{P}(\sigma_t) = 1$. This says that $a \models \sigma_t$ and, therefore, that $t$ embeds in $a$. Accordingly, $T(K)$ satisfies the JEP, by Proposition 5.5, and $\Pi_K^{\mathrm{u}}$ is then complete by Proposition 6.10. □

**8.4.** REMARKS. (i) The two probability measures $\mathrm{P_b}$ and $\mathrm{P_f}$, defined for a frequency stable tree $T$, can be viewed as dual to one another in the following sense. Imagine a Galton board (or pinball machine) with channels in the form of the tree $T$. If $t \in T$ and the tree is vertically mounted with the base at the top, then $\mathrm{W_b}(t)$ is the probability that, when a ball is released and channelled into the base of $T$, it will pass along a path of channels going through $t$. If the base of the tree is now at the bottom, and a released ball is channelled randomly to the topmost channels of $T$, it will pass along a path going through $t$ with probability $\mathrm{W_f}(t)$.

(ii) Another way to view branching probability, as well as other positive probabilities, is via an infinite game of chance: At each node $t \in T$ there is a die, unbiased in the case of $\mathrm{P_b}$, whose faces are in one-to-one correspondence with the members of $\mathrm{sc}(t)$. How one moves up the tree is determined by a roll of the appropriate die. Define $\mathrm{P}(t^{\#})$ to be the probability that, starting at the base node, we get to $t$ by playing this game.

(iii) Frequency probability is more problematic than branching probability, in that its very existence is not assured (see Example 1.11). We know that balanced trees are frequency stable; but we have no other reasonable criteria for deciding when a tree is frequency stable, even when the trees are of the form $T(K)$. It would be interesting to see whether the AP or JEP bears somehow on the issue.

(iv) It is easy to devise positive probabilities which do not satisfy the zero-one law, even though the JEP holds. For instance, let $L = \{R\}$, $R$ unary, and let $K = T_\omega$. The tree $T(K)$ is the infinite binary tree. Fix $b \in T_\omega$, and weight the segments of the branch determined by $b$ according to the sequence $1 - 1/2^2$, $1 - 1/3^2$, $1 - 1/4^2, \ldots$. It is clearly possible to weight the other nodes of $T(K)$ in a positive manner, and the result is that

$$\mathrm{P}(\{b\}) = \prod_{n=2}^{\infty}\left(1 - \frac{1}{n^2}\right) = \prod_{n=2}^{\infty}\frac{(n-1)(n+1)}{n^2} = \frac{1\cdot 3}{2\cdot 2}\cdot\frac{2\cdot 4}{3\cdot 3}\cdot\frac{3\cdot 5}{4\cdot 4}\cdots = \frac{1}{2}.$$

Now arrange for $b$ to be the canonical model of $\forall x \neg Rx$. Then $\exists x Rx$ is a dense $\Sigma_1^0$-sentence whose P-probability is $\frac{1}{2}$ (see also [28]).

Using a variation on the construction in Remark 8.4(iv), we obtain the following codicil to Theorem 8.3.

**8.5.** THEOREM. *Let* $K \subseteq T_\omega$ *be an invariant set whose evolution tree satisfies the JEP. Then there is a positive probability measure* P *on* $T_\omega(K)$ *such that* $\Pi_K^{\mathrm{P}}$ *is complete. Moreover*:

(i) P *may be chosen so that* $\Pi_K^{\mathrm{P}} = \Pi_K^{\mathrm{u}}$.

(ii) *If* $\Pi_K \cup \Sigma_K$ *is an incomplete theory,* P *may be chosen so that* $\Pi_K^{\mathrm{P}}$ *and* $\Pi_K^{\mathrm{u}}$ *are incomparable.*

(iii) *If $T_\omega(K)$ is a self-dense topological space, then we may choose* P *to be continuous in* (i) *and* (ii) *above.*

PROOF. Assume $T_\omega(K)$ is self-dense, and let $b$ be any universal model for $T_\omega(K)$, i.e., a member of $[\![\Pi_K \cup \Sigma_K]\!]$. For $n < \omega$, we let $t_n = b \upharpoonright n$. We seek to construct a continuous probability measure P on $T_\omega(K)$ so that $\mathrm{P}(b^+) = 1$; the construction has two main steps.

*Step* 1. We construct levels $n_1 < n_2 < \cdots$ with sets $B_{n_i} \subseteq T_{n_i}(\omega)$ and isomorphisms $\varphi_t\colon t \to t_{n_i} \subseteq b$ for all $t \in B_{n_i}$, inductively such that if $t \in B_{n_i}$ and $t < t' \in B_{n_{i+1}}$, then the isomorphisms $\varphi_{t'}\colon t' \to t_{n_{i+1}}$ extend $\varphi_t\colon t \to t_{n_i}$, and such that for each $t \in T_{n_i}(\omega)$ there are at least two nodes $t' \in B_{n_{i+1}}$ with $t < t'$. The existence of the isomorphisms $\varphi_t$ follows from the universality of $b$. The self-denseness of $T_\omega(K)$ guarantees the existence of two extensions $t' \cong t_{n_{i+1}}$ for each $t \in T_{n_i}(K)$, provided we choose $n_{i+1}$ large enough.

*Step* 2. For each $t \in T_{n_i}(K)$, let $C_t$ be the set $\{t' \in T_{n_{i+1}}(K)\colon t < t'\}$. By induction on $i$, we may assign positive weights $W(t)$ for all $t \in T_{n_i}(K)$ such that:

(i) $\sum_{t \in T_{n_i}(K)} W(t) = 1$;
(ii) $\sum_{t' \in C_t} W(t') = W(t)$;
(iii) $\sum_{t' \in C_t \cap B_{n_{i+1}}} W(t') \geq W(t) \cdot (1 - 1/i^2)$; and
(iv) $W(t') \leq \frac{1}{2} \cdot W(t)$ for all $t' \in C_t$.

Except for $i = 1$, condition (i) follows from condition (ii). Clearly there is a unique continuous probability measure P on $T_\omega(K)$ such that if $t \in T_n(K)$ then $\mathrm{P}(t^\#) = \sum\{W(t')\colon t \leq t'$ and $t' \in T_{n_i}(K)$, where $i$ is the least such that $n \leq n_i\}$. Let $F_i = \{c \in T_\omega(K)\colon c \upharpoonright n_j \in B_{n_j}$ for all $j > i\}$. Because of the existence of the isomorphisms $\varphi_t$ described above, we have $F_i \subseteq F_{i+1} \subseteq b^+$ for all $i$. Now $\mathrm{P}(F_i) \geq \prod_{j \geq i}(1 - 1/j^2) = 1 - 1/i$. Thus $\mathrm{P}(b^+) = 1$, and hence $\Pi_K^{\mathrm{P}}$ is a complete theory; in fact the theory $\mathrm{Th}(b)$. Since $\Pi_K^{\mathrm{u}}$ is complete and extends $\Pi_K \cup \Sigma_K$, $\Pi_K^{\mathrm{u}} = \mathrm{Th}(c)$ for some $c \in [\![\Pi_K \cup \Sigma_K]\!]$. If we choose $b = c$, then we have arranged matters so that $\Pi_K^{\mathrm{P}} = \Pi_K^{\mathrm{u}}$. If $\Pi_K \cup \Sigma_K$ is incomplete and we choose $b$ so that $b$ and $c$ are not elementarily equivalent, then $\Pi_K^{\mathrm{P}}$ and $\Pi_K^{\mathrm{u}}$ are incomparable.

