Theories of Truth without Standard Models and Yablo's Sequences[1]

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Abstract.

In this paper, I argue that Yablo's sequence of sentences introduces new boundaries to the expressive capacity of languages with their own truth-predicate. Specifically, I show that adding an alleged truth-predicate to the language of first order aritmethic would cause enough expressive disorders as to make ponder whether the mentioned predicate was, after all, a truth-predicate of that language.

Keywords:

Yablo's Paradox – Non-Standard Models - ω-conservativeness - Theories of Truth

The discussion around *Yablo's Paradox* has launched a conceptual change in the expressive conditions that a language should fulfill in order to take the risks of comitting contradictions. Yablo's[2] main point is that if a language counts with enough resources as to quantify over infinite series of sentences, it is possible to formulate, with its own resources, a linguistic construction unable to receive a classic valuation consisting in truth-values. [3] In a nutshell, pace what was traditionally held, it would be possible to obtain a paradox involving semantic concepts without self-refefence playing any part whatsoever, apparently, in the language. The initial formulation of the paradox consists in an infinite set of sentences linearly ordered, each of them claims that all the following sentences are not true. This series doesn't seem to involve, at least in a superficial level, any kind of self-reference.[4] According to Yablo, this set would be incapable of having a model and, therefore, it would be inconsistent. Of course,

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[2] Yablo, S. [29] and [31].

[3] Peter Schlenker argues that Yablo's result is a special case of a much more general fact: under certain conditions, *any semantic phenomenon that involves self-reference can be emulated without self-reference* (this can be seen as a generalization of Cook, R. [6] Cfr. Schlenker, P. [22].

[4] Far from a general consensus on this, there are many who claim that Yablo didn't manage to show that there is not some kind of circularity involved in his sequence. Mainly, Priest and Beall belong to this group and Bueno, Collyvan, Sorensen and Yablo have maintained that Yablo's list generates a semantic paradox without circularity. Cfr Priest, G. [20], Beall, JC [1], Bueno, O. & Colyvan, M. [4], Sorensen, R. [24], and Yablo, S. [29].

there have been doubts about this result, first in a particular work by Priest and then in Hardy [5] and Ketland's[6] works. In particular, the latter has tried to show, starting from a formulation of the series expressed in the language of aritmethics, that it is possible to find a non-standard model for Yablo's sequence. Bueno and Colyvan have tried to answer to Ketland's point by admitting non-standar numbers not only in the domains of the models but also in the series of Yablo's sentences itself.[7]

In this paper I argue that Yablo's sequence of sentences introduces new boundaries to the expressive capacity of languages with their own truth-predicate. Specifically, I show that adding an alleged truth-predicate to the language of first order aritmethic would cause enough expressive disorders as to make ponder whether the mentioned predicate was, after all, a truth-predicate of that language.

I.-

Yablo's paradox involves a infinite sequence of sentences Y_k each one of them establishing that not all the following sentences are true.

(Y_0) For all $k > 0$, Y_k is not true.
(Y_1) For all $k > 1$, Y_k is not true.
(Y_2) For all $k > 2$, Y_k is not true.
(Y_3) For all $k > 3$, Y_k is not true.
And so on.

Informally, Yablo describes what happens here with his sequence in the following way[8]: suppose, for contradiction, that some particular sentence of the sequence is indeed true. For example, let's take Y_n as true. Considering what Y_n expresses, then for all $k > n$, Y_k is not true.For this reason, two consequences follow:

Y_{n+1} is not true,

and

For every $k > n+1$, Y_k is not true.

[5] Hardy, J. [13].
[6] See Ketland, J. [14].
[7] Bueno, O. & Colyvan, M. [5]
[8] Yablo, S. [29].

But, from the latter it can be concluded that what Y_{n+1} says is in fact the case. For this reason, contradicting the first assumption, Y_{n+1} is true. Thus Y_n doesn't have to be false after all. Due to this, no sentences of the sequence are true. But then, the sentences following any of the sentences given in the sequence are all not true. Therefore Y_n is true after all, which cannot be the case.

