Penultimate draft of the paper forthcoming in the *Journal of Philosophical Logic*

Some changes may be reflected in the published version

A Hierarchy of Classical and Paraconsistent Logics

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Abstract

In this article, we will present a number of technical results concerning Classical Logic, **ST** and related systems. Our main contribution consists in offering a novel identity criterion for logics in general and, therefore, for Classical Logic. In particular, we will firstly generalize the **ST** phenomenon, thereby obtaining a recursively defined hierarchy of strict-tolerant systems. Secondly, we will prove that the logics in this hierarchy are progressively more classical, although not entirely classical. We will claim that a logic is to be identified with an infinite sequence of consequence relations holding between increasingly complex relata: formulae, inferences, metainferences, and so on. As a result, the present proposal allows not only to differentiate Classical Logic from **ST**, but also from other systems sharing with it their valid metainferences. Finally, we show how these results have interesting consequences for some topics in the philosophical logic literature, among them for the debate around Logical Pluralism. The reason being that the discussion concerning this topic is usually carried out employing a rivalry criterion for logics that will need to be modified in light of the present investigation, according to which two logics can be non-identical even if they share the same valid inferences.

Keywords: Substructural Logics, Cut Rule, Metainference, Classical Logic

1 Background and aim

What identifies Classical Logic as such? As easy and straightforward as answering this question may appear, in this article we devote ourselves to discussing a number of overlooked subtleties that, we argue, need to be taken into account in providing a cogent answer to it. The main contribution of our work consists, therefore, in offering a novel identity criterion for logical systems in general and, therefore, for Classical Logic.

The aforementioned subtleties have their origin in recent debates in the philosophical logic literature, more particularly, in a number of observations revolving around a project carried out by the collective formed by Pablo Cobreros, Paul Egré, David Ripley and Robert van Rooij. With the goal of solving paradoxes coming from the semantic, set-theoretic and vagueness corners, these authors have put forward a number of conceptual and technical arguments that support them backing a nice, innovative and quite revolutionary stance towards these riddles: the strict-tolerant approach, which led them to entertain the non-transitive logic **ST**.[1] The main advantage of this choice lies, allegedly, in keeping Classical Logic as the underlying inferential framework, even in what pertains to the problematic phenomena referred above. Thus, these authors claim to have shown that Classical Logic—actually, **ST**, but both systems are identical, according to them—can non-trivially deal with these otherwise problematic phenomena.

In this regard, it may be instructive to discuss a number of natural reasons that we think point towards justifying that **ST** is not, after all, Classical Logic. First, embracing an essentially non-transitive logic requires understanding logical consequence between formulae in terms of something other than truth-preservation—as acknowledged in many works revolving around **ST** and other substructural logics, such as [6]. But, certainly, the notion of logical consequence featured in Classical Logic can be presented in terms of truth-preservation. Thus, even if there is nothing essentially wrong or mistaken in taking logical consequence to be defined as something different than the preservation of truth from premises to conclusion, this immediately tells **ST** and Classical Logic apart.

Secondly, identifying **ST** and **CL** would require accepting that transitivity is inessential to the characterization of classical reasoning. But, as argued in [4], it appears to be a rather central feature, allowing us to conduct cumulative reasoning, transitioning from lemmas to theorems and later to corollaries—regardless of the subject-matter under discussion. In other words, if cumulative deductive progress is not guaranteed, we cannot be sure that lemmas are knowledge checkpoints. But this seems quite an important property of classical reasoning, while it certainly is not a feature of reasoning carried out in **ST**.

Moreover, there are some other widely shared intuitions about what Classical Logic ought to be, and about which logical principles are distinctively valid in it. Explosion, Excluded Middle, Modus Ponens, Disjunctive Syllogism, etc. are only a few of these, that every proper presentation of Classical Logic should comply with. Additionally, if these are part of the core of such a logical system, then they should be expected to hold in the context of all kinds of inferences carried out in it. That is to say, they should hold within regular inferences between formulae, but also in the context of what we call *metainferences*—namely, inferences between inferences. However, as remarked in [4] and [28], it is illustrative to notice that **ST** fails to validate what might be understood as certain metainferential forms of Explosion, Modus Ponens, Disjunctive Syllogism, and so on. This, if anything, seems to add more grounds to the extent that **ST** and Classical Logic are not identical.

Furthermore, there is the important fact that Classical Logic is widely believed to be

[1] A non-exhaustive list of some works of this collective where **ST** is discussed is [8], [7], and [10].

prone to trivialization when faced with transparent truth, vague phenomena and much more, while **ST** does not fall into such troubles. Interestingly, to an argument of this sort advocates of their identity may respond that **ST** is nothing but a different *mode of presenting* Classical Logic.[2] Just like Classical Logic can be presented by means of two-valued models or valuations, it can also be presented by means of three-valued Strong Kleene models or valuations, as we will see below. In this regard, they could argue that which consequences can be drawn from an arbitrary piece of content or a theory by closing it under Classical Logic—and whether cumulative reasoning is unrestrictedly allowed—crucially depends on which presentation of Classical Logic we implement.

Faced with a reasoning of this sort, we cannot but express our substantial disagreement. Take, for example, the case of closing arbitrary theories under numerically different logical systems, and consider the case of theories of transparent truth. Looking at their consequences, and whether or not the resulting systems lead to triviality, provides a way—albeit a rather *indirect* one—to assess their strength, to compare them. As a limit case, if two logics render exactly the same consequences, we think it is fair to assume that, by all intents and purposes, they are the same logic.[3] This is not the case, according to the widely accepted beliefs about Classical Logic, of Classical Logic and **ST**. Thus, this is why we think it is necessary to have a more *direct* way of identifying and differentiating them, by providing the identity criterion that we outlined previously.

Let us clarify, though, that our aim in this article is not to develop a knock-down philosophical argument to establish that Classical Logic and **ST** are different logics. For what is worth, it is not obvious for us that there is even a way of establishing such a thing without running into some sort question-begging reasoning. Our work here is not intended to convince believers in the identity of Classical Logic and **ST** that they are wrong. Instead, we aim at offering technical developments and precise arguments to sustain in a formal way the rather loose and informal feeling that some people had so far, to the extent that Classical Logic and **ST** are not identical.

In order to do this, these systems should be told apart by some formal and philosophically motivated criterion. In this article we advance a way of doing so, by proposing a novel identity criterion for logical systems, which will take into account not only those inferences between formulae which are valid in a given logical system—but also several other inferences, e.g. between inferences themselves, which are valid in a given logical system. As we said previously, contrary to what it seems, this is by no means a trivial task. In fact, a number of details that have not received much attention in the literature, until now, make this an especially intricate business.

The first difficulty appearing in our route to finding an appropriate identity criterion for logical systems, is that the one we often use—implicitly employed by the advocates of **ST** to argue it is nothing but Classical Logic—is the nowadays standard Tarskian conception of a logic as a consequence relation between collections of formulae. Such an account, although subject to some generalizations in the last decades, is the ruling

[2]This way to understand **ST** was presented to us, in conversation, by Dave Ripley.

[3]An option along these lines has been both suggested and explored in what pertains to Tarskian logics—particularly, Classical and Intuitionistic Logic—in [27].

standard for us to distinguish logical systems. The problem is, thus, evident. We need a different account to individuate logics and, in particular, to identify Classical Logic.

A versed reader in the recent literature revolving around the strict-tolerant approach might point to a tempting solution, reflecting upon the set of metainferences which are valid in these logics. As mentioned earlier and as pointed out e.g. by Barrio, Rosenblatt and Tajer in [4], a number of classically valid metainferences—such as meta Modus Ponens, meta Explosion and others, on which more below—fail to be valid in **ST**. Thus, one might be tempted to say that it differs from Classical Logic because, even if both systems coincide at the inferential level, they do not agree at the metainferential level.[4] However promising this solution may appear, and however easily it seems to address the problem at hand, it cannot provide a definitive answer.

The reason for this is tied to the second difficulty we stumble upon when trying to find an appropriate identity criterion for logics, namely that the previously discussed phenomenon can be replicated. By this we mean that—just like **ST** is a system which coincides with Classical Logic at the inferential level, but not at the metainferential level—it is possible to obtain a system which coincides with Classical Logic at the inferential and the metainferential level, but which does not coincide with it at the level of the metametainferences, i.e. of the inferences between metainferences.[5] Furthermore, if the search of an extensional identity criterion is expanded to cover this case too, it is still possible to design another **ST**-like logic which also complies with it, albeit non-classical with regard to its metametametainferences, and so on and so forth.

In other words, we claim the phenomenon incarnated by the **ST** approach is pervasive. In fact, it is possible to obtain a sequence of logics which can be progressively ordered in terms of their degree of classicality. To argue for this we will present a collection, a hierarchy of systems which coincide with Classical Logic in more and more inferential levels as we move in the sequence—although for all of them there is some inferential level at which they start to behave non-classically. This constitutes the *pars destruens* of the article. The *pars construens* will be devoted to presenting our own identity criterion for logical systems, and considering possible objections to it.

To this extent, the article is structured as follows. In Section 2 we introduce some preliminary definitions that will be of use in the remainder of the article, including the definition of metainference, metametainference, and metainference of an arbitrary large level. Section 3 reviews the strict-tolerant phenomenon incarnated in **ST**, as it is featured in the discussions appearing in the literature so far. Section 4 includes our main technical and philosophical contributions: the hierarchy of **ST**-related logics, and our novel identity criterion for logical systems. On the one hand, this section contains

[4]An anonymous reviewer wonders whether we know which metainferences are valid in Classical Logic beforehand this investigation, i.e. whether we have some means of ruling on these matters that we are not stating upfront. In fact, throughout this article we will adopt a semantic stance towards metainferential validity, as seen in the various definitions appearing in Section 4. This does not mean that these issues cannot be established and explored by proof-theoretical means, but for matters of space these will be left for another occasion.

[5]Another way to fully recover Classical Logic, starting from **ST**, is through the introduction of a so-called consistency or recovery operator. This path was explored by us in detail in [1].

a discussion that deepens the understanding of the **ST** phenomenon by generalizing it to infinitely many different logical systems, and an impossibility result concerning the extensional characterization of Classical Logic, i.e. the identification of Classical Logic with its set of valid inferences of some inferential level. On the other, it contains an analysis of the conceptual importance of the results associated to this hierarchy, by providing and defending a novel identity criterion for logical systems that is able to overcome the difficulties of the available ones and their modifications. Section 5 considers some objections to our account and our replies to them. Finally, Section 6 wraps up our work by making some concluding remarks.

