Relative Randomness and Cardinality

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Question

Given an oracle B, what is the cardinality of the class

$$\{A \mid \mathrm{MLR}^B \subseteq \mathrm{MLR}^A\}$$

where MLR^X is the class of Martin-Löf random sets relative to X?

The Cantor space

- 2^ω is the space of infinite binary strings: the *reals*
- $2^{<\omega}$ is the space of finite binary strings
- The standard topology on 2^ω is induced by the basic open sets: $[\sigma] = \{\sigma X \ : \ X \in 2^\omega\}$ for all $\sigma \in 2^{<\omega}$.
- Lebesgue measure on the Cantor space: the measure of a basic open set $[\sigma]$ is $\mu([\sigma]) = 2^{-|\sigma|}$

Martin-Löf Randomness

- Identify finite binary strings with intervals in 2^ω: $\sigma \rightarrow [\sigma]$
- Prefix-free sets of finite binary strings correspond to independent (basic open) sets of reals

Definition

A Martin-Löf test $\mathcal{M}$ is a uniform sequence (E_i) of c.e. sets of binary strings such that $\mu(E_i) \leq 2^{-i}$. A real α avoids $\mathcal{M}$ if some for i, $\alpha \notin E_i$. A real number is called random if it avoids all Martin-Löf tests. W.l.o.g. assume $E_{i+1} \subset E_i$.

Martin-Löf tests

- Martin-Löf tests and randomness relativize to any oracle.
- There is a universal Martin-Löf test.

Basic fact (Kjos-Hansen)

The following are equivalent:

- $\mathrm{MLR}^B \subseteq \mathrm{MLR}^A$
- For every $\Sigma_1^{0,A}$ class T^A of measure < 1 there is a $\Sigma_1^{0,B}$ class V^B of measure < 1 such that

$$T^A \subseteq V^B.$$

- For some member U^A of a universal Martin-Löf test relative to A there is $V^B \in \Sigma_1^{0,B}$ with $\mu V^B < 1$ and

$$U^A \subseteq V^B.$$

Back to the Question

Given an oracle B, what is the cardinality of the class

$$\mathcal{C}^B := \{A \mid \mathrm{MLR}^B \subseteq \mathrm{MLR}^A\}$$

where MLR^X is the class of Martin-Löf random sets relative to X?

Note

- The reals in $\mathcal{C}^{\emptyset}$ are also known as *low for random*.
- The relation $\mathrm{MLR}^B \subseteq \mathrm{MLR}^A$ is also known as $A \leq_{LR} B$.

Known Facts

- If $B = \emptyset$ then $\mathcal{C}^B \subset \Delta^0_2$, so $|\mathcal{C}^B| = \aleph_0$ (Nies).
- If $B = \emptyset'$ then $|\mathcal{C}^B| = 2^{\aleph_0}$ (Barmpalias, Lewis, Soskova).
- If $(B \oplus \emptyset')' <_T B''$ then $|\mathcal{C}^B| = 2^{\aleph_0}$ (Barmpalias, Lewis, Soskova).
- there is a c.e. B such that $B' \leq_{tt} \emptyset'$ and $|\mathcal{C}^B| = 2^{\aleph_0}$ (Barmpalias, Lewis, Stephan).
- If B is random relative to $\emptyset'$ then $|\mathcal{C}^B| = \aleph_0$ (Miller).
- So $|\mathcal{C}^B| = \aleph_0$ for almost all oracles B.

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The result

Theorem

Let B be Δ^0_2. Then $|\mathcal{C}^B| = \aleph_0$ iff B is low for random (i.e. $B \in \mathcal{C}^\emptyset$).

Moreover, Δ^0_2 is the largest arithmetical class for which the theorem holds.

Corollary

Let B be Δ^0_2 such that $|\mathcal{C}^B| = 2^{\aleph_0}$. Then $\mathcal{C}^B$ contains a perfect Π^0_1 set of reals.

