Pocklington Equation via Circuit Theory

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In this paper it is shown a circuit-theory approach for the integral equation for thin wire antennas, from which Pocklington's equation can be deduced as a special case. In this way, when solving it via method of moments, impedance, current and voltage matrix acquire meaning [1]. It is shown that a thin wire can be considered as an infinite-port electric network, in which the goal consists in finding out the current in each port. The following approach is based on Aharoni's theory [2].

Keywords: Circuit theory, complex inductance, complex capacitance, Pocklington's integral equation.

Introduction

Circuit Theory is a fundamental area in Electrical Engineering, since lets determine currents as well as voltages in electrical networks, distinguished by their concentrated parameters: resistances, inductances and capacitances. However, just as it is, theory is just a correct approximation under certain conditions [3].

Circuit Theory can provide several analyses in which networks are represented by black-boxes, characterized by their transfer functions, such that for a given excitation a response is gotten. Generally, transfer function is secured by Laplace Transform, presenting a complex function $H(s)$ of one complex variable $s = \sigma + j\omega$, where σ is the attenuation coefficient (it is due because $\sigma < 0$ generally) and ω is the signal's angular frequency [4].

Another representation is secured by Fourier Transform, obtaining a complex function $H(\omega)$, which provides information for harmonic excitations. Since an input signal can be modeled with harmonic ones, in agreement with Fourier's theorem, $H(\omega)$ can be used for getting the output for any harmonic input [5]. Hence, harmonic case is important in Electrical Engineering not only for its simplicity and ease of use, but because almost signals, fields, sources and charges vary harmonically in time.

Circuit Theory's exactness is established upon network's dimensions regarding the wavelength. If it is large enough, current and voltage in any network branch could be assumed changeless along it. For instance, an electronic circuit working at $10\ \text{KHz}$, has a wavelength of $30\ \text{Km}$ approximately, which is much larger than any electronic device's dimensions. The set of equations which represents a network in the harmonic case is:

$$\sum_{l=1}^{n}\left(j\omega L_{kl} + \frac{1}{j\omega C_{kl}} + R_{kl} \right) I_l = V_k \ , \qquad k = 1, 2, \cdots, n \ , \tag{1}$$

where n is the number of meshes, L_{kl}, C_{kl} and R_{kl} are the mutual inductances, capacitances and resistances between k-th and l-th meshes, respectively, V_k is the voltage in k-th mesh and I_l, the unknown, is the circulating current in l-th mesh [6].

Network eqs. (1) satisfy Kirchhoff's voltage and current laws in each mesh and node, respectively:

$$\sum V_i = 0 \ , \qquad \sum I_i = 0 \ . \tag{2}$$

However, as the network dimensions are nearly comparable with wavelength, electromagnetic induction occurs along the meshes and nodes, transforming eq. (2) as:

$$\sum V_i = -j\omega L_s I \ , \qquad \sum I_i = -j\omega C_s V \ , \tag{3}$$

where L_s is the inductance mesh and I its circulating current, and C_s the capacitance node and V its voltage.

As the network dimensions are equal or higher than wavelength, electric parameters can not be considered concentrated in certain points but distributed along the network. Therefore, for finding out current and voltage distributions, Maxwell equations must be solved for certain boundary conditions, which lead to an integral equation for the current. As a special case, this integral equation corresponds to Pocklington's one for thin wires, used frequently in antennas.

The Electromagnetic Field

Electromagnetic field obeys Maxwell equations, which in differential form in the harmonic case are:

$$\nabla \times \mathbf{E} = -j\omega\mathbf{B}\,, \qquad \nabla \times \mathbf{H} = j\omega\mathbf{D} + \mathbf{J}\,,$$
$$\nabla \bullet \mathbf{D} = \rho\,, \qquad \nabla \bullet \mathbf{B} = 0\,, \tag{4}$$

where sources ρ and $\mathbf{J}$ satisfy a continuity equation:

$$\nabla \bullet \mathbf{J} = -j\omega\rho\,. \tag{5}$$

For solving eqs. (4), a pair of potential functions satisfying certain wave equations, are defined. They are referred as retarded potentials, and are, strictly speaking, mathematical tools without physical meaning, which let simplify the problem's solution. The functions are the magnetic vector potential $\mathbf{A}$ and the electric scalar potential Φ, defined by:

$$\mu\mathbf{H} = \nabla \times \mathbf{A}\,, \qquad \mathbf{E} = -\nabla\Phi - j\omega\mathbf{A}\,. \tag{6}$$

Accordingly to Helmhöltz's theorem, not only $\nabla \times \mathbf{A}$ should be defined, but also $\nabla \bullet \mathbf{A}$ should be established for setting up $\mathbf{A}$ in a unique way [7]. This is reached by means Lorenz gauge:

$$\nabla \bullet \mathbf{A} = -j\omega\mu\varepsilon\Phi\,, \tag{7}$$

which also sets up Φ in a unique way. Therefore, when substituting potentials in Maxwell equations, a pair of relations represents the electromagnetic field as a wave:

$$\nabla^2\mathbf{A} + \omega^2\mu\varepsilon\mathbf{A} = -\mu\mathbf{J}\,,$$
$$\nabla^2\Phi + \omega^2\mu\varepsilon\Phi = -\rho/\varepsilon\,, \tag{8}$$

whose solutions, gotten by Green's function technique, satisfy radiation condition at infinitum:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\iiint_v \frac{\mathbf{J}(\mathbf{r}')e^{-jkR}}{R}dv', \qquad \Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon}\iiint_v \frac{\rho(\mathbf{r}')e^{-jkR}}{R}dv', \tag{9}$$

where $\mathbf{r}'$ is the source point and $\mathbf{r}$ the field point, as shown in Fig. 1.