2 Preliminary definitions

To understand and carry our investigation, it will be essential to have an accurate grasp of the received view about inferences, metainferences and consequence relations. In turn, to analyze these matters, it will be useful to fix some terminology. In what follows, we will be working with a propositional language $\mathcal{L}$ equipped with the connectives $\neg, \wedge, \vee$, to be interpreted as negation, conjunction and disjunction, letting $FOR(\mathcal{L})$ be the set of recursively generated well-formed formulae of $\mathcal{L}$. As usual, we will let Γ, Δ, and other Greek capital letters represent sets of formulae, or sets of inferences, and Roman capital letters A, B, C represent formulae themselves, or inferences themselves.[6]

An inference $\Gamma \Rightarrow \Delta$ on $\mathcal{L}$ is an ordered pair $\langle \Gamma, \Delta \rangle$ where $\Gamma, \Delta \subseteq FOR(\mathcal{L})$, letting $SEQ^0(\mathcal{L})$ be the set of all inferences on $\mathcal{L}$, to which we might occasionally refer to as $SEQ(\mathcal{L})$.

According to a nowadays standard approach, a logic is usually identified with a set of inferences. This is very much incarnated in the definition of logic as a relation over collections of formulae, included in $SEQ^0(\mathcal{L})$, and of a Tarskian logic as a logic complying with the following structural rules—where $\Gamma, \Delta \subseteq FOR(\mathcal{L})$ and $A \in FOR(\mathcal{L})$.[7]

$$\frac{}{A \Rightarrow A}\ \text{Reflexivity} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma', \Gamma \Rightarrow \Delta, \Delta'}\ \text{Weakening} \qquad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \text{Cut}$$

However, the main point we discuss in this article consists—precisely—in defining what a logic *is*. For this reason, we will not embrace the aforementioned identification here. In other words, we will not assume that a logic is a relation over collection of formulae and, therefore, we will not assume that a logic is a relation of this sort complying with this or that structural rules. Structural rules, in fact, are schematic metainferences. But, then, given many of the critical points we will rise in this investigation—concerning plausible identity criteria for logical systems—rely on debates involving metainferences of some kind or another, we *cannot assume* structural rules are built in the definition of a logical system. That would mean ending the investigation before even starting it.

[6]Though we hope the context will make things clear enough, we will always clearly state whether, for example, Roman capital letters A, B, C represent formulae themselves, or inferences themselves.

[7]Often, the requirement is also asked that consequence relations are substitution-invariant, i.e. that if $\Gamma \Rightarrow \Delta$ is valid, and σ is a substitution on $FOR(\mathcal{L})$, then $\{\sigma(A) \mid A \in \Gamma\} \Rightarrow \{\sigma(B) \mid B \in \Delta\}$ is valid.

These considerations, however, call for a clarification concerning metainferences. Indeed, we have not yet even defined what a metainference actually is. To solve this we will say that, intuitively, a metainference is an inference between a collection of inferences, and an inference. To illustrate this observe the following, where the one on the left is an inference, whereas the one on the right is a metainference.

$$p, \neg p \Rightarrow q \qquad\qquad \dfrac{\emptyset \Rightarrow p \quad \emptyset \Rightarrow p \supset q}{\emptyset \Rightarrow q}$$

In fact, just like the one on the left above is an instance of the schematic inference usually called Explosion, the one on the right is an instance of the schematic metainference known as meta Modus Ponens—respectively depicted below.

$$A, \neg A \Rightarrow B \qquad\qquad \dfrac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow A \supset B, \Delta}{\Gamma \Rightarrow B, \Delta}$$

Of very much interest for the upcoming discussions is the following generalization of the notion of metainference, to a metainference of an arbitrary large (finite) level. In this vein, we will say a metainference $\Gamma \Rightarrow^n A$ of level n on $\mathcal{L}$ (for $1 \leq n < \omega$) is an ordered pair $\langle \Gamma, A \rangle$ where $\Gamma \subseteq SEQ^{n-1}(\mathcal{L})$ and $A \in SEQ^{n-1}(\mathcal{L})$. $SEQ^n(\mathcal{L})$ is the set of all metainferences of level n on $\mathcal{L}$.

To get a feeling of these observe the following, from which the one on the left is a metainference, the one on the middle is a metametainference (or metainference of level 2), while the one on the right is a metametametainference (or metainference of level 3). The reader can produce further examples of metainferences of arbitrary large levels, just by using her imagination.

$$\dfrac{p \Rightarrow r \quad q \Rightarrow r}{p \vee q \Rightarrow r} \qquad \dfrac{\dfrac{r \Rightarrow s}{p \Rightarrow q} \quad \dfrac{q \Rightarrow p}{t \Rightarrow u}}{\dfrac{s \Rightarrow t}{r \Rightarrow u}} \qquad \dfrac{\dfrac{\dfrac{r \Rightarrow s}{p \Rightarrow q} \quad \dfrac{q \Rightarrow p}{t \Rightarrow u}}{\dfrac{s \Rightarrow t}{r \Rightarrow u}} \quad \dfrac{\dfrac{r \Rightarrow s}{p \Rightarrow q} \quad \dfrac{q \Rightarrow p}{t \Rightarrow u}}{\dfrac{s \Rightarrow t}{r \Rightarrow u}}}{\dfrac{\dfrac{r \Rightarrow s}{p \Rightarrow q} \quad \dfrac{q \Rightarrow p}{t \Rightarrow u}}{\dfrac{s \Rightarrow t}{r \Rightarrow u}}}$$

Observe that the above definitions allow to recast e.g. all inferences as metainferences, by considering them as metainferences with empty premise sets. Similarly e.g. for metainferences, which can be recast as metametainferences with appropriately empty premise sets. This can be also done for formulae in an analogous manner, i.e. by considering them as inferences with no premises.[8] This is the technical reason why two systems characterized by the same set of inferences will be forced to have the same set of theorems, but might not necessarily be characterized by the same set of metainferences. This also explains why two logics with the same metainferences of level n will coincide regarding the metainferences and inferences of any level m lower than n, but might not necessarily coincide regarding the metainferences of level k greater than n.

Metainferences are taken to be—as Dicher and Paoli put it in [12]—*bona fide* objects, i.e. rightful linguistic items themselves, constituted by concrete inferences which are in

[8]This approximation follows the spirit of some remarks made in [12].

turn constituted by pairs of collections of formulae. This marks a difference not only with the stance adopted by Barrio, Rosenblatt and Tajer in [4], but also reflects a view opposed to Cobreros, Egré, Ripley and van Rooij's conception of these entities. The former consider metainferences to be *schemata* and not objects themselves, whereas the latter take them to be supervening *properties* under which the valid inferences of a certain logical system might or might not happen to be closed.[9]

We can, and sometimes will, refer to particular metainferences and to schematic metainferences. Two of the main schematic metainferences that will draw our attention in what follows are corresponding versions of Cut and Explosion of each metainferential level. These we will call, without loss of generality, meta Cut and meta Explosion and—more particularly—we will denote them by $\underbrace{\text{meta...meta}}_{n}$ Cut and $\underbrace{\text{meta...meta}}_{n}$ Explosion, depending on the inferential level at which they appear. In fact, each instance of the former is a metainference of level $n-1$, and each instance of the latter is a metainference of level n. More concretely, the following metainferences

$$\dfrac{\dfrac{s \Rightarrow t}{p \Rightarrow q, r} \quad \dfrac{s \Rightarrow t}{p, r \Rightarrow q}}{\dfrac{s \Rightarrow t}{p \Rightarrow q}} \qquad\qquad \dfrac{\dfrac{s \Rightarrow t}{p \Rightarrow q, r} \quad \dfrac{s \Rightarrow t}{p \Rightarrow q, \neg r}}{\dfrac{s \Rightarrow t}{p \Rightarrow q, u}}$$

are instances of the schematic metainferences appearing below, called meta Cut and metameta Explosion, respectively. The reader may use her imagination to design appropriate versions of these for greater metainferential levels.

$$\dfrac{\dfrac{\Gamma_1 \Rightarrow \Delta_1 \quad \dots \quad \Gamma_k \Rightarrow \Delta_k}{\Sigma \Rightarrow \Pi, A} \qquad \dfrac{\Gamma_1 \Rightarrow \Delta_1 \quad \dots \quad \Gamma_k \Rightarrow \Delta_k}{\Sigma, A \Rightarrow \Pi}}{\dfrac{\Gamma_1 \Rightarrow \Delta_1 \quad \dots \quad \Gamma_k \Rightarrow \Delta_k}{\Sigma \Rightarrow \Pi}}$$

$$\dfrac{\dfrac{\Gamma_1 \Rightarrow \Delta_1 \quad \dots \quad \Gamma_k \Rightarrow \Delta_k}{\Sigma \Rightarrow \Pi, A} \qquad \dfrac{\Gamma_1 \Rightarrow \Delta_1 \quad \dots \quad \Gamma_k \Rightarrow \Delta_k}{\Sigma \Rightarrow \Pi, \neg A}}{\dfrac{\Gamma_1 \Rightarrow \Delta_1 \quad \dots \quad \Gamma_k \Rightarrow \Delta_k}{\Sigma \Rightarrow \Pi, B}}$$

With these technicalities at hand, let us delve into the **ST** phenomenon and the complications it poses in offering an appropriate identity criterion for Classical Logic.[10]

[9] See e.g. [21, p. 354] and [10, §3.3].

[10] We should remark we will not consider metainferences with multiple conclusions, as with regular inferences. This is not because we think there is something wrong with them, but because it will unnecessarily complicate the presentation of the systems that we will introduce below. In this vein, we will also not consider *mixed* metainferences, i.e. metainferences with premises belonging to different inferential levels. Although there is nothing conceptually wrong about such metainferences, yet again, working with them will make the different proofs unnecessary complicated. An additional clarification concerns the kind of schematic metainferences that we will be taking into account to exemplify our investigations. These will be presented in an additive instead of a multiplicative way, i.e. they will—actually, their instances will—have shared contexts. For more on these distinctions, see e.g. [23].

3 The ST phenomenon, as we know it

To better understand the logic **ST** we will review some of its properties, pointing out its relations with Classical Logic (**CL** hereafter) and with a certain non-classical logic, i.e. Graham Priest's system **LP**. After this, we will show how this peculiar system can be seen as only one instance of what we call "the **ST** phenomenon", that is, the definability of systems agreeing with **CL** in successively many inferential levels, although not in all of them. It is with these tools at hand that we will argue that the **ST** phenomenon poses a threat to the current identity criterion for logical systems (which identifies them with their set of valid inferences) and to any natural extension thereof.

The logic **ST** is sometimes discussed by its advocates as essentially motivated by inferentialist concerns, whence it is usually presented as a sequent calculus. However, this system can be equally well-motivated from a semantic point of view. As such, logical consequence is defined in **ST** with one of the many—sixteen, as remarked by Wintein [26]—different definitions of logical consequence which happen to coincide in **CL**. In the case of this strict-tolerant system, logical consequence is defined by claiming that $\Gamma \Rightarrow \Delta$ is valid if and only if there is no assignment of truth-values making the premises true while at the same time making the conclusion false.

