Topology and the Physical Properties of Electromagnetic Fields

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Beginning with G.W. Leibniz in the 17th, L. Euler in the 18th, B. Reimann, J.B. Listing and A.F. Möbius in the 19th and H. Poincaré in the 20th centuries, "*analysis situs*" (Riemann) or "topology" (Listing) has been used to provide answers to questions concerning what is most fundamental in physical explanation. That question itself implies the question concerning what mathematical structures one uses with confidence to adequately "paint" or describe physical models built from empirical facts. For example, differential equations of motion cannot be fundamental, because they are dependent on boundary conditions which must be justified—usually by group theoretical considerations. Perhaps, then, group theory is fundamental. Group theory certainly offers an austere shorthand for fundamental transformation rules. But it appears to the present writer that the final judge of whether a mathematical group structure can, or cannot, be applied to a physical situation is the topology of that physical situation. Topology dictates and justifies the group transformations.

So for the present writer, the answer to the question of what is the most fundamental physical description is that it is a description of the topology of the situation. With the topology known, the group theory description is justified and equations of motion can then be justified and defined in specific differential equation form. If there is a requirement for an understanding more basic than the topology of the situation, then all that is left is verbal description of visual images. So we commence an examination of electromagnetism under the assumption that topology defines group transformations and the group transformation rules justify the algebra underlying the differential equations of motion.

For some time, the present writer has been engaged in showing that the spacetime topology defines electromagnetic field equations[1]—whether the fields be of force or of phase. That is to say, the premise of this enterprise is that a set of field equations are only valid with respect to a set defined topological description of the physical situation. In particular, the writer has addressed demonstrating that the $\mathbf{A}_\mu$ potentials, $\mu = 0, 1, 2, 3$, are not just a mathematical convenience, but—*in certain well-defined situations*—are measurable, *i.e.*, physical. Those situations in which the $\mathbf{A}_\mu$ potentials are measurable possess a topology, the transformation rules of which are describable by the *SU(2)* group; and those situations in which the $\mathbf{A}_\mu$ potentials are not measurable possess a topology, the transformation rules of which are describable by the *U(1)* group.

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Table 1		
	***U(1)* Symmetry Form(Traditional Maxwell Equations)**	***SU(2)* Symmetry Form**
Gauss' Law	$\nabla \bullet \mathbf{E} = J_0$	$\nabla \bullet \mathbf{E} = J_0 - iq(\mathbf{A} \bullet \mathbf{E} - \mathbf{E} \bullet \mathbf{A})$
Ampère's Law	$\frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} - \mathbf{J} = 0$	$\frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} - \mathbf{J} + iq\left[A_0, \mathbf{E}\right] - iq\left(\mathbf{A} \times \mathbf{B} - \mathbf{B} \times \mathbf{A}\right) = 0$
	$\nabla \bullet \mathbf{B} = 0$	$\nabla \bullet \mathbf{B} + iq\left(\mathbf{A} \bullet \mathbf{B} - \mathbf{B} \bullet \mathbf{A}\right) = 0$
Faraday's Law	$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$	$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} + iq\left[A_0, \mathbf{B}\right] = iq\left(\mathbf{A} \times \mathbf{E} - \mathbf{E} \times \mathbf{A}\right) = 0$