In the event $T_\omega(K)$ is not self-dense, we may carry out the above construction, except that in Step 1 we are not assured the existence of two extensions $t'$; nor can we be assured of condition (iv) in Step 2. $\square$

An immediate consequence of Proposition 6.10 and Theorems 8.3 and 8.5 is:

**8.6.** THEOREM. *Assume $K \subseteq T_\omega$ is an invariant set. The following are equivalent*:

(i) *$T(K)$ satisfies the JEP.*

(ii) *$\Pi_K^{\mathrm{u}}$ is a complete theory.*

(iii) *There is a positive probability measure, continuous if $T_\omega(K)$ is self-dense, such that $\Pi_K^{\mathrm{P}}$ is complete.* $\square$

We now concentrate on zero-one laws for branching and frequency probabilities. Let $K \subseteq T_\omega$ be an invariant set, and let P be a Borel probability measure on $T_\omega(K)$. We say that P is *finitely symmetric* if, whenever $t_1, t_2 \in T(K)$ and $t_1 \cong t_2$, then $\mathrm{P}(t_1^\#) = \mathrm{P}(t_2^\#)$. The measure $\mathrm{P_f}$ is always finitely symmetric, when defined, but $\mathrm{P_b}$ may fail in this regard; see, e.g., Example 7.4, in which $T(K)$ is unbalanced. It is possible to show that this tree actually is frequency stable. We say that P is *first order symmetric* if, given any formula $\varphi(x_1,\ldots,x_m)$ from $L$, and two sequences

$\mathbf{k}, \mathbf{l} \in \omega^m$ that are compatible, i.e., $k_i = k_j$ if and only if $l_i = l_j$, then the substitution instances $\varphi[\mathbf{k}]$ and $\varphi[\mathbf{l}]$ are equiprobable. Obviously, first order symmetric probability measures are finitely symmetric. The following states the reverse.

**8.7.** LEMMA. *Let $K \subseteq T_\omega$ be an invariant set, and suppose* P *is a finitely symmetric Borel probability measure on $T_\omega(K)$. Then* P *is first order symmetric. Moreover, if $\varphi(\mathbf{x}) \in L(\omega)$ is a formula with $n$ free variables and constants $k_1, \ldots, k_m$, and $A \subseteq \omega$ is an infinite set containing these constants, then* $\mathrm{P}(\forall \mathbf{x}\varphi(\mathbf{x})) = \mathrm{P}(\bigwedge_{\mathbf{k} \in A^n} \varphi[\mathbf{k}])$ *and* $\mathrm{P}(\exists \mathbf{x}\varphi(\mathbf{x})) = \mathrm{P}(\bigvee_{\mathbf{k} \in A^n} \varphi[\mathbf{k}])$.

PROOF. Recall from Proposition 3.1 that for any $L(\omega)$-sentence $\sigma$, $[\![\sigma]\!]$ is Borel in $T_\omega(K)$. Moreover, if $\sigma$ is quantifier-free, then $[\![\sigma]\!]$ is a clopen set. Hence $[\![\sigma]\!] = F^\#$ for some $F \subseteq T_n(K)$. Clearly $n$ may be taken arbitrarily large.

Assume that $\varphi(\mathbf{x})$ and $\mathbf{k}, \mathbf{l} \in \omega^m$ are given as in the definition of first order symmetry. Assume further that $\varphi$ is in prenex normal form. We will induct on the number of alternations of quantifier blocks in the prenex of $\varphi$. If $\varphi$ is quantifier-free, let $n > \max\{k_1, \ldots, k_m, l_1, \ldots, l_m\}$ and let $F, G \subseteq T_n(K)$ be such that $F^\# = [\![\varphi[\mathbf{k}]]\!]$ and $G^\# = [\![\varphi[\mathbf{l}]]\!]$. Let $\pi\colon \omega \to \omega$ be the permutation which interchanges $k_i$ and $l_i$, $1 \le i \le m$, and leaves all else fixed. Let $\bar{\pi}$ be the induced mapping on $\bigcup_{n \le \alpha \le \omega} T_\alpha$. Now $\bar{\pi}(t) \cong t$, and $\bar{\pi}$ takes $F^\#$ to $G^\#$. Thus $\mathrm{P}(F^\#) = \mathrm{P}(G^\#)$. This settles the quantifier-free case.

Now assume $\varphi(\mathbf{x})$ is $\forall \mathbf{y}\psi(\mathbf{x}, \mathbf{y})$, where $\psi$ does not begin with a universal quantifier. A simplifying but inessential restriction is to let $\mathbf{x}$ and $\mathbf{y}$ be single variables $x$ and $y$. Let $\mathbf{k} = k$ and $\mathbf{l} = l$ be given, and let $\pi$ exchange $k$ and $l$ as above. For each $r < \omega$ let $\eta_r(x, y_0, \ldots, y_r)$ be the conjunction $\bigwedge_{i \le r} \psi(x, y_i)$. Then $[\![\eta_0[k, 0]]\!] \supseteq [\![\eta_1[k, 0, 1]]\!] \supseteq \cdots$, and the intersection of the chain is $[\![\varphi[k]]\!]$. Also, each $\eta_r$, when put into its prenex form, has fewer blocks of quantifiers than $\varphi$, so our induction hypothesis applies. The sequences $(k, 0, \ldots, r)$ and $(l, \pi(0), \ldots, \pi(r))$ are compatible; thus

$$\mathrm{P}(\varphi[k]) = \operatorname*{Inf}_{r<\omega} \mathrm{P}(\eta_r[k, 0, \ldots, r]) = \operatorname*{Inf}_{r<\omega} \mathrm{P}(\eta_r[l, \pi(0), \ldots, \pi(r)]) = \mathrm{P}(\varphi[l]).$$

