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Elementary differences between the degrees of unsolvability and degrees of compressibility[☆]

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ABSTRACT

Given two infinite binary sequences A, B we say that B can compress at least as well as A if the prefix-free Kolmogorov complexity relative to B of any binary string is at most as much as the prefix-free Kolmogorov complexity relative to A, modulo a constant. This relation, introduced in Nies (2005) [14] and denoted by $A \leq_{LK} B$, is a measure of relative compressing power of oracles, in the same way that Turing reducibility is a measure of relative information. The equivalence classes induced by $\leq_{LK}$ are called LK degrees (or degrees of compressibility) and there is a least degree containing the oracles which can only compress as much as a computable oracle, also called the 'low for K' sets. A well-known result from Nies (2005) [14] states that these coincide with the K-trivial sets, which are the ones whose initial segments have minimal prefix-free Kolmogorov complexity.

We show that with respect to $\leq_{LK}$, given any non-trivial Δ^0_2 sets X, Y there is a computably enumerable set A which is not K-trivial and it is below X, Y. This shows that the local structures of Σ^0_1 and Δ^0_2 Turing degrees are not elementarily equivalent to the corresponding local structures in the LK degrees. It also shows that there is no pair of sets computable from the halting problem which forms a minimal pair in the LK degrees; this is sharp in terms of the jump, as it is known that there are sets computable from $\mathbf{0}''$ which form a minimal pair in the LK degrees. We also show that the structure of LK degrees below the LK degree of the halting problem is not elementarily equivalent to the Δ^0_2 or Σ^0_1 structures of LK degrees. The proofs introduce a new technique of permitting below a Δ^0_2 set that is not K-trivial, which is likely to have wider applications.

1. Introduction

Algorithmic randomness of strings or streams can be mathematically defined on the basis of the intuitive idea of 'incompressibility'. This approach was introduced by Solomonoff [19] and independently by Kolmogorov [11] for strings (finite binary sequences), and extended to streams (infinite binary sequences) by Levin [13] and independently by Chaitin [7]. According to this approach, an infinite binary sequence is random if its initial segments are hard to describe. Descriptions of strings are given by Turing machines, operating on strings, that have prefix-free domains.[1] A basic fact is the existence of a universal prefix-free machine, i.e. one that gives optimal (i.e. shortest) descriptions to all strings, modulo a constant. The prefix-free complexity of a string σ is the length of its shortest description with respect to a fixed universal prefix-free

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[1] A set of strings is called prefix-free if for every two distinct strings in that set, the first string is neither the extension nor the prefix of the second string. A string τ is a description of a string σ with respect to a prefix-free machine M if $M(\tau) = \sigma$.

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machine, and is denoted by $K(\sigma)$. A stream Z is random if its initial segments cannot be described by strings which are shorter than the segments themselves (modulo a constant); in symbols, $K(Z \restriction n) \geq n - c$ for some constant c and all $n \in \mathbb{N}$.

The study of the 'descriptive' complexity of strings and streams has naturally lead to the study of relativized complexity (where the Turing machines used have access to external information) in the same way that the theory of computability [20] lead to the theory of relative computation and unsolvability [21]. For example, a set A was called 'low for K'[2] if the prefix-free complexity relative to A is the same (modulo a constant) as the unrelativized prefix-free complexity. This means that A contains no information which could help to achieve a better compression on the binary strings. This notion was studied thoroughly in [14], where it was shown that it coincides with two other notions: K-triviality and lowness for randomness. A set A is K-trivial if its initial segments have minimal prefix-free complexity, i.e. no more (modulo a constant) than the complexity of a trivial sequence like 0^∞. Moreover, A is low for random if any random sequence is also random relative to A. In the following, we will mostly use the name 'K-trivial' to refer to any of its equivalent formulations. Based on the notion of 'low for K', Nies [14] defined the partial order $\leq_{LK}$ on the Cantor space: we say that $A \leq_{LK} B$ for two sets A, B if the prefix-free complexity relative to A is at least as much (modulo a constant) as the one relative to B. In other words, B can compress at least as well as A, and in symbols $K^B(\sigma) \leq K^A(\sigma) + c$ for a constant c and all strings σ.

This partial ordering defines an equivalence relation on the Cantor space which groups different oracles in a single class provided that they are capable of the same level of compression. These equivalent classes are usually called *LK* degrees but we also call them *degrees of compressibility*. We note that two oracles may contain mutually disjoint information (in the sense that they form a minimal pair in the degrees of unsolvability) yet be in the same LK degree.[3] An apparently weaker partial order is obtained if we only require that the random streams relative to B (i.e. the streams whose initial segments cannot be compressed using information from B) are also random relative to A. This partial order was also introduced in [14], was denoted by $\leq_{LR}$ and the induced structure of equivalent classes was called the LR degrees. Remarkably, Kjos-Hanssen/Miller/Solomon [9] (also see [15] for a presentation of this result) have shown that $\leq_{LR}$ coincides with $\leq_{LK}$.

In [4,5] the LR degrees were studied both locally and globally, and a number of similarities were discovered with the Turing degrees, both with respect to algebraic features of the partially ordered structures and in terms of the methods used to prove them. The applicability of methods from the Turing degrees to the study of the LR degrees was, to some degree, expected as $\leq_{LR}$ (and $\leq_{LK}$) is a natural extension of $\leq_T$ (the Turing reducibility) of the same arithmetical complexity. In the same papers a quite special feature of $\leq_{LR}$ was discovered, namely the uncountable predecessor property, which provided a dramatic difference with the structure of Turing degrees. On the other hand, this property is not elementary (it is not a first order property) and it does not seem to play an important role in the study of the local structures of the LR/LK degrees, for example the Σ^0_1 or the Δ^0_2 degrees. Here an LR/LK degree is called Σ^0_1/Δ^0_2 if it contains a Σ^0_1/Δ^0_2 set respectively (similar definitions hold for higher arithmetical classes).[4]

In this paper we provide the first elementary differences between the local structures of the Turing and the LR/LK degrees. We first show the following.

Theorem 1.1. *Let X be a Δ^0_2 set which is not K-trivial. Then there exists a c.e. set A which is not K-trivial such that $A \leq_{LR} X$.*

The proof of this result uses a new method for permitting[5] below a Δ^0_2 set which is not K-trivial. This is an original technique for studying relative randomness and has no analogue in the theory of Turing degrees.[6] An early, restricted version of this method was used in [6] to show that a c.e. set is not K-trivial iff it computes a c.e. set which cannot be split into two disjoint c.e. sets of the same LR degree. In [2] the same idea was extended to Δ^0_2 sets to show that a Δ^0_2 set is K-trivial iff it has $2^{\aleph_0}$ many LR predecessors. The present proof though, goes beyond these early versions of the method and demonstrates this permitting technique in full generality. It is likely that this method will have wider applications to problems related to K-triviality and computable approximations. Theorem 1.1 has interesting consequences.

Corollary 1.2. *The Δ^0_2 structure of LR/LK degrees is downward and upward dense. Also, the Δ^0_2 structures of LR/LK degrees and the Turing degrees are not elementarily equivalent.*

Proof. Let A be a Δ^0_2 set of non-trivial LR degree. If the LR degree of A is not Σ^0_1, then Theorem 1.1 implies that there is a Δ^0_2 (in fact, Σ^0_1) non-trivial LR degree strictly below the LR degree of A. On the other hand, if the LR degree of A is Σ^0_1, the existence of a non-trivial LR degree strictly below the degree of A follows from the downward density of the Σ^0_1 structure of the LR degrees which was proved in [4] (in the form of a c.e. splitting theorem) and in [5] (in the form of a more general partial density theorem).

2 This notion was defined by Andrej A. Muchnik during a seminar in 1999.

3 This follows by the fact that there is a promptly simple set $A \leq_{LK} \emptyset$ (see [15]) since every promptly simple set computes a minimal pair in the Turing degrees and $\leq_{LK}$ is an extension of $\leq_T$. For an example where the LK degree is non-trivial we refer to [1]. In that paper we show that there is a minimal pair of Turing degrees inside the *LK* degree of the halting problem.

4 Recall that Σ^0_1 sets or degrees are also called computably enumerable, or c.e. for short.

5 We call this technique 'permitting' as the enumeration or not of numbers into the constructed set A depends on certain features of the approximation to the set X, below which we are building A.