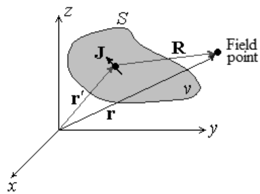

Figure 1. Source of the electromagnetic field.

In wireless communication systems, electromagnetic fields transport energy and information faraway the sources by means waving processes. For supporting them, generators are used for transferring energy between system's ends. In practice, since it is impossible to consider their detailed nature, generators are modeled by a field of force $\mathbf{F}$, whose dimensions are those of $\mathbf{E}$, and whose labor is impressing an electric current $\mathbf{J}_0 = \sigma\mathbf{F}$ which must be added to Ampere-Maxwell equation:

$$\nabla\times\mathbf{H} = \sigma(\mathbf{E}+\mathbf{F}) + j\omega\varepsilon\mathbf{E}, \qquad \sigma(\mathbf{E}+\mathbf{F}) = \mathbf{J}. \tag{10}$$

From Maxwell equations it is possible to obtain a relation which establishes an energetic balance for the electromagnetic field in the domain v. Such relation, conveniently called the complex energy equation, is:

$$\oiint_S \left(\frac{1}{2}\mathbf{E}\times\mathbf{H}^*\right)\bullet d\mathbf{a} + j2\omega\iiint_v \left(\frac{1}{4}\mu\left|\mathbf{H}\right|^2 - \frac{1}{4}\varepsilon\left|\mathbf{E}\right|^2\right)dv$$
$$+\iiint_v \frac{\left|\mathbf{J}\right|^2}{2\sigma}dv = \iiint_v \frac{1}{2}\mathbf{F}\bullet\mathbf{J}^* dv\,. \tag{11}$$

When the electromagnetic field performs forced oscillations, there exists a continuous transformation between electric and magnetic energy and vice versa, where their average values need not necessarily be equal, since they are barely so. Given that the field is in an oscillatory state, energy density should not remain stationary at any time, implying the source absorbs and gives energy periodically; magnetic energy excess over electric one has a throbbing behavior between the source and the field. Such difference, twice multiplied by ω is equal to the throbbing average power. Appearance of 2ω in (11) is related with the quadratic nature of power.

Equation for a Uniform Current

By substituting Faraday-Maxwell and Ampere-Maxwell equations into (11), the next relation results:

$$-\iiint_v \mathbf{E}\bullet\mathbf{J}^* dv + \iiint_v \frac{\left|\mathbf{J}\right|^2}{\sigma}dv = \iiint_v \mathbf{F}\bullet\mathbf{J}^* dv\,, \tag{12}$$

which shows that the work done by the FEM for supporting the change in electromagnetic energy, is given by the volume integral of $-\mathbf{E}\bullet\mathbf{J}^*$, which, in the harmonic case, is equivalent to the quantity of radiation performed by the system. By expressing $\mathbf{E}$ in terms of the retarded potentials, the following equation raises:

$$\oiint_S \Phi \mathbf{J}_n^* da - j\omega \iiint_v \Phi \rho^* dv + j\omega \iiint_v \mathbf{A} \bullet \mathbf{J}^* dv + \iiint_v \frac{|\mathbf{J}|^2}{\sigma} dv = \iiint_v \mathbf{F} \bullet \mathbf{J}^* dv \,. \tag{13}$$

Normal surface current component $\mathbf{J}_n$ equals surface charge density on the conductor. For simplicity, surface integral term is omitted since ρ can represents volumetric charges as well as superficial ones. Therefore:

$$-j\omega \int_v \Phi \rho^* dv + j\omega \int_v \mathbf{A} \bullet \mathbf{J}^* dv + \int_v \frac{|\mathbf{J}|^2}{\sigma} dv = \int_v \mathbf{F} \bullet \mathbf{J}^* dv \,. \tag{14}$$

By expressing retarded potentials in terms of current and charge densities, it happens that:

$$-\frac{j\omega}{4\pi\varepsilon} \int_v \int_v \frac{\rho \rho^* e^{-jkr}}{r} dv dv' + \frac{j\omega\mu}{4\pi} \int_v \int_v \frac{\mathbf{J} \bullet \mathbf{J}^* e^{-jkr}}{r} dv' dv + \int_v \frac{|\mathbf{J}|^2}{\sigma} dv = \int_v \mathbf{F} \bullet \mathbf{J}^* dv \,. \tag{15}$$