Assume $\sigma \equiv \forall \mathbf{x}\varphi(\mathbf{x})$, where $\varphi(\mathbf{x}) \in L(\omega)$. We may assume $\mathbf{x}$ is the single variable $x$. Let $k_1, \ldots, k_m$ be the constants occurring in $\sigma$, and let $A \subseteq \omega$ be an infinite subset of $\omega$ including the $k_i$. Then $[\![\sigma]\!] = \bigcap_{l<\omega} [\![\bigwedge_{k \le l} \varphi[k]]\!]$. Let $\rho\colon \omega \to A$ be a bijection which fixes each $k_i$. Then by symmetry we have $\mathrm{P}(\bigwedge_{k \le l} \varphi[k)]) = \mathrm{P}(\bigwedge_{k \le l} \varphi[\rho(k)])$. Thus

$$\mathrm{P}(\sigma) = \lim_{l \to \infty} \mathrm{P}\Big(\bigwedge_{k \le l} \varphi[\rho(k)]\Big) = \mathrm{P}\Big(\bigwedge_{k \in A} \varphi[k]\Big).$$

The case for $\sigma \equiv \exists x\varphi(x)$ follows by complementation. □

The next concept we wish to discuss in preparation for a zero-one law theorem is independence. We say that P on $T_\omega(K)$ is *finitely independent* (resp. *first order independent*) if whenever $\sigma$ and $\tau$ are finite conjunctions of atomic sentences from $L(\omega)$ (resp. $\sigma$ and $\tau$ are sentences from $L(\omega)$) having no constants in common, $\mathrm{P}(\sigma \wedge \tau) = \mathrm{P}(\sigma) \cdot \mathrm{P}(\tau)$.

**8.8.** LEMMA. *Let $K \subseteq T_\omega$ be an invariant set, and suppose* P *is a first order independent probability measure on $T_\omega(K)$. Then $\Pi_K^{\mathrm{P}}$ is complete.*

PROOF. Let $\sigma$ be any $L$-sentence. Then $\mathrm{P}(\sigma) = \mathrm{P}(\sigma \wedge \sigma) = \mathrm{P}(\sigma)^2$, by first order independence. Thus $\mathrm{P}(\sigma) = 0$ or $\mathrm{P}(\sigma) = 1$. □

In order to get a workable zero-one law, we need to establish easily verified conditions that ensure independence. One might conjecture that branching probability will have this property when $T(K)$ is balanced. In view of Theorem 8.3, however, the JEP would have to hold for $T(K)$, and it is easy to cook up examples of invariant sets $K$ such that $T(K)$ is balanced but the JEP fails. (The most simple-minded example is to let $L = \{R\}$, $R$ unary, and to take $K = [\![\forall x Rx \vee \forall x \neg Rx]\!]$.)

For any tree $T$, $t \in T$, and $n < \omega$, define $\mathrm{sc}^n(t)$ to be $\{t' \in T_{rk(t)+n}: t \leq t'\}$. For an invariant set $K$, define $T(K)$ to be *strongly balanced* if it is balanced and if, in addition, it satisfies the condition that whenever $\sigma(x_1, \ldots, x_n)$ is a finite conjunction of atomic formulas, $t_1, t_2 \in T_k(K)$, and $(p_1, \ldots, p_n) \in \omega^n$ is such that, for $1 \leq i \leq n$, we have $k \leq p_i < k + n$, then

$$|\{t \in \mathrm{sc}^n(t_1): t \models \sigma[p_1 \cdots p_n]\}| = |\{t \in \mathrm{sc}^n(t_2): t \models \sigma[p_1 \cdots p_n]\}|.$$

**8.9.** Theorem. *Let $K \subseteq T_\omega$ be an invariant set.*

(i) *If $T(K)$ is strongly balanced and* P *is the branching probability* $\mathrm{P_b}$ (*or the frequency probability* $\mathrm{P_f}$, *since* $\mathrm{P_b}$ *and* $\mathrm{P_f}$ *agree on* $T_\omega(K)$), *then* P *is finitely independent (and finitely symmetric).*

(ii) *If* P *is a finitely symmetric probability measure on $T_\omega(K)$ which is finitely independent, then* P *is first order independent; hence $\Pi_K^{\mathrm{P}}$ is complete.*

Proof. (i) Assume $\sigma$ and $\tau$ are conjunctions of atomic sentences, mentioning at most the constants $k_1, \ldots, k_m$ and $l_1, \ldots, l_m$ respectively, where no $k_i$ is an $l_j$, $1 \leq i$, $j \leq m$. Since P is finitely symmetric, we can invoke Lemma 8.7 and assume further that $0 \leq k_i < m \leq l_j < 2m$ for $1 \leq i, j \leq m$. Let $r = |\{t \in T_m(K): t \models \sigma\}|$ and $s = |\{t' \in \mathrm{sc}^m(t): t' \models \tau\}|$ for any $t \in T_m(K)$, invariants of $t \in T_m(K)$ by strong balance. Since $T(K)$ is a balanced tree, we have

$$\mathrm{P}(\sigma) = r/|T_m(K)| \quad \text{and} \quad \mathrm{P}(\tau) = s \cdot |T_m(K)|/|T_{2m}(K)|.$$

But also

$$\mathrm{P}(\sigma \wedge \tau) = \frac{r \cdot s}{|T_{2m}(K)|} = \mathrm{P}(\sigma) \cdot \mathrm{P}(\tau).$$

Assume P is finitely symmetric and finitely independent. We show first that if $\sigma$ and $\tau$ are finite conjunctions of atomic and negated atomic sentences, having no constants in common, then $\mathrm{P}(\sigma \wedge \tau) = \mathrm{P}(\sigma) \cdot \mathrm{P}(\tau)$. Induct on the number of negation symbols occurring in the conjunction. If no negations occur in either $\sigma$ or $\tau$, we have our original hypothesis. Assume $\sigma$ or $\tau$ contains negations, say $\sigma \equiv \neg\alpha \wedge \sigma_1$ for some atomic $\alpha$. By induction,

$$\mathrm{P}(\alpha \wedge \sigma_1 \wedge \tau) = \mathrm{P}(\alpha \wedge \sigma_1) \cdot \mathrm{P}(\tau).$$

Then

$$\begin{aligned}\mathrm{P}(\sigma \wedge \tau) &= \mathrm{P}(\sigma_1 \wedge \tau) - \mathrm{P}(\alpha \wedge \sigma_1 \wedge \tau) = \mathrm{P}(\sigma_1) \cdot \mathrm{P}(\tau) - \mathrm{P}(\alpha \wedge \sigma_1) \cdot \mathrm{P}(\tau) \\ &= \mathrm{P}(\neg\alpha \wedge \sigma_1)\mathrm{P}(\tau) = \mathrm{P}(\sigma)\mathrm{P}(\tau).\end{aligned}$$