Table 2	
U(1) Symmetry Form (Traditional Maxwell Theory)	**SU(2) Symmetry Form**
$\rho_{\blacklozenge} = J_0$	$\rho_e = J_0 - iq(\mathbf{A} \bullet \mathbf{E} - \mathbf{E} \bullet \mathbf{A}) = J_0 + q\mathbf{J}_z$
$\rho_m = 0$	$\rho_m = -iq(\mathbf{A} \bullet \mathbf{B} - \mathbf{B} \bullet \mathbf{A}) = -iq\mathbf{J}_y$
$g_e = \mathbf{J}$	$g_e = iq\left[A_0, \mathbf{E}\right] - iq(\mathbf{A} \times \mathbf{B} - \mathbf{B} \times \mathbf{A}) + \mathbf{J} = iq\left[A_0, \mathbf{E}\right] - iq\mathbf{J}_x + \mathbf{J}$
$g_m = 0$	$g_m = iq\left[A_0, \mathbf{B}\right] - iq(\mathbf{A} \times \mathbf{E} - \mathbf{E} \times \mathbf{A}) = iq\left[A_0, \mathbf{B}\right] - iq\mathbf{J}_z$
$\sigma = \mathbf{J} / \mathbf{E}$	$\sigma = \frac{\left\{iq\left[A_0, \mathbf{E}\right] - iq(\mathbf{A} \times \mathbf{B} - \mathbf{B} \times \mathbf{A}) + \mathbf{J}\right\}}{\mathbf{E}} = \frac{\left\{iq\left[A_0, \mathbf{E}\right] - iq\mathbf{J}_x + \mathbf{J}\right\}}{\mathbf{E}}$
$s = 0$	$s = \frac{\left\{iq\left[A_0, \mathbf{B}\right] - iq(\mathbf{A} \times \mathbf{E} - \mathbf{E} \times \mathbf{A})\right\}}{\mathbf{H}} = \frac{\left\{iq\left[A_0, \mathbf{B}\right] - iq\mathbf{J}_z\right\}}{\mathbf{H}}$

Historically, electromagnetic theory was developed for situations described by the *U(1)* group. The dynamic equations describing the transformations and interrelationships of the force field are the well known Maxwell equations, and the group algebra underlying these equations is *U(1)*. There was a need to extend these equations to describe *SU(2)* situations and to derive equations whose underlying algebra is *SU(2)*. These two formulations are shown in Table 1. Table 2 shows the electric charge density, ρ_e, the magnetic charge density, ρ_m, the electric current density, g_e, the magnetic current density, g_m, the electric conductivity, σ, and the magnetic conductivity, s.

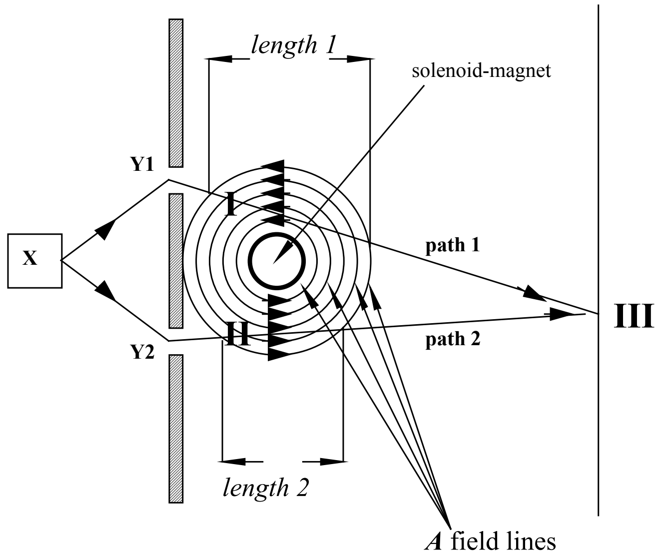

Fig 1. Two-slit diffraction experiment of the Aharonov-Bohm effect. Electrons are produced by a source at X, pass through the slits of a mask at Y1 and Y2, interact with the **A** field at locations I and II over lengths l_1 and l_2, respectively, and their diffraction pattern is detected at III. The solenoid-magnet is between the slits and is directed out of the page. The different orientations of the external **A** field at the places of interaction I and II of the two paths 1 and 2 are indicated by arrows following the right-hand rule.

As an example of the basic nature of topological explanation, we considered the Aharonov-Bohm effect. Beginning in 1959 Aharonov and Bohm[2] challenged the view that the classical vector potential produces no observable physical effects by proposing two experiments. The one which is most discussed is shown in Fig 1. A beam of monoenergetic electrons exists from a source at *X* and is diffracted into two beams by the slits in a wall at *Y1* and *Y2*. The two beams produce an interference pattern at III which is measured. Behind the wall is a solenoid, the **B** field of which points out of the paper. The absence of a free local magnetic monopole postulate in conventional *U(1)* electromagnetism ($\nabla \bullet \mathbf{B} = 0$) predicts that the magnetic field outside the solenoid is zero. Before the current is turned on in the solenoid, there should be the usual interference patterns observed at III, of course, due to the differences in the two path lengths.

