EVERY 1-GENERIC COMPUTES A PROPERLY 1-GENERIC

BARBARA F. CSIMA, ROD DOWNEY, NOAM GREENBERG, DENIS R. HIRSCHFELDT, AND JOSEPH S. MILLER

ABSTRACT. A real is called *properly* n-generic if it is n-generic but not $n+1$-generic. We show that every 1-generic real computes a properly 1-generic real. On the other hand, if $m > n \geqslant 2$ then an m-generic real cannot compute a properly n-generic real.

1. INTRODUCTION

The notions of measure and category (or in forcing terminology, random (Solovay) and Cohen forcing) have made their way into computability theory via the notions of restricted randomness and genericity. Restricted genericity for Cohen reals was introduced by Jockusch [Joc80], who studied *n-genericity*, that is, genericity where the forcing relation is restricted to n-quantifier arithmetic (as Jockusch and Posner [JP78] observed, a real is n-generic iff for all Σ^0_n sets of strings S, there is some initial segment σ of A such that $\sigma \in S$ or $\sigma \nsubseteq \tau$ for all $\tau \in S$.) Restricted genericity gives rise to a proper hierarchy (every $n+1$-generic real is also n-generic but not vice-versa). Thus, we can define a real to be *properly n-generic* iff it is n-generic and not $n+1$-generic. [A related notion, first discussed by Kurtz [Kur81], is that of *weak n-genericity*. Here a real A is weakly n-generic iff A meets all dense Σ^0_n sets of strings. Kurtz [Kur81] showed that weak genericity refines the genericity hierarchy, with n-generic $\supsetneq$ weakly $n+1$-generic $\supsetneq$ $n+1$-generic.]

The study of reals random at various levels of the arithmetical hierarchy was introduced by Martin-Löf [ML66]. A real A is called *n-random* iff for all Σ^0_n-tests $\{U_n : n \in \mathbb{N}\}$, we have $A \notin \bigcap_n U_n$. Here, a *Σ^0_n-test* is a (uniform) collection of Σ^0_n-classes $\{U_n : n \in \mathbb{N}\}$, such that $\mu(U_n) \leqslant 2^{-n}$, where μ is Lebesgue measure. (We refer the reader to Downey, Hirschfeldt, Nies and Terwijn [DHNT] for a general introduction to results relating genericity, randomness and relative computability, as well as to the forthcoming books Nies [Nie] and Downey and Hirschfeldt [DH].)

Both n-genericity and n-randomness can be relativized to a given real Z by replacing Σ^0_n objects by ones that are Σ^0_n relative to Z. For instance, a real A is n-random over Z iff $A \notin \bigcap_n U_n$ for all tests $\{U_n : n \in \mathbb{N}\}$ that are Σ^0_n relative to Z. It is easy to see that a real is n-generic iff it is 1-generic over $\emptyset^{(n-1)}$. Kurtz [Kur81] showed that this is also true for randomness; that is, a real is n-random iff it is 1-random over $\emptyset^{(n-1)}$.

There are striking similarities between the ways these two notions interact with Turing reducibility. For example, relatively 1-generic reals form minimal pairs, as

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do relatively 2-random reals. Another nice example is van Lambalgen's Theorem (van Lambalgen [VL87]) which says that $A \oplus B$ is n-random iff A is n-random and B is n-random over A; Yu [Yu06] proved the analogous statement for genericity.

There are interesting distinctions as well. For example, there are complete Δ^0_2 1-random reals, whereas all 1-generic reals are generalised low.

This paper is motivated by a result of Miller and Yu [MY]:

Theorem 1.1 (Miller and Yu)**.** *Let A be 1-random over a real Z, and let B be 1-random and computable in A. Then B is 1-random over Z.*

[This result follows from van-Lambalgen's theorem in the case that Z has 1-random degree.] In particular, letting $Z = \emptyset^{(n-1)}$, if A is n-random and $B \leqslant_{\mathrm{T}} A$, with B 1-random, then B is n-random. Asking whether the same property holds for Cohen genericity yields both a similarity and a distinction from the random case. We will show that the analogue of Miller and Yu's result holds in the generic case, if the bottom real B is 2-generic:

Theorem 1.2. *Let A be 1-generic over a real Z, and let B be 2-generic and computable in A. Then B is 1-generic over Z.*

As a result, it is impossible for, say, a 3-generic real to compute a properly 2-generic real. We mention that Theorem 1.2 may be known, but is not yet found in print, and so we include a proof here.