One difference between **CL** and **ST** lies in the scope of the candidate valuations involved in the definition of logical consequence. While for **CL** valuations are only Boolean or two-valued, in the case of **ST** these are essentially three-valued valuations respecting the Strong Kleene truth-tables appearing below—where $\supset$ and $\leftrightarrow$ are definable as usual, i.e $A \supset B =_{def} \neg A \vee B$ and $A \leftrightarrow B =_{def} (A \supset B) \wedge (B \supset A)$.

	$\neg$
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

$\wedge$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

$\vee$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

Definition 3.1. A Strong Kleene valuation (SK-valuation, hereafter) is a mapping from $FOR(\mathcal{L})$ to $\{1, \frac{1}{2}, 0\}$ that respects the Strong-Kleene truth-tables above. Similarly, a Boolean valuation is a SK-valuation whose range is $\{1, 0\}$.

Below, we define what is for a valuation to satisfy a given inference in **CL** and **ST**. As is common practice, we will define the validity of an inference (and, more generally below, of an inference of an arbitrary inferential level) in a certain logic as nothing more than satisfaction by all valuations—in our case, all SK-valuations.

Definition 3.2. A Boolean valuation v satisfies an inference $\Gamma \Rightarrow \Delta$ in **CL** ($v \vDash_{\mathbf{CL}} \Gamma \Rightarrow \Delta$) if and only if it is not the case that $v(A) = 1$ for all $A \in \Gamma$ and $v(B) = 0$ for all $B \in \Delta$. Similarly, a SK-valuation v satisfies an inference $\Gamma \Rightarrow \Delta$ in **ST** ($v \vDash_{\mathbf{ST}} \Gamma \Rightarrow \Delta$) if and only if it is not the case that $v(A) = 1$ for all $A \in \Gamma$ and $v(B) = 0$ for all $B \in \Delta$.

These definitions allow to prove the much commented result stating that the set of valid inferences of **CL** and **ST** coincide—as shown e.g. in [13] and [8].[11]

[11] Yet another case of a substructural logic which has, nevertheless, the same valid inferences than **CL** is the case of the non-contractive system discussed by Lucas Rosenblatt in [24].

Fact 3.3. *For all* $\Gamma, \Delta \subseteq FOR(\mathcal{L})$*:*

$$\vDash_{\mathbf{ST}} \Gamma \Rightarrow \Delta \quad \textit{if and only if} \quad \vDash_{\mathbf{CL}} \Gamma \Rightarrow \Delta$$

Now, it is often emphasized that the strict-tolerant division giving rise to the **ST** label is motivated by appealing to assertion and denial in their strict and tolerant moods (cf. [22]), but we will claim that this label can be equally well-motivated from a semantic point of view. This can be done by looking at the following quote:

> The logic **ST** sets different standards for satisfaction in premises and in conclusions. A "good" premise (a premise good enough to produce a sound argument) is one that takes value 1. A "good" conclusion, on the other hand (a conclusion that is not false enough to produce a counterexample) is one that takes value greater than 0. [9, p. 79]

This amounts to saying the semantic essence of the **ST** phenomenon consists in its adoption of the standards of **CL** for taking premises to be satisfied by a valuation and for taking conclusions not to be dissatisfied by a valuation—albeit in the context of three-valued valuations. Thus, the semantics for **ST** are the semantics of a strict-tolerant logic, because although the requirement for the premises is strict (i.e. that they *do satisfy* the standard to count as a premise of a sound argument in **CL**), the requirement for the conclusions is tolerant (i.e. that they *do not satisfy* the standard to count as the conclusion of an invalid argument in **CL**).

These observations portray **ST** as a system incarnating an essentially *mixed* notion of logical consequence, by means of which standards of different strength are asked for the premises and the conclusions of valid inferences. However, three-valued systems adopting homogeneous requirements for premises and conclusions seem equally motivated. In fact, taking the strict and the tolerant routes *tout court* leads, respectively, to the well-known three-valued logics **LP** and $\mathbf{K_3}$.

Definition 3.4. A SK-valuation v satisfies an inference $\Gamma \Rightarrow \Delta$ in **LP** ($v \vDash_{\mathbf{LP}} \Gamma \Rightarrow \Delta$) if and only if it is not the case that $v(A) \in \{1, \frac{1}{2}\}$ for all $A \in \Gamma$ and $v(B) = 0$ for all $B \in \Delta$. Similarly, a SK-valuation v satisfies an inference $\Gamma \Rightarrow \Delta$ in $\mathbf{K_3}$ ($v \vDash_{\mathbf{K_3}} \Gamma \Rightarrow \Delta$) if and only if it is not the case that $v(A) = 1$ for all $A \in \Gamma$ and $v(B) \in \{0, \frac{1}{2}\}$ for all $B \in \Delta$.

Letting L' and L'' be two systems characterized semantically by means of SK-valuations, we can define a *mixed* system L as in Definition 3.5—in which case we will describe L as L'/L''.[12] Thus, **ST** can be described as $\mathbf{K_3}/\mathbf{LP}$.

Definition 3.5. A SK-valuation v satisfies an inference $\Gamma \Rightarrow \Delta$ in L ($v \vDash_L \Gamma \Rightarrow \Delta$) if and only if it is not the case that $v \vDash_{L'} A$ for all $A \in \Gamma$ and $v \nvDash_{L''} B$ for all $B \in \Delta$.

Going back to our main theme, though, we must recall that even if there are some important similarities between **CL** and **ST**—indeed, even if they coincide at the inferential level—they do *not* coincide at every inferential level. More precisely, there are metainferences valid in **CL** but invalid in **ST**. To wit, the following

[12] This is, very much, the definition of mixed consequence discussed by Chemla, Egré and Spector in their recent article [6]—with changes in the definition, to adapt their notation to ours, being inessential.

$$\frac{p, q \Rightarrow r, s \quad p, q, r \Rightarrow s}{p, q \Rightarrow s} \qquad \frac{\emptyset \Rightarrow p \quad \emptyset \Rightarrow \neg p}{\emptyset \Rightarrow q}$$

are instances of the schematic metainferences depicted below, respectively called Cut and meta Explosion (following Zardini in [28], and Barrio, Rosenblatt and Tajer in [4]). All of them are valid in **CL** but invalid in **ST**, according Definition 3.6.

$$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow \neg A, \Delta}{\Gamma \Rightarrow B, \Delta}$$

To observe the disparity in this regard between **CL** and **ST**—without any appeal to a sequent calculus—it is crucial to understand how metainferences can be taken to be valid or invalid in a certain system. In this regard we take a semantic stance, borrowing from [14] and [12] the distinction between the *local* and the *global* definition of metainferential validity. Saying that a metainference is locally valid according to a certain logic L means that if the premise inferences are satisfied by a valuation in L, so is the conclusion inference. This is different to saying that a metainference is globally valid according to a certain logic L, which requires that whenever the premise inferences are valid in L, so is the conclusion. In a nutshell, the difference between local and global metainferential validity consists in nothing more than the difference between satisfaction-preservation and validity-preservation. In what remains, we will adopt the local notion, postponing our justification for this choice until Section 5. Below, we detail the definition of local metainferential validity for **CL** and **ST**.[13]

Definition 3.6. For a metainference $\Gamma \Rightarrow_1 A$, where $\Gamma \subseteq SEQ(\mathcal{L})$ and $A \in SEQ(\mathcal{L})$, a Boolean valuation v satisfies $\Gamma \Rightarrow_1 A$ in **CL** ($v \vDash_{\mathbf{CL}} \Gamma \Rightarrow_1 A$) if and only if $v \nvDash_{\mathbf{CL}} \gamma$ for some $\gamma \in \Gamma$, or $v \vDash_{\mathbf{CL}} A$. Similarly, a SK-valuation v satisfies $\Gamma \Rightarrow_1 A$ in **ST** ($v \vDash_{\mathbf{ST}} \Gamma \Rightarrow_1 A$) if and only if $v \nvDash_{\mathbf{ST}} \gamma$ for some $\gamma \in \Gamma$, or $v \vDash_{\mathbf{ST}} A$.

We close this section by calling the attention to the fact that, while the above remarks point towards the non-classical nature of **ST**—regarding its valid metainferences—they do not say much about what is particularly distinctive about its non-classicality. For all we know, **ST** could be non-classical metainferentially speaking, but there could be no recognizable pattern in its non-classical behavior. A clearer understanding of the non-classical nature of metainferential validity in **ST** is, nevertheless, delivered by a recent number of articles by Barrio, Rosenblatt and Tajer [4] and, independently, by Pynko [19]. In their work, it is clearly described how it is possible to characterize the metainferential non-classicality of **ST** in terms of the inferential non-classicality of Graham Priest's **LP**. In other words, every metainference that is invalid in **ST** has a corresponding inference that is invalid in **LP**, and viceversa.

This result is achieved, in one of its versions, through an indispensable translation function from metainferences to inferences, neatly laid out by Dicher and Paoli in [12]. It is interesting and illustrative to notice that our translation function below is intended to be inspired by theirs, while simultaneously subsuming it. Indeed, while their function

[13]For what it is worth, for a metainference $\Gamma \Rightarrow_1 A$, where $\Gamma \subseteq SEQ(\mathcal{L})$ and $A \in SEQ(\mathcal{L})$, we say it is *globally* valid in **ST** if and only if $\nvDash_{\mathbf{ST}} \gamma$ for some $\gamma \in \Gamma$, or $\vDash_{\mathbf{ST}} A$.

allows to translate metainferences to inferences and inferences to formulae, ours allows to translate—by applying essentially the same ideas—metainferences of any arbitrary large inferential level to metainferences of the immediately predecessor level, as well as metainferences to inferences, and inferences to formulae.