II.-

It is well known that from the previous informal proof Yablo intents to extract important consecuences pertaining the concept of *truth*. Since it doesn't seem to be an explicit use of any kind of self-reference, Yablo's infinite sequence of sentences seems to show that the aforementioned feature is neither a sufficient nor a necessary condition for the existence of a semantic paradox. No doubt the self-referentiality is a feature hard to characterize and its links to the concept of *circularity* are not helpful at all for this task.[9] Usually, the existence of self-referential features within the language has been assesed using techniques related to the aritmethization of formal languages. Generating liar-like sentences within the first-order aritmethic (*PA*), starting from the lemma of diagonalization, from which it is inferred the existence of fixed points for the predicates of the language, supports this path. With these techniques and from the assumption that any serious candidate for truth-predicate of aritmethics should satisfy the tarskian adequacy condition, it follows that truth cannot be defined within that language. For this reason, *Tarski's Theorem on the Indefinability of Truth*[10] constitutes a limitation to the expressive capacities of the language of aritmethics, since it implies the innability of expressing within aritmethics, reasoning classically, the interpretation of the truth-predicate.[11] For these languages, the theorem proves that the set of their true sentences is not expressable using aritmethization, if the underlying logic is the classical one. This result restricts the expressive capacity of these languages, and under these conditions, their own truth-predicates are not definable within them using their own resources. Even if they were, the diagonalization lemma implies for any formula "Tx" in which x is the only free variable, the existence of a closed formula λ such that

(The Liar) $\lambda \leftrightarrow \neg\, T(\underline{\lambda})$

[9] Even more, there are people thinking that an explanation on what the self-reference is cannot be given. Cfr. Leitgeb, H. [17].

[10] Tarski, A. [26]

[11] Recently, Hartry Field has shown with all clarity how to extend Tarski's result to different types of languages; particularly, to the language of Set Theory, showing that under the assumption that quantifiers of the language are interpreted in unrestricted way and the classicality of the underlaying logic, a hierarchy of truth predicates is generated. According to Field, this can be read in different ways: (i) claiming that there is not a unique concept of *truth* at all; (ii) claiming that there is such a concept but that it violates Tarski's T-schema or (iii) that there is such a concept but that logic must be altered in order to describe it. Cfr Field, H. [9], p. 30.

But such a formula cannot receive a classic valuation of truth-values, if “T” is interpreted as the truth-predicate of the language. Of course, this means that the resources of self-representation should be delimited. That is, if certain expressive conditions are fulfilled and if we reason clasically, the need for representing *all* the truths in a language could make place to the emerging of a hierarchy of languages, where each of their levels it would possess more resources than the immediately previous one, and where each level would be capable of representing the truth-predicate of the precedent.[12] In this way, the diagonalization lemma and the consequent existence of fixed points apply to any arithmetical predicate. But, if it is admitted with Tarski that an elemental principle of the functioning of the truth-predicate of a language is the compliance with the T-condition, the existence of a fixed point prevents the existence of a model in which the theory *PA* ∪ {This sentence is not true} could receive a consistent assignation of truth values. For this reason, it is not possible to express the truth of *PA* with the expressive resources of *PA*. Hence, if there is an unique concept of *truth* and we want to express it within some language, we should either adopt some kind of limitation to the T-schema[13] or stand aside of classic logic by adopting some sort of Kripkean model, for instance. [14]

In addition, it could be emphasized that Yablo’s sequence of sentences also pretends to limit the expressive resources of languages. Nonetheless, the idea is that these expressive limitations are the result of the capacity of expressing series of sentences not involving any kind of self-reference. Now, unlike what happens with the Liar sentence, the paradoxicallity of Yablo’s sequence of sentences has been doubted. Usually a set of sentences is considered paradoxical if, at least, there isn’t a consistent way of assigning truth-values to all of its members. In Yablo’s sequence case, an important point is directly related with the representation of the set itself. Specifically, we need to establish how the expressive resources are capable of representing the sequence of sentences.

Following this direction, Ketland has suggested[15] an answer to this question that contains a double move: first, he proposes to represent the sequence inside the language of first order arithmetic.[16] Second, he describes Yablo’s list of sentences as a sequence of biconditionals. Of course, what sustains such a movement is the idea of representing the application of the T-schema to any sentence of the type Y_n. In this way it is expected that the set of Yablo’s associated biconditionals of the form

[12] Regarding the alleged essential richness of metalanguage over the object language for which the definition of truth is effected, cfr De Vidi, D and Solomon, G. [7] and Ray, G. [21].
[13] For example, see Bolander, Th. [2] and Halbach, V. [11].
[14] Cfr Kripke, S. [15] (1975), "Outline of the Theory of Truth" *Journal of Philosophy* 72: 690–716.
[15] Ketland, J. [14], p. 295.
[16] This movement not only seems plausible because often enough truth is taken as a predicate applied to Gödelian numbers, what makes generating self-reference easier, but also because of the fact that one would expect that adding axioms or principles on the truth predicate within a consistent theory helps generating a new widened theory that inherits that consistency.