When conductor's dimensions are less than wavelength, it can be supposed that current is uniform across the conductor, as shown in **Fig. 2**, such that:

$$\mathbf{I} = I \hat{\mathbf{s}} = \iint_a \mathbf{J} da = a\mathbf{J} \,, \tag{16}$$

where a is the transverse section area, $\hat{\mathbf{s}}$ a unit vector in direction of $\mathbf{J}$, and $I = |\mathbf{I}|$ the total current which crosses the considered area. Therefore, we get:

$$-\frac{j\omega}{4\pi\varepsilon}\int_v\int_v \frac{\rho\rho^* e^{-jkr}}{r}dv dv' + \frac{j\omega\mu I^* I}{4\pi}\int_v\int_v \frac{\hat{\mathbf{s}}\bullet\hat{\mathbf{s}}' e^{-jkr}}{a^2 r}dv' dv$$
$$+I^* I\int_v \frac{dv}{a^2\sigma} = I^*\int_v \frac{\mathbf{F}\bullet\hat{\mathbf{s}}}{a}dv\,. \tag{17}$$

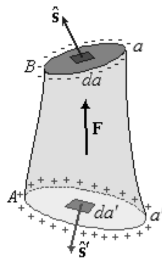

Figure 2. Oscillating charges due the field $\mathbf{F}$.

A charge density ρ is located at conductor's ends, creating a uniformly distributed superficial density ρ_S which contributes in the result by means a surface integral. In A end exists a total electric charge Q_A, while in B one exists a total electric charge Q_B, related between them by:

$$Q_A = \iint_A \rho_{SA} da_A = -Q_B = \iint_B \rho_{SB} da_B = Q\,, \tag{18}$$

where da_A and da_B are differential surface elements at the A and B ends, respectively. In this way, using the continuity equation $I = -j\omega Q$, eq. (17) can be written as:

$$\frac{I}{j\omega 4\pi\varepsilon}\left[\int_A\int_{A'}\frac{e^{-jkr}da'_A da_A}{a_A^2 r}+\int_B\int_{B'}\frac{e^{-jkr}da'_B da_B}{a_B^2 r}\right.$$
$$\left.-2\int_A\int_B\frac{e^{-jkr}da_A da_B}{a_A a_B r}\right]+\frac{j\omega\mu I}{4\pi}\int_v\int_v\frac{\hat{\mathbf{s}}\bullet\hat{\mathbf{s}}' e^{-jkr}}{a^2 r}dv'dv \qquad (19)$$
$$+I\int\frac{dl}{a\sigma}=\int F_l dl ,$$

where dl is a differential length element. Eq. (19) corresponds to Kirchhoff's voltage law for current in an RLC series circuit in steady-state [8]:

$$I\left[j\omega L-j\left(\omega C\right)^{-1}+R\right]=IZ=V , \qquad (20)$$

where Z is the circuit's complex impedance. Eq. (19) expresses a generalized complex form of Ohm's law for the harmonic case. Circuit's parameters are secured by:

$$C^{-1}=\frac{1}{4\pi\varepsilon}\left[\int_A\int_{A'}\frac{e^{-jkr}da'_A da_A}{a_A^2 r}+\int_B\int_{B'}\frac{e^{-jkr}da'_B da_B}{a_B^2 r}\right.$$
$$\left.-2\int_A\int_B\frac{e^{-jkr}da_A da_B}{a_A a_B r}\right], \qquad L=\frac{\mu}{4\pi}\int_v\int_v\frac{\hat{\mathbf{s}}\bullet\hat{\mathbf{s}}' e^{-jkr}}{a^2 r}dv'dv , \qquad (21)$$
$$R=\int\frac{dl}{a\sigma}, \qquad V=\int F_l dl .$$

From these equations it is clear that inductance and capacitance are now complex quantities instead of real ones, like occurs in ordinary circuits. Such difference is due to the appearance of the e^{-jkr} factor inside integrals, term provoked by the retard suffered by the field while propagating through the space.

This procedure can be extended for getting the n circulating currents in their respective meshes. From eq. (1), circuit's complex parameter are now:

$$C_{kl}^{-1} = \frac{1}{4\pi\varepsilon}\left[\iint \frac{e^{-jkr_{kl}}\, da_{Ak}\, da_{Al}}{a_{Ak} a_{Al} r_{kl}} + \iint \frac{e^{-jkr_{kl}}\, da_{Bk}\, da_{Bl}}{a_{Bk} a_{Bl} r_{kl}} \right.$$
$$\left. -\iint \frac{e^{-jkr_{kl}}\, da_{Ak}\, da_{Bl}}{a_{Ak} a_{Bl} r_{kl}} - \iint \frac{e^{-jkr_{kl}}\, da_{Bk}\, da_{Al}}{a_{Bk} a_{Al} r_{kl}} \right], \qquad (22)$$
$$L_{kl} = \frac{\mu}{4\pi}\iint \frac{\hat{\mathbf{s}}_k \bullet \hat{\mathbf{s}}_l e^{-jkr_{kl}}}{r_{kl} a_k a_l}\, dv_k\, dv_l\ , \quad R_{kl} = \int \frac{dl_{kl}}{\sigma a}.$$

That circuit's parameters are now complex quantities, allows considering (20) as a general theory for uniform currents where classical theory (1) is a special case.

