APPROXIMATION REPRESENTATIONS FOR REALS AND THEIR wtt-DEGREES

GEORGE BARMPALIAS

ABSTRACT. We study the approximation properties of computably enumerable reals. We deal with a natural notion of approximation representation and study their wtt-degrees. Also, we show that a single representation may correspond to a quite diverse variety of reals.

1. INTRODUCTION

We are interested in Δ_2 reals and in particular on their effective approximation properties. For some work done in this area of computable analysis we refer to the survey Zheng[7]. However this paper deals with a different approach, based on earlier work by the author (see [1],[2]). By a well known fact, these are the reals that are limits of computable sequences of rationals. To study these properties we introduced (see Barmpalias[1],[2]) a notion of an *approximation representation* of a Δ_2 real. Let $x = \lim z$ where z is a computable sequence of rationals converging symmetrically to (i.e. having infinitely many terms on each side of) x. These assumptions will be made without notice throughout this paper. We say that the set

$$A_t = \{i \mid z_i < x\}$$

is the *approximation representation* (or simply representation) of x, corresponding to z. Obviously, a real can have more than one representation. The set A_t represents the way that z approximates x. Note that we are not studying the left cuts of Δ_2 reals, but the set of indices of the terms of some z converging to x, which are on the left of x. So the work in Calude et.al. [3] is different from ours. We have shown a number of results about approximation representations in [1],[2] which we do not need to discuss here. So detail relating to any facts mentioned below which are not entirely obvious can be found in these references.

So far we have been especially interested in c.e. representations. A representation of a real is c.e. iff the real is c.e. (in the sense that there is a computable increasing sequence of rationals converging to it). So in the rest

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of the paper we assume that all reals and representations are c.e. There are three main questions we would like to ask:

- First, how do the representations of a fixed real x relate computationally to each other? We have shown that they are all T-equivalent but with respect to stronger reducibilities like wtt or m, they may (or may not, depending on the real) be very varied. Also, under a strong reducibility r they form a substructure of the r-degrees inside the T-degree of x.
- Secondly, given some representation, how do the reals which have this representation relate to each other computationally? Of course they are T-equivalent, but we show in the following that they can be quite diverse with respect to stronger reducibilities. Theorem 1 says that there is a wtt-computably independent set of c.e. reals which have a common representation A. A (countable) set of reals $\{x_i\}$ is wtt-computably independent if $x_i \not\leq_{\mathrm{wtt}} \oplus_{j\neq i} x_j$ for all i. This means that no element x_i can be computed in a wtt fashion from the rest of the elements in the set.[1]
- The third goal we want to achieve is a complete characterization of the (c.e. of course) wtt degrees which contain representations (with reference to any real). In [2] we characterized representations as the *c.e. cuts of computable orderings of order type* $\omega+\omega^*$ (here ω^* is the inverse of ω). So these sets are quite interesting in many ways, and it is natural to ask which degrees they occur in. They obviously occur in every T-degree, and so we turn to look at stronger reducibilities (such as wtt). We have shown in [1] that any non-computable representation (which is what we are really interested in) is a hypersimple set; and since the wtt-complete degree contains no hypersimple sets (by a classical result), it contains no representations. So indeed there are representation-free c.e. wtt-degrees. But are there hypersimple such degrees? In theorem 2 we show not only that there are, but also that there is a certain freedom in constructing them. In fact we construct entire upper cones of wtt-degrees, free of representations. By an upper cone (with bottom $\mathbf{a}$) we mean the set $\{\mathbf{x} \mid \mathbf{a} \leq \mathbf{x}\}$ (for a fixed notion of degree, and the order associated with it). The proof and particularly the strategy for the cone construction is especially interesting, as we have not encountered it before. In theorem 3 we show downward density of the representation wtt-degrees (i.e. the ones containing representations) in the c.e. ones. In other words, any non-computable c.e. set wtt-bounds a non-computable representation.

  In theorem 4 we construct a non-zero T-degree which bounds no bottom of a representation-free cone of wtt degrees (like the ones

[1] this is analogous to the term ‘recursively (or computably) independent’ which refers to T-reducibility.

constructed in theorem 2). The proof of this result is especially interesting as it is an infinite injury where the restraint imposed by a single requirement can tend to infinity (i.e. has no lim inf).

We assume some background in computability theory and especially relating to priority arguments (finite, infinite and tree arguments). For this we refer to [6],[4], [5]. Unexplained notation in this paper is quite standard. When we write $\Phi^A = B; \phi$ we mean that the reduction of B to A is wtt (i.e. that the use ϕ is computable).

In priority constructions (particularly) it is very helpful to have an intuitive picture of what's going on. For this reason we describe briefly how we picture the construction of a representation A. We define a sequence z (which will eventually tend symmetrically to a limit) and a non-decreasing sequence y in $[0, 1]$. Our aim is to build A_z $(= A)$ so that it satisfies certain computational properties. Whenever we want to enumerate a number $n \searrow A_z$ we wait until $z_n \downarrow$ and let y be greater than z_n. The interval $[0, y_s]$ is called the *black area* (at stage s) and so enumeration into A_z is done by expansion of the black area. The distinctive feature of representation constructions is that when you enumerate $n \searrow A_z$ you *have to* enumerate all k such that $z_k \leq z_n$ into A_z. An illustration is given in figure 1.

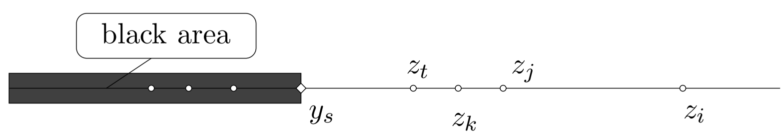

Figure 1: Representation constructions

2. Different reals with common representation

We begin with

Theorem 1. *There is a wtt-computably independent set of c.e. reals which have a common representation A.*

Proof. We want to build a representation A and symmetrically converging sequences $z^i \to x_i$ such that $A_{z^i} = A$ and $x_i \not\leq_{\text{wtt}} \oplus_{j \neq i} x_j$ for all $i \in \mathbb{N}$. Our requirements are

$$N_{\langle e,i \rangle} : \Phi_e^{\oplus_{j \neq i} x_j} \neq x_i; \phi_e$$

and we are going to build the sequences and reals in our usual framework. For each real x_i we have a sequence y^i which converges monotonically to x_i. At any stage s the interval $[0, y^i_s]$ is the i-*black area* and y^i_s is our current approximation for x_i. At all times we ensure that all A_{z^i} are equal to the representation A we are constructing. This means that if the i-black area expands and covers new z^i-terms, we *assume* that the indices of these terms are enumerated into A. Moreover we motivate the expansion of other j-black areas (i.e. for those j for which there are *defined* z^j terms outside the

j-black area) so that we preserve $A = A_{z^j}$ for all j. This *chain reaction* will happen for only finitely many j since at any given stage only finitely many z^j terms (for any j) are defined. In fact, at stage s we define z^j_s for all $j \leq s$ (so at s the defined z-terms are z^j_t for all $j \leq t \leq s$).

