From Geometry to Conceptual Relativity*

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Abstract

The purported fact that geometric theories formulated in terms of points and geometric theories formulated in terms of lines are "equally correct" is often invoked in arguments for conceptual relativity, in particular by Putnam and Goodman. We discuss a few notions of equivalence between first-order theories, and we then demonstrate a precise sense in which this purported fact is true. We argue, however, that this fact does not undermine metaphysical realism.

1 Introduction

Late 20th century philosophy witnessed a distinctive movement away from metaphysical realism. This movement, particularly evident in the work of Goodman, Putnam, and Quine, was motivated by certain examples from logic and science. Most notably, Putnam and Goodman often cited the example of Euclidean geometry, arguing that there is no answer to the question of whether the Euclidean plane is made of points, or whether points are instead derived entities. We will call this example the *argument from geometry*.

According to the argument from geometry, certain situations could equally well be described using a theory that takes points as fundamental entities, or instead using a theory that takes lines as fundamental entities. Somebody who adopts the first theory is committed to the existence of points and not lines, while somebody who adopts the second theory is committed to the existence of lines and not points. But points and lines are different kinds of things, and in general, the number of points (according to the first theory) will be different from the number of lines (according to the second theory). Since both parties correctly describe the world, but use different ontologies to do so, it follows that there is no matter of fact about what the ontology of the world is. This directly contradicts a fundamental tenet of metaphysical realism.

In responding to examples of this sort, metaphysical realists typically grant that the two theories in question involve incompatible ontological commitments

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(see Sider, 2009; Van Inwagen, 2009). They then claim, however, that at most one of the two theories can be correct, at least in a fundamental sense. The upshot of this kind of response, of course, is that a realist ontology has been purchased at the price of an epistemic predicament: Only one of the theories is correct, but we will never know which one.

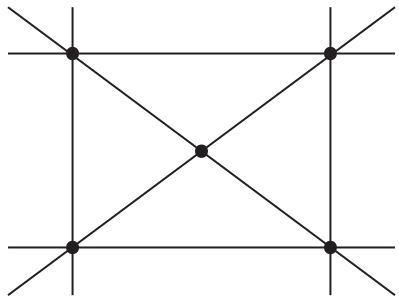
Our purpose in this paper is to suggest another reply to arguments of this sort, and specifically to the argument from geometry. We show that geometries with points can naturally be considered equivalent to geometries with lines, and we argue that this equivalence does not in any way threaten the idea that there is an objective world. In other words, since these two theories are equivalent, there is a sense in which they involve exactly the same ontological commitments. The example of geometries with points and geometries with lines does not undermine metaphysical realism in the way that Putnam and Goodman suggested.

2 Preliminaries

There are many ways to formulate a particular geometric theory, and these formulations often differ with respect to the kinds of objects that are taken as primitive. The most famous example of this phenomenon is Euclidean geometry. Tarski first formulated Euclidean geometry using open balls (Tarski, 1929), and later using points (Tarski, 1959). Schwabhäuser and Szczerba (1975) formulated Euclidean geometry using lines, and Hilbert (1930) used points, lines, planes, and angles. These formulations of Euclidean geometry all take different kinds of objects to be primitive, but despite this ostensible difference, they nonetheless manage to express the same geometric facts. Indeed, it is standard to recognize some sense in which all of these formulations of Euclidean geometry are *equivalent*. This sense of equivalence, however, has never been made perfectly precise.[1]

In fact, from a certain point of view, it might seem that these theories cannot be equivalent. Consider a simple example: Take six lines in the Euclidean plane, as in the following diagram.

[1] Beth and Tarski (1956), Scott (1956), Tarski (1956), Robinson (1959), and Royden (1959) focus on the relationships between formulations of geometry that use different primitive *predicate* symbols, but not different primitive *sort* symbols. Szczerba (1977) and Schwabhäuser et al. (1983) take crucial steps toward capturing the relationships between geometries with different sorts, but do not explicitly prove their equivalence.



On the one hand, if this diagram were described in terms of the point-based version of Euclidean geometry (T_p), then we would say that there are exactly five things. On the other hand, if this diagram were described in terms of the line-based version of Euclidean geometry (T_ℓ), then we would say that there are exactly six things.

Indeed, according to one natural notion of theoretical equivalence, the first description T_p is not equivalent to the second description T_ℓ. This notion is called *definitional equivalence*, and was introduced into philosophy of science by Glymour (1970, 1977, 1980). If two theories are definitionally equivalent, then the cardinality of their respective domains will be equal. Since the cardinality of the domains of T_p and T_ℓ are unequal, therefore, these descriptions cannot be definitionally equivalent.

This would be the end of the matter if definitional equivalence were the only legitimate notion of theoretical equivalence. However, we believe that there is a better notion of theoretical equivalence that does not prejudge issues about the cardinality of domains. We will now expend some effort explaining this more expansive notion of equivalence; to do so will require that we enter into the framework of many-sorted logic. We therefore begin with some preliminaries about this framework.[2]

A **signature** Σ is a set of sort symbols, predicate symbols, function symbols, and constant symbols. Every signature is required to contain at least one sort symbol. The predicate, function, and constant symbols in Σ are assigned **arities** constructed from sorts in Σ. The arity of a symbol specifies which sorts the symbol "applies to." The Σ-terms, Σ-formulas, and Σ-sentences are recursively defined in the standard way. The only difference from the syntax of single-sorted logic is that the quantifiers $\forall_\sigma$ and $\exists_\sigma$ that appear in Σ-formulas are indexed by sorts $\sigma \in \Sigma$. We will occasionally drop these indices, but only when it is perfectly clear what sort of variables are being quantified over.

A **Σ-structure** A is a family of nonempty and pairwise disjoint sets A_σ, one for each sort symbol $\sigma \in \Sigma$, in which the predicates, functions, and constant symbols in Σ have been interpreted. One recursively defines when elements

[2]The reader is encouraged to consult Hodges (2008) and Barrett and Halvorson (2015b) for further details.

$a_1, \dots, a_n \in A$ **satisfy** a Σ-formula $\phi(x_1, \dots, x_n)$ in the Σ-structure A, written $A \vDash \phi[a_1, \dots, a_n]$. A **$\Sigma$-theory** T is a set of Σ-sentences. The sentences $\phi \in T$ are called the **axioms** of T. A Σ-structure M is a **model** of a Σ-theory T if $M \vDash \phi$ for all $\phi \in T$. A theory T **entails** a sentence ϕ, written $T \vDash \phi$, if $M \vDash \phi$ for every model M of T.

We now have the resources to state the following preliminary criterion for theoretical equivalence.

Definition 1. Theories T_1 and T_2 are **logically equivalent** if they have the same class of models.

One can verify that theories T_1 and T_2 are logically equivalent if and only if they entail precisely the same sentences. It is therefore easy to see that logical equivalence is too strict to capture any kind of equivalence between geometries with points and geometries with lines. Theories can only be logically equivalent if they are formulated in the same signature. Since these formulations of geometry employ different sort symbols — some are formulated in signatures with a sort of lines, others are formulated in signatures with a sort of points — they cannot be logically equivalent.