On the other hand, the analogue of Miller and Yu's result *always* fails when 2-genericity is reduced to 1-genericity:

Theorem 1.3. *Every 1-generic real computes a properly 1-generic real.*

In fact we prove something somewhat stronger.

Theorem 1.4. *Every 1-generic real computes a 1-generic real that is not weakly 2-generic.*

We mention some related results: Haught [Hau86] showed that below $\mathbf{0}'$, the 1-generic degrees are downward closed; Martin showed that for $n \geqslant 2$, the n-generic degrees are downward dense (see [Joc80]). More such results are surveyed in [GM03], which gives some applications.

Several questions remain:

Question 1.5. *Can a sufficiently generic real compute a weakly 2-generic real that is not 2-generic? Must it?*

A degree is properly 1-generic if it contains a 1-generic real but no 2-generic real.

Question 1.6. *Can a sufficiently generic real compute a properly 1-generic Turing degree? Must it?*

1.1. **Notation and terminology.** We work with Cantor space 2^ω. A *class* is a subset of 2^ω. For every $\sigma \in 2^{<\omega}$, let $[\sigma]$ denote the clopen interval in 2^ω defined by σ, i.e., $\{X \in 2^\omega : \sigma \subset X\}$. For any $W \subseteq 2^{<\omega}$, we let $\mathcal{W} = \bigcup_{\sigma \in W} [\sigma]$ be the open class defined by W. An open class $\mathcal{O} \subseteq 2^\omega$ is *enumerable* by some Turing degree $\mathbf{b}$ (we write that $\mathcal{O}$ is $\Sigma^0_1(\mathbf{b})$) if it is defined by some W that is computably enumerable by $\mathbf{b}$. (So a real $X \in 2^\omega$ is weakly 2-generic iff it is an element of every dense, open class that is enumerable by $\mathbf{0}'$.) We say that $\mathcal{O}$ is *c.e.* if it is enumerable by $\mathbf{0}$. This terminology can be used up the arithmetic hierarchy; thus a $\Pi^0_1(\mathbf{b})$ class is the complement of an open set enumerable by $\mathbf{b}$ (equivalently, the set of paths through a tree computable by $\mathbf{b}$); and a $\Pi^0_2(\mathbf{b})$ class is the intersection of a countable sequence of open sets, uniformly enumerable by $\mathbf{b}$.

If $W \subseteq 2^{<\omega}$, we say that a string $\sigma \in 2^{<\omega}$ *decides* (or *forces*) W if either $\sigma \in W$ or no extension of σ is in W. Topologically, either $[\sigma] \subseteq \mathcal{W}$ or $[\sigma]$ is a subset of the complement of $\mathcal{W}$. So $X \in 2^\omega$ is 1-generic iff for every c.e. $W \subseteq 2^{<\omega}$, the real X lies in some interval $[\sigma]$ that decides W.

Turing functionals are codes of partial computable functions from 2^ω to 2^ω. Formally, a Turing functional is a c.e. set $\Phi \subset 2^{<\omega} \times 2^{<\omega}$ that is *consistent*: for $\sigma' \subseteq \sigma$, if $(\sigma', \tau') \in \Phi$ and $(\sigma, \tau) \in \Phi$, then $\tau' \subseteq \tau$. For $\sigma \in 2^{\leqslant\omega}$, let $\Phi(\sigma) = \bigcup\{\tau : \exists \sigma' \subseteq \sigma \; [(\sigma', \tau) \in \Phi]\}$.