Next we prove the main assertion for $\sigma$ and $\tau$ quantifier-free. Assume $\sigma$ and $\tau$ are in disjunctive normal form, $\sigma \equiv \sigma_1 \vee \cdots \vee \sigma_m$, $\tau \equiv \tau_1 \vee \cdots \vee \tau_n$, where each disjunct is a conjunction of atomic and negated atomic sentences. We can further arrange matters so that $[\![\sigma_i]\!] \cap [\![\sigma_j]\!] = \varnothing = [\![\tau_i]\!] \cap [\![\tau_j]\!]$, $i \neq j$. For $1 \leq i \leq m$ and $1 \leq j \leq n$, let $p_i = \mathrm{P}(\sigma_i)$ and $q_j = \mathrm{P}(\tau_j)$. Then whenever $(i,j) \neq (k,l)$ we have

$[\![\sigma_i \wedge \tau_j]\!] \cap [\![\sigma_k \wedge \tau_l]\!] = \varnothing$. Thus

$$\begin{aligned} P(\sigma \wedge \tau) &= P(\bigvee\{\sigma_i \wedge \tau_j\colon 1 \leq i \leq m, 1 \leq j \leq n\}) \\ &= \sum\{P(\sigma_i \wedge \tau_j)\colon 1 \leq i \leq m, 1 \leq j \leq n\} \\ &= \sum\{p_i \cdot q_j\colon 1 \leq i \leq m, 1 \leq j \leq n\} \\ &= \left(\sum_{i=1}^{m} p_i\right) \cdot \left(\sum_{j=1}^{n} q_j\right) = P(\sigma) \cdot P(\tau). \end{aligned}$$

Finally we assume $\sigma$ and $\tau$ in prenex normal form, and induct on the sum of the number of quantifier alternations in $\sigma$ and $\tau$. So assume $\sigma$ or $\tau$ has quantifiers, say $\sigma$ is $\forall \mathbf{x}\varphi$, and $\varphi$ does not begin with a universal quantifier. As in the proof of Lemma 8.7, we can assume $\mathbf{x}$ is the single variable $x$. Let $\{k_1, \ldots, k_m\}$ and $\mathbf{l} = \{l_1, \ldots, l_n\}$ be the constants occurring in $\sigma$ and $\tau$ respectively, where no $k_i$ is an $l_j$. Let $A = \omega \backslash \mathbf{l}$, and let $\pi\colon \omega \to A$ be a bijection which fixes each $k_i$. By Lemma 8.7,

$$P(\sigma \wedge \tau) = P(\forall x\varphi \wedge \tau) = P\left(\bigwedge_{k\in A} \varphi[k] \wedge \tau\right) = \lim_{n\to\infty} P\left(\bigwedge_{k\leq n} \varphi[\pi(k)] \wedge \tau\right).$$

By our inductive hypothesis, the term on the right is

$$\lim_{n\to\infty} P\left(\bigwedge_{k\leq n} \varphi[\pi(k)]\right) \cdot P(\tau).$$

By Lemma 8.7, this is $P(\forall x\varphi(x)) \cdot P(\tau)$, whence $P(\sigma \wedge \tau) = P(\sigma) \cdot P(\tau)$. The argument above may be dualized to handle the case when $\sigma$ is $\exists \mathbf{x}\varphi$. Invoking Lemma 8.8 finishes the proof. $\square$

**8.10.** REMARKS. (i) The second inductive argument in Theorem 8.9 is similar to the one used by H. Gaifman in §5 of [7]. However, our setting is essentially different from his.

(ii) Theorem 8.9 can be applied directly to Examples 7.1–7.3, but not to Example 7.4 because the evolution tree is not balanced. Although the AP holds in all applications of Theorem 8.9 that we know of, we do not know whether the AP for $T(K)$ necessarily follows from strong balance as does the JEP.

(iii) In the definition of "strongly balanced", one might wonder whether the formula $\sigma(x_1, \ldots, x_n)$ could be taken simply to be atomic. This weaker property does not even imply the JEP, as the following example shows. Let $L = \{R, S\}$ consist of two unary predicates, and let $K = [\![\forall x(Rx \leftrightarrow Sx) \vee \forall x(Rx \leftrightarrow \neg Sx)]\!]$. One easily verifies that $T(K)$ is "strongly balanced" in the weaker sense, but that the JEP fails. Thus, by Theorem 8.6, there is no positive probability P for which $\Pi_K^{\mathrm{P}}$ is complete.

(iv) One ploy for proving a zero-one law for P is to show that $\Pi_K^{\mathrm{P}}$ contains a known complete theory. For example, if $T(K)$ satisfies the AP and $\rho_0 = [\![\Pi_K \cup \Sigma_K \cup \Gamma_K]\!]$, one might try to prove $P(\rho_0) = 1$. While this seems to be a good general approach, it is very hard to implement except in special cases. In [7], Gaifman does this for our Example 7.1: A zero-one law for measure $m^*$ (defined in quite a different manner from the probabilities considered here) is established; then it is asserted that each Rabin-Scott sentence has measure one. R. Fagin [6] does something similar. The proof of his Theorem 2, although concerned with asymptotic limits of probabilities for finite relational structures, can be easily adapted to show that $\mathrm{P}(\rho_0) = 1$ in certain cases. What one needs is symmetry, plus a way of dealing with $\mathrm{P}(\sigma \wedge \gamma)$ when $\sigma$ and $\gamma$ are $L(\omega)$-sentences with some constants in common. These conditions are met, for instance, when $\mathrm{P} = \mathrm{P_b}$ and $K = T_\omega$ (our Example 7.1), or when $\mathrm{P} = \mathrm{P_b}$ and $K$ is the invariant set of graphs (our Example 7.3). Fagin's approach can still be used in the case of branching probability and linear orderings, Example 7.2, but some care must be taken. His proof, as is, must fail because the theory of $\eta_0$ is finitely axiomatizable. If his proof went through without modification, there would have to be a finite dense linear ordering. We will prove $\mathrm{P_b}(\eta_0) = 1$ directly later on.

A case in which no approach we have seen can be applied is Example 7.4. Symmetry and independence, two cornerstones of the approach, fail decisively here for $\mathrm{P} = \mathrm{P_b}$, and it is not even clear whether the evolution tree in this case is frequency stable. We will deal with this case, too, in the sequel.

§9. **More examples and remarks.** In this section we go over those examples in §7 for which we can make some definitive statements from the probabilistic point of view. We have nothing worth mentioning about the examples beyond Example 7.6. The following numberings parallel those in §7.

**9.1.** EXAMPLE. $L$ is arbitrary, and $K = T_\omega$. Then, by work of Gaifman and Fagin, $\mathrm{P}(\iota_0(L)) = 1$, where P is either $\mathrm{P_b}$ or $\mathrm{P_f}$. Thus $\Pi^{\mathrm{P}}_K$ is axiomatized by the $\aleph_0$-categorical theory $\Sigma_K \cup \Gamma_K$. This justifies calling the members of $\iota_0(L)$ *random L-structures.*

**9.2.** EXAMPLE. $L$ consists of one binary relation, $K$ is the canonical linear orderings, $\mathrm{P} = \mathrm{P_b} = \mathrm{P_f}$, and $\eta_0$ is the order type of the rational line. Then $\mathrm{P}(\eta_0) = 1$; hence $\Pi^{\mathrm{P}}_K$ is the theory of dense linear orderings without endpoints, the theory of random linear orderings.