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Proof

Given Nies' result, it suffices to show the following.

Theorem

Given a Δ^0_2 set B which is not low for random, the class $\mathcal{C}^B$ contains a perfect Π^0_1 set of reals.

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- the full binary tree
- with a recursive assignment of measure along its branches

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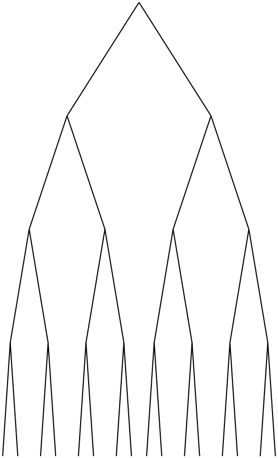

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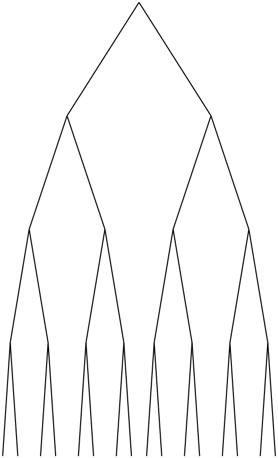

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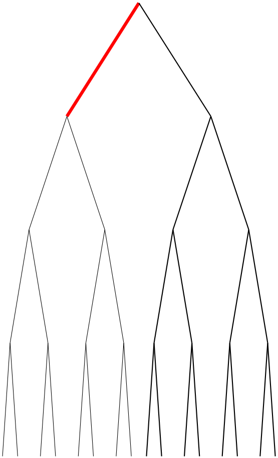

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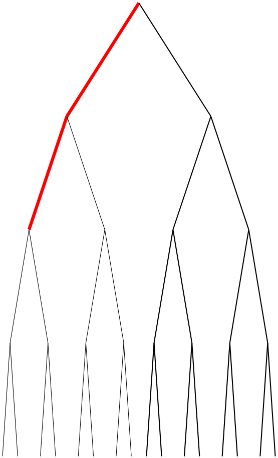

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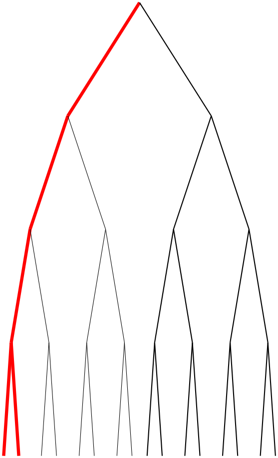

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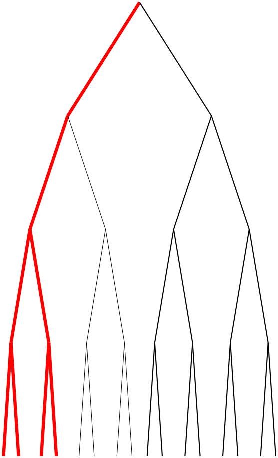

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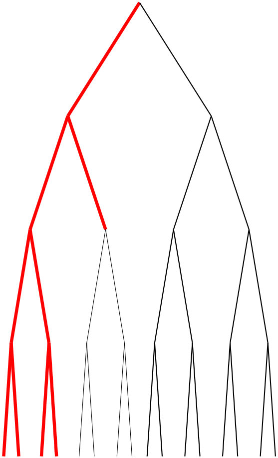

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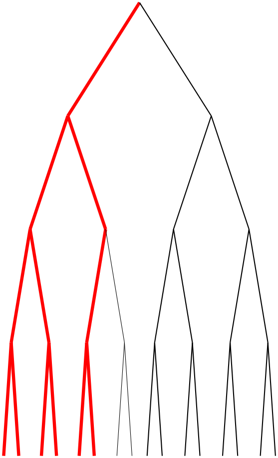