As usual, during a construction, at each stage, all expressions involving dynamic objects are evaluated according to the state of the objects (either constructed or given) at the stage. Usual conventions apply; if during a construction we approximate a Δ^0_2 set A, then the value of A on every x is carried over from one stage to the next unless we explicitly act to change that value.

If $\sigma, \tau \in 2^{<\omega}$ then $\sigma\tau$ denotes the concatenation of σ and τ. A digit $i \in \{0, 1\}$ often stands for the string $\langle i \rangle$. If $\sigma \in 2^{<\omega}$ and $k < \omega$ then σ^k is the concatenation of σ with itself k times.

We let $W_0, W_1, \ldots,$ be a uniform enumeration of all c.e. subsets of $2^{<\omega}$. The enumeration is arranged so that at every stage $s > 0$, there is exactly one string σ and one e such that σ enters W_e at stage s. We also assume that if $e \geqslant s$ then W_e is empty at stage s.

2. A positive result

We prove Theorem 1.2: Let A be 1-generic over a real Z, and let B be 2-generic and computable in A. Then B is 1-generic over Z.

Proof of Theorem 1.2. Let A be 1-generic over Z and let $B \leqslant_{\mathrm{T}} A$ be 2-generic. Let Φ be a Turing functional such that $\Phi(A) = B$. Let $W \subseteq 2^{<\omega}$ be c.e. in Z; we may assume that W is closed upwards.

Suppose that $B \notin \mathcal{W}$. Let $\tilde{W} = \{\sigma \in 2^{<\omega} : \Phi(\sigma) \in W\}$. Certainly $\tilde{W}$ is c.e. in Z. Since $A \notin \tilde{\mathcal{W}}$ and A is 1-generic over Z, we know that there is some $\sigma^* \subset A$ with no extension in $\tilde{W}$.

Let $U = \{\tau \in 2^{<\omega} : \neg\exists\sigma \supseteq \sigma^* \; [\tau \subseteq \Phi(\sigma)]\}$. The set U is co-c.e., and $B \notin \mathcal{U}$, so since B is 2-generic, there is some $\tau^* \subset B$ with no extension in U. Thus if $\tau \supseteq \tau^*$ then there is a $\sigma \supseteq \sigma^*$ such that $\tau \subseteq \Phi(\sigma)$. Since $\sigma \notin \tilde{W}$, we have $\Phi(\sigma) \notin W$. Since W is closed upwards, $\tau \notin W$. Thus τ^* has no extension in W. □

3. Computing properly 1-generic sets

In this section we prove theorem 1.4: Every 1-generic real X computes a 1-generic real that is not weakly 2-generic.

To do this, we construct a Turing functional Γ with the following properties:

(1) There is a dense $\Pi^0_2(\mathbf{0}')$ class that is contained in the domain of Γ, and whose image under Γ consists of 1-generic sets.
(2) The range of Γ is contained in a nowhere dense $\Pi^0_1(\mathbf{0}')$ class.

To see that this suffices, assume that X is weakly 2-generic (if X is not weakly 2-generic then we are of course done). Let $\mathfrak{A}$ be the class guaranteed by (1). The real X is an element of any dense $\Sigma^0_1(\mathbf{0}')$ class, hence of any countable intersection of such classes; so $X \in \mathfrak{A}$. Then $\Gamma(X)$ (which is computable by X) is 1-generic by (1), and is not weakly 2-generic because by (2), it misses a dense open set enumerable by $\mathbf{0}'$.

3.1. **Discussion.**

3.1.1. *Getting property 1.* To make the image of $\mathfrak{A}$ under Γ consist of 1-generic sets, for each $e < \omega$, we must construct a dense, open set $\mathcal{S}_e$ such that for all $\sigma \in S_e$, $\Gamma(\sigma)$ decides W_e; and further we must make the sequence $\mathcal{S}_0, \mathcal{S}_1, \dots$ uniformly enumerable by $\mathbf{0}'$, so that we can define $\mathfrak{A} = \bigcap_e \mathcal{S}_e$. We need to ensure that $\mathfrak{A}$ is dense; by Baire's theorem, it is sufficient to ensure that each $\mathcal{S}_e$ is dense.