Definition 3.7. *lower*: $\bigcup_{n\in\omega} SEQ^n(\mathcal{L}) \longrightarrow FOR(\mathcal{L}) \cup \bigcup_{n\in\omega} SEQ^n(\mathcal{L})$ is defined so that:

- $lower(\Gamma \Rightarrow \Delta) = \bigwedge \Gamma \supset \bigvee \Delta$
- for $1 \leq n$, $lower(\Gamma \Rightarrow_n A) = \{lower(\gamma) \mid \gamma \in \Gamma\} \Rightarrow_{n-1} lower(A)$

Notice that, given the above definition, if $\Gamma \neq \emptyset$ and $\Delta = \emptyset$ then $lower(\Gamma \Rightarrow \Delta) = \neg A_1 \wedge \cdots \wedge \neg A_n$ for $A_i \in \Gamma$. Similarly, if $\Gamma = \emptyset$ and $\Delta \neq \emptyset$ then $lower(\Gamma \Rightarrow \Delta) = B_1 \vee \cdots \vee B_n$ for $B_i \in \Delta$. We take, furthermore, $\Gamma \Rightarrow_0 \Delta$ to be just $\Gamma \Rightarrow \Delta$. It is, then, with the help of these definitions that we can formally rephrase the previously mentioned collapse result for **ST** and **LP**—interpreted by Barrio, Rosenblatt and Tajer in [4] as meaning that **LP** is nothing but the logic of the metainferences of **ST**.[14]

Fact 3.8. *For all* $\Gamma, \Delta \subseteq FOR(\mathcal{L})$*:*

$$\vDash_{\mathbf{ST}} \Gamma \Rightarrow_1 \Delta \quad \textit{if and only if} \quad \vDash_{\mathbf{LP}} lower(\Gamma \Rightarrow_1 \Delta)$$

As we said earlier, given these results one might be tempted to say that **ST** differs from **CL** because they do not agree at the metainferential level. Two logical systems may be claimed to be identical only if they coincide with regard not only to their valid inferences, but also with regard to their valid metainferences. But, as promising as this solution might be, it constitutes an unsatisfactory answer. The reason for this is that the case of **ST** can be generalized. In the next section, we do this by constructing a sequence of logics which coincide with **CL** in progressively many inferential levels, although all of them start behaving non-classically at some point—and when they do, they will just like **ST** behave in a way essentially tied to **LP**.

4 The ST phenomenon, generalized

In this section we will, first, present a logic which coincides with **CL** regarding its valid metainferences, although not regarding its metametainferences, or metainferences of level 2. Later, we will present a number of systems coinciding with **CL** in progressively many inferential levels although not in all of them. This, we claim, will allow us to provide an *impossibility result*: it is impossible to identify a logic with its given set of valid inferences of any level whatsoever, for it is possible to find logics coinciding with it in that respect despite being intuitively different from it.

To start up this part of the investigation, though, we need to focus on a system closely related to **ST**, namely the logic **TS**. This system incarnates a mixed notion of logical consequence that is dual to the one characteristic of **ST**—if **ST** can be described

[14] For more on the relation between **ST** and paraconsistency, see [3] and [1].

as the mixed system $\mathbf{K_3}/\mathbf{LP}$, then **TS** can be seen as the mixed system $\mathbf{LP}/\mathbf{K_3}$. With its help we are now able to define yet another logic that we call **TS/ST**, precisely because logical consequence for it is defined in a mixed way with the help of **TS** and **ST**.[15] The definition of logical consequence for this system, appearing below, is provided at the level of metainferences—ruling only derivatively over inferences and formulae.[16]

Definition 4.1. For a metainference $\Gamma \Rightarrow_1 A$, where $\Gamma \subseteq SEQ(\mathcal{L})$ and $A \in SEQ(\mathcal{L})$, we say a SK-valuation v satisfies $\Gamma \Rightarrow_1 A$ in **TS/ST** ($v \vDash_{\mathbf{TS/ST}} \Gamma \Rightarrow_1 A$) if and only if $v \nvDash_{\mathbf{TS}} \gamma$ for some $\gamma \in \Gamma$, or $v \vDash_{\mathbf{ST}} A$.

Furthermore, the system **TS/ST** can be rightfully called a **ST**ian logic, for it defines its central notion of logical consequence as outlined above by adopting the classical standard to be satisfied as the premise of a metainference, and the classical standard to be regarded as the counter-exemplifying conclusion of a metainference. This can be observed in the lemma stated below, whose proof we leave to the reader as an exercise.[17]

Lemma 4.2. *For all $\Gamma, \Theta, \Delta, \Pi \subseteq SEQ(\mathcal{L})$, and all SK-valuations v, there is a Boolean valuation v^* such that:*

$$\begin{array}{lccc} \textit{if} & v \nvDash_{\mathbf{ST}} \Gamma \Rightarrow \Delta & \textit{and} & v \vDash_{\mathbf{TS}} \Theta \Rightarrow \Pi, \\ \textit{then} & v^* \nvDash_{\mathbf{CL}} \Gamma \Rightarrow \Delta & \textit{and} & v^* \vDash_{\mathbf{CL}} \Theta \Rightarrow \Pi \end{array}$$

Similarly, for all Boolean valuations v, there is a SK valuation v^* such that:

$$\begin{array}{lccc} \textit{if} & v \nvDash_{\mathbf{CL}} \Gamma \Rightarrow \Delta & \textit{and} & v \vDash_{\mathbf{CL}} \Theta \Rightarrow \Pi, \\ \textit{then} & v^* \nvDash_{\mathbf{ST}} \Gamma \Rightarrow \Delta & \textit{and} & v^* \vDash_{\mathbf{TS}} \Theta \Rightarrow \Pi \end{array}$$

In light of these remarks, that **CL** and **TS/ST** have the same valid metainferences becomes straightforwardly provable. We will not show the details of the proof here, given it follows from the more general Theorem 4.12 appearing below.

Fact 4.3. *For all $\Gamma \subseteq SEQ(\mathcal{L})$ and $A \in SEQ(\mathcal{L})$:*

$$\vDash_{\mathbf{TS/ST}} \Gamma \Rightarrow_1 A \quad \textit{if and only if} \quad \vDash_{\mathbf{CL}} \Gamma \Rightarrow_1 A$$

Perhaps looking at **TS/ST** in this way will give the reader the feeling that, if this system coincides with **CL** regarding its metainferences (and, hence, its inferences and

[15] **TS/ST** is just one member of a wide family of metainferential logics. For more about such a family, see [16]. In addition, the introduction of **TS/ST** has, in itself, an effect in a variety of philosophical phenomena. To name a few, it poses new challenges to the issue of Logical Pluralism and the so-called Collapse Argument for it (as we discussed in [2]), it can be implemented to give a new solution to semantic paradoxes (as argued by one of us in [17]), and to suggest that paraconsistency might be something more than the mere invalidity of certain inferential forms of Explosion (as done by us in [3]).

[16] Although, as remarked by an anonymous reviewer, Definition 4.1 is an instance of Definition 4.7, the latter requires understanding the hierarchy of logics we define next whereas the former does not.

[17] An anonymous reviewer sharply points out that instead of proving conditional claims which have conjunctions on their antecedents and consequents, we could prove the conjunction of two conditionals linking, respectively, their first conjunct and their second conjunct. We would like to clarify that although this is certainly possible, in the context of the overall strategy of our proofs in this article, the current claims are sufficient enough.

valid formulae) then **TS/ST** ought to be identified with **CL**. Needless to say, knowing that **TS/ST** is meant to be an instance of the **ST** phenomenon, it is quite obvious that this conclusion must be resisted. In fact, **TS/ST** can be shown not to coincide with **CL**, as its metametainferences (or metainferences of level 2) do not coincide with those valid in **CL**. This can be checked, by paying attention to the definition of metametainferential validity in **CL** and **TS/ST**.

Definition 4.4. For a metainference $\Gamma \Rightarrow_2 A$, where $\Gamma \subseteq SEQ^1(\mathcal{L})$ and $A \in SEQ^1(\mathcal{L})$, we say a Boolean valuation v satisfies $\Gamma \Rightarrow_2 A$ in **CL** ($v \vDash_{\mathbf{CL}} \Gamma \Rightarrow_2 A$) if and only if $v \nvDash_{\mathbf{CL}} \gamma$ for some $\gamma \in \Gamma$, or $v \vDash_{\mathbf{CL}} A$. Similarly, a SK-valuation v satisfies $\Gamma \Rightarrow_2 A$ in **TS/ST** ($v \vDash_{\mathbf{TS/ST}} \Gamma \Rightarrow_2 A$) if and only if $v \nvDash_{\mathbf{TS/ST}} \gamma$ for some $\gamma \in \Gamma$, or $v \vDash_{\mathbf{TS/ST}} A$.

With the help of this notion of validity, some examples become available of the disparity between **CL** and **TS/ST**. In particular, the following metametainferences (instances, themselves, of the schematic metametainferences meta Cut and metameta Explosion referred in Section 2) are valid in the former, although they are invalid in the latter. Below, $\emptyset$ stands for the empty set of premises.

$$\dfrac{\dfrac{\emptyset}{p \Rightarrow q, r} \quad \dfrac{\emptyset}{p, r \Rightarrow q}}{\dfrac{\emptyset}{p \Rightarrow q}} \qquad\qquad \dfrac{\dfrac{\emptyset}{\Rightarrow p} \quad \dfrac{\emptyset}{\Rightarrow \neg p}}{\dfrac{\emptyset}{\Rightarrow q}}$$

Moreover, it is interesting to notice that **TS/ST** follows the path initiated by **ST**. It coincides with **CL** up to a certain inferential level at which it starts to works non-classically, but furthermore a proper characterization of its metametainferential non-classicality shows its intimate relation with **LP**. Thus, not only is **TS/ST** non-classical at the level of its metametainferences, but every metametainference valid in **TS/ST** has a translation in terms of an inference valid in **LP**, as shown in the following fact (which we will not be stopping to prove since it follows from the more general Theorem 4.16 appearing below). It can be interpreted as saying that, inasmuch as **LP** is the logic of the metainferences of **ST**, **LP** is the logic of the metametainferences of **TS/ST**.

Fact 4.5. *For all* $\Gamma \subseteq SEQ^1(\mathcal{L})$ *and* $A \in SEQ^1(\mathcal{L})$*:*

$$\vDash_{\mathbf{TS/ST}} \Gamma \Rightarrow_2 A \quad \textit{if and only if} \quad \vDash_{\mathbf{ST}} lower(\Gamma \Rightarrow_2 A)$$

In other words:

$$\vDash_{\mathbf{TS/ST}} \Gamma \Rightarrow_2 A \quad \textit{if and only if} \quad \vDash_{\mathbf{LP}} lower(lower(\Gamma \Rightarrow_2 A))$$

Thus, one might be tempted to say that **TS/ST** differs from **CL** because they do not agree at the metametainferential level. Two logical systems may be said to be identical only if they coincide with regard not only to their valid inferences and metainferences, but also with regard to their valid metametainferences. Once again, as promising this solution might be, it constitutes an unsatisfactory answer. The reason is that the **ST** phenomenon is indeed endemic, precluding us from identifying **CL** not only with its set of valid inferences, or valid metainferences, but also with any of its set of valid metainferences for any arbitrary large inferential level. We face this impossibility because we can define a **ST**-like system that coincides with **CL** pertaining those metainferences, but differs regarding the next metainferential level. Showing this requires defining a sequence of systems that are progressively more coincidental with **CL**. We turn to this task next.