6 Recall that there are minimal Δ^0_2 degrees, by Sacks [16].

The upward density of the Δ^0_2 LR degrees follows by relativizing the c.e. splitting theorem of [4] in the same way that the upward density of the Turing degrees is proved using a relativization of the Sacks splitting theorem. Indeed, let $B <_{LR} \emptyset'$ be a Δ^0_2 set. The splitting theorem of [4] ensures that every c.e. set A which is not K-trivial is the disjoint union of two c.e. sets C, D which are not of the same LR degree as A.[7] If we relativize this theorem to any oracle $X <_{LR} A$ we get that A is the disjoint union of two X-c.e. sets C, D such that $X \oplus C$ and $X \oplus D$ are not of the same LR degree as A and either $X \oplus C \not\leq_{LR} X$ or $X \oplus D \not\leq_{LR} X$.[8] Now if we let $A = \emptyset'$ and $X = B$, we get two sets $\emptyset'$-c.e. sets C, D such that

- $B \oplus C$, $B \oplus D$ are Δ^0_2.[9]
- $B \oplus C <_{LR} \emptyset'$ and $B \oplus D <_{LR} \emptyset'$
- $B \oplus C \not\leq_{LR} B$ or $B \oplus D \not\leq_{LR} B$.

This completes the argument for the upward density of the Δ^0_2 structure of LR degrees, given that $\leq_{LR}$ is an extension of $\leq_T$. The elementary difference between the Δ^0_2 structures of Turing and LR degrees is the existence/non-existence of minimal Δ^0_2 degrees respectively (where the existence of a Δ^0_2 minimal Turing degree was established in [16]). □

A remarkable feature of the permitting technique involved in the proof of Theorem 1.1 is that enumerations into A (the c.e. set we construct below the given one X) do not happen upon changes in the approximation of X, as with most permitting methods. Instead, they happen anyway (subject to some conditions) and changes in X are motivated *afterwards* by advancing a pseudo-strategy which tries to show that X is low for random. The relevant segments of X associated with enumerations into A need not always change, however they will change to a degree which ensures $A \leq_{LR} X$. This feature allows for the permitting to be applied with respect to two given non-trivial sets X, Y with a computable approximation, *simultaneously*. Hence, after a modification of the construction behind the proof of Theorem 1.1 we are able to establish the following.

Theorem 1.3. *Let X, Y be Δ^0_2 sets which are not K-trivial. Then there exists a c.e. set A which is not K-trivial such that $A \leq_{LR} X$ and $A \leq_{LR} Y$.*

Although Theorem 1.1 follows from Theorem 1.3, we choose to present a full proof of the former so that the technique is fully understood. Then the presentation of the latter one is smoother, as it is partially based on the previous sections. Theorem 1.3 has further consequences, outlined in the following corollary.

Corollary 1.4. *The Σ^0_1 structures of LR/LK degrees and the Turing degrees are not elementarily equivalent. Also, the structure of the LR/LK degrees below the LR/LK degree of the halting problem is not elementarily equivalent to the Σ^0_1 and Δ^0_2 structures of LR/LK degrees.*

Proof. The first claim follows from Theorem 1.3 and the fact from [12,22] that there are minimal pairs in the c.e. Turing degrees. The second claim follows from Theorem 1.3 and the fact from [3] that there are two sets LR below $\emptyset'$ which form a minimal pair in the LR degrees. □

Finally, we have the following.

Corollary 1.5. *Given any finite collection of Δ^0_2 sets which are not K-trivial, there is an uncountable collection of LR/LK degrees below the LR/LK degrees of all of them.*

This follows by Theorem 1.3 in combination with the result in [2] that every Δ^0_2 set which is not K-trivial LR bounds uncountably many sets and the fact from [14] (also see [17] for a different presentation) that every LR degree is a countable equivalence class.

Finally we note a further consequence of Theorem 1.3: if X, Y are relatively 1-random, it does not follow that every set LK below both of them is K-trivial. This contrasts the situation in the Turing degrees. For the proof, it suffices to consider two Δ^0_2 sets X, Y which are relatively 1-random and apply Theorem 1.3.

2. Preliminaries

In the following, we use c.e. sets of strings to generate subclasses of the Cantor space 2^ω. For example, a binary string σ is often identified with the clopen set $[\sigma] = \{X \mid \sigma \subset X\}$ and more generally, a set of strings M is often identified with the open set

$$S(M) = \{X \in 2^\omega \mid \exists n (X \upharpoonright n \in M)\}$$

[7] Notice that this does not automatically imply that both C, D are not K-trivial because $\oplus$ is not a least upper bound in the LR degrees (see [15]). However it implies that at least one of them is not K-trivial, since K-triviality is closed under join (see [8]). A stronger version of this splitting theorem is true, which guarantees that both C, D are not K-trivial, but we do not need to use this here.

[8] Notice that if both $X \oplus C \leq_{LR} X$ and $X \oplus D \leq_{LR} X$, then C, D would be K-trivial relative to X (by a result in [14]) and so $C \oplus D$ would also be K-trivial relative to X; in particular, $C \oplus D \leq_{LR} X$. This is a contradiction since $A \leq_T C \oplus D$ (because C, D split A) and $A \not\leq_{LR} X$.

[9] Given that $C \cup D = \emptyset'$, $C \cap D = \emptyset$ and $\emptyset'$ can enumerate C, D.

of the Cantor space. Also, boolean operations, inclusion and measure on sets of strings refer to the sets of reals that they represent. Thus if $M, N \subseteq 2^{<\omega}$, we define $\mu(M) := \mu(S(M))$ (where μ is the Lebesgue measure), $M \subseteq N$ iff $S(M) \subseteq S(N)$, $M \cap N := S(M) \cap S(N)$, $M \cup N := S(M) \cup S(N)$ and $M - N := S(M) - S(N)$.

An *oracle* Σ^0_1 *class* V is an oracle Turing machine which, given an oracle A, outputs a set of finite binary strings V^A representing an open subset of the space 2^ω. The oracle class V can be seen as a c.e. set of axioms $\langle \tau, \sigma \rangle$ (where $\tau, \sigma \in 2^{<\omega}$) so that

$$V^A = \{\sigma \mid \exists \tau (\tau \subset A \wedge \langle \tau, \sigma \rangle \in V)\}$$
$$V^\rho = \{\sigma \mid \exists \tau (\tau \subseteq \rho \wedge \langle \tau, \sigma \rangle \in V)\}$$

for $A \in 2^\omega$, $\rho \in 2^{<\omega}$. An oracle Σ^0_1 class V is *bounded* if $\mu(V^X) < 1$ for all $X \in 2^\omega$. We denote the finite approximation of a parameter at stage s of the universal enumeration of c.e. sets by the suffix $[s]$. An oracle Martin–Löf test (U_e) is a uniform sequence of oracle Σ^0_1 classes U_e such that $\mu(U^X_e) < 2^{-(e+1)}$ and $U^X_e \supseteq U^X_{e+1}$ for all $X \in 2^\omega$, $e \in \mathbb{N}$.

In [10] (see [4] for a different proof) it was shown that $A \leq_{LR} B$ iff for some member U of a universal oracle Martin–Löf test, there is a bounded $\Sigma^0_1(B)$ class V^B such that $U^A \subseteq V^B$. Also, this is equivalent to the property that every bounded $\Sigma^0_1(A)$ class is contained in a bounded $\Sigma^0_1(B)$ class. This is the formulation of $\leq_{LR}$ that we are going to use in the proofs below, in accordance with previous work [4] on this relation. We can choose a universal oracle Martin–Löf test (U_i) such that U^τ_i are clopen sets (i.e., finite sets of strings) that are uniformly computable in i, τ, see [4].