Consider W_0. A simple plan for meeting it would be setting $\mathcal{S}_0 = 2^\omega$ and acting as follows: if W_0 is empty, do nothing; if there is some $\tau \in W_0$, let $\Gamma(\langle\rangle) = \tau$. Now move to W_1. Of course, this plan is not effective, so we must use the priority method for our construction. Again, a naïve approach would be as follows: While W_0 is empty, do nothing, and let weaker requirements $(W_1, W_2, \dots)$ act if they want to. If some string τ enters W_0 then injure the weaker requirements and set $\Gamma(\langle\rangle) = \tau$. The problem here is that we cannot cancel the axioms that our work for $W_1, W_2, \dots$ had us enumerating into Γ, so if we want to keep Γ consistent, we cannot make the definition we like. The solution is to break up the playing ground into pieces, let weaker requirements work on some of the pieces, and make sure that there is sufficient room for the stronger requirement to act if necessary.

Here is the strategy for W_0. In the beginning, we mark the interval $2^\omega = [\langle\rangle]$ to work on W_0. We break the interval up into infinitely many disjoint subintervals whose union is dense in 2^ω, say $[1], [01], [001], [0001], \dots$. For the time being, each such subinterval believes it has met the W_0-requirement by forcing its image under Γ into the complement of $\mathcal{W}_0$, simply because $\mathcal{W}_0$ is still empty. So we can be generous and let each subinterval work for the next requirement W_1.

At a later stage, some string τ enters W_0. Only finitely many subintervals have been spoiled for W_0; so we can define Γ to be τ everywhere else. On the spoiled intervals, we need to work again for W_0; since definitions of Γ have been made on possibly small subsubintervals, we need to break the spoiled region into small intervals on which we individually work on W_0.

We let S_0 be the collection of intervals $[\sigma]$ that are "good" for W_0, which are those intervals on which we ensure that $\Gamma(\sigma)$ meets W_0, and those at which we had a correct belief that $[\Gamma(\sigma)] \cap \mathcal{W}_0 = \emptyset$. This set will in fact be d.c.e., and so certainly Σ^0_2; and reals in $\mathcal{S}_0$ will satisfy the W_0 requirement. We need to ensure that $\mathcal{S}_0$ is dense; this holds because we break it up into finer and finer subintervals (each time our hopes for an easy win are dashed).

The strategy for weaker W_e is similar, except that of course we need to take into consideration injury by stronger requirements.

3.1.2. *Getting property 2.* To ensure that the range of Γ is nowhere dense, we could, whenever we define some axiom $\Gamma(\sigma) = \tau$, pick some extension τ' of τ and declare that no value of Γ may ever extend τ'. This straightforward approach, however, interferes with the priority mechanism that ensures property (1), in the following way. Suppose that we mark some interval $[\sigma_0]$ for W_1, and later define $\Gamma(\sigma_1) = \tau$

for some $\sigma_1 \supset \sigma_0$, marking $[\sigma_1]$ for W_2. We then declare that the range of Γ must be disjoint from $[\rho]$, where $\rho \supset \tau$. A later W_0 action elsewhere invalidates $[\sigma_0]$'s marking, so we mark $[\sigma_1]$ for W_0. Then, some string extending ρ enters W_0, but W_0 is prohibited from winning by directing Γ through that string on a subinterval of $[\sigma_1]$. We will indeed direct Γ to go through some extension τ' of τ that is incomparable with ρ, and this presumably will give us another chance of attacking W_0; but this process may repeat itself, since following the straightforward approach compels us to first declare some extension ρ' of τ' disjoint from the range of Γ. After infinitely many failed attempts at meeting W_0 we have a real in the range of Γ belonging to the closure of $\mathcal{W}_0$ but not to $\mathcal{W}_0$ itself.