But before, notice that once we grasp the semantic definition of logical consequence for a given system this settles which formulae, inferences and metainferences of any arbitrary level are valid in it. If we give the definition of logical consequence in L aiming at regular *inferences*—as we did for $\mathbf{K_3}$, **LP**, **ST** and even for **TS**—this consequently settles both the issues of which formulae are valid and the issue of which metainferences are valid. It determines a definitive answer to the former, since we can understand formulae as degenerate cases of inferences with empty premises, and it determines a definitive answer to the latter, since we know how to evaluate metainferences for these logics, applying the notion of local metainferential validity. If, alternatively, we give the definition of logical consequence in L aiming at metainferences—as we did for **TS/ST**—this consequently settles the issue of which formulae and which inferences are valid, along with the issue of which metametainferences (or metainferences of level 2) are valid. All of these for reasons perfectly analogous to the previously given. It is easy to observe that if we give the definition of logical consequence for L_j aiming at metainferences of level k, then this will settle the issue of which metainferences of level m lower than k are valid, along with which metainferences of level n greater than k are valid.

Thus, to construct a sequence of **ST**-like logics we need to give the definition of logical consequence of each of these systems to aim at progressively higher inferential levels. That is, the definition of logical consequence for the first logic—namely, **ST**—will be aimed at regular inferences, for the second—namely, **TS/ST**—will be aimed at metainferences, for the third at metametainferences, and so on and so forth. This is what we intend to do with the pair of definitions appearing below.

Definition 4.6. The collection $\mathbb{ST} = \{L_i \mid i \in \mathbb{N}\}$ of logical systems is recursively defined so that $L_0 = \mathbf{LP}$, $L_1 = \mathbf{ST}$, and for $2 \leq j$, $L_j = \overline{L_{j-1}}/L_{j-1}$ (where $\overline{L_j} = L_n/L_m$ if $L_j = L_m/L_n$).

Definition 4.7. For $2 \leq j$ and $L_j \in \mathbb{ST}$, a metainference $\Gamma \Rightarrow_{j-1} A$, where $\Gamma \subseteq SEQ^{j-2}(\mathcal{L})$ and $A \in SEQ^{j-2}(\mathcal{L})$, we say a SK-valuation v satisfies $\Gamma \Rightarrow_{j-1} A$ in L_j ($v \vDash_{L_j} \Gamma \Rightarrow_{j-1} A$) if and only if $v \nvDash_{\overline{L_{j-1}}} \gamma$ for some $\gamma \in \Gamma$, or $v \vDash_{L_{j-1}} A$.

With these technical tools at hand, we show below that the hierarchy is indeed a collection of systems instantiating the **ST** phenomenon. We do this with the help of three groups of formal results, which we explain and comment after stating them.

First, we argue that the previously defined hierarchy is a set of systems instantiating the **ST** phenomenon, because each logic of the hierarchy can rightfully be called a strict-tolerant logic. In fact, each system of the hierarchy mirrors **ST**'s adoption of the classical standard to be satisfied as the premise of an inference, and the classical standard to be regarded as a counter-exemplifying conclusion.[18] Each logic L_j of the hierarchy is taken to be defined by its notion of validity for j-level metainferences, and in all such cases this

[18]What we mean by this is that in each logic, and for every valuation v, a premise is satisfied by v in

definition requires that the premises of a metainference of level $j-1$ *do satisfy* classical standard to be regarded as the premises of a sound inference of this sort (this being the strict part of the definition) and at the same time that the conclusion inference *does not satisfy* the classical standard to be regarded as a counter-exemplifying conclusion of a level $j-1$ metainference (this being the tolerant part of the definition). These facts are formalized in the statement of the following lemma.

Lemma 4.8. *For all $n \geq 1$, for all $\Gamma, \Theta \subseteq SEQ^{n-1}(\mathcal{L})$, $A, B \in SEQ^{n-1}(\mathcal{L})$, and all SK-valuations v, there is a Boolean valuation v^* such that:*

$$\begin{array}{lccc} \textit{if} & v \nvDash_{L_{n+1}} \Gamma \Rightarrow_n A & \textit{and} & v \vDash_{\overline{L_{n+1}}} \Theta \Rightarrow_n B, \\ \textit{then} & v^* \nvDash_{\mathbf{CL}} \Gamma \Rightarrow_n A & \textit{and} & v^* \vDash_{\mathbf{CL}} \Theta \Rightarrow_n B \end{array}$$

Similarly, for all Boolean valuations v, there is a SK-valuation v^* such that:

$$\begin{array}{lccc} \textit{if} & v \nvDash_{\mathbf{CL}} \Gamma \Rightarrow_n A & \textit{and} & v \vDash_{\mathbf{CL}} \Theta \Rightarrow_n B, \\ \textit{then} & v^* \nvDash_{L_{n+1}} \Gamma \Rightarrow_n A & \textit{and} & v^* \vDash_{\overline{L_{n+1}}} \Theta \Rightarrow_n B \end{array}$$

Proof. We prove the first conditional, by induction on the index of the logic.

Base case: $n = 1$. Assume that there is a SK valuation v such that $v \nvDash_{L_2} \Gamma \Rightarrow_1 A$ and $v \vDash_{\overline{L_2}} \Theta \Rightarrow_1 B$, i.e. $v \nvDash_{\mathbf{TS/ST}} \Gamma \Rightarrow_1 A$ and $v \vDash_{\mathbf{ST/TS}} \Theta \Rightarrow_1 B$. From the fact that $v \nvDash_{\mathbf{TS/ST}} \Gamma \Rightarrow_1 A$ we may infer that $v \vDash_{\mathbf{TS}} \gamma$, for all $\gamma \in \Gamma$ and that $v \nvDash_{\mathbf{ST}} A$. From the fact that $v \vDash_{\mathbf{ST/TS}} \Theta \Rightarrow_1 B$ we may infer that $v \nvDash_{\mathbf{ST}} \theta$, for some $\theta \in \Theta$, or $v \vDash_{\mathbf{TS}} B$. Furthermore, from all these facts and Lemma 4.2 we are guaranteed that there is a Boolean valuation v^* such that, on the one hand, $v^* \vDash_{\mathbf{CL}} \gamma$, for all $\gamma \in \Gamma$ and $v^* \nvDash_{\mathbf{CL}} A$ and, on the other, $v^* \nvDash_{\mathbf{CL}} \theta$, for some $\theta \in \Theta$, or $v^* \vDash_{\mathbf{CL}} B$. But, by definition, this is the same as saying that there is a Boolean valuation v^* such that $v^* \nvDash_{\mathbf{CL}} \Gamma \Rightarrow_1 A$ and $v^* \vDash_{\mathbf{CL}} \Theta \Rightarrow_1 B$.

Inductive step: $n > 1$. Assume that there is a SK valuation v such that $v \nvDash_{L_{n+1}} \Gamma \Rightarrow_1 A$ and $v \vDash_{\overline{L_{n+1}}} \Theta \Rightarrow_1 B$. From the fact that $v \nvDash_{L_{n+1}} \Gamma \Rightarrow_1 A$ we may infer that $v \vDash_{\overline{L_n}} \gamma$, for all $\gamma \in \Gamma$ and that $v \vDash_{L_n} A$. From the fact that $v \vDash_{\overline{L_{n+1}}} \Theta \Rightarrow_1 B$ we may infer that $v \nvDash_{L_n} \theta$, for some $\theta \in \Theta$, or $v \vDash_{\overline{L_n}} B$. Furthermore, from all these facts and the Inductive Hypothesis we are guaranteed that there is a Boolean valuation v^* such that, on the one hand, $v^* \vDash_{\mathbf{CL}} \gamma$, for all $\gamma \in \Gamma$ and $v^* \vDash_{\mathbf{CL}} A$ and, on the other, $v^* \nvDash_{\mathbf{CL}} \theta$, for some $\theta \in \Theta$, or $v^* \vDash_{\mathbf{CL}} B$. But, by definition, this is the same as saying that there is a Boolean valuation v^* such that $v^* \nvDash_{\mathbf{CL}} \Gamma \Rightarrow_n A$ and $v^* \vDash_{\mathbf{CL}} \Theta \Rightarrow_n B$.

The proof of the remaining conditional is similar. □

Second, the previously defined hierarchy is a set of systems instantiating the **ST** phenomenon, because every such system mimics the behavior of **ST**. By this we mean, on the one hand, that all the logics of the hierarchy, after **ST**, have the same valid inferences than **CL**—as witnessed in Lemma 4.9 and Theorem 4.10, whose proofs we

Classical Logic if and only if it is satisfied by the standard for premises, and v is a counterexample to the validity of the conclusion in Classical Logic if and only if v is a counterexample in the logic defined by the standard for the conclusion.

leave to the reader as an exercise. On the other hand, by this we mean that just like **ST** coincides with **CL** up to its valid inferences, for each inferential level j the hierarchy counts with a system L_{j+1} which has the same valid inferences than **CL** up to that inferential level. In this regard, the cumulative nature of the progression of logics in the hierarchy, i.e. the fact that whatever validities obtain in a given logic these are retained in the successive systems appearing in the hierarchy, is represented by Lemmata 4.9 and 4.11. Whereas, the fact that these validities are in fact progressively more and more classical is formalized by Theorem 4.12.

Lemma 4.9. *For all $j \geq 1$, for all $\Gamma, \Delta \subseteq FOR(\mathcal{L})$, for every SK-valuation v:*

$$v \vDash_{L_j} \Gamma \Rightarrow \Delta \quad \textit{if and only if} \quad v \vDash_{\mathbf{CL}} \Gamma \Rightarrow \Delta$$

Theorem 4.10. *For all $j \geq 1$, for all $\Gamma, \Delta \subseteq FOR(\mathcal{L})$:*

$$\vDash_{L_j} \Gamma \Rightarrow \Delta \quad \textit{if and only if} \quad \vDash_{\mathbf{CL}} \Gamma \Rightarrow \Delta$$

Lemma 4.11. *For all $j > n \geq 2$, for all $\Gamma \subseteq SEQ^{n-1}(\mathcal{L})$, $A \in SEQ^{n-1}(\mathcal{L})$, for every SK-valuation v:*

$$v \vDash_{L_n} \Gamma \Rightarrow_n A \quad \textit{if and only if} \quad v \vDash_{L_j} \Gamma \Rightarrow_n A$$

Theorem 4.12. *For all $n \geq 1$, for all $\Gamma \subseteq SEQ^{n-1}(\mathcal{L})$, $A \in SEQ^{n-1}(\mathcal{L})$*

$$\vDash_{L_{n+1}} \Gamma \Rightarrow_n A \quad \textit{if and only if} \quad \vDash_{\mathbf{CL}} \Gamma \Rightarrow_n A$$

Proof. The proof is by induction on the index of the logic.