3. Proof of Theorem 1.1

Given an effective list (V_e) of all bounded Σ^0_1 classes and a member U of a universal oracle Martin–Löf test such that $\mu(U^Z) < 2^{-2}$ for all $Z \in 2^\omega$, it suffices to construct a c.e. set A, a $\Sigma^0_1(X)$ class V^X and a $\Sigma^0_1(A)$ class $U^A_\star$ such that

$$U^A \subseteq V^X \quad \text{and} \quad \mu(V^X) < 1 \tag{3.1}$$
$$R_e : U^A_\star \not\subseteq V_e \quad \text{and} \quad \mu(U^A_\star) < 1 \tag{3.2}$$

for all $e \in \mathbb{N}$. The star '$\star$' in $U_\star$ indicates that the class is built by us (and is 'universal' according to the requirements that it satisfies) as opposed to the 'universal' class U which is given to us. Later on, we will introduce more parameters with a star subscript (in particular $F_{\star e}$ and $L_{\star e}$ of Table 2) which will be an indication that they are directly related to $U_\star$. Without loss of generality[10] we can assume that

$$\mu(V_e) < 1 - 2^{-e} \quad \text{for all } e \in \mathbb{N}. \tag{3.3}$$

Requirement (3.1) ensures that $A \leq_{LR} X$ while (3.2) ensures that A is not K-trivial. Of course we also have the hypothesis that $X \not\leq_{LR} \emptyset$. If (U_i) is a universal oracle Martin–Löf test, this hypothesis means that for every Σ^0_1 class E that we might construct, either $U^X_i \not\subseteq E$ or $\mu(E) = 1$. This is exactly the way we are going to use the fact that X is not trivial.

The strategy for $U^A \subseteq V^X$ is straightforward: at stage s we look at $U^{A\restriction s}$. If some clopen set is in it with use u and is not in $V^X[s]$, we enumerate it in $V^X[s]$ with the same use u. The challenge in (3.1) is to ensure that $\mu(V^X) < 1$. Although this constraint is trivial if we do not enumerate into A, any strategy for (3.2) requires such enumerations thus posing a threat to $\mu(V^X) < 1$. Indeed, to make A not K-trivial we need to put clopen sets from $2^\omega - V_e$ into $U^A_\star$, wait until they appear in V_e and then enumerate into A in order to eject them from $U^A_\star$, and so on. The *potential cost* of enumerating n into A at stage s is

$$cost(n, s) = \mu(\{Z \mid Z \in U^A[s] \text{ with use} \geq n\}). \tag{3.4}$$

This is the measure of the reals that will be ejected from U^A after such an enumeration, and could potentially stay in V^X, if $X \restriction n$ does not permanently change to a new configuration. In order to deal with this conflict we need to use really small pieces of measure for (3.2) and be more flexible with this strategy.

Choose some $b_e > e$ for $(3.2)_e$ (that is, (3.2) for a fixed e). When we discuss the global construction later on, an appropriate choice of those b_e will become relevant. The oracle class U_{b_e} can be seen as a computable measure assignment along the paths of the full binary tree. At any stage we have an approximation for X, which points to a particular path of the binary tree. We are about to define a procedure enumerating a Σ^0_1 class E_e which tries to cover $U^X_{b_e}$. The class E_e will be covering $U^X_{b_e}$ only as long as V_e keeps covering $U^A_\star$. Also, the measure of E_e will be $\leq \mu(V_e)$. If this procedure never stops, $\mu(E_e) = 1$ given that $X \not\leq_{LR} \emptyset$. But that would mean that $\mu(V_e) = 1$ which is a contradiction. Let

$$v_e[s] = \text{least } t[U^{X[s]\restriction t}_{b_e} - E_e[s] \neq \emptyset] \tag{3.5}$$
$$C_e[s] = U^{X\restriction v_e}_{b_e}[s] - E_e[s] \quad \text{and} \quad c_e[s] = \mu(C_e[s]). \tag{3.6}$$

In the following (and especially in the construction) a *large* number at a given stage of a procedure is a number which is greater than all the current values of the parameters of the procedure (including the current stage).

10 Starting from any effective sequence $(V^\star_i)$ of all Σ^0_1 classes , consider the sequence (V_i) where the enumeration of V_i follows the one of $V^\star_i$ up to the point where its measure reaches $1 - 2^{-i}$, in which case it stops. Then (V_i) satisfies (3.3) and by the padding lemma every bounded Σ^0_1 class occurs in (V_i). Indeed, suppose that V is a bounded Σ^0_1 class, and $\mu(V) < 1 - 2^{-e}$. Chose an index i such that $V^\star_i = V$ and $i > e$. Then $V = V_i$.

3.1. Crude strategy

We give and informal outline of the basic idea behind the strategy for $(3.2)_e$ in the following steps. The module below depends on two parameters $a_e, b_e \in \mathbb{N}$, of which a_e occurs explicitly in step (c) and b_e is involved in the definition (3.5).

(a) If at some stage s we wish to act (launch an *attack*) toward satisfying (3.2), we put into $U^A_\star$ a clopen set $D_e \subseteq 2^\omega - V_e[s]$ of size $p_e[s] := c_e[s]$ with *large* use u_e.

Notice that since $b_e > e$ we have $p_e[s] \leq 2^{-e}$ and by (3.3) we can always pick such a clopen set.

(b) Wait until a stage t where $D_e \subseteq V_e[t]$. If $X \upharpoonright v_e[s]$ changes in the meantime, cancel this attack and go to (a).

If a cancellation occurs during the wait, the last clopen set we put into $U^A_\star$ becomes junk *(i.e. unwanted, useless)*.

(c) If $cost(u_e, t) \geq a_e \cdot p_e[t]$ we choose *not to attack*; we restrain $A \upharpoonright t$ and go to (a).

The last clopen set we put into $U^A_\star$ becomes junk *(i.e. unwanted, useless). However this unwanted measure corresponds to a part of U^A (magnified by a_e) through the restraint we impose.*

If $cost(u_e, t) < a_e \cdot p_e[t]$ we *attack*: put u_e into A and enumerate a subset of $C_e[s]$ of size $p_e[t]$ into E_e.

If $(X \upharpoonright v_e)[s]$ is the real configuration of X, our attack *will produce some junk in* (3.1)*, i.e. some measure w in $V^X - U^A$ (this will correspond to a part of $U^X_{b_e}$ which is a_e times smaller than w, namely the subset of $C_e[s]$ that we put into E_e). Otherwise no junk will be produced in* (3.1) *by this attack.*

(d) Go to (a).

Rough analysis of outcomes. Suppose that we recursively follow the steps of the crude strategy outlined above, in stages s at which we also enumerate the reals of $U^A[s]$ into $V^X[s]$ with the same use as they appear in $U^A[s]$. For simplicity, assume that the cancellation described in step (b) never occurs (we will refine the strategy later in order to deal with this annoying possibility). First of all notice that

$$\mu(E_e) \leq \mu(V_e) \tag{3.7}$$

since before we put a clopen set of size p_e into E_e we have observed an increase of at least p_e on $\mu(V_e)$. Moreover

$$\mu(U^A_\star) \leq \mu(U^A)/a_e + z \tag{3.8}$$

where $z = p_e[t]$ if t is the last stage where the strategy reached step (a), and 0 if there is no such stage t. Indeed, think of the restraint applied in step (c) as a movable marker m which extends to a larger value every time that we cancel the attack in step (c). Notice that, at least in this atomic case, this restraint is always respected since new attacks choose *large* numbers for enumeration into A, in particular larger than the current value of m. If an attack is fully implemented it will not leave any measure in $U^A_\star$. If it is cancelled in step (c), the increase in $U^A_\star$ that it is responsible for is at most the increase in $U^{A\upharpoonright m}$ over the magnification parameter a_e (after m increases). Hence $\mu(U^A_\star)$ is at most $\mu(U^A)/a_e$ plus the amount of the last attack, in the case that some attack got stuck at step (b), so that it did not have the chance to either be implemented or be cancelled. By (3.8) it is clear that $\mu(U^A_\star)$ can be made as small as we like, with an appropriate arrangement of the parameters.

For (3.1) it suffices to show that $\mu(V^X) < 1$ as the other relation is straightforward. We can argue that

$$\mu(V^X) \leq \mu(U^A) + a_e \cdot \mu(U^X_{b_e}) \tag{3.9}$$

which clearly suffices, by an appropriate choice of a_e, b_e. Let J^X be the set of reals Z such that for some n and stages $s < t$ we have

- $Z \in U^A[s] - U^A[s-1]$ with use $\geq n \geq v_e[s]$
- $n \in A[t] - A[s]$
- $(X[s] = X[i] = X) \upharpoonright v_e[s]$ for all $i \in [s, t]$.

