**Epistemic Circularity: Worry, Illusion, and Determination**

The figurative observational statement "We are moving around a circle" has different extended meanings in different contexts: in the context of a rational epistemic pursuit, and, say, in the context of exploring a forest. In both contexts, the primary meaning is the same, that that we are getting nowhere (beyond the circumference of that circle). However, a tacit negative connotation adds to the extended meaning of the statement in both contexts: "We are getting nowhere (by following this path), though it *was supposed* we would get somewhere (since we have set out on a road)". Not only does a negative connotation emanate from the statement in this form, but a contradiction is formed between the supposition and the observation, thus raising a double question needing further investigation: "What is wrong with the path we have followed? Is there actually anything wrong with such paths?" Answers to these questions should be different in the two contexts.

Epistemic circularity (EC), also termed 'circular argument' or 'question-begging argument', has been defined in epistemological terms, starting with Alston (1986), as something that arises when a belief is formed about the reliability on one's own belief source by relying directly or indirectly on that source. Inferential argumentation submits to this definition as a particular case, as an argumentation a premise of which is, includes, or relies on truth of the argument or conclusion. Bergman (2004) refined this definition, saying that epistemic circularity characterizes the formation of an agent's belief in the trustworthiness or reliability of one of that agent's belief sources *X*, if the formation of that belief depends on *X*, where to 'depend upon' a belief source *X* in forming a belief *B* is for *B* either to be an output of *X* or to be held on the basis of an actually employed inference chain leading back to an output of *X*.

Contemporary literature on EC adopted such definitions, and the research focused on the attribute of EC being either a non-problematic or problematic thing in reasoning, or in other terms, justified or non-justified, as well as benign or malignant (I shall adopt this latter wording). Such a focus, of course, has as its object the tacit 'negative connotation' I mention in the figurative example of moving around a circle, also present in the epistemological definition above.

With respect to this tacit connotation, observe that EC has quite a special status as an epistemological category or concept. It is not a principle or a method (like inference to the best explanation, for instance), because a principle is something that can act like a premise, postulate, or hypothesis, either questionable or not; as such, it participates effectively within arguments and theories while EC is something just characterizing a class of arguments. It is not a logical category, since it is epistemic in nature; it is not a puzzle, since it does not require a solution, but rather a clarification. Then, what is EC? The classical definition above does not answer this question by establishing *genus et differentia* for epistemic *circularity*, but only makes the distinction between circular and non-circular arguments.

For going deeper into the nature of EC, I will start by calling it a 'worry' as a category not yet defined. I do not know at this point to what extent a 'worry' is an epistemological concept, but clearly it also has a psychological nature, since worry is a reaction of the nervous system to a perception that triggers alarms in the mind. What makes the status of this epistemic worry[1] – or intellectual worry – so special is that it does not have a well-established object (worry about *what*?) nor a clear justification (*why* the worry?). However, the worry has arisen in connection with the scientific and philosophical practice of using circular reasoning and other circular theoretical constructs, and also of investigating these circularities. The psychological and the practice aspects of the worry grant it a phenomenological nature. Phenomenologically, I would hypothesize that the cause of the worry resides in the logical aspect of EC. It is widely accepted that logical circularity and the vicious logical circle are wrong when used in reasoning, and even other circularities were found as wrong (such as a large class of circular definitions and self-references), so we are worried that circular arguments are also wrong. But how do we distinguish between logically wrong and epistemically wrong? On the other hand, there is relevant positive track record of using circularities in

[1] I qualified it as *epistemic* just because the concept has arisen in connection with EC.

science, mathematics, and philosophy[2] that counterbalances the above partial justification of the worry.

In terms of malignity/benignity, observe that investigating EC for the criteria of these qualifications assumes *a priori* the worry for a kind of epistemic malignity. The mere existence of this *a priori* worry makes EC malignant *in itself* and not only when so established through criteria of qualification applicable to particular types of EC. Think of border security: if one's name appears on a previous list of suspicious individuals, that person is assumed to be suspect even before an actual check. Not only that, but this characterization seems to induce a second-order circularity with respect to the malignity of EC.

In this paper I show that EC has several distinct "personalities" (natures) and placing it on a "list of (epistemic) suspects" can be determined only through each "personality" in part and not cumulatively; this is why EC may pass the "border control" if it is able to change its "personality" properly.

I shall start from the principle consisting of the following conjunction: 1. A property of a concept (e.g., whether EC is wrong for reasoning) can be expressed, understood, and used only in relation with the nature of that concept; 2. Investigating the properties of an unclear concept (as I take EC to be) cannot go deeper into the nature of that concept than investigating how these properties may vary with the extensions and generalizations of that concept – that is, placing the concept in a larger epistemic structure that exhibits the relations of that concept with well-established concepts, not necessarily belonging to the conceptual framework of the domain and theory where the initial concept originates.

On this line of thought, I shall pursue an extensional approach of the concept of EC (not reflected in the classical definitions of EC) by focusing on the immediate more general concept, that of *circularity*, which is not merely epistemic. Such an approach will allow us to provide a formal definition of circularity (of a structural type), instantiable in all types of known circularities (logical, epistemic, of definition, and linguistic). In the interpretation of this formal account, we should seek a reflection of the concepts of

[2] I will provide some relevant examples in the next section.

malignity of EC and 'epistemic worry'. Finding this reflection is expected to shed light on both those concepts and implicitly on EC..

In the first section, I provide a brief overview of the known types of circularities with an eye toward their *use* in the scientific and philosophical practice. Analyzing the commonalities of these types of circularities as theoretical constructs and in nature, I conclude that a general definition which covers all these types is possible and can be only structural. With respect to this structural resemblance between types of circularities and to the practice of using them, I advance the question as to why we are worried about EC more than about other types, including no-worry types.

In the second section, I extend the meaning of the key epistemological terms (belief source, reliability) referred to in the classical definition of EC to also cover other associated concepts belonging to other conceptual frameworks, such as those specific to other theoretical disciplines. With this extension, I discuss the linguistic nature of EC and conclude that the linguistic aspect is essential in determining EC, as both being foundational and having the potential to manipulate the structural elements that define circularity. I support this view with the example (as a brief case study) of the philosophical problem of applicability of mathematics in science and the empirical universe. I argue that the linguistic aspect raises the problem of *genuineness* of an EC.

In the third section, I express a formal definition of circularity in structural terms and instantiate it for each type of circularity. This formal account is built so as to accommodate with the traditional relational account of determination. Within the interpretation of this formal account, I discuss the worry problem and the malignity with respect to the nature of EC.

In the last section I draw conclusions.

## 1. Living with circularity: linguistic circularity, circular definitions, logical circularity, and epistemic circularity

In our communication, reasoning, and scientific and philosophical practice, we encounter various types of circularities. In this overview, I will not define the concept of circularity in general (this will be the task of the next sections), but I will describe it as specific for each type.

### 1.1. Linguistic circularities

Our ordinary language is circular in several ways. First, we have a permanent circularity given by the lexicographic dictionary: Words are defined lexicographically through other words, and as such, any conceptual regression through a dictionary, with the aim of complete understanding, will never end, coming at some point to words already passed through. This closure is something we can call a circle that brings us nowhere if the goal is a complete conceptual understanding. Thus, circularity is given by the repetition of the word passed twice in the process of lexicographic defining and the continuous search for a complete *determination* through word definition. Do we have any epistemic worry about this lexicographic circularity? No, generally we don't, just because the main role of the dictionary is to guide the *use* of the words with respect to sense and meaning, and such circularity does not prevent the fulfillment of this role, because the semantic aspects of the words' usage are not determined entirely by each word alone, but in whole linguistic constructions.

This characterization of ordinary language does not apply to formal languages, such as those of logic and mathematics. Mathematical definitions not only fix names and their usages, but also establish mathematical concepts. In order for these concepts to allow the obtaining of analytical truths when relating them to each other, their definitions must be founded so as to avoid that kind of lexicographic circularity specific to ordinary language. This is why one of the fundamental principles of mathematics is to grant some primary concepts (such as variable, function, relation, set) the status of indefinable.

The second kind of circularity for ordinary language – occasional – is that of pleonastic or redundant constructions. Such constructions are based on syntax and the pleonastic charge depends on both syntax and conceptual semantics. Consider the sentences:

Sun emanates light. (1)

Sun is sunny. (2)

Sentence (1) is pleonastic because the concept of light emanation is already embedded in the concept of sun (or, in Fregean wording, sun – generally, a star – falls within the concept of light emanation), therefore the predication does not bring any new knowledge beyond the description of the concept of sun. We do not have a word repetition in (1), but a concept repetition; in fact, the repetition is expressed through a syntactic relation (subject-predicate) connecting a concept (sun) with a concept (light emanation) that is constitutive for the former. Seen just as an epistemic constitutive relation, (1) does not appear as circular, but as a linguistic construct that is part of the definition of the concept 'sun'. Of course, (1) can be rephrased so as to exhibit a word repetition, for instance 'Sun, which emanates light, emanates light' (1a). In this form, the syntactic relation between the repeated words is different from that between the words expressing the repeated concepts in (1). Hence, we may say that we have a word repetition, but is there any "circle" circumscribed to it, as in the lexicographic case? In that case, the "movement around a circle" was driven by determining through definition each constitutive word: Word *A* is defined through word *B*, word *B* is defined through word *C*, and so on, until reaching *A* again. In the current case, (1a) exhibits relations that may be regressed, but not closed into a circle: light emanation is constitutive for sun – as concepts – (expressed by the attributive clause), light emanation is a property of the sun (expressed by the main clause); the two relations cannot be composed.