Integral Equation for a Non-Uniform Current

When conductor's dimensions are smaller than wavelength, current is distributed approximately uniform along conductors, from which Circuit Theory equations are valid. However, the larger the conductor's dimensions are, the less the current is uniform along them, from which an integral equation is necessary for describing it. A tri-dimensional integral equation has the following general shape:

$$\iiint K\left(x,y,z;x',y',z'\right) f\left(x',y',z'\right) dx'dy'dz'$$
$$+ f\left(x,y,z\right) = g\left(x,y,z\right), \qquad (23)$$

where $f\left(x,y,z\right)$ is an unknown function, while $g\left(x,y,z\right)$ and the kernel $K\left(x,y,z;x',y',z'\right)$ are known, the last satisfying the following symmetry property:

$$K\left(x,y,z;x',y',z'\right)=K\left(x',y',z';x,y,z\right). \tag{24}$$

Eq. (23) is an integral equation of the second kind, while in a one of the first kind the unknown does not appear outside the integral:

$$\iiint K\left(x,y,z;x',y',z'\right)f\left(x',y',z'\right)dx'dy'dz'=g\left(x,y,z\right). \tag{25}$$

Let us consider an one-dimensional integral equation of the second kind possessing only x and x'. In certain sense, it should be regarded as an infinite set of linear equations:

$$\left.\begin{aligned}\left(1+K_{11}\right)f_1+K_{12}f_2+\ldots+K_{1n}f_n&=g_1\ ,\\ K_{21}f_1+\left(1+K_{22}\right)f_2+\ldots+K_{2n}f_n&=g_2\ ,\\ &\vdots\\ K_{n1}f_1+K_{n2}f_2+\ldots+\left(1+K_{nn}\right)f_n&=g_n\ ,\end{aligned}\right\}\Rightarrow$$
$$f\left(k\right)+\sum_{l=1}^{n}K\left(k,l\right)f\left(l\right)=g\left(k\right). \tag{26}$$

In similar way when a summation becomes an integral, provided that indices become continuous, previous equation can be written as:

$$f\left(k\right)+\int K\left(k,l\right)f\left(l\right)dl=g\left(k\right). \tag{27}$$

Any infinite set of linear equations is numerable, while an integral equation suggests an infinite set of non-numerable equations. With this idea, equivalence between an integral equation and an infinite set of linear equations is useful. In such sets, it is not allowed referring of superior dimensions, but eq. (27) can be generalized into a three-dimensional integral equation:

$$\begin{aligned}&\int K\left(k_1,k_2,k_3;l_1,l_2,l_3\right)f\left(l_1,l_2,l_3\right)dl_1dl_2dl_3\\ &\quad+f\left(k_1,k_2,k_3\right)=g\left(k_1,k_2,k_3\right).\end{aligned} \tag{28}$$

Whenever indices in a set of linear equations have certain relation with a set of discontinuous points in a n-dimensional space, at the limit, a n-dimensional integral equation is gotten.

Let us assume in a given conductor exists a current distributed in stationary lines of flow, as in Fig. 3a, each one of them described by a unit tangent vector $\hat{\mathbf{s}}(x,y,z)$ which points out the direction of current flow. All of the points in each current line keep the same intensity. In general, an stationary line of flow will be described by two linear independent functions, g_1 and g_2:

$$\mathbf{J}(x,y,z,t)=\left[g_1(x,y,z)\operatorname{sen}\omega t+g_2(x,y,z)\cos\omega t\right]\hat{\mathbf{s}}(x,y,z). \quad (29)$$

In general, $\mathbf{J}$ vector draws an ellipse instead a line. However, in wires ellipses degenerate into their major axis, becoming in lines of flow essentially parallel to wire's axis.

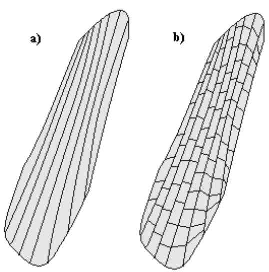

Figure 3. Stationary lines of flow.

If we assume each line to be composed of a large number of elements, as in **Fig. 3b**, along which current can be supposed uniform, without mutual resistances, the following set is reached:

$$\sum_{l=1}^{n}\bar{K}_{kl}I_l=V_k-R_kI_k\ , \qquad \bar{K}_{kl}=j\omega L_{kl}+\frac{1}{j\omega C_{kl}}. \quad (30)$$

In this way, circuit's parameters must be calculated when current elements become infinitesimal. Mutual inductance is then expressed as follows:

$$\begin{aligned} L_{kl} &= \lim_{n\to\infty}\frac{\mu}{4\pi}\iint\frac{\hat{\mathbf{s}}_k \bullet \hat{\mathbf{s}}_l e^{-jkr_{kl}}}{r_{kl}q_k q_l}dv_k dv_l \\ &= \frac{\mu}{4\pi}\frac{e^{-jkr(x,y,z;x',y',z')}\hat{\mathbf{s}}(x,y,z)\bullet\hat{\mathbf{s}}'(x',y',z')}{r(x,y,z;x',y',z')}dl_k dl_l \ . \end{aligned} \tag{31}$$