3 Morita equivalence

Logical equivalence is a particularly strict criterion for theoretical equivalence, so many other criteria for theoretical equivalence have been proposed. The criterion that we will consider here is called **Morita equivalence**.[3] The intuition behind Morita equivalence is simple, and similar criteria have been known to logicians for many years. Theories T_1 and T_2 are Morita equivalent if T_1 can define all of the vocabulary that T_2 uses, and in a compatible way, T_2 can define all of the vocabulary that T_1 uses. It takes a bit of work to make this intuition precise.

In particular, we need to formalize the concept of a definition. We begin by saying how to define new predicate, function, and constant symbols. Let $\Sigma \subset \Sigma^+$ be signatures and let $p \in \Sigma^+ - \Sigma$ be a predicate symbol of arity $\sigma_1 \times \dots \times \sigma_n$. An **explicit definition of p in terms of Σ** is a Σ^+-sentence of the form

$$\forall_{\sigma_1} x_1 \dots \forall_{\sigma_n} x_n \big(p(x_1, \dots, x_n) \leftrightarrow \phi(x_1, \dots, x_n)\big)$$

where $\phi(x_1, \dots, x_n)$ is a Σ-formula. Similarly, an explicit definition of a function symbol $f \in \Sigma^+ - \Sigma$ of arity $\sigma_1 \times \dots \times \sigma_n \to \sigma$ is a Σ^+-sentence of the form

$$\forall_{\sigma_1} x_1 \dots \forall_{\sigma_n} x_n \forall_\sigma y \big(f(x_1, \dots, x_n) = y \leftrightarrow \phi(x_1, \dots, x_n, y)\big) \quad (1)$$

[3]In addition to the criterion proposed by Glymour (1970, 1977, 1980), see Quine (1975) for another. See de Bouvére (1965), Kanger (1968), Pinter (1978), Pelletier and Urquhart (2003), Andréka et al. (2005), Friedman and Visser (2014), and Barrett and Halvorson (2015a,b,c) for some results that have been proven about different standards of equivalence. Barrett and Halvorson (2015b) provide an introduction to Morita equivalence, and Meré and Veloso (1992), Andréka et al. (2008), and Andréka and Németi (2014) contain additional discussion of it.

and an explicit definition of a constant symbol $c \in \Sigma^+ - \Sigma$ of sort σ is a Σ^+-sentence of the form

$$\forall_\sigma x\big(x = c \leftrightarrow \psi(x)\big) \tag{2}$$

where $\phi(x_1, \ldots, x_n, y)$ and $\psi(x)$ are both Σ-formulas. Note that in all of these cases it must be that the sorts $\sigma_1, \ldots, \sigma_n, \sigma \in \Sigma$.

Although they are Σ^+-sentences, (1) and (2) have consequences in the signature Σ. In particular, (1) and (2) imply the following sentences, respectively:[4]

$$\forall_{\sigma_1} x_1 \ldots \forall_{\sigma_n} x_n \exists_{\sigma=1} y \phi(x_1, \ldots, x_n, y)$$
$$\exists_{\sigma=1} x\, \psi(x)$$

These two sentences are called the **admissibility conditions** for the explicit definitions (1) and (2).

We also need to say how to define new sort symbols. Let $\Sigma \subset \Sigma^+$ be signatures and consider a sort symbol $\sigma \in \Sigma^+ - \Sigma$. The sort σ can be defined as a product sort, a coproduct sort, a subsort, or a quotient sort. In each case one defines σ using old sorts from Σ and new function symbols from $\Sigma^+ - \Sigma$. These new function symbols specify how the new sort σ is related to the old sorts in Σ. We describe in detail these four ways to define new sorts.

In order to define σ as a product sort, one needs two function symbols $\pi_1, \pi_2 \in \Sigma^+ - \Sigma$ with π_1 of arity $\sigma \to \sigma_1$, π_2 of arity $\sigma \to \sigma_2$, and $\sigma_1, \sigma_2 \in \Sigma$. The function symbols π_1 and π_2 serve as the "canonical projections" associated with the product sort σ. An explicit definition of the symbols σ, π_1, and π_2 as a **product sort** in terms of Σ is a Σ^+-sentence of the form

$$\forall_{\sigma_1} x \forall_{\sigma_2} y \exists_{\sigma=1} z(\pi_1(z) = x \wedge \pi_2(z) = y)$$

One should think of a product sort σ as the sort whose elements are ordered pairs, where the first element of each pair is of sort σ_1 and the second is of sort σ_2.

One can also define σ as a coproduct sort. In this case, one needs two function symbols $\rho_1, \rho_2 \in \Sigma^+ - \Sigma$ with ρ_1 of arity $\sigma_1 \to \sigma$, ρ_2 of arity $\sigma_2 \to \sigma$, and $\sigma_1, \sigma_2 \in \Sigma$. The function symbols ρ_1 and ρ_2 are the "canonical injections" associated with the coproduct sort σ. An explicit definition of the symbols σ, ρ_1, and ρ_2 as a **coproduct sort** in terms of Σ is a Σ^+-sentence of the form

$$\forall_\sigma z\big(\exists_{\sigma_1=1} x(\rho_1(x) = z) \vee \exists_{\sigma_2=1} y(\rho_2(y) = z)\big) \wedge \forall_{\sigma_1} x \forall_{\sigma_2} y \neg\big(\rho_1(x) = \rho_2(y)\big)$$

One should think of a coproduct sort σ as the disjoint union of the elements of sorts σ_1 and σ_2.

When defining a new sort σ as a product sort or a coproduct sort, one uses two sort symbols in Σ and two function symbols in $\Sigma^+ - \Sigma$. The next two ways of defining a new sort σ only require one sort symbol in Σ and one function symbol in $\Sigma^+ - \Sigma$.

[4] We will use the notation $\exists_{\sigma=n} x\, \phi(x)$ and $\exists_{\sigma\leq n} x\, \phi(x)$ throughout to abbreviate the sentences "There exist exactly (respectively, less than than or equal to) n things of sort σ that are ϕ."

In order to define σ as a subsort, one needs a function symbol $i \in \Sigma^+ - \Sigma$ of arity $\sigma \to \sigma_1$ with $\sigma_1 \in \Sigma$. The function symbol i is the "canonical inclusion" associated with the subsort σ. An explicit definition of the symbols σ and i as a **subsort** in terms of Σ is a Σ^+-sentence of the form

$$\forall_{\sigma_1} x \left(\phi(x) \leftrightarrow \exists_\sigma z(i(z) = x)\right) \wedge \forall_\sigma z_1 \forall_\sigma z_2 \left(i(z_1) = i(z_2) \to z_1 = z_2\right) \tag{3}$$

where $\phi(x)$ is a Σ-formula. One should think of σ as "the things of sort σ_1 that are ϕ." The sentence (3) entails the following Σ-sentence:

$$\exists_{\sigma_1} x\, \phi(x)$$

As above, we will call this Σ-sentence the **admissibility condition** for the definition (3).