All the parameters in the construction will be finite binary rationals (i.e. rational numbers with a finite binary expansion). The strategy to satisfy $N_{\langle e,i\rangle}$ is the following: we start at a stage s by choosing a finite binary sequence q such that $q0 \sqsubseteq y^i_s$ (we think of rationals both as binary expansions and binary sequences). This can be done at any stage since y^i_s is finite and can be assumed to have a suffix of any (finite) number of zeros. We impose restraints (on the growth of y^i) to ensure $q0 \sqsubseteq x_i$ and wait until $\Phi_e^{\oplus_{j\neq i} x_j}(n) \downarrow = 0; \phi_e$ where $n = |q| + 1$. If we never get this computation, our restraints will guarantee the satisfaction of $N_{\langle e,i\rangle}$.

If we get it, say at stage s_0, we would like to define $y^i = q1$ (in order to create a disagreement). But this increase in y^i may motivate an enumeration into A and so (by the chain reaction described above) a change in $\oplus_{j\neq i} x_j$ below the use. In this case we will not be able to preserve the disagreement. To deal with this problem, we first set y^i to be the largest z^i-term less than $q1$ and j-restrain $(p_j, 1)$ where

$$p_j = y^j \restriction \phi_e(n) + 2^{-s_j} \tag{1}$$

and s_j is the largest 0-position in $y^j \restriction \phi_e(n)$, for all j involved in the use (for those j that $s_j \uparrow$ we do not put any restrain). We also require any new z^j term to be defined outside $(p_j, 1)$, for those j (so that a following action $y^i = q1$ will only cause changes in the expansion of x_j outside the use). Now we wait until $\Phi_e^{\oplus_{j\neq i} x_j}(n) \downarrow = 0; \phi_e$. If we don't get it, $N_{\langle e,i\rangle}$ is satisfied as before. Otherwise the use will be the same and setting $y^i = q1$ will create disagreement without spoiling the computation. That is because if x_j changed below the use, this would be because some term z^j_t motivated a y^j expansion (due to a $t \searrow A$ enumeration). But such z^j_t terms were defined after stage s_0, and so were defined in order not to motivate any such change in the expansion of x_j below the use (which was the same as now). Hence this would lead to a contradiction.

Finally we will define z^i_s in the middle of (y^i_s, m') where

$$m' = \min\{m, z^j_k \mid k \notin A[s]\}$$

and $(m, 1)$ is the strictest j-restraint imposed currently. This ensures that the z^j-terms are defined close enough to $\lim y^j$ so that $\lim z^j = \lim y^j$. Next, we lay out the formal module for $N_{\langle e,i\rangle}$ which is actually a part of the construction. Note that here we take into account restraints imposed by higher priority requirements. During the construction each requirement imposes j-restraints for various j. Such a restraint is of the form 'don't let y^j enter $(p, 1)$'. Note these restraints imply restraints on A: if $(p, 1)$ is j-restrained and contains z^j_k then k is restrained from A.

(1) Choose a prefix of the current y^i-approximation to x_i with last digit 0, i.e. some q with $q0 \sqsubseteq y^i[s]$) such that $q1$ is not i-restrained by a higher requirement, and it doesn't sit on any defined z^j-term for any j. i-restrain $(q1, 1)$.

(2) Wait until

$$\Phi_e^{\oplus_{j\neq i} x_j}(n) \downarrow = 0; \phi_e \tag{2}$$

where $n = |q| + 1$.

(3) Let $y^i = z^i_t$ where z^i_t is the largest z^i-term less than $q1$. For each j involved in the use of (2) j-restrain $(p_j, 1)$ where p_j is defined in (1). Wait until (2) is restored. *By this action, A-enumeration occurs and so, various y^j-black areas move. This will not affect higher priority requirements because $(0, q1)$ is not restrained by them.*

(4) Drop the i restraints of step (1) and set $y^i = q1$; also i-restrain $(z^i_k, 1)$ where z^i_k is the least z^i-term $> q1$. *The use $\oplus_{j\neq i} x_j \restriction \phi_e(n)$ doesn't change because for the $k \searrow A$ by this action, z^j_k were defined after step (3) and so they are $< p_j$ (by the way we define z-terms, see below). The disagreement will be preserved by keeping the i-restraints of this step and the j-restraints of step (3).*

Now the *construction* is as follows. For all j set $y^j_0 = 0$. At $s > 0$

(a) Define z^j_s (for each $j < s$) in the middle of (y^j, m') where $m' = \min\{m, z^j_k \mid k \notin A\}$ and $(m, 1)$ is the strictest j-restraint imposed by any requirement at the moment.

(b) Let the least requirement which requires attention (i.e. is ready to move step) act and initialize lower requirements (i.e. set their modules in step (1) and cancel their restraints).

(c) For all j, k, if $y^j_{s-1} < z^j_k$ and $k \in A$ then set $y^j_s = z^j_k$ for the max such z^j_k. *This ensures $A = A_{z^j}$ for all j.*

Now we do the *verification* of the construction. It is evident that for all $n, k \notin A$,

$$n > k \iff z^j_n < z^j_k$$

for all j. And so, by step (c) of the construction, $A = A_{z^j}$ for all j. Now we prove by induction that each $N_{\langle e,i\rangle}$ is satisfied and eventually ceases requiring attention. The induction step for $N_{\langle e,i\rangle}$: assume that after s_0 no higher priority requirement requires attention. $N_{\langle e,i\rangle}$ will receive attention and step (1) of the module will be performed. Note that y^j, z^j, p_j, q all take values of finite binary rationals $\mathbb{Q}_2$ since this set is closed under addition and division by 2. So, since only finitely many restraints are imposed by higher priority requirements, q will be found in step (1). If we wait forever in step

(2) of the module, we are done since then $x_i < q1$ ($y^i \not\to q1$ because of the infinitely many requirements with empty functionals, and the restraint they impose in (1)).