PROOF. Let $t \in T_n(K)$. Then $|\mathrm{sc}(t)| = n + 1$. Since $|T_n| = n!$, we have, by Proposition 1.13, $\mathrm{W_b}(t) = 1/n!$. Now, for each $n < \omega$, let $U_n = \{r \in T_\omega(K)\colon m <^r n \text{ for some } m\}$, where $<^r$ is the interpretation of "less than" in $r$. Letting $K_1$ be the set of linear orderings with no left endpoint, we see that $K_1 = \bigcap_{n<\omega} U_n$. So we show that $\mathrm{P_b}(K_1) = 1$ by showing that $\mathrm{P_b}(U_n) = 1$ for each $n < \omega$. Let $k \geq 1$ and let $U_{n,k} = \{t \in T_{n+k}(K)\colon m <^t n \text{ for some } m\}$. Then $U^{\#}_{n,1} \subseteq U^{\#}_{n,2} \subseteq \cdots$ and $U_n = \bigcup_{k\geq 1} U^{\#}_{n,k}$. Now

$$\mathrm{P_b}(U^{\#}_{n,k}) = \mathrm{W_b}(U_{n,k}) = \frac{(n+k-1)! \cdot (n+k-1)}{(n+k)!} = \frac{n+k-1}{n+k}.$$

Thus $\mathrm{P_b}(U_n) = \sup_{k\geq 1} \mathrm{W_b}(U_{n,k}) = 1$. Similarly we show that $\mathrm{P_b}(K_2) = \mathrm{P_b}(K_3) = 1$, where $K_2$ is the set of linear orderings with no right endpoint and $K_3$ is all dense orderings. Since $\eta_0 = K_1 \cap K_2 \cap K_3$, we have $\mathrm{P_b}(\eta_0) = 1$. □

**9.3.** EXAMPLE. $L$ consists of one binary relation, $K$ is the canonical graphs, $\mathrm{P} = \mathrm{P_b} = \mathrm{P_f}$, and $\rho_0$ is the isomorphism type of the random graph. Then, by adapting methods of Fagin [6] (see our Remark 8.10(iii)), $\mathrm{P}(\rho_0) = 1$.

**9.4.** EXAMPLE. $L$ consists of one binary relation, $K$ is the canonical equivalence relations, $\varepsilon_0$ is the isomorphism type of the totally infinite equivalence relation, and $q_= \in T_\omega(K)$ is the equivalence relation in which equivalence means equality. Then $\mathrm{P_b}(\varepsilon_0) = 1$ and $\mathrm{P_f}(\{q_=\}) = 1$. Thus both $\Pi^{\mathrm{P_b}}_K$ and $\Pi^{\mathrm{P_f}}_K$, entirely different theories, are $\aleph_0$-categorical, and hence complete.

PROOF. Note at the outset that $T(K)$ is indeed an unbalanced tree: $|\mathrm{sc}(t)| = 1 +$ (the number of equivalence classes of $t$). Thus it is not immediately clear that $\mathrm{P_f}$ is even well-defined on $T_\omega(K)$. Even after we have shown that it is, we know it cannot agree with $\mathrm{P_b}$, because $\mathrm{W_f}(t)$ is an invariant of the isomorphism type of $t$. This is manifestly untrue for $\mathrm{W_b}(t)$.

To prove $\mathrm{P_b}(\varepsilon_0) = 1$, let $m, n, p < \omega$ be given, and define $U^p_{m,n} = \{b \in T_\omega(K)$: $|[p]^b| \geq m$ and $b$ has at least $n$ equivalence classes$\}$, where $[p]^b$ denotes the $b$-equivalence class containing $p$. Clearly, $\varepsilon_0 = \bigcap_{m,n,p<\omega} U^p_{m,n}$; so we must show $\mathrm{P_b}(U^p_{m,n})$ is always 1. For each $k < \omega$, define $U^p_{m,n,k} = \{t \in T_k(K)$: $|[p]^t| \geq m$ and $t$ has at least $n$ equivalence classes$\}$. Then we have $U^p_{m,n}$ as the chain union of $(U^p_{m,n,0})^\#$ $\subseteq (U^p_{m,n,1})^\# \subseteq \cdots$. Thus we have only to show that $\lim_{k\to\infty} \mathrm{W_b}(U^p_{m,n,k}) = 1$. To this end, define $A^p_{m,n,k} = \{t \in T_k(K)$: $|[p]^t| = m$ and $t$ has exactly $n$ equivalence classes$\}$. Then

$$T_k(K) \backslash U^p_{m,n,k} = \bigcup_{i=0}^{m-1} \bigcup_{j=0}^{n-1} A^p_{i,j,k},$$

a disjoint union. Thus $\mathrm{W_b}(U^p_{m,n,k}) = 1 - \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \mathrm{W_b}(A^p_{i,j,k})$, so it suffices to show that each summand on the right tends to 0 as $k$ gets large. To simplify matters, let $E_{n,k} = \{t \in T_k(K)$: $t$ has exactly $n$ equivalence classes$\}$. Then $A^p_{m,n,k} \subseteq E_{n,k}$, so we are done if we can show that $\lim_{k\to\infty} \mathrm{W_b}(E_{n,k}) = 0$ for each $n$. Let $e_{n,k} = \mathrm{W_b}(E_{n,k})$. We induct on $n \geq 1$. Clearly, $e_{1,k} = 1/2^{k-1}$, so the assertion is true for $n = 1$.

For the sake of induction, assume $\lim_{k\to\infty} e_{n-1,k} = 0$. Now $E_{n,k+1}$ has two kinds of elements: Either $t' \in E_{n,k+1}$ is a successor of some $t \in E_{n,k}$ (with relative probability $n/(n+1)$); or $t'$ is a successor of some $t \in E_{n-1,k}$ (with relative probability $1/n$). Thus

$$e_{n,k+1} = \frac{n}{n+1} \cdot e_{n,k} + \frac{1}{n} \cdot e_{n-1,k}.$$

So pick $\varepsilon > 0$, and let $k$ be large enough so that $e_{n-1,k+l} < \varepsilon$ for all $l \geq 0$. Then

$$e_{n,k+1} < \frac{n}{n+1} \cdot e_{n,k} + \frac{1}{n} \cdot \varepsilon,$$

and an easy induction on $l$ reveals that

$$e_{n,k+l} < \left(\frac{n}{n+1}\right)^l e_{n,k} + \left(\frac{1}{n} + \sum_{i=1}^{l-1} \frac{n^{i-1}}{(n+1)^i}\right) \cdot \varepsilon.$$

The coefficient of $\varepsilon$ is essentially a geometric series, and is therefore bounded above by some $N$ (depending only on $n$). Thus $\limsup_{l\to\infty} e_{n,k+l} \leq N\varepsilon$. Since $\varepsilon$ can be chosen arbitrarily small, we have $\lim_{k\to\infty} e_{n,k} = 0$, as desired.