Things are different with sentence (2), which also exhibits a word repetition. This time, the words 'sun' and 'sunny' belong to the same word family, and the subject 'sun' literally appears in the predicate. This repetition closes a "circle" in the same way the lexicographic definition does: Sun is sunny, and 'sunny' is something definable through 'sun'. The conceptual reading of (2) does not follow a pattern similar to (1) or (1a). The sentence – whether seen as observational or part of a definition – says something about the subject, which can be conceived and understood only through the subject and in terms of the subject; 'light emanation' is a property that can be determined as non-dependent on the sun, while 'sunny' is sun-dependent, and the regression is possible to the starting point, namely the concept of sun (even though 'sunny' may characterize other substantives as well, such as 'beach', 'weather', 'morning', etc.). As such, the pleonasm

(2) is circular per our common-sense qualification based on 'circular movement' through repetition.

Other pleonasms are based on other syntactic relations. Constructs such as 'friendly friend', 'loving lover', and the like are circular just as sentence (2) and the lexicographic definitions are.

Is the linguistic circularity of the pleonasm (where present) a subject of any epistemic worry? As far as language is concerned, the only rejection is for stylistic and aesthetic reasons, not for any kind of illegitimacy. As epistemology of communication is concerned, such pleonasms are seen as not advancing any new knowledge, so they are redundant but not invalid constructs.

A third type of linguistic circularity is yielded by self-reference. Language (not only ordinary language) has the potential to be self-referential, either as metalanguage or in the language itself. For the former case, the paradigmatic "This sentence is false" (or the liar's paradox simplified) is rejected as a linguistic construct for its inconsistency with the assignation of a truth value in conditions of correct syntactic and semantic rules. A construct leading to a paradox of inconsistency with a well-established foundation is a reason for rejection and reconstruction in every domain, not only logic, linguistics, or philosophy of language. And it is more than a worry – it is the ultimate unwanted event that the worry is concerned with.

Is the self-referential sentence above actually circular? A conceptual repetition seems to be that of 'sentence', which can be seen literally if rewriting it as, say, "The sentence consisting of the sentence you just now read is $P$," or "The sentence 'This sentence is $P$' is $P$," where $P$ is an arbitrary predicate, not necessarily that of 'false'. In the former variant, the second word 'sentence' from the attributive clause is supposed to determine the first word 'sentence' by pointing, or naming, or restricting the reference area. However, the concept repetition is questionable due to the Tarskian distinction between language and metalanguage. In the latter variant, we have a double word repetition, that of 'sentence' and $P$. The circular path is actually formed through the application of the predicate $P$, which is supposed to denote the same concept in both language and metalanguage; however, its roles are different in the two languages: Denoting by $S'$ the sentence in metalanguage and by $S$ the sentence in language and writing the whole construct in symbolic predicative logic, we have that $PS'SP$ or $P\left(S'\left(S\left(P\right)\right)\right)$, which reads as: $P$ is a property (predicate) of (about) $S'$, which is about $S$, which is about (depends constitutively on) $P$. Starting the path with $P$ (the repetitive concept) as a predicate and closing it with $P$ as a variable makes the circle. The composed relation $PS'SP$ can be seen as a composed or chained general relation of determination, as follows: $P$ is determined by $S'$ (a relation between predicate and subject) just as an argument or variable determines constitutively a logical predicate about it; or, $P$ is applied to $S'$; $S'$ is determined by $S$ (a relation between noun and attribute); $S$ is determined by $P$ (a relation between predicate and subject). Determination is an epistemic role of the syntax and also accounts for the kinds of the relations forming the syntactic structure. As such, the general relation of syntactic determination makes the connection with the semantics of a language and with its epistemology and is essential in detecting circularity. However, if we take the syntactic structure of the self-referential construct to have a role in the truth determination within its semantics rather than a role in epistemic (conceptual) determination, things may be different with circularity. In this setup, Yablo (2006) argued through an infinite extension of the Liar paradox that semantic paradoxes do not require circularity.

There are self-referential sentences in the same language that are not pleonastic. For example, "Paul says that Paul is a kind person". The words in repetition stand in certain syntactic relation, but this relation itself does not determine the circularity; circularity is reached when the syntactic relation becomes epistemic, and when composed with other relations, the entire composition lands back to the concept in repetition. This happens also in the previous examples of linguistic circularity. For epistemic purposes such as conceptual understanding, intelligibility, clarity of definition, justification, and gaining new knowledge, our reason operates in the background with the knowledge expressed through language by completing it through operations of a logical-epistemic nature; as such, it creates composed relations between concepts, which try to fulfill those purposes. When this extended structure is not able to fulfill a purpose due to conceptual repetition (by creating a "circle" of knowledge), we label it as circularity and worry about it. I will come back to this general view later.

Let us see whether or how our example above is circular. That sentence exhibits a syntactic relation between the subject 'Paul' ($S$) and the complement clause having also the subject 'Paul', reducible to a relation between Paul and a predication about Paul and implicitly a relation between the two words 'Paul'. Is this relation able to close any epistemic circle, as in the metalanguage example? Seen just as an observational, informative, or declarative sentence about the subject $S$ (understood as 'Paul says something, namely that he is a kind person'), any epistemic operation upon the latter predication ($P$) appears unnecessary; $S$ and $P(S)$ are connected through syntax, this relation suffices for the epistemic needs [$P(S)$ is all that counts in this respect, regardless of the fact that the former 'says' ($P'$)] and the repetition of $S$ does not raise any worry.

Now extend the sentence in this way: "Paul says that Paul is a kind person, and *this implies that/therefore* Paul is a kind person." In this form, the conjunction introducing the result clause establishes a relation between the predicates $P$ and $P'$. Linguistically, this is not a relation of subordination or dependence, as the two predicates have the same status since they form two clauses non-dependent on each other, even though the whole sentence is an implication. The first clause consists of the predication of $P'$ about $P$, so $P$ determines $P'$ both linguistically and epistemically (and logically). [Symbolically, $P'\big(P(S)\big)$, or $P \to P'$, illustrates the sense/order of this relation.] The whole sentence establishes another relation of determination between $P'$ about $P$ (although linguistically they have the same status), if we consider the *meaning* of the implication conjunction. Since $P$ from the second clause is a result of the first clause (the knowledge the latter expresses is obtained from the former as a source), where $P'$ is essential, it follows that $P$ is determined epistemically by $P'$ ($P' \to P$). Composing the relations of general determination, we have $P \to P' \to P$, which is a circularity with the repetition of $P$.

Of course, this example is in the form of the paradigmatic example of self-testimony for the EC, but the only epistemological aspect I have stressed in the current example was determination, in connection with syntactic determination.

In closing the section on linguistic circularity, let us note that in what concerns the worry, this affects only paradoxical self-reference and is justified through the paradox which leads to change of well-established foundations and conceptual frameworks, especially in precise scientific disciplines, mathematics, logic, and analytic philosophy. At the mere level of everyday usage of language for the purpose of communication and description, there is actually no worry about circularity – lexicographic circularity does not affect this usage; furthermore, not all circular self-references pose problems when imported (through ordinary language) in the mixed language of science and philosophy.

Self-reference cannot always be equated with circularity, and when it can be, except in the paradoxical situations, it is not always qualified as malignant. For instance, mathematics by its nature allows self-references and operates with them. In mathematics, a function $f$ can be applied to itself as a composition $f(f(x))$ while not posing any problem of determination as happens in the case of linguistic self-reference; this is possible because the mathematical predication (including a function) is defined including through a domain within which its variables may range. In case of the composition $f(f(x))$, a condition is imposed that the domain and co-domain of $f$ are the same. As such, the composition is not applied in a second-order domain, as was the case with the linguistic predication where – due to the freedom of language – arguments or variables (in the form of grammatical subjects) are minimally constrained. Besides, in mathematics, governed by the axiomatic method, when a self-referential construct poses problems of inconsistency, the axiomatic changes in the foundation of that theory may solve them, as was the case with the Russell paradox in set theory, solved through his theory of types and the subsequent axiomatic revisions of set theory. A generic example of self-referential mathematics is metamathematics, as a theory about mathematical theories, formed with mathematical methods, as well as category theory, which yields theories about the mathematical structures with mathematical concepts and tools.

Seen from outside mathematics, such self-references can be qualified as circular, especially when described in ordinary language, however mathematicians raised not worries about that, by trusting the methods of mathematics.

Self-reference is also present in programming science and practice, where a pragmatic approach for circular self-reference, oriented to roles and practical effects (somehow similar to the case of lexicographic circularity), is adopted by computer scientists and programmers [see, for instance, (Royer & Case, 1994) and (Case & Moelius, 2007)].