According to **Fig. 4a**, mutual capacitance is calculated as:

$$\begin{aligned} 4\pi\varepsilon C_{kl}^{-1} &= \lim_{n\to\infty}\left[\iint\frac{e^{-jkr_{kl}}da_{Ak}da_{Al}}{a_{Ak}a_{Al}r_{kl}} + \iint\frac{e^{-jkr_{kl}}da_{Bk}da_{Bl}}{a_{Bk}a_{Bl}r_{kl}}\right. \\ &\left. -\iint\frac{e^{-jkr_{kl}}da_{Ak}da_{Bl}}{a_{Ak}a_{Bl}r_{kl}} - \iint\frac{e^{-jkr_{kl}}da_{Bk}da_{Al}}{a_{Bk}a_{Al}r_{kl}}\right], \end{aligned} \tag{32}$$

where the next identity, **Fig. 4b**, allows several simplifications:

$$f\left(r'\right) - f(r) = \left[\hat{\mathbf{s}}\left(A'\right)\bullet\nabla_A f(r)\right]dl' \ . \tag{33}$$

Then, the result follows from:

$$\begin{aligned} 4\pi\varepsilon C_{kl}^{-1} &= \left(\frac{e^{-jkr'_{kl}}}{r'_{kl}} - \frac{e^{-jk\overline{r}'_{kl}}}{\overline{r}'_{kl}}\right) - \left(\frac{e^{-jkr''_{kl}}}{r''_{kl}} - \frac{e^{-jk\overline{r}''_{kl}}}{\overline{r}''_{kl}}\right) \\ &= \left[\nabla_{B_k}\left(\frac{e^{-jkr}}{r}\right) - \nabla_{A_k}\left(\frac{e^{-jkr}}{r}\right)\right]\bullet\hat{\mathbf{s}}(x,y,z)dl_k \\ &= \nabla'\left[\nabla\left(\frac{e^{-jkr}}{r}\right)\bullet\hat{\mathbf{s}}(x,y,z)\right]\bullet\hat{\mathbf{s}}'(x',y',z')dl_k dl_l \ . \end{aligned} \tag{34}$$

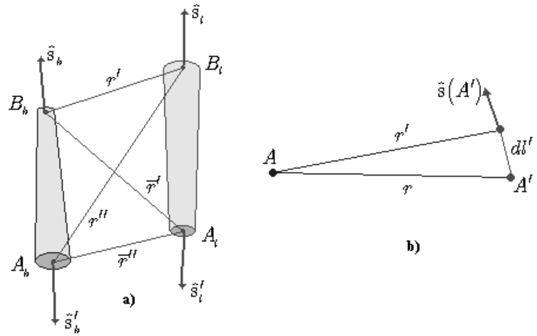

Figure 4. Calculation of mutual capacitance.

The remaining circuit's parameters are calculated with:

$$I_l = a_l \mathbf{J}_l \bullet \hat{\mathbf{s}}, \qquad R_k = \lim_{n\to\infty} \int \frac{dl_k}{\sigma a} = \frac{dl_k}{\sigma a_k}, \tag{35}$$

$$V_k = \lim_{n\to\infty} \int \mathbf{F}_k \bullet d\mathbf{l}_k = \mathbf{F}_k \bullet \hat{\mathbf{s}} dl_k ,$$

where $a_{k,l}$ is the element's cross section. Therefore the set of equation becomes:

$$\sum_{l=1}^{n} K_{kl} \mathbf{J}_k \bullet \hat{\mathbf{s}} a_k dl_k dl_l = \mathbf{F}_k \bullet \hat{\mathbf{s}} dl_k - \frac{\mathbf{J}_k \bullet \hat{\mathbf{s}} a_k dl_k}{\sigma a_k}, \tag{36}$$

$$\bar{K}_{kl} = K_{kl} dl_k dl_l .$$

Dividing by dl_k and removing the dot product, we get:

$$\sum_{l=1}^{n} K_{kl} \mathbf{J}_k a_k dl_l = \mathbf{F}_k - \frac{\mathbf{J}_k}{\sigma}. \tag{37}$$

In the limit, when $n \to \infty$, the set of equations is equivalent to an integral equation, where $a_l dl_l = dv'$:

$$\frac{\mathbf{J}(x,y,z)}{\sigma}+\int_{v}K(x,y,z;x',y',z')\mathbf{J}(x',y',z')dv' = \mathbf{F}(x,y,z),$$
$$K(x,y,z;x',y',z')=\frac{j\omega\mu}{4\pi}\frac{e^{-jkr}\hat{\mathbf{s}}\bullet\hat{\mathbf{s}}'}{r}+\frac{1}{j\omega4\pi\varepsilon}\nabla'\left(\nabla\frac{e^{-jkr}}{r}\bullet\hat{\mathbf{s}}\right)\bullet\hat{\mathbf{s}}'. \tag{38}$$

Integral equation of the second kind (38) is a very generalization of Circuit Theory, taking into account that current is not uniformly distributed in the conductor and that it has not restrictions regarding its size. Such integral is an exact consequence of Maxwell equations by considering stationary lines of flow. For thin wires, eq. (38) can be reduced into an one-dimensional integral equation with certain exactness. In several cases σ is large enough that $\mathbf{J}/\sigma$ term can be neglected; mathematically it could be a disadvantage because an integral equation of the second kind is easier to solve than one of the first kind. However, provided that $\sigma\to\infty$, current is confined in the wire's surface, and lines of flow will be known with better degree of precision. Finally, if $\mathbf{F}=\mathbf{0}$, integral equation becomes an homogeneous one which will have complex solutions for certain eigen-values of wave-number, whose real parts bring out free-oscillation frequencies and whose imaginary parts bring out attenuation constants due radiation.