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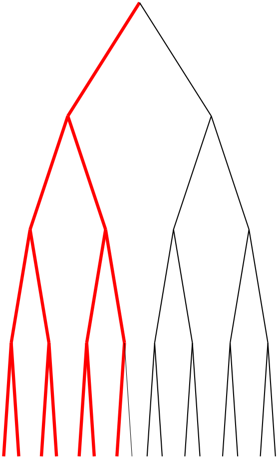

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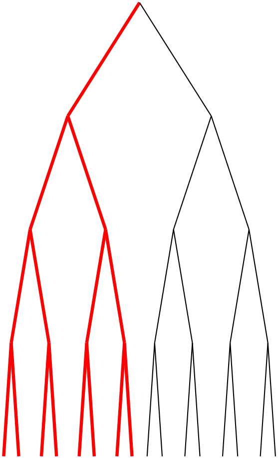

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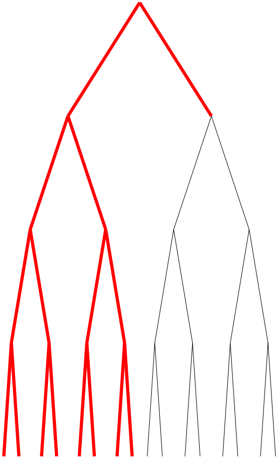

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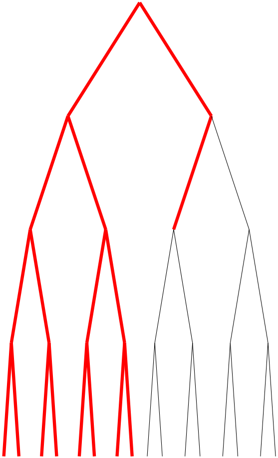

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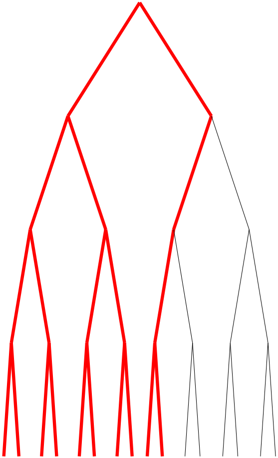

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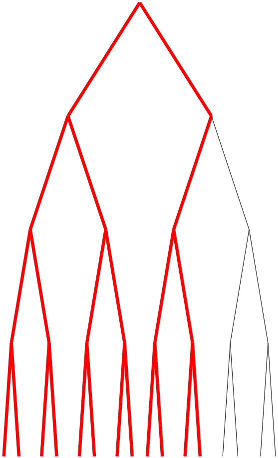

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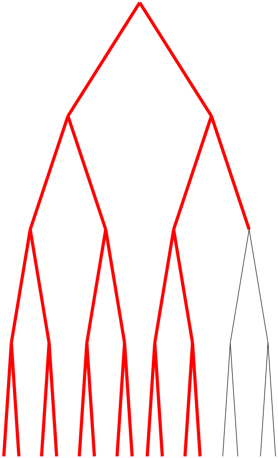

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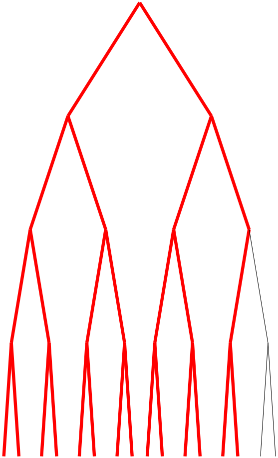

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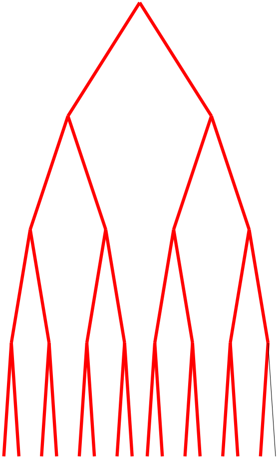