Aharonov and Bohm made the interesting prediction that if the current is turned on, then due to the differently directed **A** fields along paths 1 and 2 indicated by the arrows in Fig. 1, additional phase shifts should be discernible at III. This prediction was confirmed experimentally[3] and the evidence for the effect has been extensively reviewed[4].

The topology of this situation dictates its explanation. Therefore we must clearly note the topology of the physical layout of the design of the situation which exhibits the effect. The physical situation is that of an *interferometer*. That is, there are two paths around a central location—occupied by the solenoid—and a measurement is taken at a location, III, in the Fig 1, where there is overlap of the wave functions of the test waves which have traversed, separately, the two different paths. (The test waves or test particles are complex wave functions with phase.) It is important to note that the overlap area, at III, is the only place where a measurement can take place of the effects of the **A** field (which occurred earlier and at other locations (I and II). The effects of the **A** field occur along the two different paths and at locations I and II, but they are *inferred*, and not measurable there. Of crucial importance in this special interferometer, is the fact that the solenoid present a *topological obstruction*. That is, if one were to consider the two joined paths of the interferometer as a raceway or a loop and one squeezed the loop tighter and tighter, then nevertheless one cannot in this situation—unlike as in most situations—reduce the interferometer's raceway of paths to a single point. (Another way of saying this is: not all closed curves in a region need have a vanishing line integral, because one exception is a loop with an obstruction.) The reason one cannot reduce the interferometer to a single point is because of the existence in its middle of the solenoid, which is a positive quantity, and acts as an obstruction.

The existence of the obstruction changes the situation entirely. *Without* the existence of the solenoid in the interferometer, the loop of the two paths *can be* reduced to a single point and the region occupied by the interferometer is then *simply-connected*. But *with* the existence of the solenoid, the loop of the two paths *cannot be* reduced to a single point and the region occupied by this special interferometer is *multiply-connected*. The Aharonov-Bohm effect only exists in the multiply-connected scenario. But note that the Aharonov-Bohm effect is a *physical* effect and simple and multiple connectedness are *mathematical descriptions* of physical situations.

The topology of the physical interferometric situation addressed by Aharonov and Bohm defines the physics of that situation and also the mathematical description of that physics. If that situation *were not* multiply-connected, but simply-connected, then there would be no interesting physical effects to describe. The situation would be described by *U(1)* electromagnetics and the mapping from one region to another is conventionally one-to-one. However, as the Aharonov-Bohm situation is multiply-connected, there is a two-to-one mapping $(SU(2)/Z_2)$ of the two different regions of the two paths to the single region at III where a measurement is made. Essentially, at III a measurement is made of *the differential histories* of the two test waves which traversed the two different paths and experienced two different forces resulting in two different phase effects.

In conventional, *i.e.*, normal *U(1)* or simply-connected situations, the fact that a vector field, viewed axially, is pointing in one direction, if penetrated from one direction on one side, and is pointing in *the opposite direction*, if penetrated from the same direction, but *on the other side*, is of no consequence at all—because that field is of *U(1)* symmetry and can be reduced to a single point. Therefore in most cases which are of *U(1)* symmetry, we do not need to distinguish between the direction of the vectors of a field from one region to another of that field. However, the Aharonov-Bohm situation is not conventional or simply-connected, but special. (The physical situation has a non-trivial topology). It is a multiply-connected situation and of $SU(2)/Z_2$ symmetry. Therefore the direction of the **A** field on the separate paths is of crucial importance, because a test wave traveling along one path will experience an **A** vectorial component directed *against* its trajectory and thus be slowed down, and another test wave traveling along another path will experience an **A** vectorial component directed *with* its trajectory and thus its speed is increased. These "slowing down's" and "increases in speed" can be measured as phase changes, *but not at the time nor at the locations (I and II) where they occur along the separate paths*, but later, and at the overlap location of III. It is important to note that if measurements are attempted at locations I and II in the Fig 1, these effects will not be seen because there is no two-to-one mapping at either I and II and therefore no referents. The locations I and II are simply-connected and therefore only the conventional *U(1)* electromagnetics applies. It is only region III which is multiply-connected and at which the histories of what happened to the test particles at I and II can be measured. In order to distinguish the "sped-up" **A** field (because the test wave is traveling "with" its direction) from the "slowed-down" **A** field (because the test wave is traveling "against" its direction), we introduce the notation: $\mathbf{A}_+$ and $\mathbf{A}_-$.