This in fact *must* happen, because we made the collection of prohibited intervals a dense c.e. class, ensuring that no element of the range of Γ is even weakly 1-generic (indeed, the recursion theorem and the "slowdown lemma" imply that there is some e such that W_e is the set of prohibited intervals, and that every σ is enumerated into W_e only after it was declared prohibited). The solution is to use the priority mechanism that was introduced for getting property (1). When we define $\Gamma(\sigma) = \tau$ for meeting W_e, we define one extension to be prohibited with priority e. This prohibition can be ignored by strings $\sigma' \subset \sigma$ that are working for stronger $W_{e'}$. The whole mechanism does the work for us, so we in fact do not need to use the word "prohibited" during the construction, just to make $\Gamma(\sigma)$ long enough.

3.2. **Construction.** Here is the formal construction.

In the beginning, the entire space $2^\omega = [\langle\rangle]$ is marked *active* for W_0. (When we mark a clopen set $[\sigma]$ active for some W_e, we also say that the string σ is marked active for W_e.)

At *stage s*:

1. A string τ is enumerated into some W_e. Suppose that there is some σ that is active for W_e, such that $\Gamma(\sigma) \subseteq \tau$ (there will be at most one such σ). Do the following:

- Unmark $[\sigma]$ from being active for W_e. Choose a very large number m.
- Enumerate $(\sigma 0^s, \tau 1^m)$ into Γ. Mark $[\sigma 0^s]$ as active for W_{e+1}.
- For every $k < s$, remove all markings of strings $\sigma' \supseteq \sigma 0^k 1$. For every $\sigma' \in 2^m$ extending $\sigma 0^k 1$ for some $k < s$, mark $[\sigma']$ as active for W_e; find some $\tau_{\sigma'} \supset \Gamma(\sigma')$[1] of length m that is incompatible with $\tau 1^m$, and enumerate $(\sigma', \tau_{\sigma'}\sigma' 1)$ into Γ. [Note that the length of $\Gamma(\sigma')$ is much smaller than m because it was defined prior to this stage, and m is chosen large.]

2. Inductively for $e < s$, for every $[\sigma]$ that is active for W_e, for $k \leqslant s$, if $[\sigma 0^k 1]$ is not marked, then mark it as active for W_{e+1} and enumerate $(\sigma 0^k 1, \Gamma(\sigma) 0^k 11)$ into Γ.

This completes the construction. An illustration of a typical turn of events is given in Figures 1-3.

[1]Note again that here we mean $\Gamma(\sigma')\,[s] = \bigcup\{\rho : \exists \sigma'' \subseteq \sigma' \; [(\sigma'', \rho) \in \Gamma\,[s]]\}$; this makes sense even if there is no ρ such that $(\sigma', \rho) \in \Gamma\,[s]$. Thus $\Gamma(\sigma')$ grows at this stage.

FIGURE 1. An interval is active for W_1, some subintervals are active for W_2, and those subintervals have subintervals active for W_3:

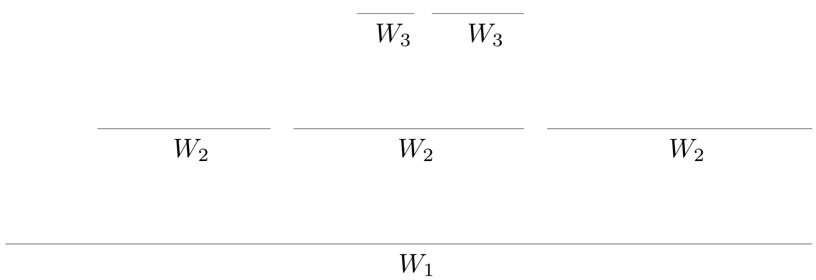


FIGURE 2. Action was taken for the middle subinterval. Its mark and the marks of its subintervals were removed. It is broken into smaller subintervals; on one we have a positive win for W_2 and so it is marked for W_3; on the rest we go back to work on W_2:

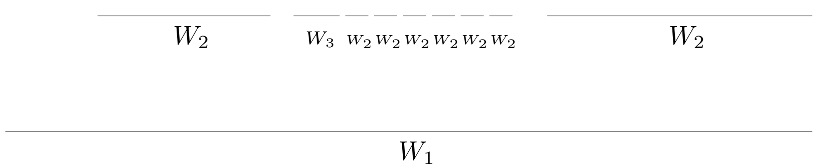
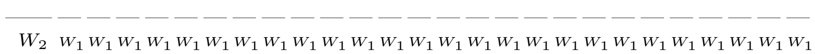


FIGURE 3. Later, action is taken for the original W_1-interval. The previous W_2 and W_3 markings are cancelled, and the original interval is broken into small subintervals:



3.3. **Verification.** Here are some facts that follow from the instructions given and may clarify the construction a little.