$\{Y_n \leftrightarrow$ For all $k > n$, Y_k is not true: $n \in \omega\}$

expressed inside first order arithmetic have a model, that is, the existence of an interpretation allowing us to assign truth conditions to every element of the set without falling into contradiction.

Ketland's point is disconcerting: unlike what we think, it is possible to find a model for such a set of sentences. However, this model cannot have as its domain the set of the standard natural numbers. For this, even when Yablo's sequence is consistent, it is at the same time ω-inconsistent.[17] In order to appreciate the point in more detail it is worth remembering that a standard model for arithmetic is any model of the set of all the sentences of the language or arithmetic that are true in the standard interpretation of arithmetic. The existence of non-standard models for arithmetic, models that are not isomorphic to the standard interpretation, is a direct consequence of the application of the theorems of Compactness and Löwenheim-Skolem for first order languages. Then, following the path opened by Hardy, Ketland has shown that the formalization of Yablo's sequence in the language of first order arithmetic is satisfiable by at least one non-standard model of arithmetic; [18] and for this reason, since Yablo's set of sentences formulated in the language of first order arithmetic has a model, it results being formally consistent.[19]

Strictly speaking, in Ketland's formalization the truth-predicate does not appear. He invites us to consider a language L_F as an extension of the language of arithmetic to which it has been added a primitive predicate F. In this extension it is formulated the theory $PA_F = PA \cup \{ F\underline{n} \leftrightarrow \forall y{>}\underline{n}\ (\neg F\ y) : n \in \omega\}$. This theory is the result of adding to PA a set of sentences that allegedly express Yablo's biconditionals. But, after all, if the truth-predicate wouldn't be involved in the formulation of Yablo's sequencie, what conceptual significance would the existence of such a sequence have? For this reason, I am interested in preserving in the formalization of Yablo's ideas the occurrence of the aforementioned semantic predicate. Of course, even when later it is shown that it is only an alleged truth-predicate without the corresponding expressive capacity.

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[17] Ketland, J. [14] p. 297.

[18] Peano's arithmetic theory formalized in first order is not categorical. That is, it can have models different from the standard (the one that has as domain the set of numbers {0, 1, 2…}). These other models (non-standard models) have domains including the standard but add another list not isomorphic of elements {0, 1, 2, …,…., -2*, -1*, 0*, 1*, 2*…}. Each one of the elements in this list is bigger to each of the standard numbers. Thus, new non-standard models can be obtained by adding new lists to the domain of interpretation. Infinite different non-standard models can be built up by including successively sets whose structures are analogous to that of whole numbers, bigger to all the rest of the previous elements. See Gaifman, H. [8].

[19] Ketland, J. [14] p. 297.

Informally, the proof on PA_F consistency is quite simple: it consists of appealing to an extended model to which not only standard natural numbers belong but also non-standard numbers, and to show that the complete list, numbered with the standard natural numbers, acquires, in this model, a consistent assignation of truth values. Obviously, the key here lies in that the new no-standard element that belongs to the constructed model is bigger than all the standard numbers with which we numbered the sentences of Yablo's sequence.[20] Nevertheless, although the sequence formalized in this way is not inconsistent, it happens to be ω-inconsistent.[21] That is, there is no model whose domain is the set of natural numbers in which Yablo's list could acquire a consistent interpretation.

A point to underline of Ketland's formalization is that the sequence of Yablo's sentences could not be well represented by a generalization like:

$\forall n$ ($Y_{\underline{n}}$ is true $\leftrightarrow$ $\forall k > \underline{n}$, Y_k is not true)

According to Ketland, as we have seen, the set of numeric instances of Yablo's sentences:

$\{Y_{\underline{n}} \leftrightarrow$ For all $k > \underline{n}$, Y_k is not true : $n \in \omega\}$

is consistent. But the schematization of the previous generalization,

$\forall x$ (x is true $\leftrightarrow$ $\forall y$ Ayx , y is not true) [22]

along with the general principles on predicates,

$\forall x \exists y$ Ayx

and

$\forall x \forall y \forall z$ ((Axy $\wedge$ Ayz) $\rightarrow$ Axz)

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[20] Another way of understanding Ketland's point consists in acknowledging that if Yablo's sequence formulated in a first order language were inconsistent, for the Theorem of Compactness, this inconsistency would carry over any finite subset of the sequence. But there are no inconsistent finite subsets.