Because of the distinction between the **A** oriented potential fields at positions I and II—which *are not* measurable and are *vectors or numbers* of *U(1)* symmetry—and the **A** potential fields at III—which *are* measurable and are *tensors or matrix-valued functions* of (in the present instance) $SU(2)/Z_2 = SO(3)$ symmetry (or higher symmetry)—for reasons of clarity we might introduce a distinguishing notation. In the case of the potentials of *U(1)* symmetry at I and II we might use the lower case, $\mathbf{a}_\mu$, $\mu = 0,1,2,3$ and for the potentials of $SU(2)/Z_2 = SO(3)$ at III we might use the upper case $\mathbf{A}_\mu$, $\mu = 0,1,2,3$. Similarly, for the electromagnetic field tensor at I and II, we might use the lower case, $\mathbf{f}_{\mu\nu}$, and for the electromagnetic field tensor at III, we might use the upper case, $\mathbf{F}_{\mu\nu}$. Then the following definitions for the electromagnetic field tensor are:

At locations I and II the Abelian relationship is:

$$\mathbf{f}_{\mu\nu}(x) = \partial_\nu a_\mu(x) - \partial_\mu a_\nu(x)\,, \tag{1}$$

where, as is well known, $\mathbf{f}_{\mu\nu}$ is gauge invariant; and at location III the nonAbelian relationship is:

$$\mathbf{F}_{\mu\nu} = \partial_\nu A_\mu(x) - \partial_\mu A_\nu(x) - ig_m\left[A_\mu(x), A_\nu(x)\right], \tag{2}$$

where $\mathbf{F}_{\mu\nu}$ is *gauge covariant*, g_m is the magnetic charge density and the brackets are commutation brackets. We remark that in the case of nonAbelian groups, such as *SU(2),* the potential field can carry charge. It is important to note that if the physical situation changes from *SU(2)* symmetry to *U(1),* then $F_{\mu\nu} \to f_{\mu\nu}$.

Despite the clarification offered by this notation, the notation can also cause confusion, because in the present literature, the electromagnetic field tensor is *always* referred to as **F,** whether **F** is defined with respect to *U(1)* and *SU(2)* or other symmetry situations. Therefore, we shall not proceed with this notation. However, its important to note that the **A** field in the *U(1)* situation is a vector or a number, but in the *SU(2)* or nonAbelian situation, it is a tensor or a matrix-valued function.

We referred to the physical situation of the Aharonov-Bohm effect as an interferometer with an obstruction and it is 2-dimensional. It is important to note that it is not a toroid. A toroid is also a physical situation with an obstruction and is also of *SU(2)* symmetry. However, the toroid effects a two-to-one mapping in not only the *x* and *y* dimensions but also in the *z* dimension, and *without* the need of an electromagnetic field pointing in two directions + and –. The physical situation of the Aharonov-Bohm effect is defined only in the *x* and *y* dimensions (there is no *z* dimension) and in order to be of $SU(2)/Z_2$ symmetry *requires* a field to be oriented differentially on the separate paths. If the differential field is removed from the Aharonov-Bohm situation, then that situation reverts to a simple interferometric raceway which can be reduced to a single point and with no interesting physics.