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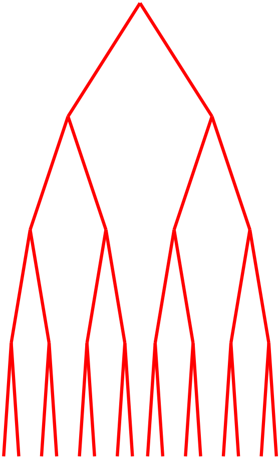


Formalization

In view of Kjos-Hansen's characterization of the relation $\mathrm{MLR}^B \subset \mathrm{MLR}^A$:

Definition

An *oracle* Σ^0_1 *class* V is an oracle Turing machine which, given an oracle A it outputs a set of finite binary strings V^A, representing an open subset of the space 2^ω. The oracle class V can be seen as a c.e. set of axioms $\langle \tau, \sigma \rangle$ (where $\tau, \sigma \in 2^{<\omega}$) so that

$$\begin{aligned} V^A &= \{\sigma \mid \exists \tau (\tau \subset A \ \wedge \ \langle \tau, \sigma \rangle \in V)\} \\ V^\rho &= \{\sigma \mid \exists \tau (\tau \subseteq \rho \ \wedge \ \langle \tau, \sigma \rangle \in V)\} \end{aligned}$$

for $A \in 2^\omega, \rho \in 2^{<\omega}$.

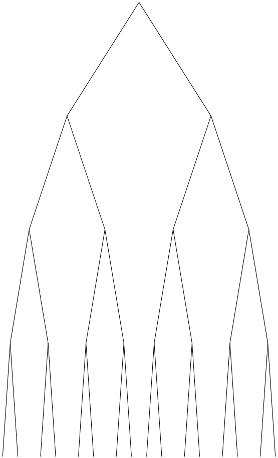
Oracle Martin-Löf tests

- An oracle Martin-Löf test is a uniform sequence of oracle Σ^0_1 classes such that the measure assigned on any path by the eth class is less than 2^{-e}.
- There is a universal oracle Martin-Löf test. Fix a member of it U.

The plan

It suffices to construct a perfect Π^0_1 class P and an oracle Σ^0_1 class V which assigns max measure 1/2 to any path of the full binary tree, such that

$$\cup_{\beta \in P} U^{\beta} \subseteq V^{B}.$$



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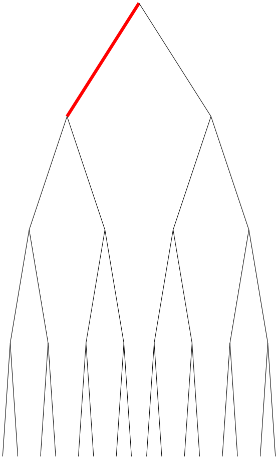

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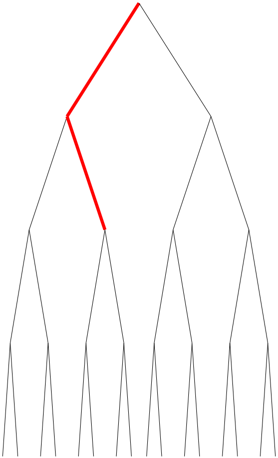

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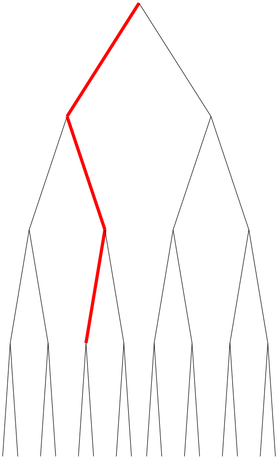

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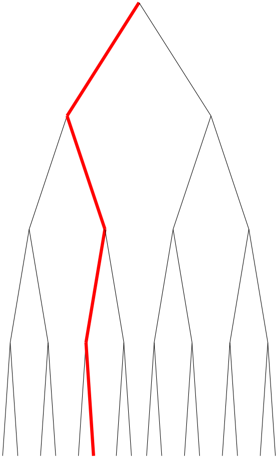

If we could control B

Lemma

For all $\epsilon > 0$ there exists σ such that for all $\beta \supset \sigma$

$$\mu(U^\beta - U^\sigma) < \epsilon.$$

Then, given that by changing B we can eject any unnecessary measure from V^B, it suffices to make P such that $\cup_{\beta \in P} U^\beta < 1$.