Base case: n = 1. From left to right, let us suppose that $\nvDash_{\mathbf{CL}} \Gamma \Rightarrow_1 A$, from which we infer that there is a Boolean valuation v such that $v \vDash_{\mathbf{CL}} \gamma$, for all $\gamma \in \Gamma$, and yet $v \nvDash_{\mathbf{CL}} A$. But, since Boolean valuations are a subset of SK valuations, we know by the definition of satisfaction in **TS** and **ST** that v is also a SK valuation such that $v \vDash_{\mathbf{TS}} \gamma$, for all $\gamma \in \Gamma$, and yet $v \nvDash_{\mathbf{ST}} A$. Thus, by the definition of validity of a metainference of level 1 in **TS/ST**, we know that $v \nvDash_{\mathbf{TS/ST}} \Gamma \Rightarrow_1 A$, whence $\nvDash_{\mathbf{TS/ST}} \Gamma \Rightarrow_1 A$. From right to left, let us suppose that $\nvDash_{\mathbf{TS/ST}} \Gamma \Rightarrow_1 A$, from which we infer that there is a SK valuation v such that $v \vDash_{\mathbf{TS}} \gamma$, for all $\gamma \in \Gamma$, and yet $v \nvDash_{\mathbf{ST}} A$. By Lemma 4.2 we know that there is a Boolean valuation v^* such that $v^* \vDash_{\mathbf{CL}} \gamma$, for all $\gamma \in \Gamma$, and yet $v^* \nvDash_{\mathbf{CL}} A$. Therefore, we know that $v^* \nvDash_{\mathbf{CL}} \Gamma \Rightarrow_1 A$, whence $\nvDash_{\mathbf{CL}} \Gamma \Rightarrow_1 A$.

Inductive step: $n > 1$. From left to right, let us suppose that $\nvDash_{\mathbf{CL}} \Gamma \Rightarrow_1 A$, from which we infer that there is a Boolean valuation v such that $v \vDash_{\mathbf{CL}} \gamma$, for all $\gamma \in \Gamma$, and yet $v \nvDash_{\mathbf{CL}} A$. By Lemma 4.8 we know that there is a SK valuation v^* such that $v^* \vDash_{\overline{L_n}} \gamma$, for all $\gamma \in \Gamma$, and yet $v^* \nvDash_{L_n} A$. Therefore, we know that $v^* \nvDash_{L_{n+1}} \Gamma \Rightarrow_n A$, whence $\nvDash_{L_{n+1}} \Gamma \Rightarrow_1 A$. From right to left, let us suppose that $\nvDash_{L_{n+1}} \Gamma \Rightarrow_1 A$, from which we infer that there is a SK valuation v such that $v \vDash_{\overline{L_n}} \gamma$, for all $\gamma \in \Gamma$, and yet $v \nvDash_{L_n} A$. By Lemma 4.8 we know that there is a Boolean valuation v^* such that $v^* \vDash_{\mathbf{CL}} \gamma$, for all $\gamma \in \Gamma$, and yet $v^* \nvDash_{\mathbf{CL}} A$. Therefore, we know that $v^* \nvDash_{\mathbf{CL}} \Gamma \Rightarrow_n A$, whence $\nvDash_{\mathbf{CL}} \Gamma \Rightarrow_1 A$. $\square$

Third, we claim that the previously defined hierarchy is a set of systems instantiating the **ST** phenomenon, because each system of the hierarchy mimics **ST** in coinciding with **CL** up to a certain inferential point, and then starting to behave non-classically in a way that is essentially tied to **LP**. The results below witness this by representing—via the *lower* translation—those inferences valid in the logics of the hierarchy, in terms of inferences of lower inferential levels which are valid in systems appearing previously in the hierarchy. With regard to regular inferences, this is featured in Lemma 4.13 and Theorem 4.14, whose proofs we leave to the reader as an exercise. In this respect, just like the metainferences valid in **ST** (which are different from those valid in **CL**) can be characterized in terms of valid inferences of **LP** via the *lower* translation, it is possible to characterize the metainferences of level j which are valid in the logic L_{j+1} of the hierarchy—and which are, again, different from those valid in **CL**—in terms of valid inferences of **LP**, via successive applications of the *lower* translation, as in Lemma 4.15 and Theorem 4.16, which immediately follows from it.

Lemma 4.13. *For all j, for all $\Gamma, \Delta \subseteq FOR(\mathcal{L})$, for every SK-valuation v:*

$$v \vDash_{L_{j+1}} \Gamma \Rightarrow \Delta \quad \textit{if and only if} \quad v \vDash_{L_j} \textit{lower}(\Gamma \Rightarrow \Delta)$$

Theorem 4.14. *For all j, for all $\Gamma, \Delta \subseteq FOR(\mathcal{L})$:*

$$\vDash_{L_{j+1}} \Gamma \Rightarrow \Delta \quad \textit{if and only if} \quad \vDash_{L_j} \textit{lower}(\Gamma \Rightarrow \Delta)$$

Lemma 4.15. *For all j, for all $n \geq 1$, for all $\Gamma \subseteq SEQ^{n-1}(\mathcal{L})$, $A \in SEQ^{n-1}(\mathcal{L})$, for every SK-valuation v:*

$$v \vDash_{L_{j+1}} \Gamma \Rightarrow_n A \quad \textit{if and only if} \quad v \vDash_{L_j} \textit{lower}(\Gamma \Rightarrow_n A)$$

Proof. We prove this via two nested inductions. The principal induction is on the index of the logic, and the secondary induction is on the inferential level.

<u>Outer base case:</u> $j = 0$. Recall that $L_0 = \mathbf{LP}$ and $L_1 = \mathbf{ST}$.

<u>Inner base case:</u> $n = 1$. Suppose $v \vDash_{\mathbf{ST}} \Gamma \Rightarrow_1 A$. By Definition 4.7, this happens if and only if either $v \nvDash_{\mathbf{ST}} \gamma$, for some $\gamma \in \Gamma$, or $v \vDash_{\mathbf{ST}} A$. In turn, letting γ be of the form $\Delta \Rightarrow \Sigma$, the former happens if and only if $v(\delta) = 1$, for every $\delta \in \Delta$, and $v(\sigma) = 0$, for every conclusion $\sigma \in \Sigma$. Additionally, letting A be of the form $\Pi \Rightarrow \Lambda$, the latter happens if and only if $v(\pi) \in \{0, \frac{1}{2}\}$, for some $\pi \in \Pi$, or $v(\lambda) = \{\frac{1}{2}, 1\}$, for some $\lambda \in \Lambda$. Moreover, the former is the case if and only if $v(\bigwedge \Delta \supset \bigvee \Sigma) = 0$. Whereas, the latter is the case if and only if $v(\bigwedge \Pi \supset \bigvee \Lambda) \in \{\frac{1}{2}, 1\}$. Moving forward, the first can be alternative understood as $v \nvDash_{\mathbf{LP}} \textit{lower}(\gamma)$, while the second can be understood as $v \vDash_{\mathbf{LP}} \textit{lower}(A)$. Finally, either these two facts entails that $v \vDash_{\mathbf{LP}} \{\textit{lower}(\gamma) \mid \gamma \in \Gamma\} \Rightarrow \textit{lower}(A)$, which by definition amounts to $v \vDash_{\mathbf{LP}} \textit{lower}(\Gamma \Rightarrow_1 A)$. The other direction just follows by reversing the same reasoning.

<u>Inner inductive step:</u> $2 \leq n$. Suppose $v \vDash_{\mathbf{ST}} \Gamma \Rightarrow_n A$. This obtains if and only if either $v \nvDash_{\mathbf{ST}} \gamma$, for some $\gamma \in \Gamma$, or $v \vDash_{\mathbf{ST}} A$. Notice that $\gamma \in \Gamma$ and A are inferences of level $n - 1$. Hence, by the Inductive Hypothesis, we can establish that either $v \nvDash_{\mathbf{LP}}$

$lower(\gamma)$, for some $\gamma \in \Gamma$, or $v \vDash_{\mathbf{LP}} lower(A)$. In either of these cases, we have that $v \vDash_{\mathbf{LP}} lower(\Gamma \Rightarrow_n A)$. The remaining direction just follows by reversing the same reasoning.

Outer inductive step: $1 \leq j$.

Inner base case: $n = 1$. Suppose $v \vDash_{L_{j+1}} \Gamma \Rightarrow_1 A$. We know, by Lemma 4.11, that $v \vDash_{L_{j+1}} \Gamma \Rightarrow_1 \Delta$ if and only if $v \vDash_{\mathbf{TS/ST}} \Gamma \Rightarrow_1 \Delta$. This, in turn, happens if and only if for some $\gamma \in \Gamma$, $v \nvDash_{\mathbf{TS}} \gamma$, or $v \vDash_{\mathbf{ST}} A$. Without loss of generality, let γ be of the form $\Delta \Rightarrow \Sigma$. Thus, $v \nvDash_{\mathbf{TS}} \Delta \Rightarrow \Sigma$ if and only if for every $\delta \in \Delta$, $v(\delta) \in \{1, \frac{1}{2}\}$, and for every $\sigma \in \Sigma$, $v(\sigma) \in \{\frac{1}{2}, 0\}$. Similarly, letting A be of the form $\Pi \Rightarrow \Lambda$, $v \vDash_{\mathbf{ST}} \Pi \Rightarrow \Lambda$ implies that for some $\pi \in \Pi, v(\pi) \in \{0, \frac{1}{2}\}$, or for some $\lambda \in \Lambda$, $v(\lambda) \in \{\frac{1}{2}, 1\}$. The former obtains if and only if $v(\bigwedge \Delta \supset \bigvee \Sigma) \in \{\frac{1}{2}, 0\}$, whereas the latter obtains if and only if $v(\bigwedge \Pi \supset \bigvee \Lambda) \in \{\frac{1}{2}, 1\}$. Moving forward, the first can be alternative understood as $v \nvDash_{\mathbf{K_3}} lower(\gamma)$, while the second can be understood as $v \vDash_{\mathbf{LP}} lower(A)$. Moreover, either these two facts entails that $v \vDash_{\mathbf{ST}} \{lower(\gamma) \mid \gamma \in \Gamma\} \Rightarrow lower(A)$, which by definition amounts to $v \vDash_{\mathbf{ST}} lower(\Gamma \Rightarrow_1 A)$. Finally, by Lemma 4.9, this implies $v \vDash_{L_j} lower(\Gamma \Rightarrow_1 A)$. The other direction just follows by reversing the same reasoning.