How does the topology of the situation affect the explanation of an effect? A typical previous explanation[5] of the Aharonov-Bohm effect commences with the Lorentz force law:

$$\mathbf{F} = e\mathbf{E} + e\mathbf{v} \times \mathbf{B} \tag{3}$$

The electric field, **E**, and the magnetic flux density, **B**, are essentially confined to the inside of the solenoid and therefore cannot interact with the test electrons. An argument is developed by defining the **E** and **B** fields in terms of the **A** and ϕ potentials:

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi, \quad \mathbf{B} = \nabla \times \mathbf{A}\,. \tag{4}$$

Now we can note that these conventional *U(1)* definitions of **E** and **B** can be expanded to *SU(2)* forms:

$$\mathbf{E} = -\left(\nabla \times \mathbf{A}\right) - \frac{\partial \mathbf{A}}{\partial t} - \nabla\phi, \quad \mathbf{B} = \left(\nabla \times \mathbf{A}\right) - \frac{\partial \mathbf{A}}{\partial t} - \nabla\phi\,. \tag{5}$$

Furthermore, the *U(1)* Lorentz force law, Eq 3, can hardly apply in this situation because the solenoid is electrically neutral to the test electrons and therefore **E** = 0 along the two paths. Using the definition of **B** in Eq 5, the force law in this *SU(2)* situation is:

$$F = e\mathbf{v}\times\mathbf{B} = e\mathbf{v}\times\left(\left(\nabla\times\mathbf{A}\right) - \frac{\partial \mathbf{A}}{\partial t} - \nabla\phi\right), \tag{6}$$

but we should note that Eqs 3 and 4 are *still valid* for the conventional theory of electromagnetism based on the *U(1)* symmetry Maxwell's equations provided in Table 1 and associated with the group *U(1)* algebra. They are *invalid* for the theory based on the modified *SU(2)* symmetry equations also provided in Table 1 and associated with the group *SU(2)* algebra.

The typical explanation of the Aharonov-Bohm effect continues with the observation that a phase difference, δ, between the two test electrons is caused by the presence of the solenoid:

$$\Delta\delta = \Delta\alpha_1 - \Delta\alpha_2 = \frac{e}{\hbar}\left(\int_{l_2} A \bullet dl_2 - \int_{l_1} A \bullet dl_1\right) = \frac{e}{\hbar}\int_{l_2 - l_1} \nabla\times\mathbf{A}\bullet d\mathbf{S} = \frac{e}{\hbar}\int \mathbf{B}\bullet d\mathbf{S} = \frac{e}{\hbar}\varphi_M\ , \tag{7}$$

where $\Delta\alpha_1$ and $\Delta\alpha_2$ are the changes in the wave function for the electrons over paths 1 and 2, **S** is the surface area and φ_M is the magnetic flux defined:

$$\varphi_M = \iint \mathbf{A}_\mu(x)dx^\mu = \iint \mathbf{F}_{\mu\nu}d\sigma^{\mu\nu}\,. \tag{8}$$

Now, we extend this explanation further, by observing that the local phase change at III of the wavefunction of a test wave or particle is given by:

$$\Phi = \exp\left[ig_m \iint A_\mu(x)dx^\mu\right] = \exp\left[ig_m\varphi_M\right]. \tag{9}$$

Φ, which is proportional to the magnetic flux, φ_M , is known as the *phase factor* and is gauge covariant. Furthermore, Φ, the phase factor measured at position III is the *holonomy* of the *connection*, $\mathbf{A}_\mu$; g_m is the *SU(2)* magnetic charge density.

We next observe that φ_M is in units of volt-seconds *(V.s)* or $kg.m^2/(A.s^2) = J/A$. From Eq 7 it can be seen that $\Delta\delta$ and the phase factor, Φ, are dimensionless. Therefore we can make the prediction that if the magnetic flux, φ_M , is known and the phase factor, Φ, is measured (as in the Aharonov-Bohm situation), the magnetic charge density, g_m, can be found by the relation:

$$g_m = \ln\left(\Phi\right)/\left(i\varphi_M\right). \tag{10}$$

Continuing the explanation: as was noted above, $\nabla \times \mathbf{A} = 0$ outside the solenoid and the situation must be redefined in the following way. An electron on path 1 will interact with the **A** field oriented in the positive direction. Conversely, an electron on path 2 will interact with the **A** field oriented in the negative direction. Furthermore, the **B** field can be defined with respect to a local stationary component $\mathbf{B}_1$ which is confined to the solenoid and a component $\mathbf{B}_2$ which is either a standing wave or propagates:

$$\begin{aligned} \mathbf{B} &= \mathbf{B}_1 + \mathbf{B}_2, \\ \mathbf{B}_1 &= \nabla \times \mathbf{A}, \\ \mathbf{B}_2 &= -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi. \end{aligned} \tag{11}$$

The magnetic flux density, $\mathbf{B}_1$, is the confined component associated with *U(1) and SU(2)* symmetry and $\mathbf{B}_2$ is the propagating or standing wave component associated *only* with *SU(2)* symmetry. In a U(1) symmetry situation, $\mathbf{B}_2 = 0$.