Since every attack starts at some stage s with a big witness n (larger than all current uses of computations in U^A) and it does not enumerate n into A at stage $t > s$ unless $(X[s] = X[i]) \upharpoonright v_e[s]$ for all $i \in [s, t]$, it follows that $V^X - U^A \subseteq J^X$.[11] That is,

$$V^X \subseteq U^A \cup J^X. \tag{3.10}$$

Every time an attack starts at stage s (step (a) in the crude strategy) and is implemented at stage $t > s$ (through the second case of (c) in the crude strategy) the reals that are added in J^X correspond to the set $C_e[s]$. Moreover, according to the second case of (c) above, the measure of the reals that are added in J^X is $\leq a_e \cdot \mu(C_e[s])$. Since at the end of an attack the set $C_e[s]$ is enumerated into E_e, for different fully implemented attacks the sets $C_e[s]$ are disjoint, and the same holds for different enumerations into J^X. On the other hand, for each enumeration into J^X, the corresponding set $C_e[s]$ is a subset of $U^X_{b_e}$, by the definition of J^X and the crude strategy. This shows that $\mu(J^X) \leq a_e \cdot \mu(U^X_{b_e})$ which, along with (3.10) gives (3.9). This finishes the verification of the crude strategy, based on the unreasonable assumption that the cancellation on step (b) of the strategy never occurs. We note that in this crude case, all of $C_e[s]$ is enumerated into E_e. This is held over from the full strategy.

11 If an attack enumerates n into A at stage s, the reals ejected from U^A will have been registered in V^X with various uses (depending on the approximations to X that occurred during the attack). However by the construction, all those configurations of X will extend $X[s] \upharpoonright v_e[s]$ which has remained constant from the stage that the attack was launched to the stage where it was completed. So the ejected reals will be a part of V^X only if $X[s] \upharpoonright v_e[s] \subset X$, but in this case they will be in J^X as well.

Table 1
Parameters in the construction.

U	Given member of a universal Martin–Löf test
$U_\star$	Oracle Σ^0_1 class constructed by us
E_e	Σ^0_1 classes constructed by us
V_e	Given Σ^0_1 class of measure $< 1 - 2^{-e}$
m_e	Restraint on A
a_e	Magnification parameter in step (c) of the R_e module
b_e	Index of the universal oracle Martin–Löf test used in (3.6)

3.2. *Refined strategy*

We discuss a modification of the strategy of Section 3.1 which meets the requirements without relying on any extra assumptions. Given that there is a possibility that $X \upharpoonright v_e[s]$ changes during the wait in step (b) of the crude strategy, we have to consider another kind of cost, i.e. the measure we put in $U^A_\star$ during an attack which is cancelled at step (b) (this will be counted by means of an auxiliary set $L_{\star e}$ which will collect all such clopen sets, see Table 2). The way to bound this cost is to 'slow down' the construction, i.e. work with smaller amounts of measure. In this way, the cost of an attack which is cancelled at step (b) will be small as well. We have a parameter r_e which counts the number of such cancellations, and if an attack starts at stage s we put into $U^A_\star$ a clopen set of size $2^{-r_e[s]-e-4} \cdot c_e[s]$. In this way this additional measure in $U^A_\star$ cannot exceed 2^{-e-3}. The apparent danger of such a 'slow down' of the construction is that the strategy module may run indefinitely, although we may not be able to argue (as in the verification of Section 3.1) that E_e covers $U^X_{b_e}$ because $r_e \to \infty$ and this makes us choose smaller and smaller amounts of U^X_i for enumeration into E_e. But in that case the parameter v_e from (3.5) would reach a limit, and so would the approximation to $X \upharpoonright v_e$. So r_e would reach a limit, and this is a contradiction. Hence this scenario is not possible, and the modification we described provides a successful strategy.

The R_e strategy can be seen as a sequence of cycles which wish to enumerate $U^{X\upharpoonright n}_{b_e}$ into E_e, for each n (of course they do this in coordination with enumerating into $U^A_\star$ and according to some rules, but we do not wish to emphasize this aspect right now). A cycle is a maximal interval of stages $[t, s]$ during which the parameter v_e of (3.5) and the approximation to $X \upharpoonright v_e$ remain constant. For each stage s we let $t_e[s]$ be the first stage of the cycle that s is in, i.e.

$$t_e[s] = \mu t \leq s\,[\forall i \in [t, s]\ (v_e[i] = v_e[t]\ \wedge\ X_i \upharpoonright v_e[t] = X_t \upharpoonright v_e[t])]. \tag{3.11}$$

The point of this definition is that at any stage s the goal of the current cycle is to enumerate $C_e[t_e[s]]$ into E_e. Notice that this goal may be partially achieved at stage s, i.e. part of $C_e[t_e[s]]$ may already be in E_e. The current cycle will automatically be completed upon achievement of the goal, but it may also be completed upon a change in the approximation to $X \upharpoonright v_e$.

For reference in the construction we lay out the formal strategy module for the satisfaction of R_e. This may be called by the construction at certain stages $s + 1$, in which case it will execute the step it is currently at. The following steps should be understood in this context. In order to accommodate the analysis in the verification of the construction, let us use a special $U_\star$ for R_e, which we denote by $U_{\star e}$ and let

$$U_\star = \cup_e U_{\star e}. \tag{3.12}$$

Given that the strategy involves both enumeration into A and a restraint m_e on A there will be injury amongst different requirements in the global construction. In the full construction there will be a form of 'initialization' of strategies, in order to control the interaction of several strategies working in parallel. Without defining the exact meaning of initialization at this point (this will be defined in Section 3.3) we let $d_e[s]$ be the number of times that R_e has been initialized by stage s. All the parameters below (see Tables 1 and 2) are thought to have a current value even when this is not explicitly denoted, but in the atomic construction (involving a single R_e strategy) the parameters d_e, a_e, b_e remain constant.

According to the rough analysis of Section 3.1 and the verification of the R_e module below, certain parts of $U^A_{\star e}$ are associated to parts of U^A (of equal or larger measure) which are only $1/a_e$ of the total measure in U^A. In order to assist the presentation of the analysis for the proof of (3.8), or a similar inequality, we use auxiliary sets F^A_e which will contain the part of U^A associated with $U^A_{\star e}$; also, the part of $U^A_{\star e}$ that has been associated with F^A_e will be enumerated into a set $F^A_{\star e}$ (see Table 2). The construction will ensure that every time the first clause of step (c) below applies, the set D_e which becomes permanent in $U^A_\star$ is matched with a new subset of F^A_e of the same size.

R_e **module for Theorem 1.1 at stage** $s + 1$. [12]

(a) Put into $U^A_{\star e}$ a clopen set $D_e \subseteq 2^\omega - V_e[s]$ of size

$$p_e[s] := 2^{-r_e[s]-e-6} \cdot c_e[t_e[s]] \tag{3.13}$$

with *large* use u_e.

[12] This module, as every module in this paper, should be thought of as being operated by a global construction which calls it 'requires attention', i.e. whenever it is ready to move to the next step. In this context the 'wait' instructions should be interpreted as follows: the module returns control to the construction and resumes (or requires attention again) if and when the associated search halts. That is why after step (c) we still refer to $cost(u, s)$ instead of $cost(u, t)$; because when (c) is visited we are at stage $s + 1$.

Table 2
More parameters in the construction.

r_e	Number of cancellations that occur in step (b) of R_e
d_e	Number of initializations of R_e
$L_{\star e}$	Junk measure in $U^A_{\star e}$ produced by R_e through cancellation in step (b)
F^A_e	Part of U^A mapped to a subset of $U^A_\star$ in step (c)
$F^A_{\star e}$	Subset of $U^A_\star$ which is mapped to F^A_e
J^X_e	Junk measure in V^X produced by R_e by enumerations in step (c)

Notice that $p_e[s] \le 2^{-e}$, so that by (3.3) we can always pick such a clopen set.

(b) Wait until a stage t where $U^A_{\star e} \subseteq V_e[t]$. If $X \upharpoonright v_e[s]$ changes in the meantime, put $D_e[s]$ into $L_{\star e}$, let $r_e[s+1] = r_e[s] + 1$ and go to (a). Here 'wait' means that we go to the next stage and, as long as $X \upharpoonright v_e[s]$ does not change, in the following stages $t + 1$ we remain on this step unless $U^A_{\star e} \subseteq V_e[t]$ (in which case we go to (c) of module R_e at stage $t + 1$).

If r_e increases, the current attack is cancelled and the clopen set we last put $U^A_\star$ becomes junk.