Otherwise we pass on to (3) and, if stuck forever, we are done for the same reasons. Otherwise the use of the restored computation is again $\phi_e(n)$ and $\oplus_{j\neq i} x_j \upharpoonright \phi_e(n)$ same as just after we acted in (3), due to the j-restraints and the induction hypothesis. So we pass on to (4) and the z^i-terms in $(y^i, q1)$ have indices $k > s_1$, the stage when (3) was executed. So for those k and the j involved in the use of (2), $z^j_k < p_j$ and so, enumeration into A will not spoil (2) (under step (c) of the construction). So at step (4) we put $y^i = q1$ and preserve the disagreement by restraining $(z^i_k, 1)$.

This concludes the induction step and the only thing left to show is that $\lim y^i = \lim z^i$ for all i. Fix i: y^i converges as non-decreasing and bounded. By the construction we have

$$z^i_{s+1} \leq \frac{y^i_{s+1} + \lambda^i_s}{2} \tag{3}$$

where $\lambda^i_s = \min\{z^i_k \mid k \notin A[s]\}$. Let $\{j_s\}$ be a monotone enumeration of $\mathbb{N} - A$. By (3),

$$z^i_{j_{s+1}} \leq \frac{y^i_{j_{s+1}} + z^i_{j_s}}{2}.$$

Let

$$\begin{aligned} a_0 &= z^i_{j_0} \\ a_{s+1} &= \frac{x_i + a_s}{2}. \end{aligned}$$

For all s, $a_s \geq z^i_{j_s}$; indeed, it holds for $s = 0$ and if $a_s \geq z^i_{j_s}$ then

$$a_{s+1} = \frac{x_i + a_s}{2} \geq \frac{y^i_{j_{s+1}} + z^i_{j_s}}{2} \geq z^i_{j_{s+1}}.$$

So $\lim_s a_s = \lim_s z^i_{j_s} = x_i$. Now it is easy to see that $\lim_s z^i_s = x_i$, which finishes the proof. □

3. Wtt-degrees of representations

We noticed in [1],[2] that any representation of a non-computable c.e. real is a hypersimple set. And since the wtt-complete degree contains no hypersimple sets (by a well known result), this degree contains no representations. This raises the question which c.e. wtt degrees contain representations (note that every c.e. T-degree contains such). Are they the hypersimple ones? The following theorem says that there are entire upper cones of wtt-degrees, free of representations. Moreover the bottoms of these cones can avoid any specified non-trivial upper cone of T-degrees; and can even have hypersimple wtt-degree (which means that the wtt-degrees containing representations are properly contained in the hypersimple ones).

In [2] we also noted that the representations of c.e. reals (from now on, just *representations*) are exactly the c.e. cuts of computable orderings of $\mathbb{N}$ of order type $\omega+\omega^*$. So the results below can be stated in terms of computable orderings. If A is a representation, there is a computable ordering of $\mathbb{N}$ determined by a computable function ψ (i.e. $n \prec m \iff \psi(n,m) = 1$) whose (say) left cut is A. Then we say that A *is a representation via* ψ. Let $\{D_n\}$ be an effective sequence of all finite sets.

Theorem 2. *Let C be a non-computable c.e. set. There is $A \not\geq_T C$ hypersimple such that for all c.e. $W \geq_{wtt} A$, W is not a representation.*

For theorem 2 we need to satisfy the following requirements:

$$\begin{aligned} \mathcal{N}_{\Phi,W,\psi}: &\quad \Phi^W = A; \phi \Rightarrow W \text{ not a representation via } \psi \\ \mathcal{H}_\varphi: &\quad \exists n (D_{\varphi(n)} \cap A = \emptyset) \\ \mathcal{C}_\Phi: &\quad \Phi^A \neq C \end{aligned}$$

where Φ, ϕ, ψ run over the computable functionals/functions, and W over the c.e. sets. The strategies for $\mathcal{C}, \mathcal{H}$ (guaranteeing cone-avoidance in the Turing degrees and hypersimplicity) are well known, but we will state them briefly. A new strategy is described for $\mathcal{N}$. Roughly, to satisfy $\mathcal{N}$ we will start enumerating $\overline{W}$ (via an auxiliary set D) using the hypothesis that W is a representation via ψ (and of course $\Phi^W = A; \phi$). If at some point our guess D for $\overline{W}$ fails (i.e. an element of D appears in W) then we will be able to satisfy $\Phi^W \neq A; \phi$ by creating and preserving a disagreement.

Let us discuss this plan in more detail. We are given W and ψ, and we may assume that W *is* a representation via ψ in order to try to destroy $\Phi^W = A$. If this hypothesis fails, $\mathcal{N}_{\Phi,W,\psi}$ is satisfied. So we may think W, ψ as the construction of a sequence of rationals converging symmetrically to a real, which produces the representation W of that real. In terms of our framework, the black area is controlled by the enumeration in W and the relative position of the terms of the sequence is determined by ψ (this description gives us a picture of what we are trying to control, i.e. the procedures given to us by the opponent).

The strategy consists of two recursive procedures A, B. The first one consists of potentially infinitely many cycles A_n, each of which builds upon the work done on its predecessor A_{n-1}. The purpose of A_n is to find and enumerate elements to D (so that we are closer to $D = \overline{W}$). Suppose that W is a representation via ψ. The main idea behind D-enumeration is that any $d \in \overline{W}$ has only finitely many ψ-successors. Now, considering a d which is apparently in $\overline{W}$ (i.e. has not yet been enumerated in W) we look for a set I of witnesses (intended for Φ-diagonalizations) such that the set R of their rectification codes (i.e. numbers currently outside W and below the use of at least one Φ-computation on an argument in I) which are ψ-greater than d is smaller (in cardinality) than I itself. Since W is a representation via ψ, there are only finitely many elements ψ-greater than d and so such a set I will be found provided that d is indeed member of $\overline{W}$.