The electrons traveling on paths 1 and 2 require different times to reach III from *X*, due to the different distances and the opposing directions of the potential **A** along the paths l_1 and l_2. Here we only address the effect of the opposing directions of the potential **A**. The change in the phase difference due to the presence of the **A** potential is then:

$$\begin{aligned} \Delta\delta = \Delta\alpha_1 - \Delta\alpha_2 = \frac{e}{\hbar}\left[\int_{l_2}\left(-\frac{\partial \mathbf{A}_+}{\partial t} - \nabla\phi_+\right).dl_2 - \int_{l_1}\left(-\frac{\partial \mathbf{A}_-}{\partial t} - \nabla\phi_-\right)dl_1\right] \bullet d\mathbf{S} = \\ \frac{e}{\hbar}\int \mathbf{B}_2 \bullet d\mathbf{S} = \frac{e}{\hbar}\varphi_M. \end{aligned} \tag{12}$$

There is no flux density $\mathbf{B}_1$ in this equation since this equation describes events outside the solenoid, but only the flux density $\mathbf{B}_2$ associated with group SU(2) symmetry; and the "+" and "–" indicate the direction of the **A** field encountered by the test electrons—as discussed above.

We note that the phase effect is dependent on $\mathbf{B}_2$ and $\mathbf{B}_1$, but not on $\mathbf{B}_1$ alone. Previous treatments found no convincing argument around the fact that whereas the Aharonov-Bohm effect depends on an interaction with the **A** field outside the solenoid, **B,** defined in *U(1)* electromagnetism as $\mathbf{B} = \nabla \times \mathbf{A}$, is zero at that point of interaction. However, when **A** is defined in terms associated with an *SU(2)* situation, that is not the case as we have seen.

We depart from former treatments in other ways. Commencing with a *correct* observation that the Aharonov-Bohm effect depends on the topology of the experimental situation and that the situation is not simply-connected, a former treatment then erroneously seeks an explanation of the effect in the connectedness of the *U(1)* gauge symmetry of conventional electromagnetism, but for which (1) the potentials are ambiguously defined, (the *U(1)* **A** field is gauge invariant) and (2) in *U(1)* symmetry $\nabla \times \mathbf{A} = 0$ outside the solenoid.

Furthermore, whereas a former treatment again makes a correct observation that the nonAbelian group, *SU(2),* is simply-connected and that the situation is governed by a multiply-connected topology, the author fails to observe that the nonAbelian group *SU(2)* defined over the integers modulo 2, $SU(2)/Z_2$, is, in fact, multiply-connected. Because of the two paths around the solenoid it is this group which describes the topology underlying the Aharonov-Bohm effect[6]. $SU(2)/Z_2 \cong SO(3)$ is obtained from the group *SU(2)* by identifying pairs of elements with opposite signs. The $\Delta\delta$ measured at location III in Fig. 1 is derived from a single path in *SO(3)* because the *two* paths through locations I and II in *SU(2)* are regarded as a *single* path in *SO(3)*. This path in $SU(2)/Z_2 \cong SO(3)$ cannot be shrunk to a single point by any continuous deformation and therefore adequately describes the multiple-connectedness of the Aharonov-Bohm situation. Because the former treatment failed to note the multiple connectedness of the $SU(2)/Z_2$ description of the Aharonov-Bohm situation, it fell back on a *U(1)* symmetry description.

Now back to the main point of this excursion to the Aharonov-Bohm effect: the reader will note that the author appealed to topological arguments to support the main points of his argument. Underpinning the *U(1)* Maxwell theory is an Abelian algebra; underpinning the *SU(2)* theory is a nonAbelian algebra. The algebras specify the form of the equations of motion. However, whether one or the other algebra can be (validly) used can only be determined by topological considerations.

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