Lemma 3.1.

(1) *Suppose that at the beginning of stage s, a string σ is already active for W_e. Then for all $k < s$, the string $\sigma 0^k 1$ is active for W_{e+1}, and no string extending $\sigma 0^s$ is active for any $W_{e'}$.*
(2) *If at stage s, compatible strings σ and σ' are marked for W_e and $W_{e'}$ respectively, and $e < e'$, then $\sigma \subset \sigma'$.*
(3) *Each string can be marked as active for at most one W_e, at most once.*

Proof. The point is that if at some stage s, action is taken for some string σ, then all marks are removed from all extensions of σ, and new marks are only given to very long extensions of σ that we didn't deal with previously. At the second part of the stage, we mark extensions of these new, long strings; and for every string σ' that was already active for W_e at the beginning of the stage (and whose mark was not removed during the stage, i.e. if $\sigma' \not\supseteq \sigma$), we mark $\sigma' 0^s 1$ (which was never dealt with before) for W_{e+1}; we also mark some of this string's extensions. □

We define S_e, the success set for W_e. At the beginning of a stage s, the approximation for this success set is $S_e[s]$. There are two ways into S_e, which give rise to a partition of S_e into two sets S_e^+ and S_e^-.

(1) A positive win for W_e occurs at stage s if at that stage we act for a string σ that is marked for W_e. We then put a W_{e+1}-mark on $\sigma 0^s$; we enumerate $\sigma 0^s$ into $S_e^+[s+1]$.
(2) The other case is when some string σ is active for W_e, and we mark some string $\sigma 0^k 1$ as active for W_{e+1} (where $k \leqslant s$), assuming that we won W_e on $\sigma 0^k 1$ in a negative fashion. We then enumerate $\sigma 0^k 1$ into $S_e^-[s+1]$.

A W_{e+1}-mark is removed from a string σ' at a stage $t > s$ if some action is taken for any string $\sigma \subset \sigma'$. Again there are two cases:

(1) If $\sigma 0^k 1 \in S_e^-[s]$ and action is taken for σ at stage s, then the assumption of a negative win on $\sigma 0^k 1$ for W_e is invalidated, and the W_{e+1} mark is removed from $\sigma 0^k 1$; we extract $\sigma 0^k 1$ from $S_e^-[s+1]$. [In this case, the W_e-mark is removed from σ, but we do not extract σ from S_{e-1}. Thus S_e does *not* equal the set of strings that are active for W_{e+1}. Also note that if $\sigma 0^t \in S_e^+[s]$ then σ is not marked for W_e anymore; action is not taken for σ again and so the instructions here refer only to strings in S_e^-.]
(2) If $\sigma 0^k 1$ or $\sigma 0^s$ is in S_e (S_e^- and S_e^+ respectively) and some action is taken for some $\sigma' \subsetneq \sigma$ then all markings are removed from extensions of σ'; in this case we extract $\sigma 0^k 1$ and $\sigma 0^s$ from S_e.

By our definitions, and by Lemma 3.1(3), each S_e is d.c.e. (uniformly). [In fact $\bigcup_e S_e$ is d.c.e.] Thus $\mathfrak{A} = \bigcap_e \mathcal{S}_e$ is a $\Pi_2^0(\mathbf{0}')$ class.

If σ is a string that at stage s becomes marked for some W_e, then at s we define $\Gamma(\sigma) = \nu_\sigma 1$ for some $\nu_\sigma \in 2^{<\omega}$. Note that this is the only time at which we enumerate (σ, τ) into Γ for any string τ; so $\Gamma(\sigma)$ does not change after stage s; again refer to Lemma 3.1.