Inner inductive step: $2 \leq n$. Suppose $v \vDash_{L_{j+1}} \Gamma \Rightarrow_n A$. This obtains if and only if either $v \nvDash_{L_{j+1}} \gamma$, for some $\gamma \in \Gamma$, or $v \vDash_{L_{j+1}} A$. Notice that $\gamma \in \Gamma$ and A are inferences of level $n-1$. Hence, by the Inductive Hypothesis, we can establish that either $v \nvDash_{L_j} lower(\gamma)$, for some $\gamma \in \Gamma$, or $v \vDash_{L_j} lower(A)$. In either of these cases, we have that $v \vDash_{L_j} \{lower(\gamma) \mid \gamma \in \Gamma\} \Rightarrow_n lower(A)$, which by definition amounts to $v \vDash_{L_j} lower(\Gamma \Rightarrow_n A)$. The remaining direction just follows by reversing the same reasoning. □

Theorem 4.16. *For all j, for all $n \geq 1$, for all $\Gamma \subseteq SEQ^{n-1}(\mathcal{L})$, $A \in SEQ^{n-1}(\mathcal{L})$:*

$$\vDash_{L_{j+1}} \Gamma \Rightarrow_n A \quad \textit{if and only if} \quad \vDash_{L_j} lower(\Gamma \Rightarrow_n A)$$

In other words:

$$\vDash_{L_{j+1}} \Gamma \Rightarrow_n A \quad \textit{if and only if} \quad \vDash_{\mathbf{LP}} \underbrace{lower...(lower}_{n\ times}(\Gamma \Rightarrow_n A)...)$$

All the previously discussed results are illustrated in the next figure. In the vertical axis we have the indication of the inferential levels being evaluated, while in the horizontal axis we have the systems of the hierarchy as they are recursively generated.

$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$	
metametainferences	w	$\neq$	w'	$\neq$	w''	$\neq$	w'''	$\neq$	w	$\dots$
metainferences	z	$\neq$	z'	$\neq$	z''	$\neq$	z	$=$	z	$\dots$
inferences	y	$\neq$	y'	$\neq$	y	$=$	y	$=$	y	$\dots$
theorems	x	$=$	x	$=$	x	$=$	x	$=$	x	$\dots$
			L_0		L_1		L_2		L_3	$\dots$
	CL		**LP**		**ST**		**TS/ST**			

To sum up, as we previously remarked, the hierarchy allows to convey a sort of philosophical *impossibility result*: that it is impossible to identify a logic with its given set of valid inferences of any inferential level—for it is possible to find logics which will coincide in that respect, despite of being intuitively different from it. Such logics are brought to our attention through the **ST** phenomenon. This phenomenon, already as presented in the literature, allows to notice that the commonly used identity criterion for logics—according to which a logic is a consequence relation between (collections of) formulae—cannot be entirely right. This is because, as previously argued, even if both **ST** and **CL** have the same valid inferences they are intuitively different logics. But the **ST** phenomenon is not circumscribed only to the logic **ST**. Indeed, the logics in our hierarchy show that it is equally dissatisfying to identify a logic with a set of valid metainferences—for what it is worth, systems like **TS/ST** and **CL** agree in this respect, despite of being radically different logics. Or to identify a logic with its set of valid metametainferences, or with its set of valid metainferences of level 3, or 4, ..., or any arbitrarily large level. Whence, the need of an appropriate identity criterion for logical systems that can cope with cases of this sort.

Before moving to the further difficulties that coming with such a solution brings, we would like to comment on two related issues connected with the **ST** phenomenon. The first one is the characterization of paraconsistency, as a property of logical systems in general. While normally paraconsistency is associated with the failure of some version of Explosion—i.e. with an inference being invalid—the results hinted at here highlight the possibility of a different phenomenon, that of paraconsistency associated with the failure of metainferential versions of Explosion, which we might call substructural paraconsistency. Thus, there may be logics which are paraconsistent in the usual sense and in this new sense, just like there may be logics which are paraconsistent in only one way but not the other, i.e. logics with homogeneous policies and logics with heterogeneous policies. The existence of such systems is confirmed by the previous results, which generalize the discussion of (anonymized). In fact, the hierarchy exhibits logics which have a certain degree of classicality and a certain degree of paraconsistency. This is to be expected, given the hierarchy is constructed in the image of **ST**, which behaves classically up to the inferential level, but behaves "paraconsistently" from the metainferential level onward. Something essentially analogous happens for every logic in our hierarchy.

Particularly, for each j, the logic L_j behaves classically in the inferential levels below j, and then "paraconsistently" from the inferential level j.

The second issue on which these results have a significant echo is that of Logical Pluralism, especially as pertains to the so-called Collapse Argument. In essence, the various forms of the Collapse Argument for Logical Pluralism—as discussed e.g. in [25], [18], [20], and [15]—aim at showing how someone embracing a plurality of logics might find herself being a Logical Monist at the end of the day, given the normative guidance that one is expected to have from logical systems in general. Were a Pluralist to believe the premises of an inference that is valid according to one of the logics she embraces, but not to the other, should she believe the conclusion or not? Since there can be no pluralism about this, Logical Pluralism collapses into Logical Monism.

Interestingly, our results resonate in this debate because two different logics need not differ with regard to their valid inferences, but in other respect—e.g. concerning their valid metainferences of some level. But, if a Logical Pluralist were to embrace two logical systems with the same set of valid inferences, then she would not be affected by the current form of the Collapse Argument. This observation should lead those opposing the Pluralist view to refine their argument, in order to adapt that not only to logics with the same valid inferences and different valid metainferences, but to logics with the same valid metainferences of level n and different valid metainferences of level $n+1$. In this respect, a refined Collapse Argument could be designed, along the following lines. Suppose a Pluralist embraces two different logics with the same inferences of level n but different inferences of level $n+1$. Were she to believe the premises of an inference of level $n+1$ that is valid according to one of the logics she embraces, but not to the other, should she believe the conclusion or not? Since there can be no pluralism about this, Logical Pluralism collapses into Logical Monism once again.[19] A precise formulation of this strategy was presented by us in [2]—where in all the remaining relevant respects no argument either for or against Logical Pluralism is put forward.

Finally, let us address the elephant in the room. How are we going to identify logical systems if the usual option of identifying it with its set of valid inferences is no good, and neither is any of the intuitive fixes leading to identify it with its set of valid metainferences of some arbitrarily large level n? Our proposal is, precisely, to embrace these facts, by identifying a logic with its set of valid inferences at every inferential level. Thus, what may suffice to differentiate a pair logical systems could be their set of theorems, but that might not be sufficient for another pair, for which their set of valid inferences might be required. However, there might be yet another pair for which that could be still insufficient, requiring to draw the attention to their valid metainferences, as could be the case with a different pair of logics whose valid metainferences may be identical, thereby highlighting the need to pay attention to their valid metametainferences—i.e. their valid metainferences of level 2. And so on, and so forth. It is easy to see that no distinguished inferential level can be the stopping point, although all of them could be of some use in differentiating certain logical systems.

To put our proposal differently, then, we may claim that a logic is to be identified

[19] We would like to thank an anonymous reviewer for suggesting us to further comment on this issue.

with *an infinite sequence of consequence relations* revealing which formulae are valid, which inferences are valid, which metainferences of level 1 are valid, which metainferences of level 2 are valid, ..., and which metainferences of level n are valid—for every natural number n. What we are claiming is that however a system is defined—at least from the point of view of its semantics—this determines which formulae, inferences, and metainferences of any arbitrary large inferential level are valid. Putting this sets of validities one next to each other and forming a sequence allows to differentiate this system from other systems highly resembling it in such a clear way, which suggests taking this sequence to constitute the identity, the DNA of a logic.

Remarkably, such a criterion allows to provide an answer to our initial question: what identifies **CL** as such? It is, in fact, the inferences it validates at every inferential level that are crucial to its identity and which allow to tell it apart not only from closely related systems like **ST**, but also from systems that are even more similar to it—like **TS/ST** or any of the recursively defined systems appearing in our hierarchy, for that matter. What identifies **CL** are all its valid inferences of every inferential level, and each of those systems differ with it at some point. This clearly explains why **ST** is not **CL**, and neither is **TS/ST** or any of the systems of our hierarchy. Subsequently, this criterion makes it also easy to tell the difference between the systems in the hierarchy themselves.

A further advantage of such a criterion is that it induces an interesting scale of classicality for logical systems and, in particular, for those systems in the hierarchy. We can certainly say that a system is similar to **CL** if it agrees with it to some extent, i.e. with regard to its theorems, or inferences, or metainferences, or something else. But, more concretely, we can say that a certain logic is k-*similar* to **CL** if it agrees with it up to the metainferential level k—i.e. it coincides with regard to their valid metainferences of level $i \leq k$, as well as their valid inferences and their theorems. In this respect, for instance, **ST** would be 0-similar while **TS/ST** would be 1-similar to **CL**. This helps to cash out formally the fact that, in some intuitive sense, **TS/ST** is classical to a greater degree than **ST**, whereas **ST** itself is classical to a greater degree than e.g. **LP**. Thus, in this vein, two logics formulated over the same language will be identical in our proposed sense if and only if they are n-similar, for every natural number n.

After these remarks, in the next section we consider some objections that may be directed at our account, and provide appropriate responses for each of them.

5 Replies to some objections

One of the plausible things that can be called into attention as regards our discussion is the way in which metainferences are ruled to be either valid or invalid. In this respect, it is worth mentioning that we adopted a semantic perspective towards this issue—not that we are not interested in other understandings of these things, but we will not be commenting on them or on their relation with the semantic readings of metainferential validity in the context of this article.[20]

This being said, it is the rather different available semantic approaches to this phenomenon which should be evaluated, in particular, with the aim of telling some kind of story that explains why we decided to go with the *local* and not with another understanding of metainferential validity. Remember that, as discussed in Section 3, the difference between local and global metainferential validity consists in nothing more than the difference between satisfaction-preservation and validity-preservation in a certain logic. To wit, taking **ST** as an example, from the below the leftmost is locally and globally valid, the middle one is globally but not locally valid, and the rightmost is neither locally nor globally valid.

$$\frac{p, q \Rightarrow p \quad p, q \Rightarrow q}{p, q \Rightarrow p \wedge q} \qquad \frac{p \Rightarrow q \quad q \Rightarrow r}{p \Rightarrow r} \qquad \frac{p \wedge q \Rightarrow p}{p \Rightarrow p \wedge q}$$

In this regard, we think that there is a considerable conceptual advantage in ruling metainferential validity locally. It comes from the fact that the local reading allows to provide a unified account of validity for all kinds of inferences—that is, for inferences between formulae, between inferences, between metainferences, and so on. While ruling over the validity of regular inferences we require that, if the premises are satisfied (according to the standard for premise-formulae to be satisfied in the logic in question), then at least one of the conclusions is satisfied (according to the standard for conclusion-formulae to be satisfied in the said logic). Ruling over the validity of metainferences with a local criterion asks, precisely, the same thing. Thus, if we are going to think about logic as a discipline concerned with the study and analysis of valid inferences, adopting something different than the local understanding of metainferential validity would mean that validity is understood one way as pertaining to formulae and another way as pertaining to inferences. But this, we think, seems rather undesirable. Having a unified stance towards validity, regardless of its relata, seems like a promising and interesting endeavor, and the local reading allows for this unification. This is why we think local validity is the way to go.