Once we find I (and d is still outside W) we have reasons to believe that d is not going to appear in W later on, and we enumerate it in D. Our belief comes from the following fact: if later on $d \searrow W$, every element not ψ-greater than d will enter W (since the later is a representation via ψ) and so we hold a set I of witnesses whose overall rectification codes are less than their actual number. This means that can we start a diagonalization ripple which ensures a final $\Phi^W \neq A; \phi$ disagreement: for each I-diagonalization at least one element of R will enter W to rectify it and so there will be a final I-diagonalization which is impossible to rectify. The diagonalization procedures is the content of steps B_n.

Of course there is the possibility that during the process of searching for I, d is enumerated in W. In this case we have to pick a different d. Since $\overline{W}$ is infinite, we will eventually come up with a suitable d. Moreover when we enumerate $d \searrow D$ we can enumerate all numbers ψ-greater than d as well (since if any of these appear in W, d must also appear). This feature, along with choosing d as ψ-small as possible (see parts (a),(b),(c) of step 2 of A_n below) ensures that if this procedure is not interrupted (e.g. by $D \cap W \neq \emptyset$), it will give the whole $\overline{W}$.

So if indeed the hypotheses of $\mathcal{N}$ hold and W is a representation via ψ, $D \cap W = \emptyset$ and according to the above, $D = \overline{W}$. So W is computable. In other words, we have satisfied the requirement:

$$\mathcal{N}'_{\Phi,W,\psi} : \Phi^W = A; \phi \Rightarrow W \text{ not a representation via } \psi \text{ or } W \text{ is computable.}$$

In fact, we can let all strategies like the one described (i.e. for all Φ, W, ψ) work together without any interference. Indeed, each strategy chooses witnesses from a special set (disjoint from the sets of other strategies) and so there is no injury (the only restraints set by the strategy are on witnesses). What we achieve is the satisfaction of all $\mathcal{N}'$. But this obviously implies that A is non-computable. Using this fact it is now clear that the satisfaction of $\mathcal{N}'$ implies the satisfaction of $\mathcal{N}$ (since the computability of W and $\Phi^W = A; \phi$ implies the computability of A).

This is an interesting phenomenon: $\mathcal{N}'$ can be regarded as pseudo- requirements which are individually weaker than the main requirements and whose satisfaction is the direct outcome of our strategy. However the satisfaction of all of them (which is the direct outcome of our construction) implies the satisfaction of all of the real requirements. The 'outcome' W *is computable* can be regarded as a pseudo-outcome of $\mathcal{N}'$ since it is never the outcome of a strategy in the sense that no strategy will end up with D infinite (and so, $D = \overline{W}$ according to the above analysis). This is an implication of the fact that A ends up incomputable (and so $\Phi^W \neq A; \phi$, if W is computable, which means that we only get finitely many expansionary stages and D finite). What happens here is that the D-enumeration is a pseudo-strategy which always fails, but it pushes the satisfaction of the pseudo-requirements in different ways (diagonalization and representation property failure).

As a byproduct of this analysis we get that no strategy is going to run for ever. Each family of steps $A_0, A_1, \dots$ must stop in a final A_n (and of course the family $B_0, B_1, \dots$ does not have the potential of running for ever, see below). So, an $\mathcal{N}'$ strategy (like the one discussed above and described below) runs finitely often thus imposing only a finite restraint on numbers of its special set U. This feature allows us to add the hypersimplicity requirements $\mathcal{H}$. These strategies will always respect the higher priority $\mathcal{N}'$ strategies and when they act they will initialize the lower priority strategies. Finally the only effect that the cone avoidance strategies $\mathcal{C}$ have in the strategies discussed above is a Sacks restraint with $\liminf < \infty$ as in the usual Sacks argument done on a tree. So if we transfer our $\mathcal{N}', \mathcal{H}$ strategies on the usual tree that is used in the cone-avoidance strategy the whole construction works without any special modifications. We now formally state the strategy for $\mathcal{N}'$.

Let $n = 0$, $D = \emptyset$. We assume all functional uses increasing, and a fixed restraint r that the strategy is asked to respect by higher priority strategies (in the complete tree construction this will be the $\liminf$ of one or more Sacks restraints lying on the tree above the strategy and a fixed restraint from the nodes on the left). As mentioned above, each $\mathcal{N}'$ strategy chooses A-witnesses from a special set U disjoint from the sets of other $\mathcal{N}'$ strategies. This very strategy imposes its own restraint but this is only on numbers of its special use set U and so they only affect lower $\mathcal{H}$ requirements. The module for $\mathcal{N}'_{\Phi,W,\psi}$ is as follows (the various parameters like n, D may be reassigned values after running the module):

A_n (D-enumeration step)

(1) If $D \cap W \neq \emptyset$ then wait until some $d_k \searrow W$ and go to step B_k. *In order to start A_n we must ensure that the previous A_i steps look successful, i.e. $D \cap W = \emptyset$. If they do we proceed to the main clauses of A_n; otherwise we wait until some $d_i \searrow W$; these elements d_i control the D-enumerations in the sense that any element t in D must have appeared 'after' some d_i ψ-less than t was enumerated in D. So if $D \cap W \neq \emptyset$ and W is indeed a representation, some d_i must appear in D.*

(2) (a) Let $\ell > 0$ be the maximum such that ψ has ordered $\mathbb{N} \restriction \ell$ and wait until it takes a value greater than any previous one (including the values it took in previous A_i steps).

(b) Choose the currently ψ-minimum element d_n in $\overline{W} \upharpoonright \ell$ and ψ-less than any number currently in D. If it doesn't exist, go to (a).

(c) Find k such that for the set I_n of the next k unused elements in U above the restraint r (i.e. the first k elements of $U - \cup_{i<n} I_i$ greater than r) the following holds: if $v_n = \max_{i \in I_n} \phi(i)$ then the number of elements less than v_n and ψ-greater than d is less than k. If during this search $d_n \searrow W$, let d'_n be the ψ-minimum element in $\overline{W} \upharpoonright \ell - D$, $d_n := d'_n$ and go to (c); if it doesn't exist, go to (a). Otherwise, if the search is complete and $d_n \notin W$ go to step 3.