Integral Equation for Thin Wires

It is common that almost antennas were made with thin cylindrical conductors, which in practical are those whose diameters are less than $\lambda/100$ [9]. For avoiding singularities, wire's radio a should never be considered infinitely thin; in this way, errors of the order $2a/L$ can be committed by considering small radios but finites. For such conductors, integral eq. (38) should be adjusted for finding out cross-section current distribution as well as longitudinal distribution, the last satisfying an one-dimensional integral equation.

In a thin wire, current's flow is mainly confined along it in such a way that $\hat{\mathbf{s}}$ and $\hat{\mathbf{s}}'$ have practically the same direction, tangent to curve $\mathbf{r}(s)$ expressing wire's axis. In fact, lines of flow are slightly bent toward the wire's sides; it causes superficial charges appearing on the wire. Thin wire's geometry is shown in **Fig. 5**. Wire's axis is parameterized by its arc length s [10]:

$$s(t)=\int_0^t \sqrt{\frac{d\mathbf{r}(\xi)}{d\xi}\bullet\frac{d\mathbf{r}(\xi)}{d\xi}}\,d\xi\,. \tag{39}$$

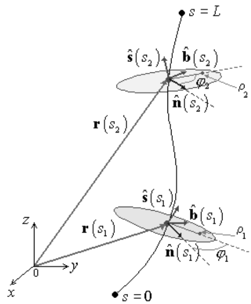

Figure 5. Wire vector representation.

At each point in curve there exist three unit vectors which can be used as an ortonormal vector base for defining a local cylindrical system of coordinates [11]. They are the tangent unit vector $\hat{\mathbf{s}}(s)$, the normal unit vector $\hat{\mathbf{n}}(s)$ and the binormal unit vector $\hat{\mathbf{b}}(s)$, which are expressed as:

$$\mathbf{r}(s) = x(s)\hat{\mathbf{i}} + y(s)\hat{\mathbf{j}} + y(s)\hat{\mathbf{k}} ,$$
$$\hat{\mathbf{s}}(s) = \frac{d\mathbf{r}(s)}{ds}, \quad \hat{\mathbf{n}}(s) = \frac{1}{K}\frac{d\hat{\mathbf{s}}(s)}{ds}, \tag{40}$$
$$\hat{\mathbf{b}}(s) = \hat{\mathbf{s}}(s)\times\hat{\mathbf{n}}(s), \quad K = \left|\frac{d\hat{\mathbf{s}}(s)}{ds}\right|,$$

where K is the wire's curvature. Vectors $\hat{\mathbf{n}}$ and $\hat{\mathbf{b}}$ form a normal plane, laying in wire's cross section, whose points, with cylindrical coordinates $(\rho, \varphi, 0)$, are determined by the next position vector, referring to the local coordinate system:

$$\mathbf{a}(\rho,\varphi) = \rho\cos\varphi\,\hat{\mathbf{n}} + \rho\,\mathrm{sen}\,\varphi\,\hat{\mathbf{b}} ,$$
$$0 \le \rho \le a , \quad 0 \le \varphi \le 2\pi . \tag{41}$$

The same points, referring to the rectangular coordinate system XYZ, have the form:

$$\mathbf{P}(s,\rho,\varphi) = \mathbf{r}(s) + \rho\cos\varphi\,\hat{\mathbf{n}}(s) + \rho\,\mathrm{sen}\,\varphi\,\hat{\mathbf{b}}(s) . \tag{42}$$

Then, the distance R between two of them in the wire is expressed by the next equation:

$$R = \left|\mathbf{r}(s_2) - \mathbf{r}(s_1) + \mathbf{a}(\rho_2,\varphi_2) - \mathbf{a}(\rho_1,\varphi_1)\right|. \tag{43}$$

Therefore, the integral equation for a thin wire is:

$$\int_0^L \int_0^{2\pi} \int_0^a \left[\frac{j\omega\mu}{4\pi} \frac{e^{-jkR}}{R} \hat{\mathbf{s}} \bullet \hat{\mathbf{s}}' + \frac{1}{j\omega 4\pi\varepsilon} \nabla' \left(\nabla \frac{e^{-jkR}}{R} \bullet \hat{\mathbf{s}} \right) \bullet \hat{\mathbf{s}}' \right] \mathbf{J}(s',\rho') \rho' d\rho' d\varphi' ds' + \frac{\mathbf{J}(s,\rho)}{\sigma} = \mathbf{F}(s,\rho) . \qquad (44)$$

As can be seen, current is not longer a function of the azimuth variable φ since a thin wire does not allow circumferential variations in its distribution. Also should be noticed that current is function of the vector radio ρ which represents the deepness in wire's radio, provided that σ is large enough but finite, provoking concentric current layers not flowing in phase. For a perfect conductor, current will be confined in its surface and in almost cases circumferential distribution will be that of static charges in an infinitely long wire.