(c) ⋆ If $cost(u_e, s) \ge a_e \cdot p_e[s]$ set $m_e[s+1] = s$. Let $x = \mu(U^{A[s]\upharpoonright s} - U^{A[s]\upharpoonright m_e[s]})$ put a clopen subset of $U^{A[s]\upharpoonright s} - U^{A[s]\upharpoonright m_e[s]}$ of size x/a_e into F^A_e with use s, put $D_e[s]$ into $F^A_{\star e}$ with use s and go to (a).

We choose not to attack and the last clopen set we put $U^A_{\star e}$ becomes junk. *However this unwanted measure corresponds to a part of U^A through the restraint we impose.*

⋆ If $cost(u_e, s) < a_e \cdot p_e[s]$, put u_e into A, enumerate

$$\{Z \mid Z \in U^A[s] \text{ with use} \ge u\}$$

into $J^X_e[s+1]$ with use $v_e[s]$ and enumerate a subset of $C_e[s]$ of size $p_e[s]$ into E_e.

We attack. *If $(X \upharpoonright v_e)[s]$ is the real configuration of X, the* attack *produces some extra measure x in $V^X - U^A$ (this will correspond to a part of $U^X_{b_e}$ of weight $p_e \ge x/a_e$). Otherwise no junk will be added in $V^X - U^A$ by this attack.*

We say that the clopen set $[\sigma]$ is in $U^A[s]$ with use u if there is some $\tau \subseteq \sigma$ such that $\tau \in U^A[s]$ with use u.

Verification of R_e module for Theorem 1.1. Consider the construction of Section 3.4 restricted to a single strategy R_e with arbitrary constant parameters a_e, b_e, d_e. That is, at stage $s+1$ consider the minimal strings τ such that $[\tau] \subseteq U^A[s] - V^{X[s+1]}$ and enumerate them into $V^X[s+1]$ with the same use as they occur in $U^A[s]$; also perform the next step of the R_e module as presented above. Then we have $U_\star = U_{\star e}$. Since m_e moves monotonically, by the basic properties of U^A we have that every clopen set enumerated into F^A_e is disjoint from the sets previously enumerated into F^A_e. Also whenever we put some intervals into F^A_e at some stage $s+1$, these intervals appear in $U^{A\upharpoonright m_e[s+1]} - U^{A\upharpoonright m_e[s]}$, and the latter set has measure a_e times larger than the measure of the new intervals put into F^A_e. This shows that

$$\mu(F^A_e[s]) \le \frac{1}{a_e} \cdot \mu(U^{A\upharpoonright m_e[s]}) \quad \text{at every stage } s \tag{3.14}$$

and $F^A_e \subseteq U^A$ (notice that F^A_e is a Σ^0_1 class). We are now going to verify that (3.1) and (3.2) hold. The first clause of (3.1) follows directly from the construction. Also, notice that (3.7) holds by the same argument as in the 'rough analysis of outcomes' of Section 3.1. By the construction we have

$$U^A_{\star e}[s] \subseteq F^A_{\star e}[s] \cup L_{\star e}[s] \cup D_e[s] \tag{3.15}$$

at every stage s, because for every clopen set that enters $U^A_\star$ one of the following happens:

- it exits $U^A_\star$ through a successful attack
- or it enters L_e through a cancellation in step (b)
- or it enters $F^A_{\star e}$ through the first clause of step (c)
- or it remains in $D_e[s]$ if R_e is stuck waiting in step (b).

By (3.13) we have that the ith enumeration into $L_{\star e}$ enumerates a clopen set of measure 2^{-i-e-4} (notice that $c_e[s] \le 2^{-1}$ for all s). Therefore

$$\mu(L_{\star e}[s]) \le 2^{-e-3} \quad \text{and} \quad \mu(D_e[s]) \le 2^{-e-4} \quad \text{for all stages } s \tag{3.16}$$

where the second clause also follows from (3.13) (notice that $L_{\star e}$ is a Σ^0_1 class). By (3.15) combined with (3.16) we have

$$\mu(U^A_\star) \le \mu(U^A)/a_e + \mu(L_{\star e}) + \mu(D_e) < 1.$$

On the other hand by the choice of U and the construction, as explained in footnote 11 of Section 3.1, we have

$$V^X \subseteq U^A \cup J^X_e \quad \text{and} \quad \mu(U^A) < 2^{-2}. \tag{3.17}$$

Also notice that at all stages s, every enumeration of a clopen set G into $J^X_e[s]$ (with some use u) is accompanied by an enumeration of a clopen set C into E_e, in such a way that

- $\mu(C) \leq a_e \cdot \mu(G)$
- If $X[s] \upharpoonright u = X \upharpoonright u$ then $C \subseteq U^X_{b_e}$.

This shows that

$$\mu(J^X_e) \leq a_e \cdot \mu(U^X_{b_e}) < a_e \cdot 2^{-b_e}. \tag{3.18}$$

Now by an appropriate choice of a_e, b_e we get $\mu(J^X_e) < 2^{-1}$ and by (3.17) we get $\mu(V^X) < 1$. It remains to show that $U^A_\star \not\subseteq V_e$. For a contradiction suppose that $U^A_\star \subseteq V_e$, so that R_e does not get stuck permanently in step (b). We show by induction that $U^{X\restriction n}_{b_e} \subseteq E_e$ for all n. Suppose that this holds for $n = k$ and let s_0 by a stage such that $U^{X\restriction k}_{b_e} \subseteq E_e[s_0]$. Also let $s_1 > s_0$ be a stage such that $X[s] \upharpoonright (k+1) = X \upharpoonright (k+1)$ for all $s > s_1$. If $U^{X\restriction(k+1)}_{b_e} \subseteq E_e[s_1]$ the induction step is complete. Otherwise, notice that as long as $U^{X\restriction(k+1)}_{b_e} \not\subseteq E_e[s]$ (for $s > s_1$) we will have (recall v_e from (3.5)) $v_e[s] = k+1$ and r_e, t_e, p_e will remain constant. This means that from now on the R_e module can pass through the second clause of step (c) at most $1/p_e[s_2+1]$ times before we have $U^{X\restriction k+1}_{b_e} \subseteq E_e$. Also, R_e can pass through the first clause of step (c) at most finitely many times before we have $U^{X\restriction k+1}_{b_e} \subseteq E_e$ since every time this happens, $\mu(U^{A\restriction m_e})$ increases by at least $a_e \cdot p_e$. Given that R_e does not get stuck permanently in step (b), and it has to pass from one of the clauses of step (c), eventually we will have $U^{X\restriction k+1}_{b_e} \subseteq E_e$ and this completes the induction step.

Now since $X \not\leq_{LR} \emptyset$ we have $\mu(E_e) = 1$. But then by (3.7) we get $\mu(V_e) = 1$ which is a contradiction. Hence $U^A_\star \not\subseteq V_e$ and this concludes the verification.

3.3. *Interaction between strategies*

The R_e module of Section 3.2 was given in sufficient generality so that the R_e strategies for $e \in \mathbb{N}$ can work simultaneously. For example, by setting the parameters a_e, b_e, r_e, d_e appropriately we can ensure that $\mu(L_{\star e})$, $\mu(V^X - U^A)$, $\mu(F^A_{\star e})$ are sufficiently small. Of course there will be a finite injury effect, namely when a strategy R_e makes an enumeration into A or increases the restraint m_e, all R_i, $i > e$ (namely, the *lower priority strategies*) have to be cancelled. If $i < j$ then we will have $m_i < m_j$ and any enumeration of R_j into A will involve numbers which are larger than m_i. However, notice that $L_{\star e}, J_e$ will be Σ^0_1 (here J_e is viewed as the oracle class behind J^X_e), and $U^A_\star$ needs to be $\Sigma^0_1(A)$ so, although the work of an injured strategy is erased upon injury, the damage (the cost) that this strategy has caused to the construction (in the form of useless measure that it has contributed to $L_{\star e}, J^X_e, U^A_\star$) is not. This is why we need to run the strategies 'arbitrarily cheaply', which is achieved by specifying appropriate values of its parameters each time the strategy is run.