The restraint r will remain the same during the life of this strategy unless it is initialized by the global construction. If ψ defines a total linear ordering of $\mathbb{N}$ of order type $\omega+\omega^*$ and W is its bi-infinite left cut, this step will be completed. Indeed, $\ell \to \infty$ (so it is impossible to be stuck on (a)) and since $\overline{W}$ has no ψ-least element, any (a)-(b)-(a) loop is only finite.

Also, no infinite loop involving (c) can occur for the following reason: any (c)-(b)-(c) loop uses a fixed ℓ and so it must be finite; so any infinite loop involving (c) must also involve (a). Now every time we visit (a), ℓ gets bigger and there will be a stage where there is an element $d' < \ell$ permanently outside W and ψ-less than any element currently enumerated in D (according to the assumptions on ψ and W). At such a stage, (b) will pick up a d_n ψ-less than or equal to d'. Now if the loop continues, (c) will have to consider d', and with this value of the parameter d_n the (c)-search cannot be interrupted.

So eventually there will be a search in (c) which is not interrupted by $d_n \searrow W$. By the assumption on ψ and W such a search must terminate; indeed, d_n is permanently outside W and so it has only finitely many ψ-successors. So, as k grows all the time and "the number of elements less than v_n and ψ-greater than d_n has an upper bound, the search will finish and we will eventually pass to the next step.

Note that if any of the assumptions on ψ and W fails, the above argument does not work and we may not be able to escape this step (but this is no problem as under these circumstances $\mathcal{N}$ is satisfied).

(3) Enumerate $d_n \searrow D$ and fix the values of d_n, v_n and I_n (as they were last defined above). Enumerate into D all elements less than ℓ that have been ψ-ordered greater than d_n and restrain the witnesses I_n from A. *Note that in the end of A_n, D only contains elements ψ-less than or equal to d_n. If we find out that some element of the current D appears in W, d_n must appear*

in W (or else W is not the cut we assumed it is). Upon such an event the construction will activate B_n which will start diagonalizing against $\Phi^W = A;\phi$ using I_n as the set of witnesses. Since $d_n \searrow W$, the rectification positions for any such diagonalization are less than $|I_n|$ according to (d) of step 2. So by the last diagonalization $\Phi^W = A;\phi$ will be destroyed.

(4) Let $n := n+1$ and go to step A_n.

B_k (D-failure step) We assume the values I_k, d_k, v_k as defined in step A_k.

(1) Wait until $\ell(\Phi^W = A;\phi) > m$ for all $m \in I_n$ and all ψ-predecessors of d_k less than v_k enter W. *If we wait forever here, it means that W is not the left cut of the computable ordering on $\mathbb{N}$ defined by ψ, and so $\mathcal{N}$ is satisfied. Note also that d_k has less than $|I_k|$ ψ-successors less than v_k (as when it was defined).*

(2) *(Diagonalization)*

(a) Wait for a Φ-expansionary stage.

(b) Put the least element of $I_n - A$ into A and go to (a).

After the first diagonalization in (b), every time we leave (a) a rectification has occurred and so the set R_n of I_n-rectification codes is reduced by one. Since initially $|R_n| < |I_n|$ and for each element leaving I_n at least one element exits R_n, this (a)-(b)-(a) loop must end up in (a), unable to get a further rectification (and so, expansionary stage).

Analysis of outcomes.

$\boxed{1}$ The module runs over all $A_0, A_1, \dots$ and never stops. This means that we get infinitely many Φ-expansionary stages (so $\Phi^W = A;\phi$) and $\ell \to \infty$ (so ψ defines a linear ordering on $\mathbb{N}$). It also means that $D \cap W = \emptyset$ and according to the second step of A_n, $D = \overline{W}$. So W is computable.

$\boxed{2}$ At some A_i we get stuck forever. Then either $\Phi^W \neq A;\phi$ (not giving us enough Φ-expansionary stages) or ψ does not define a linear ordering on $\mathbb{N}$ (not giving us enough ℓ-expansionary stages) or there is an infinite loop in the (a), (b), (c) clauses of step 2 of A_i. If the loop is (a)-(b)-(a) it means that $\overline{W}$ has a ψ-least element and so W is not a representation via ψ. Any other infinite loop must involve step (c) infinitely often and this means again that W is not a representation via ψ (e.g. see the comments following step 2 of A_n).

$\boxed{3}$ We end up on some B_k step. In this case $\Phi^W \neq A;\phi$ is guaranteed as we explained above.

The analysis of outcomes shows that $\mathcal{N}'$ is satisfied. The module for $\mathcal{H}_\varphi$ is to simply find a t such that $\min D_{\varphi(t)} > r$ (where r is the restraint inherited by higher priority requirements) and then empty $D_{\varphi(t)}$ into A and initialize lower priority $\mathcal{N}'$ requirements. The module for $\mathcal{C}_\Phi$ is to impose (to lower priority strategies) the restraint $r =$ the use of the computations $\Phi^A = C$ up to the first point of disagreement (or Φ being undefined).

We picture the construction on a (downwards expanding) tree. The nodes of the tree are effectively assigned strategies so that any infinite branch is equipped with strategies for each of our infinitely many requirements. An $\mathcal{N}'$ or $\mathcal{H}$ node has only one branch. A $\mathcal{C}_\Phi$ node has infinitely many branches corresponding to (and ordered as) the natural numbers. These are meant to be the various values that the restraint of this strategy takes during the construction.

During a stage s we successively access the nodes of a branch of length s, starting from the top node $\emptyset$ and going through the branch that is activated by the strategy that we have last accessed. For a $\mathcal{C}_\Phi$ node this is the branch corresponding to the current value of the restraint while for the others there is only one choice. If during a stage, an $\mathcal{H}$ strategy α enumerates into A, we initialize all lower priority $\mathcal{N}'$ strategies (so that they start anew). Lower priority strategies are the ones that are below α (i.e. their branch contains α) or to the left of it (with respect to the usual lexicographical ordering of the nodes induced by the ordering on the outcomes). Of course, when a node α becomes accessible, all strategies sitting on nodes to the left of α are initialized. The restraint r that a strategy α is asked to respect (often mentioned in the above modules) is the restraint imposed by nodes above or on the left of α.