Lemma 3.2. *For $s < \omega$, let $R[s]$ be the collection of all σ that at the beginning of stage s are active for some W_e or are in some S_e. If $\sigma, \sigma' \in R[s]$ are incomparable, then ν_σ and $\nu_{\sigma'}$ are incomparable.*

Proof. By induction on s. Suppose this holds for s. At stage s, say we first act for some interval σ which was marked for W_e. By the instructions given, all proper extensions of σ lose any markings they may have, and by the definition of S_e, these strings are now removed from R. If σ', σ'' are two extensions of σ that are now put into R, then we define ν_σ and $\nu_{\sigma'}$ to be incomparable. If $\sigma' \supset \sigma$ is now put into R and $\sigma'' \in R[s]$ is "old", and $\sigma' \perp \sigma''$, then $\sigma'' \perp \sigma$ and so $\nu_{\sigma''} \perp \nu_\sigma \subset \nu_{\sigma'}$.

At the second step of the stage, more intervals are added to R, but always refining intervals that are already in R; if we mark $[\sigma 0^k 1]$ for some W_{e+1} then we define $\nu_{\sigma 0^k 1} = \Gamma(\sigma) \sigma 0^k 1$ and so incomparability is assured. □

Here are some more observations.

Lemma 3.3. *For $s < \omega$, let $Q[s]$ be the downward closure of $\bigcup_{t \leqslant s} R[t]$. Then:*

1. *Every $\sigma \in Q[s]$ has some extension in $R[s]$.*
2. $2^{<s} \subseteq Q[s]$.

Proof. For (1), note that if σ is removed from R at stage t then some extension is put into R immediately. For (2), note in fact that if σ is removed from R at stage t then all extensions of σ of length t are placed into Q; and if $\sigma \in 2^{<s}$ is placed into R at stage t then both extensions of length $|\sigma| + 1$ are placed into Q at the same stage. [For the latter assertion, note that if σ is marked for some W_e then both $\sigma 1$ and $\sigma 01$ are marked for W_{e+1} at the same stage, which puts both $\sigma 1$ and $\sigma 0$ into Q at that stage.] □

Corollary 3.4. *Suppose that $\sigma \in \bigcup_e S_e$ (at the end of time). Then for no σ' do we have $\Gamma(\sigma') \supseteq \nu_\sigma 0$.*

Proof. Suppose for a contradiction that there is some σ' such that $\Gamma(\sigma') \supseteq \nu_\sigma 0$. We cannot have σ and σ' comparable; if $\sigma' \supset \sigma$ then $\Gamma(\sigma') \supset \Gamma(\sigma) = \nu_\sigma 1$; and if $\sigma' \subset \sigma$ then $\Gamma(\sigma') \subseteq \Gamma(\sigma)$.

Without loss of generality assume that we mark σ' at stage s_0; and suppose that we mark σ at stage s_1. We cannot have $s_1 \leqslant s_0$, because then $\sigma, \sigma' \in R[s_0 + 1]$ and so, by Lemma 3.2, $\nu_\sigma \perp \nu_{\sigma'}$. But we also cannot have $s_0 \leqslant s_1$, because then, even if $\sigma' \notin R[s_1]$, there is some $\sigma'' \supseteq \sigma'$ which is in $R[s_1 + 1]$, and $\Gamma(\sigma'') \supset \nu_\sigma 0$, contradicting $\nu_\sigma \perp \nu_{\sigma''}$. □

We now are ready to verify properties (1) and (2). For simplicity of notation, we let (at every stage) $S_{-1} = \{\langle\rangle\}$.

Lemma 3.5. *Each $\mathcal{S}_e$ is dense, and so $\mathfrak{A} = \bigcap_e \mathcal{S}_e$ is dense too.*

Proof. We take some $\sigma \in S_{e-1}$ and show that $\mathcal{S}_e$ is dense in $[\sigma]$. Suppose that σ is put into S_{e-1} at stage s_0.