### 1.2 Circular definitions

In the traditional account of definition, there are postulated two intuitive criteria through which an internal legitimacy of a definition is established – namely, *conservativeness* (the definition should not be able to establish, by means of itself, new knowledge) and the *use* (the definition should fix the use of the defined expression, and it should be the only definition available to guide us in the use of the defined expression; in other words, it should fix the meaning of the *definiendum*). Starting from these intuitions, a theory of definition has been developed on the basis of three principles: 1) definitions are generalized identities, 2) their structure is sentential, and 3) the reduction principle: the use of any formula containing the defined term is explained by reducing it to a formula in the ground language[3]. The reduction principle conjoined with the sentential one leads to a strong version of the *use* criterion, called the *eliminability* criterion: the definition must reduce each formula containing the defined term to a formula in the ground language.

Lexicographic definitions obviously obey the conservativeness and use criteria. This qualification and the linguistic nature of definition would justify discussing the lexicographic case in the current section. However, seen more as linguistic and less as epistemic constructs, lexicographic definitions do not assume a distinction between a ground language and an expanded language, as it is assumed in a theory of definition. Thus, I have chosen to place lexicographic definitions in a separate discussion with the aim of distinguishing between types of circularity, on the ground that the linguistic and epistemic natures of the concepts, although related, are still different. With this distinction in mind, I shall consider the theoretical concept of definition more epistemic than linguistic, as a theoretical entity participating in rational constructions characterized by an objectivity higher than that of ordinary language (such as in scientific/mathematical/philosophical theories).

[3] In the terminology of theory of definition, the ground language $L$ is that used in stating the definition, less the defined term ($L$ is applicable to *definiens*), and the expanded language $L^+$ is obtained by adding the defined term to the ground language $L$; in this context, "formula" is the term for sentences and sentence-like things with free variables.

It is known that the conservativeness and eliminability criteria can be expressed formally in model-theoretic terms [Urbaniak & Hämäri, 2012]; however, testing a definition against those criteria is relative to the ground language, since each language has its own systems of proof and classes of interpretation (Gupta, 2019). As such, the two criteria are not absolute criteria for a legitimate definition (in the traditional sense), and the linguistic nature of definition is essential in this respect. The nature of ordinary language also prevents lexicographic definitions from obeying the eliminativity criterion.

Circular definitions have been studied in connection with this concept of legitimacy of a definition. A definition is traditionally called circular if it uses the term(s) being defined in the *definiens* or assumes a prior understanding of the term being defined. This definition can be extended if the 'usage of the term(s)' is not understood literally, but rather in the sense of conceptual connection. This extension covers the situations in which a word from the family of words of the *definiendum* is used in the *definiens*; as additionally, it covers those cases in which the *definiendum* is not a single word but a group of words with internal syntax and meaning and one or more words from this group appears in the *definiens* and grounds its meaning. Consider the following definitions as examples:

a) A definition is called circular if it exhibits circularity.

b) Subprogram is a part of a program that can be designed and tested independently.

c) A finite set is a set that has a finite number of elements. An infinite set is a set that is not finite (in naïve set theory).

d) Naïve set theory is a non-axiomatized theory about sets.

Apparently, example a) is a circular definition, as 'circularity' is a word derivation of 'circular'. The concept of circularity is defined through the word 'circular', and the general concept of circle or circular path. If the *definiendum* 'circular' imports the meaning of the concept of circle or circular path, then this meaning grounds the understanding of the *definiens* and we do have circularity. However, such an import can run only via ordinary language. If we just ignore the ordinary meaning of 'circular' and see it instead as a new word in the language of the domain in which the definition is stated (say, analytical philosophy), or simply imagine it as another word or an arbitrary sequence of symbols, the circularity (as a conceptual repetition) vanishes.

Example b) is not a circular definition. Even though we have a word derived from the *definiendum* in the *definiens*, that word repetition does not close any circle and the understanding of the concept 'program' does not depend upon the *definiendum*. The new term is actually defined through *genus et differentia*, where *genus* is represented by 'all parts of a program' and *differentia* by the described specific property. As such, the word derivation (through the prefix "sub") just reflects this differentiation.

The definitions from example c) are circular because the concepts of finiteness and infiniteness are epistemically dependent upon each other. Even though we may rewrite the *definiens* as "we can count and finish counting" to describe a finite set, the concept of 'finishing counting' is still determined by infiniteness. The fact that axiomatized set theories define finiteness and infiniteness of a set in terms of inclusion, one-to-one correspondence and/or ordering does not cancel the circular qualification of the naïve definitions from c), because they remain relative to their ground language, in which the concepts of set theory are described.

Apparently, definition d) falls within the same principle of *genus et differentia*, as b) does. But I argue that d) is circular because of the repetition of 'set'. Since it is assumed that the definition belongs to a context that discuss and distinguishes between theories about sets, the concept of 'set' (defined differently in each of these theories, in both language and principles) should be represented in the *definiens* as a *general* concept within which all the various definitions of set do fall. As such, a 'theory about sets' from the *definiens* should refer to *any* theory that deals with the general concept of set, including the particular theory that the *definiendum* refers to when naming it "naïve". We therefore have a cyclic relation of determination for the concept of 'set'.

What happens if we rephrase the definition d) as d1): "Define naïve as a set theory that is non-axiomatized"? *Definiendum* no longer contains 'set', and the definition fixes the use of the word 'naïve' for all set theories. As such, the definition acquires the same status as definition b) and is not circular.

Schematically, the circular examples above exhibit their circularity as follows: For a definition $D =_{def} \left(D_i\right)_i$, where $D$ is the *definiendum*, $D_i$ the definiens (which do not form necessarily a sentential conjunction), and $D_1$ one of the *definiens* that depends on *D*, then we have the cyclic closure $D \to D_1 \to D$ or $D = D_1 \to D$ (in case *D* is $D_1$).

The traditional account of definition makes a tight connection between circularity and legitimacy of a definition, in which circularity is defined in terms of understanding and not necessarily of conceptual repetition and cyclicity. Weakening the conditions of legitimacy, we can find circular definitions that provide some guidance in the use of the defined term, and therefore, they have semantic value, being also logically valid. On the other hand, there are apparently circular definitions that do obey all the requirements of the traditional account – for instance, the inductive and recursive definitions from mathematics and logic [Moschovakis, 1974].

Gupta [1988/1989] shows that circular definitions do not obey the eliminability criterion and suggests that given the strong parallelism between the logical behavior of the concept of truth and that of concepts defined by circular definitions, since truth is a legitimate concept, so also are concepts defined by circular definitions. Following this line of argument, Gupta and Belnap [1993] developed the *revision theory of definitions* in which a circular definition provides the defined term with a meaning that is hypothetical in character; the semantic value of the defined term is a rule of revision, while non-circular definitions hold a rule of application. The fundamental idea of the revision theory is that the derived interpretation of the meaning of the defined term is better than the hypothetical one, and the semantic value that the definition confers on the defined term is not an extension, but a revision rule. The revision processes help provide a semantics for circular definitions. Under this theory, the logic and semantics of non-circular definitions remain the same as in the traditional account, and revision stages are dispensable. Circular definitions do not disturb the logic of the ground language; conservativeness holds, but eliminability fails to hold, even though the weaker use criterion does hold [Gupta & Belnap, 1993].

In conclusion, qualifying circular definitions as non-legitimate with respect to the criteria of the traditional account does not necessarily grant them a malignant character, in an *external* sense – that is, a circular definition does not affect the logic and the main epistemic values of either the ground language or of the theory, rational arguments, or discourse that employs such definition. Instead, such a qualification establishes their membership to a narrower class defined through internal properties, one of which is that of not obeying the eliminability criterion.

This argument for benignity is well supported by the scientific and mathematical practice of using circular definitions. Mathematics abounds in circular definitions, whose circularity is more or less visible, and which were found not to affect in any way the principles of the axiomatic method and the flow of the analytical truth. I limit myself to mentioning only the following: inductive and recursive definitions; the systematic foundational definitions of Euclidian geometry that define the point, line, and plane through each other; the concept of classical (Laplacian) mathematical probability (which is grounded on the primary concept of *equally-possible* elementary events and is fundamentally constitutive for the concept of the Kolmorogovian probability as a measure of physical possibility); and finally, the method of mathematical induction. Circular (non-mathematical) definitions are also used in theories of physics and other scientific disciplines, but this is not the place to review such examples. All of these mathematical and scientific circular definitions are not considered to affect in any way the theories employing them – that is, what would be an external malignity – nor to be suspicious in respect of their own validity – that is, what would be an internal malignity. In other words, mathematicians and physicists (and perhaps a wide majority of philosophers) have developed no worry about their possible malignity.

Things stand differently with the next kind of circularity.

### 1.3 Logical circularity

Simply stated, logical circularity characterizes a logical derivation in which the conclusion is embedded directly or indirectly in one of the premises. In terms of propositional logic, such "embedding" is actually participating in the usual logical operations between the propositions forming the premise or the conjunction of more premises.

In a logically circular derivation, the relation of logical consequence between the set of premises ($P_i$) and the conclusion ($C$) closes a vicious circle starting and ending with the conclusion:

$C \rightarrow C$ in the case of one premise being the conclusion; $C \rightarrow P \rightarrow C$ in case of one premise with the conclusion indirectly embedded in; $C \wedge P_2 \wedge \ldots \wedge P_n \rightarrow C$ in case of several premises with the conclusion directly embedded in one of them; $C \rightarrow P_1; P_1 \wedge P_2 \wedge \ldots \wedge P_n \rightarrow C$ in case of several premises with the conclusion indirectly embedded in one of them.