First we verify that there is an infinite leftmost infinitely often accessible path f and $\mathcal{C}$, $\mathcal{N}'$ are satisfied. Inductively suppose that the branch $f \upharpoonright n$ is defined (and satisfies the 'leftmost' properties). If node $f \upharpoonright n$ is $\mathcal{H}$ or $\mathcal{N}'$ then we easily see that $f \upharpoonright n+1$ defined by extending through the unique branch of the node, satisfies the 'leftmost' properties. If it is $\mathcal{C}$ then assuming that there is no leftmost edge infinitely often accessible we show the usual Sacks contradiction, that C is computable. So there is such edge and $f \upharpoonright (n+1)$ is defined by adding this edge to f. This also shows that $\mathcal{C}$ is satisfied. Now that we know that f is infinite (and so it contains nodes for each $\mathcal{N}'$) we show that any $\mathcal{N}'$ strategy on f succeeds. Suppose that $\mathcal{N}'$ is not initialized anymore (such stage exists since f is leftmost and there are only finitely many $\mathcal{H}$-nodes above $\mathcal{N}'$). Then the strategy will work without any distraction (lower priority $\mathcal{H}$ requirements respect it and other $\mathcal{N}'$ requirements use different witnesses) and will deliver one of the outcomes justified in the *analysis of outcomes* above. So $\mathcal{N}'$ is satisfied.

Now, as explained above, since all $\mathcal{N}'$ are satisfied, A is non-computable. So all $\mathcal{N}$ are satisfied and also no $\mathcal{N}'$ strategy runs forever (going through $A_0, A_1, \dots$); in other words outcome $\boxed{1}$ is never realized. This means that each $\mathcal{N}'$ only imposes a finite restraint to lower priority $\mathcal{H}$ requirements, and so the later are satisfied. This completes the proof of the theorem. On the other hand we have

Theorem 3. *Every non-computable c.e. wtt-degree bounds a non-zero wtt-degree containing representations.*

To prove this theorem we combine our usual construction of a real with non-computable representation (see e.g. [1]) with permitting. We build a sequence z which converges symmetrically to a real x, and a non-decreasing sequence y which converges monotonically to x. Let A be a non-computable set; $A_z = \{k \mid z_k < x\}$ will be our desired representation, bounded by A. We want to satisfy:

$$\mathcal{P}_\Phi : \Phi \neq A_z$$

So we carry on defining z-terms in decreasing order *outside* $[0, y_s]$ (which we often call the *black area*). When we are ready to attack some $\mathcal{P}_\Phi$ (of least priority requiring attention) we define the current term y_s up to z_k where k is the index we want to enumerate into A_z (thus expanding the black area); and so on and so forth. The observation here is that we can easily add permitting: we don't want to enumerate an index k unless some number less than k enters A at the current stage. As usual every $\mathcal{P}_\Phi$ will require attention infinitely many times unless satisfied. Now note that such an action for satisfying $\mathcal{P}_\Phi$ may enumerate into A_z numbers other than k (namely the indices of terms less than z_k which have not yet entered the black area). The crucial point is that all these will be greater than k (according to the way we define z) and so they will be A-permitted whenever k is so. Finally we need to keep an order on the witnesses: lower positive requirements hold larger unrealized witnesses k (i.e. with $\Phi(k)\uparrow$) and a new witness is chosen for $\mathcal{P}_\Phi$ whenever the previous one has been realized (i.e. $\Phi(k)\downarrow$). This will give a standard finite injury effect to the construction (since whenever a new witness is chosen for $\mathcal{P}_\Phi$, all lower requirements have to change theirs).

4. Non-bounding bottoms of representation-free wtt upper cones

Our last result has perhaps the most interesting proof.

Theorem 4. *There is a non-zero c.e. Turing degree which bounds no wtt-degree whose upper cone is free of representations.*

The requirements are

$$\begin{aligned} \mathcal{Q}_{\Phi,W} :&\quad \Phi^C = W \Rightarrow \exists A \text{ representation}(W \leq_{\text{wtt}} A) \\ \mathcal{P}_\Phi :&\quad \Phi \neq C \end{aligned}$$

and we attempt $W \leq_{\text{wtt}} A$ in $\mathcal{Q}_{\Phi,W}$ by enumerating a functional Γ with computable use γ, trying to preserve and expand the agreement $\Gamma^A = W; \gamma$.

In order to ensure that A, the set we are constructing for the sake of $\mathcal{Q}_{\Phi,W}$, is a representation, we construct a sequence z of rationals in the usual way such that $A_z = A$ (with an increasing 'black area' controlling the enumeration into A_z). By the characterization of representations as left cuts of computable orderings of type $\omega + \omega^*$ (see [2]) we only need to specify the position of each z-term relative to the others, when constructing z (we are not concerned with its convergence).

We define γ on numbers which are currently outside A (i.e. A_z) and make it increasing. The z-terms are defined as usual in decreasing order *outside the black area*. Now the problem is that if the black area expands up to $z_{\gamma(k)}$ (for the sake of enumerating $\gamma(k) \searrow A$) all the defined $\gamma(n)$ with $n \geq k$ will enter A. When a part of $\mathbb{N} \upharpoonright \gamma(k)$ enters A, it is not good news because our opportunities to change computation $\Gamma(k) \downarrow$ (after a possible $k \searrow W$) become fewer (as the use $\gamma(k)$ is fixed, once defined). To make things clear, we use a Γ-*marker* Γ_k for each k, which initially sits on the position (i.e. value) of $\gamma(k)$. In general, it sits on the largest number (i.e. smallest z-term) outside the black area and less than or equal to $\gamma(k)$. *The values that* Γ_k *takes are decreasing* and it could happen that eventually it has nowhere to sit (i.e. it is undefined). This is exactly what we want to avoid. We want each Γ_k to eventually rest on a number outside A (so that if k appears in W we are able to rectify the Γ-computation by enumerating the current position of Γ_k into A). Hence Γ_k being defined means that we are able to rectify Γ on k, if needed.

Now as explained above, an enumeration of some $\gamma(k)$ into A may result in the enumeration of other $\gamma(n)$ into A. This means that during the construction, many Γ-markers may occupy the same position. So if Γ_k loses its current position (to move to a smaller one) it may not be because $k \searrow W$ (but because of some other W-enumeration). So Γ_k may lose all of the positions that is allowed to have (thus ending up undefined) and still k not have appeared in W. A subsequent $k \searrow W$ will result in Γ being wrong and us being unable to rectify it.