Lastly, in order to define σ as a quotient sort one needs a function symbol $\epsilon \in \Sigma^+ - \Sigma$ of arity $\sigma_1 \to \sigma$ with $\sigma_1 \in \Sigma$. An explicit definition of the symbols σ and ϵ as a **quotient sort** in terms of Σ is a Σ^+-sentence of the form

$$\forall_{\sigma_1} x_1 \forall_{\sigma_1} x_2 \left(\epsilon(x_1) = \epsilon(x_2) \leftrightarrow \phi(x_1, x_2)\right) \wedge \forall_\sigma z \exists_{\sigma_1} x(\epsilon(x) = z) \tag{4}$$

where $\phi(x_1, x_2)$ is a Σ-formula. This sentence defines σ as a quotient sort that is obtained by "quotienting out" the sort σ_1 with respect to the formula $\phi(x_1, x_2)$. The sort σ should be thought of as the set of "equivalence classes of elements of σ_1 with respect to the relation $\phi(x_1, x_2)$," and the function symbol ϵ is the "canonical projection" that maps an element to its equivalence class. And indeed, one can verify that the sentence (4) implies that $\phi(x_1, x_2)$ is an equivalence relation. In particular, (4) entails the following Σ-sentences:

$$\forall_{\sigma_1} x\, \phi(x, x)$$
$$\forall_{\sigma_1} x_1 \forall_{\sigma_1} x_2 (\phi(x_1, x_2) \to \phi(x_2, x_1))$$
$$\forall_{\sigma_1} x_1 \forall_{\sigma_1} x_2 \forall_{\sigma_1} x_3 \left((\phi(x_1, x_2) \wedge \phi(x_2, x_3)) \to \phi(x_1, x_3)\right)$$

These Σ-sentences are the **admissibility conditions** for the definition (4).

Now that we have described the four ways of defining new sort symbols, we can define the concept of a Morita extension. One can think of a Morita extension of a theory T as a theory that results from adding "abbreviations" or "shorthand" to the theory T in the form of new defined symbols. Let $\Sigma \subset \Sigma^+$ be signatures and T a Σ-theory. A **Morita extension** of T to the signature Σ^+ is a Σ^+-theory

$$T^+ = T \cup \{\delta_s : s \in \Sigma^+ - \Sigma\}$$

that satisfies the following three conditions. First, for each symbol $s \in \Sigma^+ - \Sigma$ the sentence δ_s is an explicit definition of s in terms of Σ. Second, if $\sigma \in \Sigma^+ - \Sigma$ is a sort symbol and $f \in \Sigma^+ - \Sigma$ is a function symbol that is used in the explicit definition of σ, then $\delta_f = \delta_\sigma$. (For example, if σ is defined as a product sort with projections π_1 and π_2, then $\delta_\sigma = \delta_{\pi_1} = \delta_{\pi_2}$.) And third, if α_s is an admissibility condition for a definition δ_s, then $T \vDash \alpha_s$.

As we will discuss in section 5, there is a natural sense in which a Morita extension of a theory "says no more" than the original theory. Indeed, one can show that if T^+ is a Morita extension of T, then T^+ is a conservative extension of T (Barrett and Halvorson, 2015b, Theorem 4.2). Using the concept of a Morita extension, we have the machinery to define Morita equivalence.

Definition 2. Let T_1 be a Σ_1-theory and T_2 a Σ_2-theory. T_1 and T_2 are **Morita equivalent** if there are theories $T_1^1, \ldots, T_1^n$ and $T_2^1, \ldots, T_2^m$ that satisfy the following three conditions:

- Each theory T_1^{i+1} is a Morita extension of T_1^i,
- Each theory T_2^{i+1} is a Morita extension of T_2^i,
- T_1^n and T_2^m are logically equivalent Σ-theories with $\Sigma_1 \cup \Sigma_2 \subset \Sigma$.

We will often call the theory T_1^n (or similarly, T_2^m) the "common Morita extension" of T_1 and T_2. Two theories are therefore Morita equivalent if they have a common Morita extension in this precise sense. The intuition behind Morita equivalence is fairly simple: T_1 and T_2 are Morita equivalent if they each can, in compatible ways, define all of the vocabulary that the other uses. Morita equivalence captures a sense in which two theories are "intertranslatable." Indeed, one can show that if T_1 and T_2 are Morita equivalent, then there is a natural way to convert models of T_1 into models of T_2, and vice versa (Barrett and Halvorson, 2015b, Theorem 5.1). And in addition, there is a natural way to translate sentences between the two theories (Barrett and Halvorson, 2015b, Theorem 4.3).

4 Equivalent formulations of geometry

With the concept of Morita equivalence in hand, we can now discuss these different formulations of geometry. All of the geometries that we will consider are formulated using (some subset of) the following vocabulary.[5]

- The sort symbols σ_p and σ_ℓ will indicate the sort of points and the sort of lines, respectively. We will use letters from the beginning of the alphabet like a, b, c to denote variables of sort σ_p, and letters from the end of the alphabet like x, y, z to denote variables of sort σ_ℓ.
- The predicate symbol $r(a, x)$ of arity $\sigma_p \times \sigma_\ell$ indicates that the point a lies on the line x.
- The predicate symbol $s(a, b, c)$ of arity $\sigma_p \times \sigma_p \times \sigma_p$ indicates that the points a, b and c are colinear.
- The predicate symbol $p(x, y)$ of arity $\sigma_\ell \times \sigma_\ell$ indicates that the lines x and y intersect.

[5]We follow Schwabhäuser et al. (1983) in this regard.

- Lastly, the predicate symbol $o(x, y, z)$ of arity $\sigma_\ell \times \sigma_\ell \times \sigma_\ell$ indicates that the lines x, y and z are compunctual, i.e. that they all intersect at a single point.

We now prove two theorems that capture the equivalence between geometries with points and geometries with lines. We then provide three examples that illustrate the generality of these results.

4.1 Two theorems

Suppose that we are given a formulation of geometry T that uses both of the sort symbols σ_p and σ_ℓ. The two theorems that we will prove in this section show that, given some natural assumptions, the theory T is Morita equivalent to a theory T_p that only uses the sort σ_p and to a theory T_ℓ that only uses the sort σ_ℓ. In this sense, therefore, the geometry T can be formulated using only points, only lines, or both points and lines.

Our first theorem captures a sense in which the geometry T can be formulated using only points. In order to prove this theorem, we will need the following important result. The proof of this proposition is given by Schwabhäuser et al. (1983, Proposition 4.59).

Proposition 1 (Elimination of line variables)**.** *Let T be a theory formulated in the signature $\Sigma = \{\sigma_p, \sigma_\ell, r, s\}$, and suppose that T entails the following sentences:*

1. $(a \neq b) \rightarrow \exists_{=1} x\, (r(a, x) \wedge r(b, x))$
2. $\forall x \exists a \exists b\, (r(a, x) \wedge r(b, x) \wedge (a \neq b))$
3. $s(a, b, c) \leftrightarrow \exists x\, (r(a, x) \wedge r(b, x) \wedge r(c, x))$

Then for every Σ-formula ϕ without free variables of sort σ_l, there is a Σ-formula ϕ^*, whose free variables are included in those of ϕ, that contains no variables of sort σ_ℓ, and such that $T \vDash \forall \vec{a}(\phi(\vec{a}) \leftrightarrow \phi^*(\vec{a}))$.

We should take a moment here to unravel the intuition behind this proposition. The theory T can be thought of as a geometry that is formulated in terms of points and lines, using the basic notions of a point lying on a line and three points being colinear. Since the theory T is a geometry, the sentences 1, 2, and 3 are sentences that one should naturally expect T to satisfy. Given these assumptions on T, Proposition 1 simply guarantees that Σ-formulas ϕ can be "translated" into corresponding formulas ϕ^* that do not use the apparatus of lines.[6] With this proposition in hand, we have the following result.