This being said, the global understanding could, in principle, be equally suitable for a cogent analysis of this sort. However, global validity as applied to metainferential validity has a considerable flaw, i.e. it vastly overgenerates the set of valid metainferences. In fact, it is enough to have a metainference whose premise-set contains an invalid inference, for the corresponding metainference to be globally valid—albeit in a trivial sense. This is not the case with the local criterion, even if the premise-inferences are invalid. Nevertheless, this does not represent in itself a sufficient reason for discarding global validity. A further ground for the rejection of global validity is its uselessness for the study of validity as a unified phenomenon. Indeed, were we to adopt that global reading for the study of all kinds of inferences, we would need to do this for regular inferences too. This would require of valid inferences that, instead of preserving truth from premises to conclusion, they preserve the property of being a tautology. But this is

[20] We mention this not because there is a special connection between ruling metainferences to be either valid or invalid and adopting a semantic approach, but because there are other available ways of doing so both semantically (e.g. global validity) and proof-theoretically (e.g. admissibility, derivability).

seriously non-standard, as it would deem that the inference $p \Rightarrow q$ is valid—just because p is not a tautology—as it would deem any inference with an invalid premise-formulae, as valid. Given this, we think enough reasons have been presented to favor the local instead of the global understanding.

An additional objection, claiming that our criterion for the identity of logical systems is wrong, can be extracted from Dicher and Paoli's work, [12] and [11]. These authors favor the identification of logics with a certain class of abstract consequence relations—as defined by Blok and Jónsson in e.g. [5]. This, together with looking at **ST** as a consequence relation holding between sets of inferences and inferences, renders the idea—built on the results of [4]—that **ST** is not **CL**, but **LP**.

For Dicher and Paoli to conclude this, it is essential that a logic shall be identified with the logical system characterizing the valid inferences between the intended relata of the system—which in the case of **ST** are sequents or inferences, insofar as it is mainly motivated as a sequent calculus. This can be illustrated by the following quote.

> Notice, however, that in a sequent calculus all of the action takes place at the level of sequent-to-sequent rules, whereby from one or more sequents (intuitively understood as 'inferences') we derive more sequents (i.e., more 'inferences'). Which is to say, the action takes place at the level of metainferences. [12, p. 8]

Whence, the system to be highlighted is the one characterized by the valid inferences of this sort, i.e. the valid metainferences. This system happens to be Graham Priest's **LP** and, therefore, **ST** is not **CL** but **LP**.

Before we start discussing their conclusion, let us notice the following. There is no doubt that Dicher and Paoli's utilization of the Blok-Jónsson approach has, at least, one advantage. For what it is worth, it allows to establish something we have an intuitive grasp of, i.e. that there can be and that there are different presentations of the same logic. This is indeed something we informally assume, for instance, when we talk about various proof-systems (Hilbert-style axiomatizations, natural deduction calculi, sequent calculi, etc.) for a given logic—implicitly admitting that the consequence relation in question can hold between different relata (formulae, equations, sequents, etc.)

However useful this may be, we think Dicher and Paoli's point can be legitimately resisted. The strategy we are going to apply, for this purpose, is going to be fair and clear. Their claims can be disputed, because their identity criterion for logics can be. This, furthermore, is possible due to the fact that the said criterion severely undergenerates, i.e. it does not qualify as logics many systems which a great deal of people take to be genuinely so. In rejecting their criterion we will subsequently reject their utilization of it and, furthermore, the idea that **ST** is **LP**.

The most important kind of logics that Dicher and Paoli's identity criterion dismisses are substructural logics. Why so? Precisely, because the Blok-Jónsson definition of an abstract consequence relation is required to satisfy the Tarskian axioms. Thus, if for something to be counted as a genuine logic it must be a class of abstract consequence relations satisfying the Tarskian axioms, then substructural logics would be rendered as non-existent. In fact, something similar can be said about multiple-conclusion logics. Again, if for something to be counted as a genuine logic it must be a class of single-conclusion abstract consequence relations satisfying the Tarskian axioms, then multiple-conclusion logics would be automatically rendered as non-existent.

Interestingly, these authors may argue that a generalization of the Blok-Jónsson abstract approach to logical consequence could help reinstate the logical status to substructural (and multiple-conclusion) systems. Such modification would only require of an abstract consequence relation over an arbitrary collection A to be a relation without asking for any of the Tarskian axioms to be satisfied—except, presumably, for substitution invariance. This, we think, will still give unintuitive results. In particular, considering how sequent calculi are associated to logical systems according to Dicher and Paoli, this will classify *some* systems as substructural (or multiple-conclusion) although it will *not* classify as such the ones that we usually take to be so.

To observe this notice that, as discussed in Section 2 the failure of e.g. Reflexivity, Monotonicity, and Transitivity for the logic associated to a sequent calculus are usually characterized by the failure or non-derivability of the metainferences discussed in Section 2. However, if we accept Dicher and Paoli's idea that in the context of a sequent calculus the action happens at the level of the sequent-to-sequent derivations—i.e. the inferences are from collections of sequents to sequents themselves—all of the above are not faithfully represented by the metainferences depicted above.

For starters, Reflexivity is usually understood as the property by means of which any inference having the same object as premise and as conclusion is valid. But, if inferences are only sequent-to-sequent derivations, then according to the abstract approach to logical consequence as applied to sequent calculi Reflexivity is better expressed by the following metainference—which properly speaking non-reflexive logics should invalidate.

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

Furthermore, Monotonicity is normally understood as the property by means of which any valid inference can have its premise-set or its conclusion-set legitimately augmented without leading to an invalid inference. Nevertheless, if inferences are only sequent-to-sequent derivations, then according to the abstract approach as applied to sequent calculi, then Monotonicity is better expressed by the following metainference of level 2—which properly speaking non-monotonic logics should invalidate.

$$\frac{\dfrac{\Gamma_1 \Rightarrow \Delta_1 \quad \dots \quad \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}}{\dfrac{\Gamma_1 \Rightarrow \Delta_1 \quad \dots \quad \Gamma_n \Rightarrow \Delta_n \quad \Sigma \Rightarrow \Pi}{\Gamma \Rightarrow \Delta}}$$

Finally, when applied to single-conclusion inferences, Cut is commonly understood as the property by means of which if an inference leads to a certain conclusion, and augmenting the premise-set of this inference with the said conclusion leads to a valid inference with another conclusion, then the inference with the original premise-set and the last conclusion was already valid. However, if inferences are only sequent-to-sequent derivations, then according to the abstract approach as applied to sequent calculi, then Cut (as applied to single-conclusion inferences) is better expressed by the following metainference of level 2—which genuine non-transitive logics should invalidate.

$$\dfrac{\dfrac{\Gamma_1 \Rightarrow \Delta_1 \quad \dots \quad \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta} \qquad \dfrac{\Gamma \Rightarrow \Delta \quad \Gamma_1 \Rightarrow \Delta_1 \quad \dots \quad \Gamma_n \Rightarrow \Delta_n}{\Sigma \Rightarrow \Pi}}{\dfrac{\Gamma_1 \Rightarrow \Delta_1 \quad \dots \quad \Gamma_n \Rightarrow \Delta_n}{\Sigma \Rightarrow \Pi}}$$

We certainly think the previously mentioned are some seriously disappointing technical and conceptual consequences of applying a Tarskian version of the abstract approach to logical consequence, in order to individuate logical systems.[21] For what it is worth, if this is how the story goes with the identity criterion applied by Dicher and Paoli to conclude that **ST** is **LP**, we consider that these shortcomings are enough to disregard it and, therefore, to discredit their identification of **ST** with Priest's logic.

Concerning **ST**, then, we accept that part of the action takes place at the level of sequent-to-sequent inferences, but just like a part of the expressive richness of a phenomenon would be lost if we identified a logic with the consequence relation holding between formulae, another part would be lost if we only considered the consequence relation holding between inferences. This is something our construction gives the opportunity to reflect upon.

6 Conclusion

In this article we proved a number of positive and negative results, and discussed certain philosophical consequences of them. Concerning the positive results, we first generalized the **ST** phenomenon, thereby obtaining a recursively defined hierarchy of strict-tolerant systems. Secondly, we proved that the systems of the hierarchy are progressively more classical. Thirdly, we showed that when the systems of the hierarchy stopped behaving classically, they started to behave paraconsistently in a way essentially tied to **LP**. Concerning the negative results, we argued that the previously mentioned facts suggest the impossibility of identifying **CL** with its set of valid inferences, or its set of valid metainferences of level 1, or its set of valid metainferences of any arbitrary large inferential level, for that matter—since there will always be a system in the hierarchy which will coincide with **CL** in this regard, although not in all inferential levels.

Drawing inspiration from the above, we proposed an alternative criterion to characterize **CL**, identifying it with an infinite sequence of consequence relations pertaining inferences, metainferences, metametainferences, and so on and so forth—suggesting this criterion should be applied to all logical systems, with full generality. It is important to notice that the discussed hierarchy constitutes only the beginning of a number of

[21] Moreover, for this approach to represent multiple-conclusion logics associated with sequent calculi, it would not be enough to allow sequents whose succedents have greater-than-one cardinality. Instead, it would require to have metainferences whose conclusion is not a single sequent, but a set of sequents as below—where, it might be assumed to keep the symmetry with the ongoing discussion, such a transition should be read as allowing to infer at least one of the conclusion sequents.

$$\dfrac{\Gamma_1 \Rightarrow \Delta_1 \quad \dots \quad \Gamma_n \Rightarrow \Delta_n}{\Sigma_1 \Rightarrow \Pi_1 \quad \dots \quad \Sigma_n \Rightarrow \Pi_n}$$

explorations. There is an endless amount of different, equally interesting hierarchies that can be constructed and put to good use to motivate and clarify many philosophical debates. We have reflected upon the impossibility of identifying **CL** extensionally with its set of valid inferences of any arbitrary large inferential level, but a similar move could also be done with other logics—thereby requiring to look at other hierarchies, yet to be defined.

Finally, we have decided to only look at the finite levels of the hierarchy. Thus, the problem of giving a definition of logical consequence for the system located at the first limit ordinal of the hierarchy was beyond our scope. This, and many other issues, represent deeply interesting topics which we hope to discuss in future work.

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