To avoid this situation we use Φ^C to restrain W. Whenever we define $\gamma(k)$ on some number n, we make sure that the agreement $\Phi^C = W$ is higher than n and so we can restrain a subsequent movement of Γ_k (due to $k \searrow W$). More generally, whenever we place Γ_k in a new position, we make sure that we can restrain Γ_k from further movement (i.e. we wait until $\ell(\Phi^C = W)$ is big enough before enumerating the new Γ-axiom on k). Of course this strategy results $\mathcal{Q}_{\Phi,W}$ imposing a restraint r with $\liminf r = \infty$. This conflicts with the satisfaction of the $\mathcal{P}$ requirements, which can only accept a finite restraint (or at least with $\liminf < \infty$). If we were to ensure that beyond some stage, r is not violated anymore, then we would have that almost all Γ-markers never move from their initial position. We have space to be more flexible. We describe the situation of a $\mathcal{Q}_{\Phi,W}$ with highest priority and all $\mathcal{P}$ requirements (priority-ordered in some effective way) below it. After we deal with this case, the rest of the $\mathcal{Q}$ strategies can be added with only a finite injury effect (though the atomic case has infinitary nature).

We will spread out r to the lower $\mathcal{P}$ requirements. So r will be violated by lower priority requirements infinitely often, but in a nice way. In particular we define r_n (n-restraint, for $n > 0$) to be the use of $(\Phi^C = W) \upharpoonright (\ell_n + 1)$ where ℓ_n is the index of the largest Γ-marker sitting on the n-th position (i.e. number—in order of magnitude) outside the black area. If there is currently no n-th position outside the black area, or the length of agreement is less than $\ell_n + 1$, let $r_n = 0$.

Now the n-th $\mathcal{P}$-requirement below $\mathcal{Q}_{\Phi,W}$ listens to the r_m restraints for $m \leq n$; i.e. it respects $R_n = \max_{m \leq n} r_m$. To give an idea of the construction and the movement of the Γ-markers, once we state the strategy for $\mathcal{Q}_{\Phi,W}$ it will be easy to verify inductively that at any stage

$$n < m \iff \text{the } \Gamma\text{-markers on } m \text{ have bigger index than those on } n$$

(iff $z_n > z_m$) for all n, m not (yet) in A. Also it is obvious that each position permanently outside the black area, will carry at least one Γ-marker. And for any n, the markers sitting on n are protected from losing their position by restraint r_n of $\mathcal{Q}_{\Phi,W}$ (which may be violated, but only finitely often). The strategy for $\mathcal{Q}_{\Phi,W}$ is as follows:

(1) (*z-definition*) Let n be the least such that $z_n \uparrow$. Define z_n outside the black area and less than any z-term outside the black area.

(2) (*Γ-definition*) Let n be the least such that $\Gamma(n) \uparrow$. If $\gamma(n) \downarrow$, enumerate the axiom $\Gamma(n) = W(n)$ with use $\gamma(n) \downarrow$.

 If $\gamma(n) \uparrow$, wait until $\ell(\Phi^C, W) > n$ and there is a (largest) z_k outside the black area which carries no markers. Then, if k is the t-th (in order of magnitude) number outside the black area, define $\gamma(n) = k$ (thus putting Γ_n on z_k) and t-restrain the C-use of $\Gamma^C \upharpoonright (n+1)$ (since only Γ_n sits on z_k). *The t restraint r_t will automatically be applied according to its explicit definition.*

(3) (*Γ-rectification)* Let k be the least such that

$$\Gamma^A(k) \downarrow = 0 \neq W(k)$$

(if there is no such, do nothing). Then:

- *Expand* the black area up to the position (say n) of marker Γ_k. *By this action we remove (temporarily) all Γ-markers with index $\geq k$ from the line. Later we will put them all on a single position; namely on the largest number $\leq n$ which is outside the black area (i.e. outside A).*
- *Wait* until $\ell(\Phi^C, W)$ becomes larger than $\max_t(\gamma(t) \downarrow)$ (i.e. the maximum argument for which we have ever enumerated an axiom). *Before we enumerate axioms for the arguments $\geq k$ and so place the corresponding Γ-markers back on line, we want to ensure that we are able to keep the later on their new position (and not let them roll further down) by C-restraining.*

- Enumerate Γ-axioms for the arguments in $[k, \max_t(\gamma(t)\downarrow)]$. *This action puts the Γ-markers back on line and also activates the C-restraint of their new position.*

The module above functions as follows: when it is called for first time (or after an initialization) it starts from step 1. Each time it is called, we say that it executes one round. It starts a new round from the point it last stopped. If it has stopped on the end of some step, then it starts from the beginning of the next one (the next of step 3 is 1). In one round it can only execute one step. If it last stopped on a 'wait' instruction, in the next round it checks whether the relevant test is satisfied and waits further or moves on accordingly.

Note that according to the definition of Γ_k given above, *any marker rolling to a new position must have come (and been 'allowed' to roll down) from the next higher position.* The strategy for $\mathcal{P}$ is simply to hold a witness x from its use-set, larger than the restraint imposed on it and wait until $\Phi(x)\downarrow= 0$ (when it requires attention). Then it puts $x \searrow C$ and initialize higher priority ($\mathcal{Q}$-) requirements (and stops requiring attention). Assume an effective listing of all the requirements like

$$\mathcal{P}_0 > \mathcal{Q}_0 > \mathcal{P}_1 > \mathcal{Q}_1 \dots$$

Above we defined r_n (the n-restraint of a $\mathcal{Q}$-requirement) and the restraint to which a positive requirement listens, in the simpler case of a single $\mathcal{Q}$-requirement above (i.e. higher than) an infinite list of positive requirements. In the full case each $\mathcal{Q}$ requirement has its own n-restraints (defined in exactly the same way) and the restraints imposed on some $\mathcal{P}$ on the list are defined analogously. Namely $\mathcal{P}_t$ listens to the i-restraint for all $0 < i \leq (t-k)$, of $\mathcal{Q}_k$ for each $k < t$. Remember that there are no 0-restraints.