The circularity is made in the virtue of the transitivity of the relation of logical consequence, but its viciousness is not merely logical in nature. Actually, the proposition $C \rightarrow C$ is tautologically true; however, it *does not prove C*, and this qualification – although not equivalent to 'does not justify' – is also epistemic. The worry for and rejection of logical circularity are not grounded on the lack of "warranty" for a premise (as in the case of EC), nor on the generation of a logical truth with possibly false premises, but rather on the question of whether *C* (or *P* implied by *C*) *is* a valid premise at all, for if it were not, the binary relation of consequence would not exist, since one of the *relata* is missing. This no-premise characterization can also be described in terms of (non-)determination between hypothesis and conclusion.

### 1.4 Epistemic circularity

Epistemic circularity is defined classically in terms of justification, belief, and trustworthiness or reliability[4] and covers both beliefs and arguments. Defined in terms of 'trustworthiness of a source' as being the object of the belief or argument, it apparently looks narrower in its range of usage and frequency in scientific and philosophical practice than the other types of circular constructs. In the next section, within an extensional approach, I will argue that this is not the case and actually that its range is larger.

According to the definition of Bergmann (2004, p. 711), EC is malignant if it prevents beliefs infected by it from being justified. Apparently circular (for the concepts *preventing* and *malignity* are related in determination), this definition has been adopted by contemporary theorists about EC. The first proponents of benignity were the

[4] As I already stated the classical definition of EC in the introduction and the next section is fully dedicated to it, I won't reserve a special subsection for it here, and I will limit myself to illustrate what I have called thus far 'circularity per the common sense', in the case of EC.

reliabilists, as being committed to accept the track-record arguments, a thesis also confirmed by Alston (1993). Bergmann (2004), an important proponent of the benignity of EC, proposed a contextualization of EC, by identifying contexts and large categories of EC malignant in itself, arguing that for the rest of the categories – the wide majority – EC is benign. His argument in favor of benignity is built on the fundamentalist thesis that there can be non-inferentially justified beliefs, and its basic principle is that one who accepts this thesis must accept that track-record arguments are not something bad[5]. His thesis is also supported by the primary Reidian principle that human faculties are reliable, and this belief in non-inferential. Bergmann identifies malignant EC in what he calls 'questioned source context', namely that context in which the agent begins by doubting or being unsure of his/her source's trustworthiness and look for a second opinion (independent of his/her perception) on that source.

Other persuasive approaches of EC – pro benignity – worth mentioning are in brief: Goldman's (2003), for whom the evaluation of arguments and argumentation must be done exclusively epistemologically and not syntactically [in the vein of Sorensen's (1991) thesis] and who argues that EC is not formally defective, but it may be epistemologically objectionable; Brown's (2004), who claims that we might have to accept EC reasoning if it were shown to follow from an epistemic commitment which is unavoidable; Alexander's (2011), who denies the No Self-Support Principle, on the reason that it has the skeptical consequence that the trustworthiness of all our sources ultimately depends upon the trustworthiness of certain fundamental sources that we cannot justifiably believe to be reliable, and so we should not trust any of our sources at all.

It is not the aim of this paper to analyze in depth the benignity-malignity debate per Bergman's definition of malignity, but instead what I have called previously the epistemic worry for such qualifications of EC. In order to do that, let us see first how circularity is formed within the EC beliefs and arguments.

In the following scheme, $X$ is the belief source, $P_i$ ($i$ from 1 to $n$) denotes the premises, and $T$ is the trustworthiness of a belief source predicated about or applied to source $X$. Premise $P_1$ is an output of the source $X$ (or depends on $X$ is some way).

[5] still in the sense of Bergmann's definition of malignity.

$$\left.\begin{array}{r} X \rightarrow P_1 \\ P_2 \\ \vdots \\ P_n \end{array}\right\} \rightarrow T(X)$$

The scheme is just illustrative and should not be read as in the common usage of the symbols. The arrow does not necessarily mean deduction or implication, nor does it mean that we have the same kind of relation between $X$ and $P_1$ as between the set of premises and $T(X)$. In general, there also exist non-inferential arguments, and per the EC classical definition, the relation between $X$ and $P$ may be of various epistemic kinds. In this scheme, the relation denoted by the arrow should be read in the general sense of determination, as in aforementioned cases in the previous sections. Also, the set of the premises should not be read necessarily as a propositional conjunction, as in the logical circularity case. Epistemologically, the premises of an argument may support the conclusion not only through logical entailing, but also by supporting in specific ways the *methods* of reasoning within the argument, which are not only logical rules of inference. Of course, when the arrow is understood as logical consequence and the set of premises as a propositional conjunction, the scheme reverts to that of the logical circularity.

In this schematic form, EC does not exhibit any cyclic closure of determination in its only visible repetitive element, namely source $X$. The fact that $X$ ultimately determines $T(X)$ indirectly (according to the transition through the set of premises) is not contradictory at all and does not entail any infinite regression. It is a relation of determination consistent with the status of $X$ being an argument or variable for the function or predicate $T$. However, we may want to take a closer look at the meaning of the relation of determination between $X$ and $P_1$. How should we understand the involvement of source $X$ in establishing premise $P_1$? The classical definition of EC uses the terms *dependence* and *output.* However, neither such dependence nor output is in this case understood in a mere logical sense allowing us to write $X \rightarrow P_1$ (as a consequence within the same first-order context) or $P_1 = P_1(X)$. In establishing premise $P_1$, source $X$ is actually *used* or *applied* to a context in which the argument belongs, and this interpretation is consistent with the status of $X$ being a belief source. Moreover, the justification for such a use or application can be acquired only as result of qualifying the source as trustworthy, and it is this qualification that determines the application. As such, our scheme is extended "in the background" to the left as follows: $T(X) \rightarrow X(C) \rightarrow P_1 \rightarrow T(X)$, where $C$ is the context of application of $X$, and premises $P_i$ ($i$ from 2 to $n$) were ignored for simplicity, as not influencing the sense of determination between $P_1$ and $T(X)$.

The symbolic part in the left side of $P_1$ reflects a distinction between the logical circularity and the inferential EC, by showing the richer epistemology of the latter. In this new form, the scheme reveals that it is not $X$ that closes the circle of determination, but $T$, a predication about $X$.

Of course, the symbolism of the scheme is predisposed to interpretation, especially as concerns what is actually used or applied in the circular argumentation, source $X$ or its trustworthiness $T$. The cyclic closure of determination remains if one claims that $X$ is not something to be applied, reflected in the symbolic scheme through the possible removal of the term $X(C)$. One may also argue that there are different trustworthinesses of $X$ – the one that is inferred ($T$) and the one that is applied (another property of $X$, call it $F$), and we cannot talk about trustworthiness *simpliciter* when it comes to use or application. As such, property $F$ of $X$ is that which justifies (determines) the application $X(C)$, or $X$ is applied to $C$ in the virtue of $F$. If this is the case, circularity depends on the relation between $T$ and $F$. If $F$ and $T$ are in a relation of determination ($T \rightarrow F$), again the cyclic closure of determination remains in $T$. If the determination is inverse or there is no determination between $T$ and $F$, then the circle is not closed. For instance, for the trivial EC example of self-testimony, from the trustworthiness of an agent one can extract an independent trustworthiness to be used in the argumentation – say, that of spoken sincerity (on a par with a written sincerity), and thus circularity can be claimed only after establishing the relationship between the two properties.

It is time now to take stock of all the categories and examples of circularity understood according to the common-sense concept of cyclic closure of determination.

Before discussing what all types have in common, it is fair to justify two foundational concepts used in our formal descriptions, namely those of determination and predication.

I have argued from the introduction that investigation of circularity necessitates the analysis of the epistemic worry about circularity, either present or not in specific cases. In its psychological/mental nature, this worry can be addressed in terms of *conceptual* determination, which is able to link philosophical theoretical inquiry and human cognitive structures and assets. Our brains like and seek safety just as any material entity seeks a state of equilibrium – including concerns of intellectual inquiry and the means of that inquiry. This safety can be acquired only when the concepts we deal with are *well* determined and established, while the lack of safety induces intellectual stress as a state of non-equilibrium within which the concept of epistemic worry does fall. As for predication, our judgments with concepts are judgments *about* concepts or *about* relations between concepts, and as such, the logical predication should be reflected in any formal account of these judgments. The logical nature of our judgments imposes second-order predication to be employed as a primary concept in any theoretical account dealing with unclear concepts – which I take to include circularity.

Theoretical accounts of determination (in terms of determinables and determinates) in their early stage assumed determination to hold between properties and property types (which can be seen as predications about concepts); then they were extended to hold between relations (relational concepts) and also between entities of other ontological categories beyond the traditional monadic types. A justification for the principle that there are no good reasons for any restriction on the nature of the entities standing in a relation of determination (except conjuncts and their conjunction) is given in (Johansson, 2000), developed on the idea that there are ontological determinables and conceptual determinables. Call this the no-restriction principle for determination.