[6] This translation eliminates the line variables from every Σ-formula in two steps. First, one uses the fact that every line is uniquely characterized by two non-identical points lying on it to replace equalities between line variables with more complex expressions using the predicate r. Second, one replaces instances of the predicate $r(a, x)$ by using complex expressions involving the colinearity predicate $s(a, b, c)$. The reader is encouraged to consult Schwabhäuser et al. (1983, Proposition 4.59) for details.

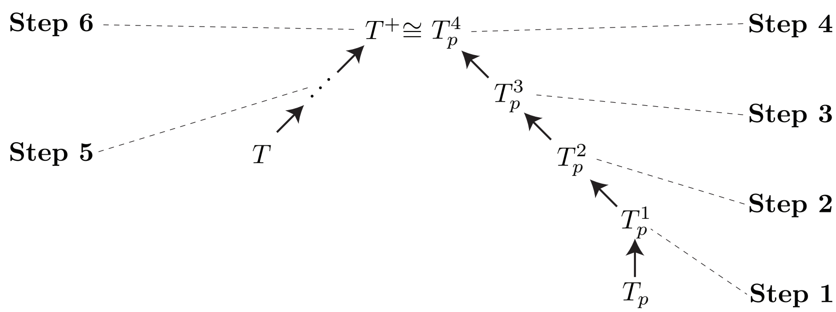
Theorem 1. *Let T be a theory that satisfies the hypotheses of Proposition 1. Then there is a theory T_p in the restricted signature $\Sigma_0 = \Sigma - \{\sigma_\ell, r\}$ that is Morita equivalent to T.*

Theorem 1 captures a sense in which every geometry that is formulated with points and lines could be formulated equally well using only points. The idea behind the proof of Theorem 1 should be clear. Consider the Σ_0-theory defined by

$$T_p = \{\phi^* : T \vDash \phi\},$$

where the existence of the sentences ϕ^* is guaranteed by the fact that T satisfies the hypotheses of Proposition 1. The theory T_p can be thought of as a theory that "says the same thing as T," but uses only the apparatus of points. One proves Theorem 1 by showing that this theory T_p has the resources to define the sort σ_ℓ of lines.[7]

Proof of Theorem 1. It suffices to show that the theories T and T_p are Morita equivalent. The following figure illustrates the structure of our argument:



We begin on the right-hand side of the figure by building four theories T_p^1, T_p^2, T_p^3, and T_p^4. The purpose of these theories is to define, using the resources of the theory T_p, the symbols σ_ℓ and r.

Step 1. The theory T_p^1 is the Morita extension of T_p obtained by defining a new sort symbol $\sigma_p \times \sigma_p$ as a product sort (of the sort σ_p with itself). We can think of the elements of the sort $\sigma_p \times \sigma_p$ as pairs of points. The theory T_p^1 is a Morita extension of T_p to the signature $\Sigma_0 \cup \{\sigma_p \times \sigma_p, \pi_1, \pi_2\}$, where π_1 and π_2 are both function symbols of arity $\sigma_p \times \sigma_p \to \sigma_p$.

Step 2. The theory T_p^2 is the Morita extension of T_p^1 obtained by defining a new sort symbol σ_s as a subsort of $\sigma_p \times \sigma_p$. The elements of sort σ_s are the elements (a, b) of sort $\sigma_p \times \sigma_p$ such that $a \neq b$. One can easily write out the defining formula for the subsort σ_s to guarantee that this is the case. We can think of the elements of sort σ_s as the pairs of distinct points, or more intuitively, as the "line segments formed between distinct points." The theory

[7] Note that in the following proof we abuse our convention and occasionally use the variables x, y, z as variables that are not of sort σ_ℓ. But the sort of variables should always be clear from context.

T_p^2 is a Morita extension of T_p^1 to the signature $\Sigma_0 \cup \{\sigma_p \times \sigma_p, \pi_1, \pi_2, \sigma_s, i\}$, where i is a function symbol of arity $\sigma_s \to \sigma_p \times \sigma_p$.

Step 3. The theory T_p^2 employs a sort of "line segments", but we do not yet have a sort of lines. Indeed, we need to take care of the fact that some line segments determine the same line. We do this by considering the theory T_p^3, the Morita extension of T_p^2 obtained by defining the sort symbol σ_ℓ as a quotient sort of σ_s using the formula

$$s(\pi_1 \circ i(x), \pi_1 \circ i(y), \pi_2 \circ i(y)) \wedge s(\pi_2 \circ i(x), \pi_1 \circ i(y), \pi_2 \circ i(y))$$

Using the fact that T is a conservative extension of T_p, one can easily verify that T_p^2 satisfies the admissibility conditions for this definition, i.e. the above formula is an equivalence relation according to T_p^2. The idea here is simple: Two line segments (a_1, a_2) and (b_1, b_2) determine the same line just in case the points a_1, b_1, b_2 are colinear and the points a_2, b_1, b_2 are too. The theory T_p^3 simply identifies the line segments that determine the same line in this sense. We have now defined the sort σ_ℓ of lines. The theory T_p^3 is a Morita extension of T_p^2 to the signature $\Sigma_0 \cup \{\sigma_p \times \sigma_p, \pi_1, \pi_2, \sigma_s, i, \sigma_\ell, \epsilon\}$, where ϵ is a function symbol of sort $\sigma_s \to \sigma_\ell$.

Step 4. All that remains on the right-hand side of the figure is to define the predicate symbol r. The theory T_p^4 is the Morita extension of T_p^3 obtained by defining the predicate $r(a, z)$ using the formula

$$\exists_{\sigma_p \times \sigma_p} x \exists_{\sigma_s} y (\pi_1(x) = a \wedge i(y) = x \wedge \epsilon(y) = z)$$

The idea here is again intuitive. A point a is on a line z just in case there is another point b such that the pair of points (a, b) determines the line l. (In the above formula, one can think of the variable x as playing the role of this pair (a, b).) The theory T_p^4 is a Morita extension of T_p^3 to the signature $\Sigma_0 \cup \{\sigma_p \times \sigma_p, \pi_1, \pi_2, \sigma_s, i, \sigma_\ell, \epsilon, r\}$.

Step 5. We now turn to the left-hand side of our organizational figure. The theory T is formulated in the signature Σ, so it needs to define all of the new symbols that we added to the theory T_p in the course of defining σ_p and r. The theory T defines the symbols $\sigma_p \times \sigma_p, \pi_1, \pi_2, \sigma_s, i$ in the obvious manner. For example, it defines $\sigma_p \times \sigma_p$ as the product sort (of σ_p with itself) with the projections π_1 and π_2.

We still need, however, to define the function symbol ϵ. The function ϵ intuitively maps a pair of distinct points to the line that they determine. This suggests that we define $\epsilon(x) = y$ using the formula

$$r(\pi_1 \circ i(x), y) \wedge r(\pi_2 \circ i(x), y)$$

Intuitively, this formula is saying that a pair of points $x = (x_1, x_2)$ determines a line y just in case x_1 is on y and x_2 is on y. We call the theory that results from defining all of these symbols T^+.

Step 6. All that remains now is to show that the theory T_p^4 is logically equivalent to the theory T^+. This argument is mainly a tedious verification.