General Pocklington Equation

Pocklington's integral equation fits a hypothetical model where current density $\mathbf{J}$ is concentrated in a filament on wire's surface. Such equation, proposed by H. C. Pocklington in 1897 [12] and based on Hertz theory [13], investigated current oscillations in straight thin wires. Following integral equation is a natural generalization for any wire's geometry, the straight form being a particular case. It is based in the following hypotheses:

1. The wire's material can be considered a perfect conductor since $\sigma \to \infty$, such that the $\mathbf{J}/\sigma$ term in **Error! Reference source not found.** must be neglected. In practice, such hypothesis is a good approximation for

materials from which antennas are made, such as aluminum, cooper or silver, where σ is in the order of 1×10^7 S/m.

2. Wire's current is totally confined on its surface, where $\mathbf{J}(s,a)=\mathbf{J}(s)$. It is a natural consequence of the former hypothesis, since in perfect conductors electromagnetic fields vanish inside them, such that $\mathbf{E}=\mathbf{H}=\mathbf{0}$. In practice, such hypothesis is a good approximation, since due skin effect, current tends to pile up in conductor's surface, forming a thin current coat whose deepness δ is equal to:

$$\delta=\sqrt{2/\omega\mu\sigma}\ . \tag{45}$$

3. Circumferential current variations can be neglected, i.e. current density is not longer function of azimuth variable φ. Relatively slow variations of $\mathbf{J}$ with respect to φ and ρ in a distance along s of the order of $2a$, justify such hypothesis.
4. Current density can be represented by a current filament $\mathbf{I}$ on wire's surface

$$\mathbf{I}=\pi a^2\mathbf{J}\ . \tag{46}$$

   Since $\mathbf{J}$ forms a uniform current coat, we can suppose it collapsing in an infinitesimal line on wire's surface, which is a parallel curve to the wire's axis. In this way, volume occupied by the wire is considered part of propagation medium.
5. Electric field boundary condition needs to be forced just in axial direction. Since wire is thin enough, electric field

boundary condition needs not to be forced around its cross section provided that current varies mainly along s.

By applying these hypotheses to eq. (44), it is possible to transform it into an one-dimensional integral equation. First term, corresponding to reactive capacitance, can be written as:

$$\nabla'\left(\nabla\frac{e^{-jkR}}{R}\bullet\hat{\mathbf{s}}\right)\bullet\hat{\mathbf{s}}'=\nabla'\left(\frac{\partial}{\partial s}\frac{e^{-jkR}}{R}\right)\bullet\hat{\mathbf{s}}'=\frac{\partial^2}{\partial s\partial s'}\frac{e^{-jkR}}{R}, \qquad (47)$$

where R is a function of s only. Then:

$$\frac{1}{\pi a^2}\int_0^L\int_0^{2\pi}\int_0^a\left[\frac{j\omega\mu}{4\pi}\frac{e^{-jkR}}{R}\hat{\mathbf{s}}\bullet\hat{\mathbf{s}}'\mathbf{I}(s') + \frac{1}{j\omega 4\pi\varepsilon}\frac{\partial^2}{\partial s\partial s'}\frac{e^{-jkR}}{R}\right]\rho' d\rho' d\varphi' ds' = \mathbf{F}(s). \qquad (48)$$

Since integrand is just a function of s, it is permitted to perform integrations with respect to ρ and φ, from which we get:

$$\int_0^L\left[\frac{j\omega\mu}{4\pi}\frac{e^{-jkR}}{R}\hat{\mathbf{s}}\bullet\hat{\mathbf{s}}' + \frac{1}{j\omega 4\pi\varepsilon}\frac{\partial^2}{\partial s\partial s'}\frac{e^{-jkR}}{R}\right]\mathbf{I}(s')ds' = \mathbf{F}(s). \qquad (49)$$

In this equation, $\mathbf{F}$ field corresponds to minus the scattered electric field tangential component, in such a way that:

$$\mathbf{I}(s') = I(s')\hat{\mathbf{s}}, \qquad \mathbf{F}(s) = -E_s^S(s')\hat{\mathbf{s}}, \qquad (50)$$

where s sub-index denotes the tangential component while S super-index denotes the scattered field. By applying electric field boundary condition on wire’s surface, we get:

$$-E_s^S(s,a) = E_s^I(s,a), \qquad (51)$$

where E_s^I is the impressed electric field tangential component. Therefore, the following equation can be conveniently called, general Pocklington's equation:

$$E_s^I(s) = \frac{1}{j\omega\varepsilon}\int_0^L \left[\frac{\partial^2}{\partial s \partial s'}\frac{e^{-jkR}}{4\pi R} - k^2 \hat{\mathbf{s}} \bullet \hat{\mathbf{s}}' \frac{e^{-jkR}}{4\pi R}\right] I(s')\, ds' . \qquad (52)$$