In order to initialize the counter of the cost that R_e has produced at a stage where its module is initialized, every time R_e is injured we will enumerate $F^A_{\star e}$ and D_e into $L_{\star e}$, so that we still count this cost (i.e. superfluous measure in $U^A_\star$) in the total calculation. Parameter a_e was introduced in Section 3.2[13] in order to be able to keep the measure of $L_{\star e}$ small, even in this hostile injury environment. We set

$$a_e[s] = 2^{e+d_e[s]+6} \quad \text{and} \quad b_e[s] = 2(e + d_e[s] + 6) \tag{3.19}$$

so that we keep the measure J^X_e sufficiently small. To *initialize* R_e means to set $r_e[s+1] = r_e[s]+1$, $d_e[s+1] = d_e[s]+1$, empty $F^A_{\star e}[s]$, $D_e[s]$ into $L_{\star e}$ (so that $F^A_{\star e}$, $D_e[s]$ become empty and the lost contents now appear in $L_{\star e}$) and empty E_e (set $E_e = \emptyset$). We say that R_e *requires attention at stage* $s+1$ if one of the following holds

(i) $t = 0$
(ii) R_e executed step (b) at stage t and $X_t \upharpoonright v_e[t] \neq X_{s+1} \upharpoonright v_e[t]$
(iii) R_e executed step (b) at stage t and $U_{\star e}(d_e) \subseteq V_e[s+1]$
(iv) R_e executed step (c) at stage t

where t is the largest stage $\leq s$ where R_e was called, and 0 if there is no such.

3.4. *Construction for Theorem 1.1*

At stage $s+1$ consider the minimal strings τ such that $[\tau] \subseteq U^A[s] - V^{X[s+1]}$ and enumerate them into $V^X[s+1]$ with the same use as they occur in $U^A[s]$. Also, check if there is some e such that R_e requires attention. In that case pick the least such e and execute the corresponding clause of the R_e module, according to the (least) clause through which it requires attention:

(i) Execute step (a)
(ii) Set $r_e[s+1] = r_e[s] + 1$ and execute step (a)
(iii) Execute step (c)
(iv) Execute step (a).

Initialize all R_i, $i > e$.

[13] Notice that the atomic strategy works for $a_e = 1$, and indeed this choice for a_e is the obvious one if we only have to deal with a single strategy.

3.5. Verification of the construction of Section 3.4

First of all, notice that by the argument in Section 3.1 about (3.10) (also see footnote 11 of Section 3.1), we have

$$V^X \subseteq U^A \cup J^X, \quad \text{where } J^X = \cup_e J_e^X \tag{3.20}$$

and (3.15), for the same reasons that were given in the analysis of the R_e module in Section 3.2 as well as the definition of initializations of R_e. Also, V^X is $\Sigma_1^0(X)$ and $U_\star^A$ is $\Sigma_1^0(A)$. The verification of the construction amounts to showing $\mu(U_\star^A) < 1$, $\mu(V^X) < 1$ and that every R_e is satisfied. Given (3.12), (3.15), (3.20) and $\mu(U^A) < 2^{-2}$, for the first two conditions it suffices to show that $\mu(J_e^X[s]) < 2^{-e-2}$, $\mu(L_{\star e}[s]) < 2^{-e-3}$ and $\mu(F_{\star e}^A[s]) < 2^{-e-3}$, since we know by the definition of b_e in (3.19) and (3.5), (3.6), (3.13) that

$$\mu(D_e[s]) < 2^{-e-d_e[s]-7} \quad \text{for every } e, s \in \mathbb{N}. \tag{3.21}$$

We say that an interval $[s, t]$ of stages is injury-free with respect to R_e, if strategy R_e is not injured during stages from s to t. We are going to show that for each $e \in \mathbb{N}$ and each maximal injury-free interval $[s, t]$ with respect to R_e, the following hold:

$$\mu(F_e^A[n]) \leq \frac{1}{a_e[s]} \cdot \mu(U^{A\restriction m_e[n]}) \quad \text{for every } n \in [s, t] \tag{3.22}$$

$$\mu(F_{\star e}^A[n]) \leq \mu(F_e^A[n]) \quad \text{and} \quad F_e^A[n] \subseteq U^{A\restriction m_e}[n] \quad \text{for every } n \in [s, t] \tag{3.23}$$

$$\mu(L_{\star e}[n] - L_{\star e}[s-1]) \leq 2^{-d_e[s]-e-5}. \tag{3.24}$$

The proof of (3.22) is exactly as the proof of (3.14) of Section 3.2, given that R_e is not injured in the interval $[s, t]$. Also, (3.23) follows from step (c) of the R_e module of Section 3.2, since F_e^A always gets reals from $U^{A\restriction m_e}[n]$ at stage n, and every time that we enumerate into $F_{\star e}^A$ we also enumerate at least the same measure into F_e^A. For (3.24) notice that during an injury-free interval with respect to R_e the only contributions to $L_{\star e}$ come from cancellations in step (b) of the R_e module. But according to (3.13), the ith such contribution occurring in the interval $[s, t]$ is of measure less than 2^{-i-e-6} (since every time a cancellation occurs in step (b), r_e increases by 1). By construction, $d_e[k] \leq r_e[k]$ for all $k \in \mathbb{N}$, hence (3.24).

Now notice that the maximal injury-free intervals of stages (with respect to R_e) cover all stages except the stages where R_e is injured. During such injury stages $F_{\star e}^A$ and D_e are emptied and their contents are enumerated in $L_{\star e}$. According to (3.19), (3.21), (3.22) and (3.23) (given that d_e increases by 1 after each injury of R_e) the clopen set enumerated into L_e at the ith injury of R_e has measure at most

$$2^{-e-i-6} + 2^{-e-i-6} = 2^{-e-i-5}$$

which along with (3.24) shows that $\mu(L_{\star e}) < 2^{-e-3}$.

When at the second clause of step (c) of the R_e module we enumerate some clopen sets $[\sigma]$ into $J_e^X[s+1]$ with some use v_e, we can view this action as an enumeration of axioms $\langle X[s+1] \restriction v_e, \sigma \rangle$ into an oracle Σ_1^0 class J_e. This amounts exactly to the enumeration of those clopen sets in $J_e^X[s+1]$ with use v_e, according to the definitions we gave in Section 2. Now let us denote by $J_{e,k}$ the current version of the oracle Σ_1^0 class J_e (which is a c.e. set of axioms) at stage k of the construction. Exactly as in Section 3.2 (see (3.18)) it follows that for every injury-free (with respect to R_e) interval $[s, t]$ we have

$$\mu(J_{e,n}^{X[n]} - J_{e,s-1}^{X[n]}) < a_e[s] \cdot 2^{-b_e[s]} \quad \text{for every stage } n \in [s, t]. \tag{3.25}$$

Since at stages where R_e is injured there is no enumeration of axioms into J_e, from (3.25) it follows that

$$\mu(J_e^X[s]) \leq \sum_{s \in I_e} a_e[s] \cdot 2^{-b_e[s]} = \sum_{s \in I_e} 2^{-d_e-e-6} \leq 2^{-e-5}$$

where I_e is the set of stages where R_e is injured. Therefore by (3.20) we have $\mu(V^X) < 1$ and it remains to show that R_e is satisfied for all $e \in \mathbb{N}$. This, along with the fact that R_e stops requiring attention after some stage s_0, follows by induction on e by the same argument that was detailed in Section 3.2 for the atomic construction, placed in a co-finite segment of stages $[t, \infty]$ where R_e is not injured.

4. Proof of Theorem 1.3

In order to prove Theorem 1.3 we have to use the permitting method of Section 3 below two sets X, Y *simultaneously*. We know from classical computability theory that most permitting methods do not work below two sets simultaneously, and this is exactly the reason for the existence of minimal pairs, e.g. in the c.e. Turing degrees. In the case of the usual permitting method for c.e. sets, for example (see [18]), there is no reason to assume that two different c.e. sets will give permission for enumeration into our set *at the same time*. The exploitation of this phenomenon is sometimes called 'gap–cogap' strategy in priority constructions, and is used in order to obtain negative results (obstructions to extensions of embeddings) in the Turing degrees.