The only non-trivial part of the argument is the following: One needs to show that $T_p^4 \vDash \phi$ for every sentence ϕ such that $T \vDash \phi$. One does this by verifying that T_p^4 itself entails the three sentences 1, 2, and 3 in the statement of Proposition 1. This means that T_p^4 entails the sentences $\phi \leftrightarrow \phi^*$ for every Σ-sentence ϕ. In conjunction with the fact that $T_p^4 \vDash \phi^*$ for every consequence ϕ of T, this implies that $T_p^4 \vDash \phi$. The theories T_p^4 and T^+ are logically equivalent, so T_p and T must be Morita equivalent. □

Our second theorem is perfectly analogous to Theorem 1. It captures a sense in which a geometry T can be formulated using only lines. As with Theorem 1, we will need a preliminary result. The proof of the following proposition is given by Schwabhäuser et al. (1983, Proposition 4.89).

Proposition 2 (Elimination of point variables). *Let T be a theory formulated in the signature $\Sigma = \{\sigma_p, \sigma_\ell, r, p, o\}$, and suppose that T implies the following sentences:*

1. $(x \neq y) \rightarrow \exists_{\leq 1} a(r(a, x) \wedge r(a, y))$
2. $\forall a \exists x \exists y((x \neq y) \wedge r(a, x) \wedge r(a, y))$
3. $o(x, y, z) \leftrightarrow \exists a(r(a, x) \wedge r(a, y) \wedge r(a, z))$
4. $p(x, y) \leftrightarrow ((x \neq y) \wedge s(x, y, y))$
5. $p(x, y) \leftrightarrow ((x \neq y) \wedge \exists a(r(a, x) \wedge r(a, y)))$

Then for every Σ-formula ϕ without free variables of sort σ_p, there is a Σ-formula ϕ^*, whose free variables are included in those of ϕ, that contains no variables of sort σ_p, and such that $T \vDash \forall \vec{x}(\phi(\vec{x}) \leftrightarrow \phi^*(\vec{x}))$.

Proposition 2 is perfectly analogous to Proposition 1. One again thinks of the theory T as a geometry, and so the sentences 1–5 are sentences that one naturally expects T to satisfy. Proposition 2 guarantees that Σ-formulas can be "translated" into formulas ϕ^* that do not use the apparatus of points.[8] With Proposition 2 in hand, we have the following result.

Theorem 2. *Let T be a theory that satisfies the hypotheses of Proposition 2. There is a theory T_ℓ in the restricted signature $\Sigma_0 = \Sigma - \{\sigma_p, r\}$ that is Morita equivalent to T.*

Proof. The proof is analogous to the proof of Theorem 1, so we will not go into as much detail. Consider the Σ_0-theory T_ℓ defined by $T_\ell = \{\phi^* : T \vDash \phi\}$, where the existence of the sentences ϕ^* is guaranteed since T satisfies the hypotheses of Proposition 2. One shows that the theory T_ℓ is Morita equivalent to T. The

[8] Analogous to Proposition 1, one proves this proposition by showing that variables of sort σ_p can be eliminated in the following manner. One first replaces equalities between these variables, and then interprets $r(a, x)$ in terms of $o(y, z, x)$, where y and z have a as their intersection point. The reader is invited to consult Schwabhäuser et al. (1983, Proposition 4.89) for further details.

theory T_ℓ needs to define the sort symbol σ_p. It does this by first defining a product sort of "pairs of lines," and then a subsort of "pairs of intersecting lines." The sort of points is then the quotient sort that results from identifying two pairs of intersecting lines (w, x) and (y, z) just in case both w, x, y and w, x, z are compunctual. The theory T_ℓ also needs to define the symbol r. It does this simply by requiring that $r(a, x)$ holds of a point a and a line x just in case there is another line y such that the pair of lines (x, y) intersect at the point a. As in the proof of Theorem 1, T defines the symbols of T_ℓ in the natural way. □

4.2 Three examples

Theorem 1 shows that every geometry formulated using points and lines could be formulated equally well using only points; Theorem 2 shows that it could be formulated equally well using only lines. These two results together capture a robust sense in which geometries with points and geometries with lines are equivalent theories.

Theorems 1 and 2 are quite general. Indeed, one can verify that many of the theories that we usually think of as geometries satisfy the hypotheses of the two theorems. We provide three examples here. We begin by revisiting a simple geometric theory that we considered earlier.

Example 1. Recall the above diagram of six lines and five points in the Euclidean plane. By interpreting the symbols $\sigma_p, \sigma_\ell, r, s, p$, and o in the natural way, one can easily convert this diagram into a $\{\sigma_p, \sigma_\ell, r, s, p, o\}$-structure M. We now consider the geometric theory $\text{Th}(M) = \{\phi : M \vDash \phi\}$. One can verify by inspection that $\text{Th}(M)$ satisfies the hypotheses of both Theorems 1 and 2. Theorem 1 implies that this diagram can be fully described using only the apparatus of points, while Theorem 2 implies that it can be fully described using only the apparatus of lines. ⌟

In our next two examples, we consider more general geometric theories: Projective geometry and affine geometry.

Example 2 (Projective geometry)**.** Projective geometry is a theory T_{proj} formulated in the signature $\{\sigma_p, \sigma_\ell, r\}$, where all of these symbols are understood exactly as above. The theory T_{proj} has the following three axioms (Barnes and Mack, 1975).

- $a \neq b \rightarrow \exists_{=1} x(r(a, x) \wedge r(b, x))$
- $x \neq y \rightarrow \exists_{=1} a(r(a, x) \wedge r(a, y)$
- There are at least four points, no three of which lie on the same line.

(One can easily express the third axiom as a sentence of first-order logic, but we here refrain for the sake of clarity.)

Projective geometry satisfies the hypotheses of both Theorems 1 and 2. We consider Theorem 2. In order to apply this result, we need to add the following two axioms that define the symbols p and o:

$$p(x,y) \leftrightarrow (x \neq y \wedge \exists a(r(a,x) \wedge r(a,y))) \qquad (\theta_p)$$

$$o(x,y,z) \leftrightarrow \exists a(r(a,x) \wedge r(a,y) \wedge r(a,z)) \qquad (\theta_o)$$

One can easily verify that the $\{\sigma_\ell, \sigma_p, r, p, o\}$-theory T^+_{proj} obtained by adding the definitions θ_p and θ_o to the axioms of T_{proj} satisfies the sentences 1–5 of Proposition 2. Theorem 2 then implies that there is a theory in the restricted signature $\{\sigma_\ell, p, c\}$ that is Morita equivalent to T^+_{proj}. Projective geometry can therefore be formulated using only the apparatus of lines. One argues in a perfectly analogous manner to show that Theorem 1 also applies to projective geometry, so it can also be formulated using only the apparatus of points. ⌟

Example 3 (Affine geometry)**.** Affine geometry is a theory T_{aff} formulated in the signature $\{\sigma_\ell, \sigma_p, r\}$, where all of these symbols are again understood exactly as above. The theory T_{aff} has the following five axioms (Veblen and Young, 1918, p. 118).