In the formal description of our examples of circular constructs, I have assumed that a relation of determination holds between a variable or argument (as the determinate) and a function or predicate of it (as the determinable). Although it is difficult to categorize this relation in terms of ontological or epistemological (conceptual) types falling within the concept of determination per the traditional or even contemporary accounts of determination, I will defend my choice with the following argument: It is clear that a certain kind of relation holds between the two entities, usually called an *internal* relation. (A relation is internal if and only if one of the *relata* would not be what it is without standing in that relation – for instance, the relation of membership between a set and one of its elements is an internal relation, which is reducible logically to a relation between a variable and its predicate). An internal relation satisfies the essential features that characterize determination in all relational accounts, namely irreflexivity, asymmetry, and transitivity[6]. In common sense conceptualization, a predication is determined by its subject or variable through the *constraint* that the predicate can be applied only to a certain range or class of variables; the mathematical correspondent of this argument is that a function is defined only on its domain, where its argument ranges; as such, the predication 'is a liar' is determined by the variable 'human' and to any particular human (although linguistically the syntactic determination is inverse!), since it *cannot* be applied to a non-human. Seen as such, the determinate determines the determinable in a *constitutive* mode, by participating foundationally to its constitution as a concept.

Despite the no-restriction principle for the categories of the determinates and determinables, the relation between a variable/argument and its predicate/function emerges as a case study for the theory of determination, its distinctive feature being that the determinate and determinable belong to different logical categories (the determinable is of a higher order). However, my formal account of circularity will employ only the three properties of a relational account of determination (mentioned above), with this special case of the nature of the *relata* (a relation that I will call second-order constitutive relation) taken as unproblematic for our purpose here[7].

### 1.5 Conclusions of the overview

We have started from the common sense conception of a circular construct of knowledge as being a construct which exhibits a cyclic determination for one of its

[6] For the particular case of set membership relation, transitivity holds modulo the transitive closure of a set, a convention that is not necessary for the logical predication.
[7] The higher-order case is expected not to verify all the features of the traditional account of determination. For instance, the increased-specificity feature (to be the determinate is to be the determinable, in a specific way) seems not to hold.

essential concepts, which prevents the full understanding, establishing, or justification of either the target concept of the construction (the cases of lexicographic and circular definitions) or the means of a judgment about a concept (the cases of logical and epistemic circularities).

Let us notice that the cyclic determination is not *actually* an infinite regress of an epistemic action – that is, the regression may be seen as potentially infinite in its steps; however, there are rather few finite number of elements that stand on the closed circle. Among these elements (words, concepts, predicates), there is one responsible for the closure through its repetition. We noticed it as a word in the lexicographic definitions, as a word or concept (including predications) in circular definitions, as a proposition in logically circular deductions, and as a predication in epistemically circular arguments. Let us call this repetitive element *the circularity element* and observe that it acts in principle like a truth maker: just as there is a certain kind of relation between a truth maker and its truth bearer that qualifies the bearer as true, so there is a relation between the circularity element and the entities standing in relations of determination that qualifies the determination as circular. The latter relations express a certain "position" (in a topological sense) that the circularity element must hold in the chain of determination for this latter to be qualified as circular; the mere repetition is not enough for such qualification. I have described all the relations between the elements of such a chain in terms of determination. Among all these relations of determination, there is one main relation that the circular construct is built upon and that is specific for each type of circularity – namely, a semantic relation for the lexicographic definitions, a sort of identity relation for lexicographic definitions and circular (theoretical) definitions, a logical-consequence relation for the logical circularity, an inferential relation for the inferential EC. The relational chain is completed with secondary relations which may be constitutive, applicative or predicative, inferential, deductive, and basically with no restriction on their nature, as long as they can be determinative. Among these secondary relations, a special type has been discussed – that of the second-order constitutive relation between a variable and its predicate – with respect to the traditional relational accounts of determination. Not any circularity type shows such a relation. In our overview, we have detected it in the case of self-referential linguistic circularity and EC. Logical circularity is free of second-order constitutive relations, as being inconsistent with the principles of logic, which does not allow a consequence between a variable and its predicate or vice versa, due to the logical-order difference. As for circular definitions, the general flat form of the determination chain ($D \to D_1 \to D$) seems free of that kind of relation. Assuming a definition exists in which $D_1$ is expressed as a predication of *D*, then the second determination in $D \to P(D) \to D$ would make no sense, and as such we can no longer qualify the definition as circular on the basis of the determination chain. We can fairly assume that useless definitions for which $D_1 = P(D)$ can be formulated and presented as illustrative examples for the case[8], but this is not a sufficient reason for rejecting the claim that circular definitions are free of second-order constitutive relations.

All the observations and considerations above suggest clearly that investigation of circularity and any formal account of it can only be structural, just as any account of determination cannot ignore the relational nature of determination.

An important observation is to be made on the natures of the circularity types. No claim can be made that linguistic and definition circularities are merely linguistic in nature, or that logical circularity is merely logical, or that EC is merely epistemic. First, logic is grounded on language, and EC arguments do have a logic, and not only inferential arguments have logic, since they all submit to the category of reasoning, which uses the basic principles of logic. Second, each of the described types of circularity has its own epistemology, including logical circularity; the fact that $C \to C$ does not prove *C* assumes *a priori* that an epistemic goal of proving exists for the logical constructs. Over all these arguments for the multiple nature of circularity of any type, note that the concept of determination, which we employed in describing circularity, is epistemic.

Not only can resemblances and distinctions be drawn upon the circularity types from this overview, but also, they may shed a first light on what I have called the epistemic worry about circularity. Two options seem worth investigating: Is there a worry about non- or improper determination, and if yes, where does it come from? Is it a worry

[8] Besides, one of the consequences of the criteria of a legitimate definition per the traditional account is that *definiens* and *definiendum* must belong to the same logical category.

that a certain kind of circularity is somehow equivalent or reducible to a logical circularity, which most of us reject? Might there be both? Can we answer these questions in a non-empirical non-phenomenological setup, and what would be the suitable conceptual framework for that?

Besides lexicographic definitions, which we all find useful, and besides a large class of circular definitions that guide us semantically, I have already mentioned that scientists, mathematicians, and experts have no worry for some circularities in what concerns their *specific* domains. Still, some of them, although having no justification for rejecting circularity, take action against it in their practice – they avoid it and look for non-circular alternatives, if possible, while circularity is usually a reason for objection to competing theories – and thus a worry still exists for them, too. Should philosophers be worried on their behalf or more worried than they and on what grounds and justification? Anyway, answering why some types of circularities are accepted and other are not assumes accounting for how malignity of circularity is understood and this is a philosopher's task.

Now consider that not only is our language circular, but neural paths in the brain are circular, biological processes and evolution are circular, we have circular cosmologies, and the life-death cycle at the cosmic scale is circular being based on the recyclable stellar matter and energy. Moreover, the ultimate circularity from our scientific and philosophical practice is that we investigate the human mind by using the same (trustworthy) mind. We live in a circular world and still we worry about circularity in our reasoning. Is there any incompatibility between these two facts?

### 2. Conceptual language, meaning, and epistemic circularity

We have seen that EC is grounded on language, and this linguistic nature is shared by all types of circularities. We have seen in the examples given that the circularity element is not always a word or a phrase, but a concept meant by those words. The conceptual circularity element is put forward when we recreate the determination chain from the *meaning* of the formulation of the argument. Since we have described circularity in terms of epistemic determination – which operates over concepts rather than language – and the *relata* of the determination are established through the semantics of the language of the argument, it follows that circularity is grounded on the conceptual language – that is, circularity is both on the paper and in our minds. This makes the linguistic aspect essential in an epistemological account of circularity, and also raises a question about the degree in which circularity is language-dependent, and even whether language is able to manipulate EC. If the answer to the latter question is positive, then we can pose the problem of *genuineness* of circularity in arguments, independent of its alleged malign/benign character.

Take for example the standard self-testimony circular argument: A person *X* says that s/he is trustworthy; therefore, *X* is trustworthy.

In this descriptive formulation, the visible circularity element seems to be *X*, but I have argued in section 1.4 that it is a predication about *X* (*X*'s trustworthiness) and not *X* that closes the determination circle, according to scheme $T(X) \rightarrow X(C) \rightarrow P \rightarrow T(X)$, where *P* is the premise of the argument (*C* is the context in which source *X* is used or applied). This is visible if we rewrite the argument thus: "Since *X* is trustworthy, we can use *X* by taking for granted all that s/he is saying, and given that *X* says s/he is trustworthy, then *X* is trustworthy," which fits our scheme of cyclic determination. Now imagine that a philosopher with certain views (say, a proponent of the linguistic nature of scientific theories and an external-world realist) reads the argument *conceptually,* meaning that it is not source *X* that we *use* or employ in our argument, but the mere content of the knowledge provided by *X*, i.e. *what* it provides. Put this way, the statement '*X* is trustworthy' becomes independent of *X*, for anyone else may provide the same knowledge. The formulation of the argument in this view would be " '*X* is trustworthy'; therefore, *X* is trustworthy". This formulation is circular, but it is not epistemically circular, per the classical definition; it is linguistically circular. Furthermore, seen as an implication and written as " '*X* is trustworthy'; therefore, '*X* is trustworthy'," it is logically circular.