The restraint imposed on some $\mathcal{P}$ may change only finitely many times; and each time it changes we make sure that $\mathcal{P}$ is initialized (so that it picks up a new appropriate witness). In particular, whenever r_n of some $\mathcal{Q}_k$ changes value (according to the way we defined it) we assume that all positive requirements which listen to it, are initialized. In this case these are the $\mathcal{P}_i$ for $i \geq n+k$.

To sum up, positive requirements initialize the lower $\mathcal{Q}$ requirements, when they act. And once they've acted they don't act again and so each of them can only cause initialization at most once. When $\mathcal{Q}$ is initialized, it starts working anew (with a new, completely empty undefined Γ, γ etc.). A $\mathcal{Q}_k$ requirement causes initialization every time one of its n-restraints changes value; so it could cause initialization infinitely often. However, each of its r_n changes only finitely many times (since the residents of its n-th position stabilize). And since a change of r_n initializes only the $\mathcal{P}_i$ with $i \geq n+k$, each positive requirement is initialized finitely often. Notice that the steps in $\mathcal{Q}$'s module that can cause a change on its restraints are steps 1 and 3.

Construction. At stage s we first successively access each of $\mathcal{Q}_i$ strategies for $i < s$ and run them (as described above). Then we choose the highest $\mathcal{P}$ which requires attention and satisfy it.

Notice that each $\mathcal{Q}$ involves infinitary activity and so it must be visited infinitely many times. A $\mathcal{Q}$ requirement only enumerates in its own set A and not any set (like C) related to other requirements. Also it can be initialized only finitely many times since it has finitely many $\mathcal{P}$ predecessors.

Verification. First we need to show the following

Lemma 1. *Assume for some $\mathcal{Q}_k$ that $\Phi_k^C = W_k$. Then there are infinitely many positions which permanently stay outside the black area of $\mathcal{Q}_k$, and each of them has only finitely many (and at least one) permanent residents (i.e. Γ-markers). Also, for any position there is a stage beyond which it is not given additional Γ-markers.*

Proof. By induction: assume that it holds for the first $n-1$ positions outside the black area. Notice that z-positions on the real line are enumerated from right to the left. So, when the positions are still outside the black area, the ones with the smaller indices are on the right with respect to the ones with the bigger indices. Say that after s_0 no $\mathcal{P}_i$ with $i \leq n+k$ acts and no additional Γ-marker ever occupies one of the first $n-1$ positions (permanently) outside the black area.

Since $\Phi_k^C = W_k$, the module of $\mathcal{Q}_k$ doesn't get stuck on a 'wait' instruction and so it keeps on running its steps forever. After s_0 we keep on enumerating positions with initial residents successive arguments for which Γ (of $\mathcal{Q}_k$) was previously undefined. The markers sitting on the *current* n-th position after s_0 are not going to be moved. Indeed, according to the construction these markers are restrained from moving by r_n. And since no $\mathcal{P}_i$ with $i \leq n+k$ acts, r_n is not going to be violated anymore. For the same reason, r_{n+1} cannot be violated, and so no additional markers will move to the n-th position (coming from the $(n+1)$-th position). This completes the induction step. The base of the induction (i.e. the case for the 1-st position) is done in the same way, since after $\mathcal{Q}_k$ is initialized for the last time, r_1 is never violated. In particular, no Γ-marker can end up undefined. □

Now suppose that $\Phi_k^C = W_k$. It follows from the construction and the above proof that Γ of $\mathcal{Q}_k$ is total. Indeed, axioms are being enumerated infinitely often, and the use γ for each of them remains the same throughout the construction. In particular, Γ is a wtt-reduction. It is also correct. Step 3 of $\mathcal{Q}$'s module ensures that any wrong computations are being corrected; and this is always possible since Γ-markers are always defined and they always sit on a number outside A.

As a result of lemma 1 and the definition of r_n, any n-restraint of a $\mathcal{Q}$ requirement reaches a limit. This means that each $\mathcal{P}$ requirement has only a finite restraint to deal with, and so it is eventually satisfied. This concludes the proof of the theorem.

We would like to note that all representations A built in the above proof are (automatically, as a result of the construction itself) C-computable. So we actually build A within the C-ideal, as pictured in the first illustration of figure 2. Furthermore, we can replace the positive requirements for C with the $\mathcal{N}$-requirements of section 3 to guarantee that the *wtt*-cone above C is representation free. The effect will be the same since $\mathcal{N}$ involves no more than finite enumeration. If we also consider just the wtt-ideal below C (instead of the Turing ideal we considered in the proof) thus assuming $W \leq_{\mathrm{wtt}} C$ in the $\mathcal{Q}$-requirements, the construction will work as before with the additional effect that all A built will be wtt-reducible to C (instead of T-reducible as before). This situation is summarised in the following theorem.

Theorem 5. *There is a wtt-degree* $\mathbf{c}$ *such that for all* $\mathbf{w} < \mathbf{c}$ *the interval* $(\mathbf{w}, \mathbf{c})$ *contains representations (where everything is meant to be in the structure of c.e. wtt-degrees).*

This theorem is illustrated in the second part of figure 2, where the cone, the ideals and the degrees are c.e. wtt.

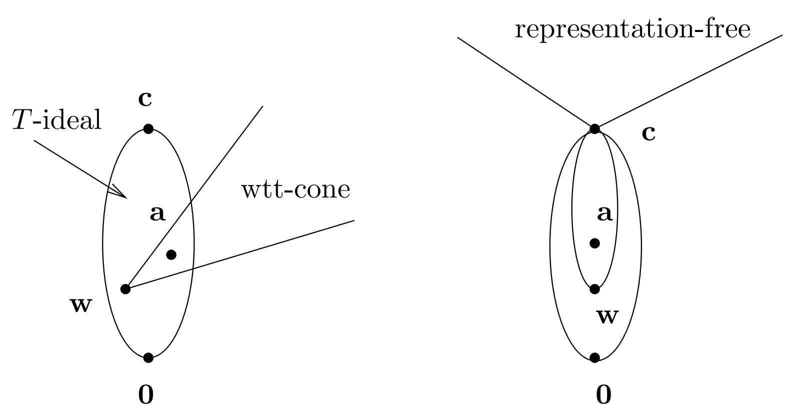

Figure 2: Degrees of Representations

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School of Mathematics, University of Leeds, Leeds LS2 9JT, U.K.
E-mail address: georgeb@maths.leeds.ac.uk