- $a \neq b \rightarrow \exists x(r(a,x) \wedge r(b,x))$
- $\neg r(a,x) \rightarrow \exists_{=1} y(r(a,y) \wedge \forall b(r(b,y) \rightarrow \neg r(b,x)))$
- $\forall x \exists a \exists b(a \neq b \wedge r(a,x) \wedge r(b,x))$
- $\exists a \exists b \exists c(a \neq b \wedge a \neq c \wedge b \neq c \wedge \neg \exists x(r(a,x) \wedge r(b,x) \wedge r(c,x)))$
- Pappus' theorem (Veblen and Young, 1918, p. 103 and Figure 40).

The fifth axiom can easily be written as a first-order sentence in the signature $\{\sigma_\ell, \sigma_p, r\}$, but since this axiom is not used in the following argument, we leave its translation to the reader. (Indeed, one only needs the first, third, and fourth axioms of T_{aff} to complete all of the following verifications.)

Affine geometry satisfies the hypotheses of both Theorems 1 and 2. We consider Theorem 1. In order to apply this result, we need to add one additional axiom to T_{aff} that defines the symbol s as follows:

$$s(a,b,c) \leftrightarrow \exists x(r(a,x) \wedge r(b,x) \wedge r(c,x)) \qquad (\theta_s)$$

It is now trivial to verify that the sentences 1–3 of Proposition 1 are satisfied by the $\{\sigma_\ell, \sigma_p, r, s\}$-theory T^+_{aff} that is obtained by adding the sentence θ_s to the axioms of T_{aff}. Theorem 1 therefore implies that there is a theory in the restricted signature $\{\sigma_p, s\}$ that is Morita equivalent to T^+_{aff}, capturing a sense in which affine geometry can be formulated using only the apparatus of points. In a perfectly analogous manner, one can apply the Theorem 2 to the case of affine geometry. This captures a sense in which affine geometry can also be formulated using only lines. ⌟

Example 3 is more general than it might initially appear. Indeed, affine geometry serves as the foundation for many of our most familiar geometries. For example, by supplementing the affine geometry with the proper notion of orthogonality, one can obtain two dimensional Euclidean geometry or two dimensional Minkowski geometry.[9] Theorems 1 and 2 therefore capture a sense in which both Euclidean geometry and Minkowski geometry can be formulated using either points or lines.

5 Morita equivalence and ontological commitment

Our discussion has shown that a geometric theory can be formulated using points or using lines, and furthermore, that these two formulations are perfectly equivalent. Putnam himself clearly agrees that point-based and line-based geometries are equivalent.[10] He must therefore be committed to a standard of equivalence that is more liberal than both logical equivalence and Glymour's definitional equivalence. We suggest that Morita equivalence is, in fact, a reasonable standard of equivalence, and one that Putnam should be willing to adopt.

One might worry, however, that Morita equivalence is too liberal. In particular, one might worry that Morita equivalence begs the question in favor of conceptual relativism, since it allows new sorts (and hence new objects) to be defined out of old ones. We do not find this worry to be particularly troublesome. In fact, there is a sense in which Putnam goes wrong when he suggests that the two geometric descriptions (in terms of points and lines, respectively) involve different ontological commitments. If T and T' are Morita equivalent theories, then there is a natural way to view them according to which they make precisely the same ontological commitments.

It will suffice to restrict our attention to the specific case of Morita equivalence where T' is a Morita extension of T. Suppose that the theory T is formulated in a signature Σ with two sort symbols σ and τ, and that T' is a Morita extension of T to the signature $\Sigma' \supset \Sigma$. The theory T' might therefore add to T the following new sorts: subsorts of σ, the product $\sigma \times \tau$, the coproduct $\sigma + \tau$, or quotients of equivalence classes of elements of σ.[11] We will argue that, despite the fact that T' introduces these new sort symbols, there is a natural sense in which the ontological commitments of T' do not exceed those of T. Indeed, one can easily view these new sorts of objects as *logical constructions* (in the sense of Russell) from the objects of sorts σ and τ.

We show this by describing how each of the new quantifier phrases "There is a thing of the new sort ..." in the language of T' exactly paraphrases (or ab-

[9] See Coxeter (1955), Szczerba and Tarski (1979), and Szczerba (1986, p. 910) for details.

[10] For example, see Putnam (1977, 489-91), Putnam (1992, 109, 115-20), and Putnam (2001).

[11] T' could also add product and coproduct sorts $\sigma \times \sigma$, $\tau \times \tau$, $\sigma + \sigma$, and $\tau + \tau$, along with subsorts and quotient sorts of τ. We omit discussion of these cases, however, since they are essentially the same as the ones that we do discuss.

breviates, or is shorthand for) an old quantifier phrase "There are things of the old sorts ..." in the language of T. Adding these new sort symbols and associated quantifiers to the theory T, therefore, does not increase one's ontological commitments. Rather, it is just a way of making more explicit the ontological commitments of the original theory T. Precisely how this paraphrasing works depends on what kind of "new sort" appears in the new quantifier phrase, so we consider the four cases in detail.[12]

Subsorts. The first case is straightforward. Let ϕ be a Σ-formula and suppose that T' defines a new sort symbol σ_ϕ as the subsort containing those objects of sort σ that are ϕ. Consider the quantifier phrase "There is a thing of sort σ_ϕ that is ψ" in the language of the theory T'. This expression can naturally be thought of as paraphrasing the expression "There is a thing of sort σ that is ϕ and ψ" in the restricted language of the theory T. In fact, there is a way to make this intuition perfectly precise. If $\psi(x)$ is a Σ'-formula with the variable x of sort σ_ϕ, then one can show that

$$T' \vDash \exists_{\sigma_\phi} x\, \psi(x) \leftrightarrow \exists_\sigma y\, (\psi^*(y) \wedge \phi(y)),$$

where here ψ^* is a particular Σ-formula that can be thought of as the "translation" of the Σ'-formula ψ. This shows that the new existential quantifier $\exists_{\sigma_\phi}$ that accompanies the new sort σ_ϕ can be defined in terms of the existential quantifier $\exists_\sigma$ that was "already there" in the theory T. Introducing the relativized quantifier $\exists_{\sigma_\phi}$ is simply a convenient shorthand. Indeed, the new sort σ_ϕ does not contain "new objects" that are independent of the old objects. It instead just provides us with a new way of talking about some of the objects of sort σ, namely those objects that are ϕ.

Product sorts. The same idea holds in the case where T' adds a new product sort $\sigma \times \tau$. The theory T implicitly quantifies over objects of sort $\sigma \times \tau$, and all expressions about objects of sort $\sigma \times \tau$ can be translated into expressions about objects of the corresponding individual sorts σ and τ. In particular, the new quantifier phrase "There is a thing of sort $\sigma \times \tau$ that is ψ" (in the language of T') can be thought of as paraphrasing the old quantifier phrase "There is one thing of sort σ and another of sort τ such that the pair is ψ" (in the language of T). One makes this idea precise in exactly the same manner as above. If $\psi(x)$ is a Σ'-formula with the variable x of sort $\sigma \times \tau$, then one can show that

$$T' \vDash \exists_{\sigma\times\tau} x\, \psi(x) \leftrightarrow \exists_\sigma y_1 \exists_\tau y_2\, \psi^*(y_1, y_2),$$

where again ψ^* is a Σ-formula that is the "translation" of ψ into the language of the theory T. All quantifier phrases in the new language of T' can be thought of as simply paraphrasing more complex quantifier phrases in the old language

[12] We are here simply unraveling the idea behind Barrett and Halvorson (2015b, Theorem 4.3). Indeed, all of the following offset equations are simple corollaries to this theorem. The reader is encouraged to consult the result and its proof for additional details.

of the theory T. Indeed, we have just seen how the new existential quantifier $\exists_{\sigma\times\tau}$ can be defined in terms of the existential quantifiers $\exists_\sigma$ and $\exists_\tau$ from the language of the theory T.