Of course, such an example is rudimentary in making the point that language is able to manipulate circularity, by either eliminating it or changing its type (and its benignity/malignity), in the respect that the imagined assumption is highly sensible to interpretations and objections[9]. The strong influence of both uttered language and conceptual language was also stressed in the examples of section one, and the example from the next section (applicability of mathematics) will also support the point. The fruitfulness of the trivial example above manifests actually in another direction. It gives us a suggestion for our investigation of circularity. That suggestion is that the key epistemological terms in which EC was defined and addressed (belief source, agent, trustworthiness) might have too narrow a palette of senses and meanings. Besides, the term 'output' (of a source) has an unclear meaning. An extensional approach and a clarification of the adequate meaning of these terms might be unificatory in the following respects: 1) by preventing the situation of switching through language between the various types of circularity for the same argument, which also affects the benign/malign qualification; 2) by extending the domain in which the concept of EC applies to cover the entire intellectual practice (scientific, mathematical, theoretic-philosophical in addition to general epistemology); 3) by making the concept of circularity definable as non-domain-specific on the basis of its structural nature. The following example tests this suggestion.

### 2.1 Epistemic circularity in the 'mapping account' of the philosophical problem of applicability of mathematics

In the last two decades, philosophy of mathematics moved its focus from the traditional ontological concerns to the epistemology of mathematics and the relations of mathematics with natural sciences and physical reality. In an influential paper, physicist Eugene Wigner (1960) suggested that the overall success of the application of mathematics in the natural sciences is something unexplained and perhaps unexplainable. As Wigner put it in general terms, both the applicability of mathematics and the very high rate of success of applied mathematics require an explanation; if no explanation were given, the effectiveness of mathematics might be called "unreasonable." The general inquiry of whether the effectiveness of mathematics is reasonable or not, (or whether this is even a genuine problem at all), has benefited from the interest of philosophers, mathematicians and physicists; the related catchword is 'Wigner's puzzle.'

[9] Although it would be interesting to discussing this further, this is not the place for such development.

Around Wigner's puzzle, philosophy of mathematics and of science together developed a new field of research, that of philosophy of applicability of mathematics, trying to answer the following general questions: *Why* is mathematics applicable in sciences and physical reality, *how* do we rationally justify the use of mathematical models in the investigation of physical phenomena, and *how* do we explain their high rate of success, given that the source and target domain of application are of different ontologies, epistemologies, languages, and logical categories?

From the various attempts to provide a solution to this general problem, I will, for the purpose of this paper, focus on the structural one, called the 'mapping account' of application of mathematics.

Pincock (2004) declared as necessary the existence of an *external* relation between the mathematical domain and the modeled physical situation[10]. Pincock bases his account on the idea of *analogy* between mathematical structures and certain structures of or associated with the physical context; this analogy can be mathematically represented through the notions of homomorphism or isomorphism[11] as a map that preserves structures between two different domains. The mapping structural formalism is developed further by Bueno & Colyvan (2011) in their theoretic model based on structure-preserving maps, called by the authors "the inferential conception of the applicability of mathematics" (I shall abbreviate it ICAM.) The core principle of ICAM is that the fundamental role of applied mathematics is inferential, and this role ultimately depends upon the ability of the model to establish *inferential relations* between the empirical phenomena and the mathematical structures. In Bueno & Colyvan's terms, ICAM consists of a three-step scheme:

1. (Immersion) Establishing a homo/isomorphic mapping from the idealized empirical context to convenient mathematical structures; this mapping serves to connect the relevant aspects of the empirical situation to the mathematical context appropriate for the application.[12]

[10] Models based on the internal relation were also developed, starting with the semantic account of Frege.
[11] depending on the case of the mathematical application
[12] The mapping is not unique, and choosing the right one is a contextual problem in the mathematician's responsibility depending on the specificity of the application.

2. (Derivation) Derivation of the consequences through mathematical formalism within a specific mathematical theory, using the structures evidentiated through immersion.
3. (Interpretation) Interpretation of the consequences obtained at the derivation step in terms of the empirical context, by establishing a homo/isomorphic mapping from the mathematical structures to the initial empirical context.[13]

Both these mapping accounts are based on the classical set-theoretical notion of relation. A classical structure is thus a set of objects/nodes/positions together with a family of connections between them. In this system, any structure $S$ is given as a pair $S = \langle D, R \rangle$, where $D$ is a subset of a given universal set and $R$ is a family of relations over $D$ of various arities. In this set-theoretic model of application of mathematics, an unknown relation in the target domain (the empirical) is inferred on the basis of the definition of structural morphism, i.e., if $n$ nodes stand in a relation in the source structure (the mathematical structure, where all the relations are *known* as mathematically defined), then their correspondents in the target structure stand in a relation of the same arity. The interpretation step expresses the inferred (unknown) relation in terms of the empirical context by providing a statement about the context, which may be a prediction, an optimization, a description, or an explanation.

The common objections to the mapping accounts subscribe to the general objections regarding structural representation in science, and |I won't mention them here. In response to these objections, the authors of the mapping accounts advanced the notions of surplus structure, partial structures, and partial isomorphism/homomorphism (see Bueno & Colyvan 2011) and an iterated model of ICAM with partial mappings [see (Bueno & French 2012)].

Instead, I will focus on the objection of circularity and show two different kinds of circularities with which the mapping account is "infected", while analyzing to what extent these can be categorized as epistemic, per the EC definition.

### 2.1.1 The assumed-structure circularity

First, let us notice that a mapping account is not only a structural model representing any arbitrary *application* of mathematics, but also a metamodel having as its

[13] not necessarily the inverse of the immersion mapping

goal the representation and justification of a general-universal applicability of mathematics covering *any* application in the investigation of empirical systems..

The mapping accounts are developed around the idea that a structural correspondence is established via structure-preserving mappings; but then, if we use the mapping account to investigate the applicability of mathematics, we must *assume* that there *is* a structure of the world suitable for this mapping – that is, the world is suitably structured for both the application of mathematics and the theoretical accounts that address the applicability. Somehow, mathematics has already been applied in the target domain *prior* to the application of mathematics. This (linguistic, at this point) circularity can be traced back to the objections raised concerning the modal-structuralist solution to the problem of applicability (Hellman, 1989). In particular, Bueno (2014) revealed the circularity. The circularity as objection in relation to the "assumed structure" problem was also revealed in general terms by Räz & Sauer (2015, pp. 11-12). A similar case of circularity was noted by Ritchie (2003) within the theories of metaphor comprehension that are based on mapping accounts.

Take now the definition of EC and see how it fits the assumed-structure circularity. First, let us observe that outside epistemology, we are not talking here about an argument, but a model or method of investigation. Still, the circular model fits structurally the EC definition, as follows: Source $X$ is now the mapping account (theory or model) of the application of mathematics; the premise $P$ is that the empirical context in its idealized form is so structured as to allow the application of mathematics by reflecting (preserving) the mathematical structures from the source domain (mathematics) which will be applied, or in other terms, the empirical context can be "*mathematized*"; the trustworthiness $T$ of $X$ is a theoretical confirmation that the model works – that is, it is rigorous, consistent, and confirmed for the *general* application of mathematics – which is actually the goal of the theoretical pursuit of creating the model. With this interpretation, we have an instance of the formation of a belief in the trustworthiness of a source through a premise that depends on or is an output of the source (both options work in this case). That is, the assumed-structure circularity fits structurally the definition of EC, but the elements in the structure are not expressed in epistemological terms. Seen as an argument, we can qualify it as non-inferential.

In terms of determination, $T(X)$ determines the premise $P$ because the structuring of the target domain for "mathematization" is a result of the confirmation that the mapping account works and thus application of mathematics runs as the mapping account describes it structurally [$T(X) \to P \to T(X)$].

### 2.1.2 The methodological circularity

Let us now focus on what I have called earlier the structural metamodel of applicability of mathematics (a mathematical model of application and applicability of mathematics, for justifying the prefix 'meta'). That is, we have the same structures (source and target) standing in a homo/isomorphic external relation as before; however, the goal of the model is not only a representation of every application, but a justification for a general applicability of mathematics, or at least any sort of inference about this applicability.

In this setup, the metamodel (as an argument, this time inferential) fits the definition of EC, as follows: Source $X$ is mathematics (as a discipline or method); the trustworthiness $T$ of $X$ is the applicability of mathematics; the premise $P$ is the mathematical (set-theoretical) setup and form of the investigated *theoretical* context, that is, an integrated structure (source and target structures together) through the external morphism. (This nature of the setup allows the inferences via the ICAM scheme.) With this interpretation, we have an instance of the formation of a belief in the trustworthiness of a source from a premise that is obtained through a *method* that depends on $T$ (the method is application of mathematics, depending on the applicability of mathematics to *any* context, including the theoretical one in this case), which again fits the definition of EC. The scheme of determination is $T(X) \to X(C) \to P \to T(X)$, and I would call this kind of circularity a methodological circularity because the first determination sub-chain from the left to $P$ is formed through the application of a method preceded by a justification for that method. Observe that the current case cannot be generalized as 'investigating mathematics through mathematics' in the sense that this is a methodological circularity also. Mathematics is known as auto-applicative, and one of the results of such auto-application is metamathematics. Neither the more general formulations 'investigating a method through that same method' or 'investigating a discipline through that same discipline' is in general methodologically circular. One of the hidden specificities of mathematics among disciplines seems to be auto-applicability. If one method or discipline is not auto-applicative by its nature, then neither of the previous formulations makes sense. Instead, the formulation 'investigating *applicability* of mathematics through mathematics does make sense and is methodologically circular, as I showed, and the responsible element for these distinctions (as well as for circularity) is the second-order predication [$T(X)$] appearing in the same formulation with its variable ($X$)[14].