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Lemma

For all $\epsilon > 0$ there exists σ such that for all $\beta \supset \sigma$

$$\mu(U^{\beta} - U^{\sigma}) < \epsilon.$$

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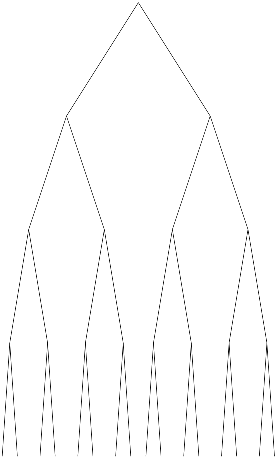

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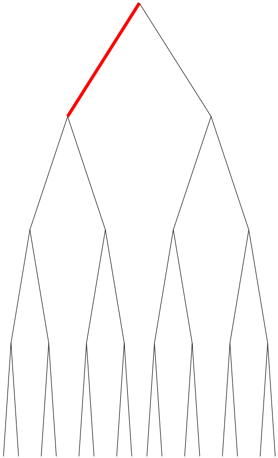

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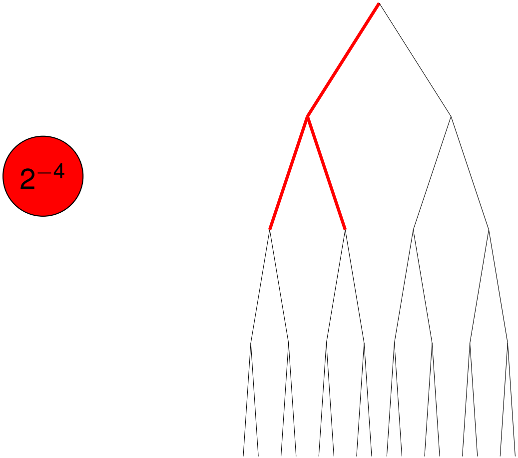

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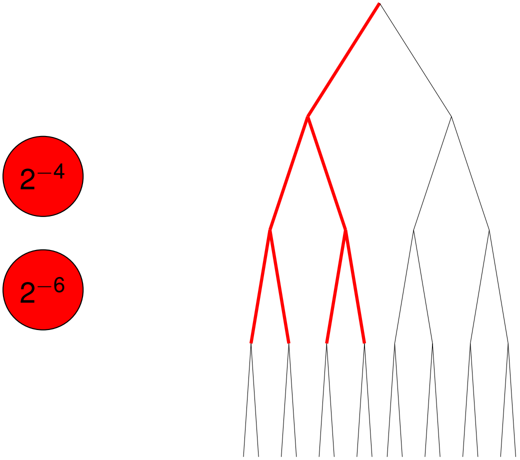
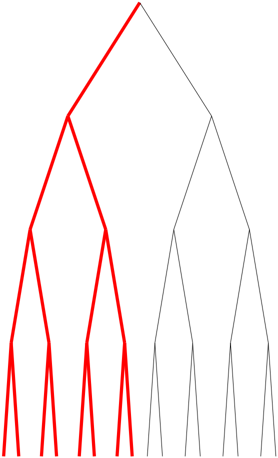