As a specific example, suppose that T is a theory about persons, marriage, and net income. Suppose in particular that T tells us which people are married, and what each individual's net income is. Without expanding one's ontology in the slightest bit, one can easily extend T to a theory T' that includes a sort σ' for married couples, and a predicate of sort σ' that applies to those married couples whose joint income is greater than \$100,000 per year. This new theory T' may be more convenient for calculating tax debts, but it does not say anything more than the original theory T.[13]

Coproduct sorts. The situation is again the same when T' adds a coproduct sort $\sigma+\tau$. Introducing coproduct sorts simply allows one to unify distinct domains of objects into a common domain. If T is a theory about two different kinds of things, of sorts σ and τ, then one can extend T to a theory T' built on the "common sort" $\sigma+\tau$. Once again, all expressions about objects of sort $\sigma+\tau$ can be understood as shorthand for expressions about the objects of sorts σ and τ. In particular, the quantifier phrase "There exists something of sort $\sigma+\tau$ that is ψ" (in the language of T') paraphrases the expression "Either there is something of sort σ that is ψ or there is something of sort τ that is ψ" (in the language of T). One makes this thought precise in the now familiar way. If $\psi(x)$ is a Σ^+-formula with x a variable of sort $\sigma+\tau$, then one can show that the new existential quantifier $\exists_{\sigma+\tau}$ can be defined in terms of the quantifiers $\exists_\sigma$ and $\exists_\tau$ in the following manner:

$$T' \vDash \exists_{\sigma+\tau} x\, \psi(x) \leftrightarrow \big(\exists_\sigma y_1\, \psi^*(y_1) \vee \exists_\tau y_2 \psi^{**}(y_2)\big),$$

where again the Σ-formulas ψ^* and ψ^{**} can be thought of as the "translations" of ψ into the signature Σ.

Quotient sorts. This final case might seem the most controversial, since introducing quotient sorts seems to allow one to quantify over classes. This maneuver is permitted, however, because the equivalence classes under consideration are always required to be definable using the resources of first-order logic. Suppose that T' adds a new quotient sort σ_θ, i.e. the sort of equivalence classes (of things of sort σ) with respect to the equivalence relation θ. In this case the quantifier phrase "There exists something of sort σ_θ that is ψ" (in the language of T') can be understood as paraphrasing the more complex expression "There exists something of sort σ such that it, and everything θ-related to it, is ψ" (in

[13]A naive metaphysical realist might be tempted to argue that despite the fact that we can add a sort $\sigma\times\tau$, there remains a sense in which objects of sort $\sigma\times\tau$ are derived entities — i.e. they depend on objects of sorts σ and τ. In the case of geometric theories, this response misses the mark. The upshot of our results above is that lines can be defined as (equivalence classes of) pairs of points, and points can be defined as (equivalence classes of) pairs of lines. Thus, there is no clear sense in which the line sort is "derived" and the point sort is "fundamental", or vice versa.

the language of T). One makes this precise again by showing that if $\psi(x)$ is a Σ'-formula with x a variable of sort σ_θ, then

$$T' \vDash \exists_{\sigma_\theta} x\, \psi(x) \leftrightarrow \exists_\sigma y \left(\psi^*(y) \wedge \forall_\sigma z(\theta(y,z) \to \psi^*(z))\right),$$

where again the Σ-formula ψ^* results from "translating" ψ into the signature Σ. As in all of the above cases, this captures a precise sense in which the new existential quantifier $\exists_{\sigma_\theta}$ can be defined in terms of the old existential quantifier $\exists_\sigma$.

In each of these cases, we have described how the new existential quantifier phrases of T' merely paraphrase more complex existential quantifier phrases of T. (It is easy to convince oneself that the same holds for the universal quantifier phrases of T'.) In this sense, therefore, one need not think of the quantifiers $\exists_{\sigma_\phi}$, $\exists_{\sigma\times\tau}$, $\exists_{\sigma+\tau}$, and $\exists_{\sigma_\theta}$ as "new" to the theory T'. Rather, there is a sense in which they were implicitly there in the theory T to begin with. The theory T' does allow one to use these new symbols $\exists_{\sigma_\phi}$, $\exists_{\sigma\times\tau}$, $\exists_{\sigma+\tau}$, and $\exists_{\sigma_\theta}$, but this does not increase the expressive power of T' over that of T. The two theories allow one to "quantify over" precisely the same things; they simply use different languages to do so.

This discussion allows us to recognize a natural sense in which moving from the theory T to its Morita extension T' does not increase ontological commitments. Indeed, if one is inclined to think that the ontological commitments of a theory can be "read off" from what the theory quantifies over, then T' and T make precisely the same ontological commitments.

6 Reconsidering conceptual relativity

Putnam and Goodman often cite the case of geometry with points T_p and geometry with lines T_ℓ as an example of "incompatible" but "equally correct" theories.[14] For example, Putnam (1992, 115-6) describes Goodman's view as follows:

> Goodman regards these two versions of [geometry] as 'incompatible'. At the same time, he regards them as both right. And since incompatible versions cannot be true of the same world, he concludes that they are true of different worlds 'if true of any.'

In their argument from geometry, Putnam and Goodman claim that the existence of incompatible but equally correct theories like T_p and T_ℓ forces one to abandon metaphysical realism. Since the two theories are incompatible, metaphysical realism would require that at most one of them can provide a correct description of the world. But the two theories are *equally* correct, so metaphysical realism itself must be false.

[14]See Putnam (1977, 489-91), Putnam (1992, 109, 115-20), Putnam (2001), and Goodman (1975, 1978). For further discussion of these arguments see Wright (1997, 300) and Cohnitz and Rossberg (2014, 216).

We disagree with the claim that T_p and T_ℓ are incompatible. To the contrary, the reason for supposing that T_p and T_ℓ are equally correct is because they *are* compatible, in the following precise sense implied by Theorems 1 and 2:

$$T_p \text{ and } T_\ell \text{ have a common Morita extension } T^+.$$

This common Morita extension T^+ is a theory that quantifies over *both* points *and* lines. The theories T_p and T_ℓ are simply convenient ways of expressing the geometric facts that are more fully expressed by the comprehensive theory T^+. The argument from geometry was based from the start on misconceptions about why we should consider these theories to be "equally correct." These theories are equally correct because they are equivalent. Far from being incompatible, the two are actually *the same theory*.

This leads us to a concluding remark. In order to recognize the sense in which geometries with points and geometries with lines are equivalent, one relies on the resources of many-sorted logic, and in particular, the notion of Morita equivalence. The many-sorted framework has, unfortunately, been mostly ignored by philosophers during the last half-century. This attitude towards many-sorted logic can be traced to an argument of Quine's. Quine famously suggested that the many-sorted framework was dispensible, and that philosophers were licensed to ignore it altogether.[15] His idea was that many-sorted logic and single-sorted logic are "intertranslatable," so nothing of real philosophical significance could turn on the use of one of the frameworks rather than the other.