[21] Formally, if a set of formulas is ω-inconsistent, it can be proved that even when each of the natural numbers fulfills a condition, there is a number that doesn't. Of course, we can be unable of proving that there is a specific number that doesn't fulfill the condition, but we can prove that there is one that doesn't fulfill it. Semantically it means that this set doesn't have as a model a structure that has as domain the set of natural numbers.

[22] Ketland dubbs *The Uniform Homogeneous Yablo Scheme* to this Principle.

happen to be inconsistent. However, the previous inconsistency, according to Ketland, is not related to the concept of *truth*, since the aforementioned general principles are not specific of this concept.

To summarize, Yablo's sequence of sentences formulated in the first order language of arithmetic is not strictly inconsistent, since it can be proved that there is at least one model, composed by non-standard numbers, in which it is possible to assign an interpretation without contradiction to the elements of the mentioned series. As we have seen, starting from this formulation of PA_F, Ketland shows that this theory has a model. His strategy consists in showing that there is a domain that includes non-standard elements with which we can build a model of the theory PA_F. Nevertheless, there is no model of PA_F whose domain are natural standard numbers en which the sequence achieves an interpretation with this characteristic. For which, even when the PA_F theory that includes the list of Yablo's sentences is not strictly paradoxical (since it has non-standard models), it cannot be interpreted using a structure in which numbers appearing in each of the biconditionals are standard natural numbers. For this reason, even if the sequence is consistent, it turns out to be ω-inconsistent. From what it follows, according to Ketland, that Yablo's Paradox is not strictly a paradox but actually a ω-paradox.

III.-

Ketland's result is directly connected to the meta-theoretic properties of first order languages. These are compact and satisfy Löwenheim-Skolem. Several options could follow, increasing the expressive capacity of the language in which it is whished to represent the sequence. An alternative mentioned by Priest[23] would be to increasing the inferential capacity of the language by adding to the usual inference rules the ω-rule.[24] Such a resource would allow to obtain a contradiction starting from the formulation of Yablo's sequence of sentences within a first order language. However, and even when Tarski[25] himself considered the aforementioned rule as an obvious case of transmission of the truth from premises to conclusion, the mentioned rule is not correct if we confine ourselves to the models of first order languages.

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[23] Priest, G. [20], p. 239.

[24] The ω-rule is formulated thus: be *P* any monadic predicate whatosever. Then, from P(1), P(2), P(3), …, it follows that $\forall$k, P(n). Regarding Yablo's sequence, the application of the ω-rule allows us to infer that $\forall$k, $\neg$T(k), from $\neg$T(1), $\neg$T(2), $\neg$T(3), …, .

[25] Tarski provides a similar inference to the ω-rule as a counterexample of the explanation of the notion of logical consequence in terms of derivability. Cfr. Tarski, A. [25], p. 410.

Of course, a possible option explored by Thomas Forster[26] would consist in adopting a first order language with other infinitary resources capable of expressing Yablo's list as an infinite conjunction:

$$\bigwedge_{n \in \omega} (Y_{\underline{n}} \text{ is true} \leftrightarrow \bigwedge_{k > \underline{n}} (Y_k \text{ is not true}))$$

A similar path has been explored[27] also by Roy Cook, who has proved that Yablo's paradox can be obtained inside a language with infinitarian respurces and without quantification. Of course, even when an infinitarian proof of contradiction is easy to obtain, notwithstanding, as Forster points out, "everything that we would obtain is a new case of failure of compactness in an inifinitary language".[28]

The compactness theorem and Löwenheim-Skolem's as well, imply that no denumerable series can be characterized categorically by means of formulas of a first order language.[29] And even when no first order quantifier is capable of preventing non-standard interpretations, this expressive flaw can be repaired using quantifications whose scope is the set of certain subsets of a denumerable set, instead of being the members of a denumerable set (as it happens with those of first order). In shor words, the motivation would be to consider that *the language of first order arithmetic is inadequate for expressing correctly Yablo's sequence of sentences.* And, of course, given the links between Yablo's list and natural numbers, the temptation of using a *higher order language,* where the models of formalized arithmetic within those languages are categorical, emerges immediately. We know that concepts like *well-order* cannot be expressed inside the boundaries of first order languages.[30] And since all the models of arithmetic expressed in second order are isomorfic, the existence of non-standard models is discarded. For which, for each of Ketland's bicondicional there is a corresponding element of the domain (a standard natural number). There is no danger like the existence of a domain with an element bigger to all the standard natural numbers serving as a model for the sequence. For this, Yablo's sequence formalized within second order arithmetic is unsatisfiable. [31] Of course, everything that this formalization would show is a new case of compactness failure, now related to higher level languages.