Notice the minus sign in kernel, which in the straight case is replaced by a plus sign [14]. The $e^{-jkR}/4\pi R$ term is the Green's function for free space. Having R as denominator provokes a singularity when the field point is that of the source point. For avoiding it, field points are located conveniently on wire's axis while source points are located conveniently on wire's surface. If $\mathbf{r}(s)$ represents wire's axis, then the parallel curve which represents current filament is, **Fig. 6**:

$$\mathbf{r}'(s) = \mathbf{r}(s) + a\mathbf{n}(s) = x'(s)\hat{\mathbf{i}} + y'(s)\hat{\mathbf{j}} + z'(s)\hat{\mathbf{k}} , \qquad (53)$$

which is gotten from (42) by doing $\rho = a$ and $\varphi = 0$. Therefore, distance between a source point and a field point is:

$$\begin{aligned} R &= \left|\mathbf{r}(s) - \mathbf{r}'(s')\right| \\ &= \sqrt{\left[x(s) - x'(s')\right]^2 + \left[y(s) - y'(s')\right]^2 + \left[z(s) - z'(s')\right]^2} . \end{aligned} \qquad (54)$$

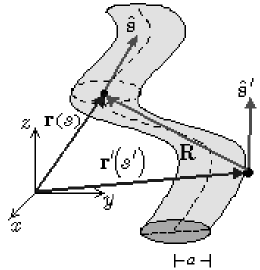

Figure 6. Wire's geometry and its associated curves.

The General Pocklington's equation can be written in a simpler form by performing the iterated differentiation of Green's function:

$$\frac{\partial^2}{\partial s \partial s'} \frac{e^{-jkR}}{4\pi R} = -\frac{\left(R + jkR^2\right)\frac{\partial^2 R}{\partial s \partial s'} + \left(k^2 R^2 - 2 - 2jkR\right)\frac{\partial R}{\partial s}\frac{\partial R}{\partial s'}}{4\pi R^3} e^{-jkR}. \tag{55}$$

From (54) the partial derivatives can be developed like:

$$\frac{\partial R}{\partial s} = \frac{\mathbf{R} \bullet \hat{\mathbf{s}}}{R}, \qquad \frac{\partial R}{\partial s'} = -\frac{\mathbf{R} \bullet \hat{\mathbf{s}}'}{R},$$
$$\frac{\partial^2 R}{\partial s \partial s'} = -\frac{R^2 \hat{\mathbf{s}} \bullet \hat{\mathbf{s}}' - \left(\mathbf{R} \bullet \hat{\mathbf{s}}'\right)\left(\mathbf{R} \bullet \hat{\mathbf{s}}\right)}{R^3}, \tag{56}$$

hence, the reactive capacitance term can be expressed as:

$$\frac{\partial^2}{\partial s \partial s'} \frac{e^{-jkR}}{4\pi R} = \frac{R^2\left(1 + jkR\right)\hat{\mathbf{s}} \bullet \hat{\mathbf{s}}' - \left[3 + 3jkR - k^2 R^2\right]\left(\mathbf{R} \bullet \hat{\mathbf{s}}'\right)\left(\mathbf{R} \bullet \hat{\mathbf{s}}\right)}{4\pi R^5} e^{-jkR}, \tag{57}$$

and the general Pocklington equation is:

$$E_s^I(s) = -\frac{1}{j\omega\varepsilon}\int_0^L \Big[\left(k^2R^4 - R^2 - jkR^3\right)\hat{\mathbf{s}}\bullet\hat{\mathbf{s}}' + \left(3 + 3jkR - k^2R^2\right)\left(\mathbf{R}\bullet\hat{\mathbf{s}}'\right)\left(\mathbf{R}\bullet\hat{\mathbf{s}}\right)\Big]\frac{e^{-jkR}}{4\pi R^5}\,Ids' . \tag{58}$$

In the particular case for a straight wire laying centered in Z axis, we have the following equation commonly found in specialized literature:

$$E_s^I(s) = -\frac{1}{j\omega\varepsilon}\int_0^L \Big[\left(1 + jkR\right)\left(2R^2 - 3a^2\right) + \left(kaR\right)^2\Big]\frac{e^{-jkR}}{4\pi R^5}\,I(z')\,dz' , \tag{59}$$

where vector relations are:

$$\mathbf{R} = (z - z')\hat{\mathbf{k}} - a\hat{\mathbf{j}} , \quad \hat{\mathbf{s}} = \hat{\mathbf{s}}' = \hat{\mathbf{k}} ,$$
$$\left(\mathbf{R}\bullet\hat{\mathbf{s}}'\right)\left(\mathbf{R}\bullet\hat{\mathbf{s}}\right) = (z - z')^2 = R^2 - a^2 . \tag{60}$$

Conclusions

Kirchhoff's laws establish relations between current and voltage in electric networks where electrical parameters are considered concentrated in certain points in meshes. They lead to Circuit Theory, which is an engineering area that models electric networks accurately enough while the mesh dimensions are less than wavelength. However, as the dimensions become greater than or equal to the wavelength, the more electrical parameters are distributed through the network. In this way, Circuit Theory should be modified in order to model accurately distributed phenomena and current-voltage relations.

Integral equations model current distribution in distributed electric networks, for given electric field distribution. In the particular case of thin wires, integral equation corresponds to Pocklington's equation. By knowing wire's current distribution, its electrical behavior can be predicted in the same way when circulating currents are determined in meshes of classical electric circuits.

Distributed phenomena are commonly found in transmission lines, wave cavities and antennas, where their current distributions are frequently guessed. Integral equations provide a secure way for finding out their real current distribution and therefore their correct electric behavior.

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