Although there is a precise sense in which the many-sorted and single-sorted frameworks are intertranslatable (Barrett and Halvorson, 2015c, Theorem 2), our results here demonstrate that Quine was mistaken about the upshot of this logical fact. It does not imply that many-sorted logic is dispensible. As our discussion of geometric theories shows, the many-sorted framework allows us to better recognize different standards of equivalence between theories. The single-sorted framework does not allow one to capture, for example, any sense in which Euclidean geometry with points is the same as Euclidean geometry with lines. But with Morita equivalence in hand, the many-sorted framework does allow one to capture the equivalence between point-based and line-based geometric theories. When Quine argues that nothing of philosophical significance turns on the use of many-sorted as opposed to the standard single-sorted framework, therefore, he is mistaken. We ignore sorts at our own peril. By doing so, we blind ourselves to the variety of ways in which theories in different signatures can be equivalent.

[15]Quine (1960, 229) says: "All in all, I find an overwhelming case for a single unpartitioned universe of values of bound variables, and a simple grammar of predication which admits general terms all on an equal footing. Subsidiary distinctions can still be drawn as one pleases, both on methodological considerations and on considerations of natural kind; but we may think of them as distinctions special to the sciences and unreflected in the structure of our notation." For other expressions of this same attitude, see Quine (1951, 69–71), Quine (1963, 267–8), and Quine and Carnap (1990, 409).

References

Andréka, H., Madaraśz, J., and Németi, I. (2008). Defining new universes in many-sorted logic. *Manuscript*.

Andréka, H., Madarász, J. X., and Németi, I. (2005). Mutual definability does not imply definitional equivalence, a simple example. *Mathematical Logic Quarterly*.

Andréka, H. and Németi, I. (2014). Comparing theories: the dynamics of changing vocabulary. A case-study in relativity theory. In *Johan van Benthem on logic and information dynamics*. Springer.

Barnes, D. W. and Mack, J. M. (1975). *An Algebraic Introduction to Mathematical Logic*. Springer-Verlag.

Barrett, T. W. and Halvorson, H. (2015a). Glymour and Quine on theoretical equivalence. *Forthcoming in the Journal of Philosophical Logic*.

Barrett, T. W. and Halvorson, H. (2015b). Morita equivalence. *Manuscript.*, URL: http://philsci-archive.pitt.edu/id/eprint/11511.

Barrett, T. W. and Halvorson, H. (2015c). Quine's conjecture on many-sorted logic. *Manuscript*, URL: http://philsci-archive.pitt.edu/11660/.

Beth, E. and Tarski, A. (1956). Equilaterality as the only primitive notion of Euclidean geometry. *Indagationes mathematicae*, 18.

Cohnitz, D. and Rossberg, M. (2014). *Nelson Goodman*. Routledge.

Coxeter, H. S. M. (1955). The affine plane. *Scripta Mathematica*, 21.

de Bouvére, K. L. (1965). Synonymous theories. In *Symposium on the Theory of Models*. North-Holland Publishing Company.

Friedman, H. and Visser, A. (2014). When bi-interpretability implies synonymy. *Manuscript*.

Glymour, C. (1970). Theoretical realism and theoretical equivalence. In *PSA: Proceedings of the Biennial Meeting of the Philosophy of Science Association*.

Glymour, C. (1977). The epistemology of geometry. *Nous*.

Glymour, C. (1980). *Theory and Evidence*. Princeton.

Goodman, N. (1975). Words, works, worlds. *Erkenntnis*, 9(1):57–73.

Goodman, N. (1978). *Ways of Worldmaking*. Hackett.

Hilbert, D. (1930). *Grundlagen der Geometrie*. Teubner.

Hodges, W. (2008). *Model Theory*. Cambridge University Press.

Kanger, S. (1968). Equivalent theories. *Theoria*.

Meré, M. C. and Veloso, P. A. S. (1992). On extensions by sorts. *Monografias em Ciência da Computação*.

Pelletier, F. J. and Urquhart, A. (2003). Synonymous logics. *Journal of Philosophical Logic*.

Pinter, C. C. (1978). Properties preserved under definitional equivalence and interpretations. *Mathematical Logic Quarterly*.

Putnam, H. (1977). Realism and reason. *Proceedings and Addresses of the American Philosophical Association*, 50(6).

Putnam, H. (1992). *Renewing Philosophy*. Harvard University Press.

Putnam, H. (2001). Reply to Jennifer Case. *Revue internationale de philosophie*, 4(218).

Quine, W. V. O. (1951). On Carnap's views on ontology. *Philosophical Studies*.

Quine, W. V. O. (1960). *Word and Object*. MIT.

Quine, W. V. O. (1963). *Set Theory and Its Logic*. Harvard University Press.

Quine, W. V. O. (1975). On empirically equivalent systems of the world. *Erkenntnis*.

Quine, W. V. O. and Carnap, R. (1990). *Dear Carnap, Dear Van: The Quine-Carnap Correspondence and Related Work*. Berkeley, CA: University of California Press.

Robinson, R. (1959). Binary relations as primitive notions in elementary geometry. In *The Axiomatic Method With Special Reference to Geometry and Physics*. North-Holland.

Royden, H. L. (1959). Remarks on primitive notions for elementary Euclidean and non-Euclidean plane geometry. In *The Axiomatic Method With Special Reference to Geometry and Physics*. North-Holland.

Schwabhäuser, W. and Szczerba, L. (1975). Relations on lines as primitive notions for Euclidean geometry. *Fundamenta Mathematicae*.

Schwabhäuser, W., Szmielew, W., and Tarski, A. (1983). *Metamathematische Methoden in der Geometrie*. Springer.

Scott, D. (1956). A symmetric primitive notion for Euclidean geometry. *Indagationes mathematicae*, 18.

Sider, T. (2009). Ontological realism. In Chalmers, D., Manley, D., and Wasserman, R., editors, *Metametaphysics*, pages 384–423. Oxford University Press, New York.

Szczerba, L. (1977). Interpretability of elementary theories. In *Logic, Foundations of Mathematics, and Computability Theory*. Springer.

Szczerba, L. (1986). Tarski and geometry. *The Journal of Symbolic Logic*, 51(4).

Szczerba, L. and Tarski, A. (1979). Metamathematical discussion of some affine geometries. *Fundamenta Mathematicae*.

Tarski, A. (1929). Les fondements de la geometrie des corps. *Ksiega Pamiatkowa Pierwszego Polskiego Zjazdu Matematyków*.

Tarski, A. (1956). A general theorem concerning primitive notions of Euclidean geometry. *Indagationes mathematicae*, 18.

Tarski, A. (1959). What is elementary geometry? In *The Axiomatic Method With Special Reference to Geometry and Physics*. North-Holland.

Van Inwagen, P. (2009). Being, existence, and ontological commitment. In Chalmers, D., Manley, D., and Wasserman, R., editors, *Metametaphysics*, pages 472–506. Oxford University Press, New York.

Veblen, O. and Young, J. W. (1918). *Projective Geometry Volume 2*. Ginn and Company.

Wright, J. (1997). *Realism and Explanatory Priority*. Springer.