Now let us leave the formal stance about this example and focus on the conceptual interpretation. Philosophers or mathematicians have not succeeded in providing a rigorous definition of mathematics, and its nature is still unclear. The mapping account has a mere set-theoretical nature – all the notions involved such as sets, relations, functions, isomorphisms, and homomorphisms are essential notions for set theory. Thus, what the metamodel has used as a method or source applied to the current theoretical context was actually the set theory. However, methodological circularity was detected through the assumption that we used *mathematics* to investigate its applicability, application that was justified by the same applicability. Although set theory is part of and foundational for mathematics as a whole, the circular determination is dependent upon the interpretation of whether set theory is actually a mathematical theory, or better said, specific for mathematics. For if it is not, then even if we may apply it as mathematical to the theoretical context, it is not a general applicability of mathematics that determines this application, but an applicability of (the non-mathematics-specific) set theory. In this latter interpretation, the circularity element vanishes from the determination chain. The idea that sets, functions, and analogic correspondences are not specific to mathematics is not new. The idea stresses the point that these concepts are fundamental concepts reflecting primary cognitive processes of our regular mental activity and are foundational for other disciplines as well. As such, mathematics is not a compact stand-alone discipline, but rather something that dissolutes in all the products of our reason. This is not the place to

[14] The two terms appear as having different logical orders in the linguistic form of the argument, however in the determination chain they are of the same order, as $T(X)$ and $X(C)$, not meaning that $T(X)$ determines $X$, but $T(X)$ determines the application of $X$ to $C$. As such, they have the same logical and epistemic status, as predicates or properties, yet over variables of different natures.

review the literature on this topic, but I would just mention the pioneering research on perceptual mathematics [(Teissier, 2005), (Ye, 2010), and (Mujumdar & Singh, (2016)] as well as the studies of the dissolution of abstract structures and the 'blurring problem' in ontic structuralism [(French & Ladyman, 2003, 2011) and (Cao, 2003)].

Observe that a similar argument does not apply to assumed-structure circularity. In that case, the circularity element remains even if interpreting set theory as non-mathematics specific because the premise is formulated in terms of structures and application and thus is dependent on set theory and not on highly complex mathematics.

### 2.2 Conclusions of the case study

We have already noticed in the linguistic examples from section one that the language in which an argument is formulated is able to manipulate circularity by making (or not) a word repetition that can be interpreted as an element of circularity. If considering circularity in terms of epistemic determination, this manipulation may also apply to non-linguistic types of circularity, because certain words have interpretable meanings, and the conceptual language is dependent on meaning. In brief, word language may manipulate through repetition, while conceptual language may both manipulate repetitions and affect the conceptual determination. We have encountered the latter situation in the previous case study. While the assumed-structure case did not even exhibit a word repetition to be taken as an element of circularity (although finding one through reformulation is just a matter of linguistic technique), in the methodological-circularity case, we had the word repetition, but it was the conceptual interpretation of the language that changed circularity in what concerns the cyclic determination.

Since we cannot ignore the linguistic nature of EC, the manipulation issue raises the problem of genuineness of circularity. However, we have seen in the methodological-circularity example that genuineness can be established only as dependent on certain theories and philosophical views about the involved concepts (recall the problem of set theory being mathematics-specific), and not through a pre-established set of context-specific metatheoretical criteria. It is debatable whether the genuineness issue and qualification as (in)genuine entail a commitment to an anti-realist position about circularity. Circularity also has other natures besides the linguistic, and within the epistemic nature, such a position would revert to an anti-realist position about the concept of determination. Instead, the genuineness problem is an additional argument for a *structural* description of circularity and just warns us to be careful, especially with the conceptual interpretation, when investigating circularity. This carefulness not only assumes to choose, revise, or adapt the meaning of the terms of the argument, but also to make clear distinctions between and delimitations of the concepts employed in the determination chain. As we have seen in the methodological-circularity case, such distinctions and delimitations may be dependent on theories external to the argument questioned for EC (see the set theory-mathematics distinction/inclusion problem). As such, I see at this point two possible kinds of ingenuinenesses for circularity, one technical (as result of a linguistic manipulation) and one conceptual (of which one sub-type would be that dependent on external theories).

Note that both cases fit structurally the classical definition of EC in terms other than epistemological and the description of circularity in terms of determination. This fact suggests that the definition ought to be generalized to cover a wider range of concepts employed in constructs for which a circularity charge can be made, possibly the entire spectrum of intellectual practice.

'Belief source', as a key concept in the definition of EC, is expressed through a term which, in the epistemological theoretical context, limits its range of senses and meanings to personal sources, in which perception, introspection, memory, reason, and testimony are traditionally included. However, the case study revealed that impersonal sources fit as well when talking about general circularity, and these may be models, theories (scientific or non-scientific), disciplines, theorems, properties, hypotheses, axioms or postulates, data, other arguments, etc. The main distinction between 'source' in the epistemological context of the classical definition of EC and 'source' as the proposed complement for its extension is that the former is something given, which provides knowledge as a product that cannot influence in any way the source itself, while the latter is something that may be still in process of construction, revision, confirmation, or debate, and whose product can eventually influence the source at a later time. In the assumed-structure case, the mapping account as the source is a theoretical model *proposed* to represent the application of mathematics.

The meaning of the 'source' influences the meaning of the attribute of the premise of the argument as being 'dependent on' or 'an output of' the source. With the new proposed meaning for 'source', dependence on the source does not make much sense. I have already suggested in section one the adjectives 'used' or 'applied' as a replacement, and now this suggestion is supported by the proposed extension of 'source'. Thus, I will take the source as something *applied* to a suitable context in order to entail the premise; this makes sense for both personal and impersonal belief sources and also covers the general meaning of the premise being "an output of" the source (as the result of the application). Accordingly, I will replace 'trustworthiness' of a source with 'applicability' of that source; a source is to be applied if it is applicable, and saying that knowledge (as either source – in the added meaning – or product of a source, as either content or method) is 'applied' in reasoning does make sense. Even the meaning of the term 'agent' (who is forming the belief) can be extended in the impersonal realm to cover, for instance, theories against which the new beliefs are tested; however, I won't propose a new term for this extension, only the conceptual extension. It is worth noting here that keeping a human personality for the concept of epistemic agent influences decisively the conceptual framework in which malignity of EC is investigated. This occurs, for example, by employing concepts like 'epistemic disagreement' and 'rational persuasion' – terms used by Lynch and Silva (2016) to argue that circular arguments fail to be persuasive as doxastic state changers; further, circularity as an attribute of the arguments is thinker-to-thinker dependent – or 'empathy' and 'cognitive-state sharing' – terms used by Sorensen (1999) to argue in a similar line of thought that circularity is a side-effect of the rational persuasion of one another. However, traditional views of knowledge as a mental state cannot be ignored in relation to the proposed extension of the meaning of 'agent'; even those philosophers who oppose that view still accept that knowledge incorporates a mental state, that of the agent's state of belief (Nagel, 2013).

Reformulating the definition of |EC in terms of application and applicability would sound like this: 'Epistemic circularity characterizes the formation of an agent's belief about the applicability of one of agent's belief sources *X* if the formation of that belief is the result of an application of $X$ to the context of the argument' with the key terms in their extended meaning.

It is also the characterization of EC in terms of determination that supports the conceptual extension detailed above, given the no-restriction principle for the natures of the determinates and determinables. The generality of the extended definition should be reflected in a formal structural account of circularity, motivated by the circularity element, the existence of distinct types of circularity, the description in terms of determination, and the genuineness problem. This account will be the matter of the last section.

## 3. The relational account of circularity

The following formal account is based on the usual notion of classical set-theoretical structure. Let $S = \left\langle A, (R_i)_i \right\rangle$ be a structure, with $A$ the set of nodes and $(R_i)_i$ the family of relations over $A$ (each $R_i$ is a set of connections of nodes of the same arity).

Definition 1: The finite ordered string of binary relations $(R_1, R_2, \dots, R_n)$, not necessarily distinct, where $n > 1$, is called *non-homogenously transitive* (nh-transitive) iff:

$\forall\ j = \overline{2,n}$, the composition $R_1 \circ R_2 \circ \cdots \circ R_j$ is not empty and $\exists\, k = \overline{1, j}$ such that $R_1 \circ R_2 \circ \cdots \circ R_j = R_k$.

(The composition of relations is meant in its usual sense, that is $R_1 \circ R_2 = \{(a,c) \in A \times A, \exists\, b \in A \text{ such that } (a,b) \in R_1 \text{ and } (b,c) \in R_2\}$ and a composition of any number of relations is defined recursively).

Intuitively, the definition states that in an nh-transitive composition, the successive heads of the compositional chain are connected through one of the relations present in the partial chain:

$$\underbrace{\underbrace{x_1 \xrightarrow{R_1} x_2 \xrightarrow{R_2} x_3}_{R_1 \text{ or } R_2} \cdots \xrightarrow{R_{n-1}} x_n}_{R_1 \text{ or } R_2 \text{ or } \ldots \text{ or } R_{n-1}}$$

Observations:

1. The definition is not vacuous. Take for example $R_1$ as the order relation < and $R_2$ as the equality relation = over a subset of naturals. Then $R_1 \circ R_2 = R_1$.