To handle frequency probability, it clearly suffices to show that, for any $t \in E_{n,k}$,

$$\mathrm{W_f}(t) = \begin{cases} 1 & \text{if} \quad n = k, \\ 0 & \text{if} \quad n < k. \end{cases}$$

This implies immediately that $\mathrm{P_f}(\{q_=\}) = 1$.

Suppose $t \in E_{n,k}$ and $t' \in \mathrm{sc}(t) \cap E_{n,k+1}$. Then, letting $\mathrm{sc}^l(t) = \{u \in T_{k+l}(K): t \leq u\}$, we have $|\mathrm{sc}^l(t)| = |\mathrm{sc}^l(t')|$ since the subtree emanating from $t$ is isomorphic to the subtree emanating from $t'$: How many immediate successors a node has depends only on the number of equivalence classes. Now,

$$F_{k+l}(t) = \frac{|\mathrm{sc}^l(t)|}{|T_{k+l}(K)|}$$

and

$$F_{k+1+l}(t') = \frac{|\mathrm{sc}^l(t')|}{|T_{k+1+l}(K)|} = F_{k+l}(t) \cdot \frac{|T_{k+l}(K)|}{|T_{k+1+l}(K)|} \leq \frac{|T_{k+l}(K)|}{|T_{k+1+l}(K)|}.$$

Assume, for the moment, that the fraction on the right goes to zero as $l$ gets large. Then $\mathrm{W_f}(t') = 0$. Suppose $t \in E_{n,k}$ with $n < k$. Then there are $t_1 < t_2 \leq t$ with $t_2 \in \mathrm{sc}(t_1)$, and such that $t_1$ and $t_2$ have the same number of equivalence classes. By the argument above (with our momentary assumption still in effect), $\mathrm{W_f}(t_2) = 0$; hence $\mathrm{W_f}(t) = 0$. On the other hand, if $t \in E_{n,n}$ then for each $t' \in T_n$ different from $t$, $t'$ has fewer than $n$ equivalence classes. Consequently $\mathrm{W_f}(t') = 0$ and thus $\mathrm{W_f}(t) = 1$. We are done, therefore, once we have proved that

$$\lim_{k \to \infty} \frac{|T_k(K)|}{|T_{k+1}(K)|} = 0.$$

We are grateful to Michael Slattery for providing the following proof [21].

Set $S_k^n = |E_{n,k}|$ and $B_k = |T_k(K)|$. The numbers $S_k^n$ are the so-called "Stirling numbers of the second kind," and $B_k$ is the $k$th "Bell number" (see [2]). We now prove the following.

Lemma (M. Slattery [21]). $\lim_{k \to \infty} B_k / B_{k+1} = 0$.

Proof of the Lemma. We first show that if $n^2 + n \leq k$, then $S_k^n \leq S_k^{n+1}$. To see this, let $E_0 = E_{n,k}$ and let $E_1$ consist of those $t \in E_{n+1,k}$ such that at least one equivalence class is a singleton. Let $G$ be the graph whose vertices are elements of $E_0 \cup E_1$ and whose edges join $t_0$ and $t_1$ just in case $t_0 \in E_0$, $t_1 \in E_1$, and $t_0$ can be obtained from $t_1$ by taking an element in an equivalence class that is a singleton, and making it equivalent to some other element. Let $e$ be the number of edges of $G$. We can get lower and upper estimates on $e$ as follows. On the one hand, if $t_0 \in E_0$ then the number of edges incident to $t_0$ is at least $k - n$. (These are "extra" elements and can be used in the making of new singleton equivalence classes.) Thus $(k - n) \cdot |E_0| \leq e$. On the other hand, if $t_1 \in E_1$ then the number of edges incident at $t_1$ is at its greatest when all equivalence classes of $t_1$ are singletons. In any event, this number cannot exceed $n^2$. Thus, $e \leq |E_1| \cdot n^2$, whence

$$\frac{k - n}{n^2} \cdot |E_0| \leq |E_1|.$$

Since $n^2 + n \leq k$, we have $|E_0| \leq |E_1|$.

Now, fix $m \geq 1$ and assume $k \geq (2m)^2 + 2m$. Since $B_k = \sum_{n=1}^k S_k^n$ and each member of $E_{n,k}$ has $n + 1$ immediate successors, we have $B_{k+1} = \sum_{n=1}^k (n + 1) \cdot S_k^n$.

Thus

$$\begin{aligned}\frac{B_k}{B_{k+1}} &= \frac{\sum_{n=1}^{m} S_k^n}{\sum_{n=1}^{k}(n+1)\cdot S_k^n} + \frac{\sum_{n=m+1}^{k} S_k^n}{\sum_{n=1}^{k}(n+1)\cdot S_k^n}\\ &\le \frac{\sum_{n=1}^{m} S_k^n}{\sum_{n=m+1}^{2m}(n+1)\cdot S_k^n} + \frac{\sum_{n=m+1}^{k} S_k^n}{\sum_{n=m+1}^{k}(n+1)\cdot S_k^n}\\ &\le \frac{\sum_{n=1}^{m} S_k^n}{(m+2)\cdot\sum_{n=m+1}^{2m} S_k^n} + \frac{\sum_{n=m+1}^{k} S_k^n}{(m+2)\cdot\sum_{n=m+1}^{k} S_k^n}.\end{aligned}$$

By the remarks above, $S_k^1 \le \cdots \le S_k^{2m}$, so $B_k/B_{k+1} \le 1/(m+2) + 1/(m+2) = 2/(m+2)$. Hence,

$$\limsup_{k\to\infty} \frac{B_k}{B_{k+1}} \le \frac{2}{m+2}$$

for all $m \ge 1$, and the proof of the lemma, and of the assertion that $\mathrm{P_f}(\{q_=\}) = 1$, is complete. $\square$

**9.5.** Example. $L$ consists of one binary relation, $K$ is the canonical partial injections, $\beta_0$ is the isomorphism type of the canonical total bijections in which there are no infinite orbits and in which there are infinitely many orbits of each finite positive length, and $p_\perp \in T_\omega(K)$ is the totally undefined partial injection. We would like to be able to report that $\mathrm{P_b}(\beta_0) = 1$ and $\mathrm{P_f}(\{p_\perp\}) = 1$. However, we are able to offer no more than a small amount of evidence in support of the first assertion. Although we do not even know the value of $\mathrm{P_b}(\exists x Rxx)$, we can show, at least, that

$$\mathrm{P_b}(\forall x \exists y Rxy \wedge \forall x \exists y Ryx) = 1.$$

Proof. First note that if $t \in T_n(K)$, then $|\mathrm{sc}(t)|$ depends on the number $k = k(t)$ of elements not in the domain of $R^t$; that is, $|\{m < n\colon R^t m \perp\}|$, where $R^A m \perp$ means that $R^A mp$ for no $p$ in the domain of $A$. Now, there is only one $t' \in \mathrm{sc}(t)$ for which $Rnn$ is true; and if $Rnx$ is true for one of the $k+1$ possible values of $x \ne n$ (including $\perp$) then at most one $m < n$, not in the domain of $R^t$, can be assigned the value $n$ in $t'$. This can happen in $k+1$ ways; hence $|\mathrm{sc}(t)| = (k(t)+1)^2 + 1$.