However the permitting argument of Section 3 is very different, in that enumerations into A are not triggered by changes in X. Instead, they happen *in advance* of such changes (under certain conditions) and only after they happen do we motivate X to (permanently) change configuration. This quality of the permitting of Section 3 allows it to be used with respect to two non-trivial Δ^0_2 sets X, Y simultaneously, as we demonstrate in the following. Let X, Y be two Δ^0_2 sets which are not low for random and have computable approximations (X_s), (Y_s) respectively. We wish to construct a c.e. set $A \not\leq_{LR} \emptyset$ such that $A \leq_{LR} X$ and $A \leq_{LR} Y$.The parameters of the construction will be the same as in Section 3, but in some cases we need to have two versions, one corresponding to X and one for Y. We are going to construct three oracle Σ^0_1 classes $U_\star, V_x, V_y$ such that

$$U^A \subseteq V^X_x \quad \text{and} \quad \mu(V^X_x) < 1 \tag{4.1}$$

$$U^A \subseteq V^Y_y \quad \text{and} \quad \mu(V^Y_y) < 1 \tag{4.2}$$

$$R_e : U^A_\star \not\subseteq V_e \quad \text{and} \quad \mu(U^A_\star) < 1 \tag{4.3}$$

where (V_e) is as in Theorem 1.3. Parameter $cost(n, s)$ is given again by (3.4) but instead of a single E_e (for each e) we have E^x_e, E^y_e, corresponding to sets X, Y. In general, if a parameter has x or y as a subscript or superscript, this is an indication that it is related to X or Y respectively. Hence (3.5), (3.6) become

$$v^x_e[s] = \mu t[U^{X[s]\restriction t}_{b_e} - E^x_e[s] \neq \emptyset] \quad \text{and} \quad v^y_e[s] = \mu t[U^{Y[s]\restriction t}_{b_e} - E^y_e[s] \neq \emptyset] \tag{4.4}$$

$$C^x_e[s] = U^{X\restriction v^x_e}_{b_e}[s] - E^x_e[s] \quad \text{and} \quad C^y_e[s] = U^{X\restriction v^y_e}_{b_e}[s] - E^y_e[s] \tag{4.5}$$

$$c^x_e[s] = \mu(C^x_e[s]) \quad \text{and} \quad c^y_e[s] = \mu(C^y_e[s]) \tag{4.6}$$

and we also let

$$c_e[s] = \min\{c^x_e[s], c^y_e[s]\}. \tag{4.7}$$

Likewise, (3.11) is replaced by

$$t^x_e[s] = \mu t \leq s\ [\forall i \in [t, s]\ (v^x_e[i] = v^x_e[t] \ \wedge\ X_i \restriction v^x_e[t] = X_t \restriction v^x_e[t])] \tag{4.8}$$

$$t^y_e[s] = \mu t \leq s\ [\forall i \in [t, s]\ (v^y_e[i] = v^y_e[t] \ \wedge\ X_i \restriction v^y_e[t] = X_t \restriction v^y_e[t])] \tag{4.9}$$

$$t_e[s] = \max\{t^x_e[s], t^y_e[s]\}. \tag{4.10}$$

Recall that the main argument for the satisfaction of R_e in Section 3.2 was that if $U^A_\star \subseteq V_e$ then $U^X_{b_e} \subseteq E_e$ (and due to certain properties of the construction and the fact that X is not K-trivial, this implies $1 = \mu(E_e) \leq \mu(V_e)$ which is a contradiction). In this section, the argument becomes as follows: if $U^A_\star \subseteq V_e$ then *either* $U^X_{b_e} \subseteq E^x_e$ *or* $U^Y_{b_e} \subseteq E^y_e$. In the first case we have $1 = \mu(E^x_e) \leq \mu(V_e)$ and in the second $1 = \mu(E^y_e) \leq \mu(V_e)$, both of which lead to a contradiction. In the following we lay out the R_e strategy for Theorem 1.3.

4.1. R_e module for Theorem 1.3 at stage $s + 1$

(a) Put into $U^A_{\star e}$ a clopen set $D_e \subseteq 2^\omega - V_e[s]$ of size

$$p_e[s] := 2^{-r_e[s]-e-6} \cdot c_e[t_e[s]] \tag{4.11}$$

with *large* use u_e.

(b) Wait until a stage t where $U^A_{\star e} \subseteq V_e[t]$. If $X \restriction v^x_e[s]$ or $Y \restriction v^y_e[s]$ changes in the meantime, put $D_e[s]$ into $L_{\star e}$, let $r_e[s + 1] = r_e[s] + 1$ and go to (a). Here 'wait' means that we go to the next stage and, as long as $X \restriction v^x_e$, $Y \restriction v^y_e$ do not change, in the following stages $t + 1$ we remain on this step unless $U^A_{\star e} \subseteq V_e[t]$ (in which case we go to (c) of module R_e at stage $t + 1$).

(c) $\star$ If $cost(u_e, s) \geq a_e \cdot p_e[s]$ set $m_e[s + 1] = s$. Let

$$z = \mu(U^{A[s]\restriction s} - U^{A[s]\restriction m_e[s]})$$

put a clopen subset of $U^{A[s]\restriction s} - U^{A[s]\restriction m_e[s]}$ of size z/a_e into F^A_e with use s, put $D_e[s]$ into $F^A_{\star e}$ with use s and go to (a).

$\star$ If $cost(u_e, s) < a_e \cdot p_e[s]$, put u into A,

- enumerate $\{Z \mid Z \in U^A[s]$ with use $\geq u_e\}$ into $J^X_{xe}[s + 1]$ with use $v^x_e[s]$ and into $J^Y_{ye}[s + 1]$ with use $v^y_e[s]$
- enumerate a subset of $C^x_e[s]$ of size $p_e[s]$ into E^x_e
- enumerate a subset of $C^y_e[s]$ of size $p_e[s]$ into E^y_e

Remark. The comments in the captions of steps (a) to (c) of the R_e module of Section 3 also hold for the R_e module of this section, with the exception of the second clause of step (c). This caption has to be replaced with the following (which involves both X, Y): *We* attack. *If $(X \restriction v_e)[s]$ is the real configuration of X, the* attack *produces some extra measure x in $V^X_x - U^A$ (this will correspond to a part of $U^X_{b_e}$ of weight $p_e \geq x/a_e$); otherwise no junk will be added in $V^X_x - U^A$ by this attack. Likewise for Y: If $(Y \restriction v_e)[s]$ is the real configuration of Y, the* attack *produces some extra measure y in $V^Y_y - U^A$ (this will correspond to a part of $U^Y_{b_e}$ of weight $p_e \geq y/a_e$); otherwise no junk will be added in $V^Y_y - U^A$ by this attack.*

4.2. Verification of R_e module for Theorem 1.3

Consider the construction of Section 4.3 restricted to a single strategy R_e with arbitrary constant parameters a_e, b_e, d_e. That is, at stage $s+1$ consider the minimal strings τ such that $[\tau] \subseteq U^A[s] - V_x^{X[s+1]}$ and the minimal strings τ such that $[\tau] \subseteq U^A[s] - V_y^{Y[s+1]}$, and enumerate them into $V_x^X[s+1]$, $V_y^Y[s+1]$ respectively with the same use as they occur in $U^A[s]$; also perform the next step of the R_e module as presented above.

First, notice that whenever new clopen sets are put into E^x and E^y under the second clause of step (c), these are of the same measure h such that $\mu(V_e)$ has increased by at least h in the interval between steps (b) and (c). This implies

$$\mu(E_e^x) \leq \mu(V_e) \quad \text{and} \quad \mu(E_e^y) \leq \mu(V_e) \tag{4.12}$$

(which corresponds to (3.7) of Section 3.1).

Next, following the argument given in the verification of the R_e module of Section 3.2 (but with respect to the R_e module of this section) we see that $U_\star = U_{\star e}$, $F_e^A \subseteq U^A$ and (3.14), (3.15), (3.16). Hence $\mu(U_\star^A) < 1$ and in fact, with an appropriate choice of the parameters of the construction, $\mu(U_\star^A)$ can be made arbitrarily small.

Notice that at all stages s, every enumeration of a clopen set G_x into $J_{xe}^X[s]$ and a clopen set G_y into $J_{ye}^Y[s]$ (with some use u at step (c) of the module) is accompanied by an enumeration of a set C_x into E_e^x and a set C_y into E_e^y, in such a way that

- $\mu(C_x) \leq a_e \cdot \mu(G_x)$ and, if $X[s] \upharpoonright u = X \upharpoonright u$ then $C_x \subseteq U_{b_e}^X$.
- $\mu(C_y) \leq a_e \cdot \mu(G_y)$ and, if $Y[s] \upharpoonright u = Y \upharpoonright u$ then $C_y \subseteq U_{b_e}^Y$.