2. For the particular case $n = 2$ and $R_1 = R_2$, the identity from the *definiens* reverts to $R_1 \circ R_1 = R_1$, which is equivalent to the classical definition of transitivity.

3. From the definition, it follows immediately that if $(R_1, R_2, \ldots, R_n)$ is nh-transitive, any ordered substring $(R_1, R_2, \ldots, R_i)$, with $i < n$ is nh-transitive.

4. For a nh-transitive string of relations, a string obtained through a permutation of them is not necessarily nh-transitive. The example from observation 1 stands as a counterexample. Sufficient conditions for the nh-transitivity to hold for permutations are related to specific properties, such as symmetry properties, of the relations in composition.

5. If relations $R_j$ are each transitive, it does not follow that the whole string is nh-transitive. As a counterexample, take in the example from observation 1 the order relation < instead of equality =. The converse is also not true: If a string (or the composed relation) is nh-transitive, it does not follow that any of the relations in the string is transitive. Take as counterexample the order relation > and the relation *p* of coprimeness over the naturals. For instance, the composed connection 7 > 3 *p* 5 shows that connection (7, 5) is in both relations; however, *p* is not transitive.

The only case in which we can formally derive transitivity from nh-transitivity is that in which all the relations in composition are identical. As such, nh-transitivity is a property of an aggregate and not of an individual relation as is classical transitivity.

Definition 2:

A finite string of binary connections $C = \left[\left(x_0, x_1\right), \left(x_1, x_2\right), \ldots, \left(x_{n-1}, x_0\right)\right]$, with $n >$ 1, $x_i \in A, \forall\, i = \overline{0, n-1}$ is called a *circularity* of the structure *S*, if there exist the asymmetric relations $R_j,\ j = \overline{1,n}$ such that $\left(x_{j-1}, x_j\right) \in R_j$ for *j* from 1 to *n* – 1 and $\left(x_{n-1}, x_0\right) \in R_n$. Call $x_0$ the *circularity element* of *C*.

The only result derived from this formal account that interests us is the following:

Statement: If *C* is a circularity formed by connections from the relations $R_1$ to $R_n$ and their composition is nh-transitive, then not all these relations are irreflexive.

The proof is obvious, by following directly from Definitions 1 and 2: if $C = \left[\left(x_0, x_1\right), \left(x_1, x_2\right), \ldots, \left(x_{n-1}, x_0\right)\right]$ is the circularity and $R_1 \circ R_2 \circ \cdots \circ R_n = R_k, 1 \le k \le n$, then $\left(x_0, x_0\right) \in R_k$ and it follows that $R_k$ is not irreflexive[15].

The interpretation I will give to this set-theoretic formalism is the circularity of any type, described in terms of epistemic determination. In this representation, a circular string is – epistemically – a determination chain between concepts in any context.

The classical relational accounts of determination (both old and contemporary) state the three essential features of a relation of determination as asymmetry, irreflexivity, and transitivity (some making asymmetry "more essential" than the other two features). However, the focus on the binarity of this relation has ignored somehow that our reason operates more with *chains* of determinations than with isolated connections between a determinate and a determinable; within such chains, the epistemic determination must have a "flow" and a sense of flowing. While the sense of flowing is represented by asymmetry ("it goes in *one* direction"), the flow is represented by irreflexivity ("it does not stop") and transitivity ("it transports the initial and intermediary determinations").

[15] Take into account the distinction between irreflexive and anti-reflexive relation. The latter allows the existence of (*a*, *a*)-type connections, but not all to be of that type, while the former does not. The same kind of distinction holds for the asymmetric and anti-symmetric relation.

However, transitivity of determination was understood in the classical sense, which is not suitable for chains, where we may have *different kinds* of determination being composed. Under the no-restriction principle, determination can apply to any category of concepts, and as such, we have different types of determination relations between them (specificity, causal, inferential, constitutive, applicative, etc.). In these conditions of diversity, standard transitivity can no longer represent that "transportation" of the determination throughout the determination chains. For example, in a chain of determination $F \xrightarrow{C} G \xrightarrow{I} H$, where concept $F$ determines concept $G$ constitutively ($C$) and concept $G$ determines concept $H$ inferentially ($I$), even if we agree that both relations of constitution and inference are transitive, it does not follow that determination between $A$ and $C$ is of a kind or another on the basis of this transitivity. Only in the particular case of the two relations being the same kind, does classical transitivity work. However, a relation must hold between $F$ and $G$ in order for the determination to "flow" properly. Therefore, I claim that the concept of nh-transitivity is more suitable for such a role, and the assimilation of the relational account of determination into the structural account of circularity argues for that. This assimilation runs on the principle that the kinds of determination are finite in number (expressing somehow one of the limitations of our reason), and the multi-step determination in a chain is of one of the kinds that participate in the composition, that is, in a "chromosomal" mode. In the schematic example above, determination between $F$ and $H$ cannot be of another kind, say of specificity, but only $I$ or $C$. In other words, the relation of determination between $F$ and $H$ ought to somehow embed $I$ and $C$ (since it is not established from outside the chain) and "absorb" one of them.

With these conventions on the relational account of determination, the formal account of circularity says simply that circularity is a determination chain in which the irreflexive condition of determination is violated. The account covers all types of circular constructs seen as structural. If the structure is linguistic and the relations in composition are syntactic, then we have a linguistic circularity. If the structure is logico-propositional and the relations are of one single type, namely consequence, the circularity is logical. For more complex types (including the constitutive second-order relation), the circularity is epistemic.

Is the epistemic worry about circularity reflected in this account? Yes, but only in what concerns determination. Determination does not flow in the circular constructs, and our brain is worried about this just because it is used with and seeks the flow. Seeking for missing explanations follows the same pattern as a worry. The mechanisms responsible for this biological attitude relative to worry cannot yet be dealt with by philosophy or set theory. Circular constructs may have other epistemic virtues such as the semantic guidance of circular definitions. A virtue cannot cancel the determination worry, but it can fulfill a role independent of the worry. If we found no virtue for the logical circularity, another kind of worry – this time person-to-person specific – is that EC would be reducible to a logical circularity. Language can do this relatively easily through words like 'entailment', 'therefore', 'premise', and 'conclusion' used in non-logical contexts; however, the structural account of circularity limits this possibility by reflecting the diversity in the kinds of relations participating in the construct. Usually the last relations in the determination chain~~,~~ expressed in language, are mainly responsible for the formulation of the construct. However, it is not only those relations but the entire composition of *n* relations that establishes the type of circularity.

As for benignity/malignity, the relational account does not reflect such qualification relative to the circular construct *itself*, as traditional accounts of EC define it. Once determination is violated, a malignity does exist in every circular construct, but it is just the quality of the construct as being non-determinative. If we want to extend the malignity qualification beyond the relativity to determination, I think that the only available alternative is externalization – that is, to relate the qualification to the entire epistemic structure (larger arguments, theories, systems of beliefs) that the circular construct is serving and is included in, and to look for external criteria of decidability.

## 4. Conclusions

The current study has been developed around two main starting questions: 1) What is circularity and what is its nature? 2) Why are we worried about EC more than about other types of circularity that we actually use in our scientific and philosophical practice? Even in their unanswered form, these questions suggest that the investigation of circular constructs should be focused *ab initio* on the general concept of circularity rather than on the specificity of each type.

In the overview on the known types of circularities, I have concluded that a circularity has a multiple nature (not only that associated with its type – epistemic, linguistic, logical), and the linguistic aspect is essential in qualifying a construct as circular, since both word language and conceptual language may manipulate both the meaning and relationships of the concepts involved, including what I have called the circularity element; as such, it raises the problem of genuineness of a circularity. This claim is made visible in the examples provided and particularly in the case study of a circular theoretical-philosophical account, which yielded the notion of methodological circularity.

The case study also argues for an extensional approach of circularity, including the meaning of the key epistemological terms referred to in the classical definition of EC to also cover other associated concepts specific to other theoretical disciplines beyond epistemology.

By analyzing the structural resemblances of the various types of circularities, I have tried to develop a unificatory account of circularity, describing circularity in terms of relational determination. The formal structural approach in section three describes circularity as a relational chain of determination for which the irreflexive condition is violated. This assimilation raises the challenge addressed to the relational accounts of determination in regard to the replacement of the transitivity condition with what I call non-homogenous transitivity.

As for the specific worry that features circularity, it can also be addressed in terms of determination, but this time with a strong cognitive-psychological-neurophysiological component of the theoretical context. Besides epistemic determination, any worry is circumstantial, person-dependent, and language-dependent.

Overall, I wouldn't be as firm as Sorensen (1999) in saying that circularity "ain't in the head," given the genuineness problem, the core feature based on determination, and the conceptual constitutive relation between circularity and the epistemic worry for it. Instead, I would say that so many features (including the nh-transitivity!) point to similarities with processes of biological functioning of the human brain that clarifying the nature of circularity and implicitly finding the complete answers to the worry questions can be pursued only in an interdisciplinary setup with the neurosciences involved.

(14,945 words)

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