We first note that if some $[\sigma'] \subset [\sigma]$ is ever marked active for W_e at a stage $s \geqslant s_0$, then there is a subinterval of $[\sigma']$ that is (permanently) in S_e. This is because either no action is taken for σ', in which case for every $k < \omega$, the string $\sigma' 0^k 1$ is in S_e, or action is taken for σ' at some stage s, in which case $\sigma' 0^s$ is in S_e. The point is that these markings cannot be eliminated by action below σ', because such action would remove σ from S_{e-1} (note that there are no marked strings between σ and σ').

Let $\rho \supset \sigma$ and suppose for a contradiction that $[\rho] \cap \mathcal{S}_e = \emptyset$. Let $s_1 > s_0, |\rho|$, and find some $\sigma' \in S_e[s_1]$ compatible with ρ (so $\sigma' \supset \sigma$). There is a later stage s_2 at which σ' is extracted from S_e; at that stage, all successors of σ' at some level $m > |\rho|$ are marked active for W_e. At least one of these successors σ'' is compatible with ρ, and so must actually extend ρ. The string σ'' has some extension that is in S_e, a contradiction. □

Lemma 3.6. *Suppose that $X \in \mathcal{S}_e$. Then some initial segment of $\Gamma(X)$ determines W_e.*

(Note that we do not assume that $\Gamma(X)$ is total.)

Proof. Suppose that $\sigma \in S_e$ and $\sigma \subset X$. If $\sigma \in S_e^+$, then $\Gamma(\sigma)$ extends some string in W_e. If $\sigma \in S_e^-$, then no $\tau \supset \Gamma(\sigma')$ is ever enumerated into W_e (where $\sigma = \sigma' 0^k 1$ for some k; note that σ' is always marked for W_e). □

Corollary 3.7. $\mathfrak{A} \subseteq \operatorname{dom} \Gamma$ *and every* $Y \in \Gamma[\mathfrak{A}]$ *is 1-generic.*

Proof. The second part follows immediately from the lemma. For the first part, apply the lemma to the sets $W_{e_n} = 2^{\geqslant n}$ for $n \in \omega$. □

Lemma 3.8. $\Gamma[2^\omega]$ *is contained in a nowhere dense* $\Pi^0_1(\mathbf{0}')$ *class.*

Proof. Let T be the downward closure of the range of Γ, viewed as a relation on *strings*, i.e.

$$T = \{\tau : \exists \sigma \; [\tau \subseteq \Gamma(\sigma)]\}.$$

T is c.e., and so $[T]$, the class of paths through T, is a $\Pi^0_1(\mathbf{0}')$ class that contains the image of Γ on 2^ω.

The class $[T]$ is closed, so to show that it is nowhere dense, it suffices to show that it does not contain any interval. Suppose for a contradiction that $[\rho]$ is an interval contained in $[T]$, which means that every extension of ρ is in T. By the definition of T, there are some σ and τ such that $(\sigma, \tau) \in \Gamma$ and $\tau \supseteq \rho$. There is some σ' extending σ which is in $\bigcup_e S_e$. Then $\nu_{\sigma'} \supseteq \tau$, and by Corollary 3.4, $\nu_{\sigma'}0$ is not on T; this is a contradiction. □

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Department of Pure Mathematics, University of Waterloo, Waterloo, ON, Canada N2L 3G1
E-mail address: csima@math.uwaterloo.ca

School of Mathematics, Statistics and Computer Science, Victoria University, P.O. Box 600, Wellington, New Zealand
E-mail address: Rod.Downey@vuw.ac.nz

School of Mathematics, Statistics and Computer Science, Victoria University, P.O. Box 600, Wellington, New Zealand
E-mail address: greenberg@mcs.vuw.ac.nz

Mathematics Department, University of Chicago, 5734 S. University Ave., Chicago, IL 60637, U.S.A.
E-mail address: drh@math.uchicago.edu

Mathematics Department, University of Connecticut, Storrs, CT 06269, U.S.A.
E-mail address: joseph.miller@math.uconn.edu