Suppose $\sigma$ is $\forall x \exists y Rxy$. For each $m < \omega$, define $U_m = \{a \in K\colon R^a m \perp\}$. Then $[\![\neg\sigma]\!] = \bigcup_{m<\omega} U_m$; so it suffices to show that $\mathrm{P_b}(U_m) = 0$. For each $n > m$ ($m$ fixed), let $V_{n,m} = \{t \in T_n(K)\colon R^t m \perp\}$. Then $U_m = \bigcap_{m<n<\omega} V_{n,m}^{\#}$, a decreasing intersection. If $t \in V_{n,m}$, then exactly $k(t)+1$ immediate successors of $t$ satisfy $Rmn$, so we get the branching weight

$$\begin{aligned}\mathrm{W_b}(V_{n+1,m} \cap \mathrm{sc}(t)) &= \left(1 - \frac{k+1}{(k+1)^2+1}\right)\cdot \mathrm{W_b}(t)\\ &\le \left(1 - \frac{n+1}{(n+1)^2+1}\right)\cdot \mathrm{W_b}(t) \le \left(1 - \frac{1}{n+2}\right)\cdot \mathrm{W_b}(t) = \frac{n+1}{n+2}\cdot \mathrm{W_b}(t).\end{aligned}$$

Thus

$$\mathrm{W_b}(V_{n+1,m}) \le \frac{n+1}{n+2}\cdot \mathrm{W_b}(V_{n,m});$$

so for $l \geq 1$ we see that

$$\mathrm{W_b}(V_{n+l,m}) \leq \frac{n+l}{n+l+1} \cdot \frac{n+l-1}{n+l} \cdots \frac{n+1}{n+2} \cdot \mathrm{W_b}(V_{n,m}) = \frac{n+1}{n+l+1} \cdot \mathrm{W_b}(V_{n,m}).$$

Thus, $\mathrm{P_b}(U_m) = \mathrm{Inf}_{m<n<\omega} \mathrm{W_b}(V_{n,m}) = 0$. Therefore, $\mathrm{P_b}(\sigma) = 1$. Similarly, we obtain $\mathrm{P_b}(\forall x \exists y Ryx) = 1$. □

**9.6.** EXAMPLE. $L$ is arbitrary, $K$ is the invariant set of total algebras, and $a_\perp$ is the totally undefined canonical partial algebra. Then $\mathrm{P_b}(K) = 1$ and $\mathrm{P_f}(\{a_\perp\}) = 1$. Thus, $\Pi_K^{\mathrm{P_f}}$ is complete.

PROOF. Let $L = \{R_1, \ldots, R_l\}$, where $R_i$ is $(n_i + 1)$-ary. Let $\mathbf{k} \in \omega^{n_i}$, and set $U_{i,\mathbf{k}} = \{a \in T_\omega(K)\colon R_i^a \mathbf{k} \perp\}$. Since $T_\omega(K) \backslash K = \bigcup \{U_{i,\mathbf{k}}\colon 1 \leq i \leq l$ and $\mathbf{k}$ is an $n_i$-tuple from $\omega\}$, it suffices to prove that $\mathrm{P_b}(U_{i,\mathbf{k}}) = 0$. For each $n < \omega$, let $V_n = \{t \in T_n(K)\colon R_i^t \mathbf{k} \perp\}$ (if $n \leq \max\{k_1, \ldots, k_{n_i}\}$, set $V_n = T_n(K)$). Assuming $n$ large enough, if $t \in V_n$ and $t' \in \mathrm{sc}(t)$ then, because $t'$ extends $t$, either $R_i^{t'} \mathbf{k} \perp$ or $R_i^{t'} \mathbf{k} n$. Since each outcome occurs exactly half of the time, we have that $\mathrm{W_b}(V_{n+1} \cap \mathrm{sc}(t)) = \frac{1}{2} \cdot \mathrm{W_b}(t)$; hence $\mathrm{W_b}(V_{n+1}) = \frac{1}{2} \cdot \mathrm{W_b}(V_n)$. Now $U_{i,\mathbf{k}} \subseteq V_n^\#$ for each $n < \omega$. From this it is immediate that $\mathrm{P_b}(U_{i,\mathbf{k}}) = 0$, and we infer that $K$ is of $\mathrm{P_b}$-measure one.

Now for simplicity let $L = \{R\}$, where $R$ is $(m + 1)$-ary, and let $t \in T_n(K)$. We first compute $F_{n+k}(t)$ for each $k \geq 1$. The denominator of this fraction is just $|T_{n+k}(K)| = (n + k + 1)^{((n+k)^m)}$. The numerator depends also on the number $0 \leq x \leq n^m$ of $m$-tuples $\mathbf{I}$ from $\{0, \ldots, n-1\}$ such that $R^t \mathbf{I} \perp$, and is easily seen to be $(n+k+1)^{((n+k)^m - n^m)} \cdot (k + 1)^x$, so we have

$$F_{n+k}(t) = \frac{(k+1)^x}{(n+k+1)^{(n^m)}} = \frac{(k+1)^{(n^m)}}{(n+k+1)^{(n^m)} \cdot (k+1)^{(n^m - x)}};$$

whence

$$\mathrm{W_f}(t) = \begin{cases} 0 & \text{if } x < n^m, \\ 1 & \text{if } x = n^m. \end{cases}$$

This assertion easily extends to arbitrary finite $L$; so the measure $\mathrm{P_f}$ is concentrated at the totally undefined partial algebra $a_\perp$. □

**9.7.** REMARKS. (i) The reason we conjecture that $\mathrm{P_f}(\{p_\perp\}) = 1$ in Example 9.5 is that the analogous statement in Example 9.6 is true. The combinatorics in the latter case, however, are much more manageable.

(ii) In Example 9.6, if $L$ contains two or more unary predicates, then $T(K)$ fails to satisfy the JEP, whence $\Pi_K^{\mathrm{P}}$ is incomplete for any positive P, by Theorem 8.3. Since $\Pi_K^{\mathrm{P_f}}$ is complete, the positivity assumption is essential. The only case in which we know $\Pi_K^{\mathrm{P_b}}$ to be complete is where $L$ consists of exactly one unary predicate ($T(K)$ as depicted in Example 1.12).