How to do that

2^{-4}

2^{-6}

2^{-2n}



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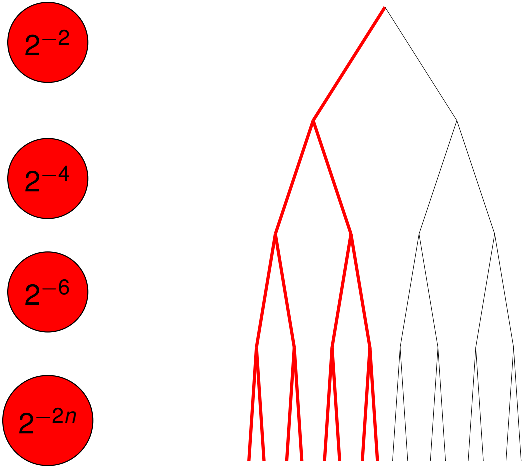

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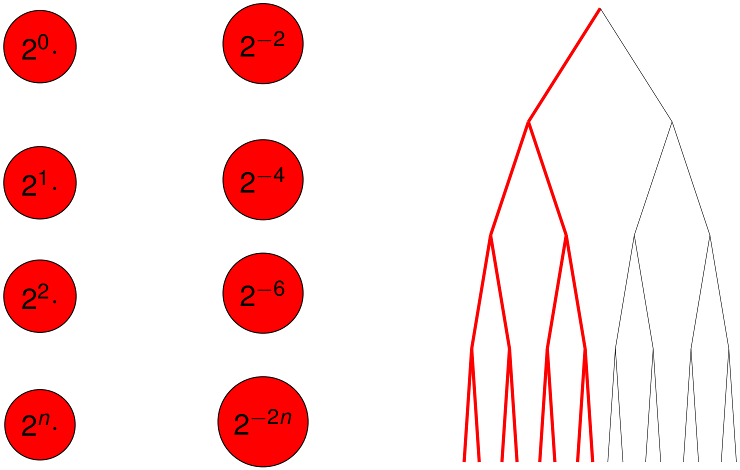

- Think of the construction dynamically.
- We construct the nodes T_σ, $\sigma \in 2^{<\omega}$ of a perfect Π^0_1 class
- Every T_σ is associated with number $2^{-2|\sigma|}$.
- Every time T_σ is redefined, either some T_τ, $\tau \subset \sigma$ is redefined or U has gained measure $2^{-2|\sigma|}$.
- Inductively, every T_σ reaches a limit.

But we don't control B

- We merely have the information that B is not low for random.
- This means that U^B cannot be covered by a Σ^0_1 class of measure < 1.
- By attempting to cover U^B in this way, we have a way to force B to eject a lot of measure from a Σ^0_1 relative to B, for instance V^B.
- This is a permitting property

But we don’t control B

- We construct a Π^0_1 class by approximating T_σ monotonically.
- Everytime T_σ moves, some measure is added in a path through U but not a constant amount as before.
- Using the fact that U^B cannot be covered by a Σ^0_1 class of measure < 1 we argue that if T_σ moves infinitely often then too much measure is loaded in a single path trough U, a contradiction.

Overview of the proof

- The key is to come up with an atomic strategy for defining T_σ which can work with arbitrarily small cost, i.e. useless measure in V^B.
- There is finite injury, cost quota assignment and reassignment (after an injury).
- The argument is a demonstration of Δ^0_2 non-low-for random permitting.

Questions

- The general question of the cardinality of $\{A \mid \mathrm{MLR}^B \subseteq \mathrm{MLR}^A\}$ remains open.
- If B is Δ^0_2 and not low for random, does $\{A \mid \mathrm{MLR}^B \subseteq \mathrm{MLR}^A\}$ contain a Π^0_1 class without low for random paths?

References

- G. Barmpalias, Relative Randomness and Cardinality, preprint.
- G. Barmpalias, A. Lewis and M. Soskova, Lowness, Randomness and Degrees, to appear in JSL.
- G. Barmpalias, A. Lewis and F. Stephan, Π^0_1 classes, *LR* degrees and Turing degrees, preprint.

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