This shows that

$$\mu(J_{xe}^X) \leq a_e \cdot \mu(U_{b_e}^X) < a_e \cdot 2^{-b_e} \quad \text{and} \quad \mu(J_{ye}^Y) \leq a_e \cdot \mu(U_{b_e}^Y) < a_e \cdot 2^{-b_e}. \tag{4.13}$$

Also by the construction (see the argument in Section 3.1 about (3.10) and footnote 11 of Section 3.1) we have $V_x^X - U^A \subseteq J_{xe}^X$ and $V_y^Y - U^A \subseteq J_{ye}^Y$. Hence (also by the choice of U) we have

$$V_x^X \subseteq U^A \cup J_{xe}^X \quad \text{and} \quad V_y^Y \subseteq U^A \cup J_{ye}^Y \quad \text{and} \quad \mu(U^A) < 2^{-2}. \tag{4.14}$$

Now by an appropriate choice of a_e, b_e and (4.13) we get $\mu(J_{xe}^X) < 2^{-1}$, $\mu(J_{ye}^X) < 2^{-1}$ and by (4.14) we get $\mu(V_x^X) < 1$, $\mu(V_y^Y) < 1$. Thus we have shown (4.1), (4.2), the second clause of (4.3) and it remains to show that $U_\star^A \not\subseteq V_e$ for each e. For a contradiction suppose that $U_\star^A \subseteq V_e$, so that R_e does not get stuck permanently in step (b). We show that either $U_{b_e}^X \subseteq E_e^x$ or $U_{b_e}^Y \subseteq E_e^y$. Indeed, suppose that $U_{b_e}^X \not\subseteq E_e^x$ which, by (4.4), implies that v_e^x reaches a limit. Let s_0 be a stage after which v_e^x does not change value, and the same holds for the approximation to $X \upharpoonright v_e^x$.

By induction we show that $U_{b_e}^{Y\upharpoonright n} \subseteq E_e^y$ for all n. Suppose that this holds for $n = k$ and let $s_1 > s_0$ by a stage such that $U_{b_e}^{Y\upharpoonright k} \subseteq E_e^y[s_1]$. Also let $s_2 > s_1$ be a stage such that $Y[s] \upharpoonright (k+1) = Y \upharpoonright (k+1)$ for all $s > s_2$. If $U_{b_e}^{Y\upharpoonright(k+1)} \subseteq E_e^y[s_2]$ the induction step is complete. Otherwise, notice that as long as $U_{b_e}^{Y\upharpoonright(k+1)} \not\subseteq E_e^y[s]$ (for $s > s_2$) we will have $v_e^y[s] = k+1$ and $v_e^x, v_e^y, r_e, t_e, p_e$ will remain constant. This means that from now on the R_e module can pass through the second clause of step (c) at most $1/p_e[s_2+1]$ times before we have $U_{b_e}^{Y\upharpoonright k+1} \subseteq E_e^y$. Also, R_e can pass through the first clause of step (c) at most finitely many times before we have $U_{b_e}^{Y\upharpoonright k+1} \subseteq E_e^y$, since every time this happens $\mu(U^{A\upharpoonright m_e})$ increases by at least $a_e \cdot p_e$. Given that R_e does not get stuck permanently in step (b), and it has to pass from one of the clauses of step (c), eventually we will have $U_{b_e}^{Y\upharpoonright k+1} \subseteq E_e^y$ and this completes the induction step.

Now since $U_{b_e}^X \subseteq E_e^x$ or $U_{b_e}^Y \subseteq E_e^y$ and $X \not\leq_{LR} \emptyset$, $X \not\leq_{LR} \emptyset$, we have $\mu(E_e^x) = 1$ or $\mu(E_e^y) = 1$. But then by (4.12) we get $\mu(V_e) = 1$ which is a contradiction. Hence $U_\star^A \not\subseteq V_e$ and this concludes the verification.

4.3. Construction for Theorem 1.3

The coordination of the requirements in the global construction, as well as the definition of initialization and the parameters is exactly as in Section 3.3, only that condition (ii) in that section becomes:

(ii)* R_e executed step (b) at stage t and $X_t \upharpoonright v_e^x[t] \neq X_{s+1} \upharpoonright v_e^x[t]$, or
$Y_t \upharpoonright v_e^y[t] \neq Y_{s+1} \upharpoonright v_e^y[t]$.

At stage $s+1$ consider the minimal strings τ such that $[\tau] \subseteq U^A[s] - V_x^{X[s+1]}$ and enumerate them into $V_x^X[s+1]$ with the same use as they occur in $U^A[s]$; also consider the minimal strings τ such that $[\tau] \subseteq U^A[s] - V_y^{Y[s+1]}$ and enumerate them into $V_y^Y[s+1]$ with the same use as they occur in $U^A[s]$. The second step of stage $s+1$ is the same as the one in the construction of Section 3.4, only that we use the R_e module of Section 4.1.

4.4. Verification of the construction of Section 4.3

First of all, notice that by the same argument that was used in Section 4.2 we have

$$V_x^X \subseteq U^A \cup J_x^X, \quad \text{where } J_x^X = \cup_e J_{xe}^X \tag{4.15}$$

$$V_y^Y \subseteq U^A \cup J_y^Y, \quad \text{where } J_y^Y = \cup_e J_{ye}^Y \tag{4.16}$$

and (3.15), for the same reasons that were given in the analysis of the R_e module in Section 3.2. Also, V_x^X is $\Sigma_1^0(X)$, V_y^Y is $\Sigma_1^0(Y)$ and $U_\star^A$ is $\Sigma_1^0(A)$. The verification of the construction amounts to showing $\mu(U_\star^A) < 1$, $\mu(V_x^X) < 1$, $\mu(V_y^Y) < 1$ and that every R_e (of Section 4) is satisfied. Given (3.12), (3.15), (4.15), (4.16), and $\mu(U^A) < 2^{-2}$, for the first three conditions it suffices to show that $\mu(J_{xe}^X[s]) < 2^{-e-2}$, $\mu(J_{ye}^Y[s]) < 2^{-e-2}$, $\mu(L_{\star e}[s]) < 2^{-e-3}$ and $\mu(F_{\star e}^A[s]) < 2^{-e-3}$ (recall that $\mu(D_e[s]) < 2^{-e-3}$ for every $e, s \in \mathbb{N}$).

Define injury-free intervals with respect to R_e exactly as in Section 3.5. The argument in Section 3.5 following this definition applies to the present construction and shows that $\mu(L_{\star e}) < 2^{-e-3}$. Now viewing J_{xe}, J_{ye} as oracle Σ_1^0 classes, and denoting by $J_{xe,k}, J_{ye,k}$ the current version of them at stage k of the construction the argument of Section 3.2 (see (3.18)) applies to every injury-free (with respect to R_e) interval $[s, t]$ of the present construction and shows

$$\mu(J_{xe,n}^{X[n]} - J_{xe,s-1}^{X[n]}) < a_e[s] \cdot 2^{-b_e[s]} \quad \text{for every stage } n \in [s, t] \tag{4.17}$$

$$\mu(J_{ye,n}^{Y[n]} - J_{ye,s-1}^{Y[n]}) < a_e[s] \cdot 2^{-b_e[s]} \quad \text{for every stage } n \in [s, t]. \tag{4.18}$$

Since at stages where R_e is injured there is no enumeration of axioms into J_{xe}, J_{ye}, from (4.17), (4.18) it follows that

$$\mu(J_{xe}^X[s]) \le \sum_{s \in I_e} a_e[s] \cdot 2^{-b_e[s]} = \sum_{s \in I_e} 2^{-d_e-e-6} \le 2^{-e-5}$$

$$\mu(J_{ye}^Y[s]) \le \sum_{s \in I_e} a_e[s] \cdot 2^{-b_e[s]} = \sum_{s \in I_e} 2^{-d_e-e-6} \le 2^{-e-5}$$

where I_e is the set of stages where R_e is injured. Therefore by (4.15), (4.16) we have $\mu(V_x^X) < 1$, $\mu(V_y^Y) < 1$ and it remains to show that R_e is satisfied for all $e \in \mathbb{N}$. This, along with the fact that R_e stops requiring attention after some stage s_0, follows by induction on e by the same argument that was detailed in Section 4.2 for the atomic construction, placed in a co-finite segment of stages $[t, \infty]$ where R_e is not injured.

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