(iii) Of course, in Example 7.13, the JEP fails; so $\Pi_K^{\mathrm{P}}$ can never be complete for P positive.

**§10. A note on asymptotic relative frequencies.** Let us now take a brief look at how the probability measures $\mathrm{P_b}$ and $\mathrm{P_f}$ relate to asymptotic relative frequencies. Given an invariant set $K \subseteq T_\omega$, a sentence $\sigma$ (over a suitable language with symbols from $L$), and a number $n$, let

$$\mu_n(\sigma, K) = \frac{|\{t \in T_n(K): t \models \sigma\}|}{|T_n(K)|}.$$

This well-known notion of relative frequency goes back at least as far as R. Carnap in the 1950's (see [6]). It has also been used in the asymptotic theory of random graphs as well as in higher order logic (see [12]). Now let

$$\mu^+(\sigma, K) = \limsup_{n \to \infty} \mu_n(\sigma, K) \quad \text{and} \quad \mu(\sigma, K) = \lim_{n \to \infty} \mu_n(\sigma, K)$$

(when it exists). The main result of R. Fagin [6] is the following.

**10.1.** THEOREM (FAGIN [6]). *Let $K = T_\omega$, and let $\sigma$ be a sentence from $L_{\omega\omega}$. Then $\mu(\sigma, K)$ always exists and is either* 0 *or* 1. $\square$

**10.2.** EXAMPLE. An invariant set $K$ and a $\boldsymbol{\Pi}_2^0$-sentence $\sigma$ such that $\mu(\sigma, K)$, $\mathrm{P_b}(\sigma)$, and $\mathrm{P_f}(\sigma)$ are all distinct.

*Construction.* Let $L = \{R\}$, where $R$ is $(m+1)$-ary, and let $K$ be the canonical total $L$-algebras. Then $K = [\![\sigma]\!]$ for a $\boldsymbol{\Pi}_2^0$-sentence $\sigma$, and we saw in Example 9.6 that $\mathrm{P_b}(\sigma) = 1$ and $\mathrm{P_f}(\sigma) = 0$. For each $n < \omega$, we have

$$\mu_n(\sigma, K) = \frac{n^{(n^m)}}{(n+1)^{(n^m)}} = \frac{1}{(1 + 1/n)^{(n^m)}}.$$

Thus,

$$\mu(\sigma, K) = \begin{cases} 1 & \text{if } m = 0, \\ 1/e & \text{if } m = 1, \\ 0 & \text{if } m > 1. \quad \square \end{cases}$$

The inevitable question, at this point, is: Can $\mu^+(\sigma, K)$ ever influence $\mathrm{P_b}(\sigma)$ or $\mathrm{P_f}(\sigma)$?

**10.3.** THEOREM. *Assume $K \subseteq T_\omega$ is an invariant set such that $T(K)$ is balanced. If $\sigma$ is any $\boldsymbol{\Pi}_2'^0$-sentence over $L$ and $\mu^+(\sigma, K) = 1$, then* $\mathrm{P_b}(\sigma) = 1$.

PROOF. Assume first that $\sigma$ is of the form $\forall x_1 \cdots x_m \bigvee_{k<\omega} \varphi_k$, where each $\varphi_k$ is of the form $\exists y_1 \cdots y_{m_k} \psi_k(x_1, \ldots, x_m, y_1, \ldots, y_{m_k})$, and each $\psi_k$ is quantifier-free. For each $m$-tuple $\mathbf{n} = (n_1, \ldots, n_m) \in \omega^m$, let

$$U_{\mathbf{n}} = \left[\!\!\left[ \bigvee_{k<\omega} \exists y_1 \cdots y_{m_k} \psi_k[\mathbf{n}](y_1, \ldots, y_{m_k}) \right]\!\!\right].$$

It suffices to prove $\mathrm{P_b}(U_{\mathbf{n}}) = 1$, since $[\![\sigma]\!] = \bigcap_{\mathbf{n} \in \omega^m} U_{\mathbf{n}}$. Now, for each $l < \omega$, let

$$U_{l,\mathbf{n}} = \left\{ t \in T_l(K): t \models \bigvee_{k<\omega} \exists y_1 \cdots y_{m_k} \psi_k[\mathbf{n}](y_1, \ldots, y_{m_k}) \right\}.$$

Clearly, $U_{\mathbf{n}} = \bigcup_{l<n} U_{l,\mathbf{n}}^\#$, a chain union. But $\mathrm{P_b}(U_{l,\mathbf{n}}^\#) = \mathrm{W_b}(U_{l,\mathbf{n}}) = |U_{l,\mathbf{n}}|/|T_l(K)|$, by Proposition 1.13. Thus,

$$\mathrm{P_b}(U_{l,\mathbf{n}}^\#) \geq \frac{|\{t \in T_l(K): t \models \sigma\}|}{|T_l(K)|} = \mu_l(\sigma, K),$$

whence $\mathrm{P_b}(U_n) \geq \limsup_{l \to \infty} \mu_l(\sigma, K) = 1$, as desired.

Now, if $\sigma$ is a general $\Pi_2'^0$-sentence, i.e. of the form $\bigwedge_{k<\omega}\sigma_k$ where each $\sigma_k$ is as above, suppose $\mu^+(\sigma, K) = 1$. Then clearly $\mu^+(\sigma_k, K) = 1$ for each $k < \omega$; so, as we have just seen, $\mathrm{P_b}(\sigma_k) = 1$ for each $k < \omega$; hence, $\mathrm{P_b}(\sigma) = 1$. □

An immediate corollary of Theorems 10.1 and 10.3 is the following.

**10.4.** COROLLARY. *Let $\sigma$ be a $\Pi_2^0$-sentence from L, and assume $K = T_\omega$. Then either $\mu^+(\sigma, K) = 0$ or $\mathrm{P_b}(\sigma) = 1$.* □

**10.5.** REMARKS. (i) Let $K$ be the canonical linear orderings, and let $\sigma$ be the $\Pi_2^0$-definition of $\eta_0$, the order type of the rational line. Then $\mu_l(\sigma, K) = 0$ for each $l < \omega$, so $\mu(\sigma, K) = 0$. By Example 9.2, the conclusion of Theorem 10.3 still obtains, so it is too much to hope for a converse.

(ii) Let $K$ be the canonical linear orderings again, but let $\sigma$ now be the $\Sigma_2^0$-sentence which says that a linear ordering has a maximal element. Then, $\mu(\sigma, K) = 1$, but $\mathrm{P_b}(\sigma) = 0$. So the syntactic form of $\sigma$ is important in Theorem 10.3.

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