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[26] Forster, Th. [10].
[27] Cook, R. [6].
[28] Forster, [10].
[29] The property of *compactness* implies the fulfilling of the Upward Theorem of Löwenheim-Skolem and this in turn implies that infinite structures cannot be described in the language.
[30] Shapiro, S. [23], p. 106.
[31] Of course, this result doesn't show that Yablo's sequence formalized within second order arithmetic would be inconsistent.

IV.-

Tarski's Theorem limits our capacities of expressing, within *PA,* all instances of the T-schema: if the set of all the T-sentences joins to *PA*, the resulting theory will be inconsistent and, therefore, it will lack a model. This result is usually interpreted as the *incapability* of *PA* of expressing its own truth predicate. What underlies this idea is that the adding of the instances of the aforementioned schema to a consistent theory wouldn't have to cause any inconsistency necessarily. For this reason, when it does cause them, we consider that the set of this instances becomes inexpressible within *PA*. According to Ketland's result, the set *PA* $\cup$ $\{Y_n \leftrightarrow$ for all k > n, Y_k is not true: n $\in \omega\}$ is consistent, or at least it is ω-inconsistent. That is, Ketland shows that this set does have a model: if Yablo's sequence is expressed within first order arithmetic, the same is satisfied by a structure whose domain includes, besides standard numbers, non-standard natural numbers. In what follows I want to emphasize that even if the sequence of Yablo sentences could have non-standard models, it will never have a standard model anyway. From my point of view, this result has significant consequences for our capacity to express the concept of *truth*. Analogously to Tarski's theorem, it is a direct consequence of Ketland's result that the adding to *PA* of expressive resources enough to talk of the infinite ordered sequences of sentences like Yablo's limits the capacity of expressing the truth within the resulting theory. And this happens not because the theory doesn't have a model and for this reason it is inconsistent, but because even if we were able of finding models for it, the alleged truth predicate of the resulting theory would not be a good representation of the truth. That is; from my perspective, the sequence of Yablo sentences shows us that a language possessing enough resources to express infinite sequences of sentences will be a language unable to express its own truth predicate.

Every successful formal truth theory must be supported by some philosophical intuitions. If not, we fall into the risk that the alleged axiomatization of truth won't be able to express a legitimate truth. However, *PA* $\cup$ $\{Y_n \leftrightarrow$ for all k > n, Y_k is not true: n $\in \omega\}$ is far from being an acceptable truth theory, since as such is not able to preserve the intuition according to which adding an alleged theory of truth to a theory having a model wouldn't have to interfere with the pretended ontology of the base-theory. Specifically, regarding *PA*, the adding of Yablo's sequence woudn't have to disturb the arithmetic's pretended ontology. Nevertheless, as we have seen, it does disturb it. Not only because *PA* $\cup$ $\{Y_n \leftrightarrow$ for all k > n, Y_k is not true: n $\in \omega\}$ doesn't have a standard model, but mainly because it in itself lacks a standard model.

The reason why the fact that *PA* $\cup$ $\{Y_n \leftrightarrow$ for all k > n, Y_k is not true: n $\in \omega\}$ lacks a standard model prevents the alleged truth predicate of language from expressing legitimate truth is related to the fact that the theory *PA* $\cup$ $\{Y_n \leftrightarrow$ for all k > n, Y_k is not true: n $\in \omega\}$ is not ω-

conservative and therefore, that it defines truth conditions for the formulas of the theory that won't depend on PA's intended ontology.

Note that the Ketland's proposed resolution of Yablo's Paradox implies giving up on the conservativeness requirement: the truth-free part of the theory *has ω-models* for arithmetic, but the theory of truth that enlarges that theory has no ω-models. That is, *PA* is a first order theory enriched with enough expressions as to express arithmetic. Take this theory on the standard interpretation. In it, every one of the 0, 1, 2 and 3 numerals are interpreted as referring to the numbers 0, 1, 2 and 3 respectively. The same thing will happen with predicates and the function symbols expressing arithmetic concepts. And quantifiers will have the standard natural numbers as their scope. In this way, in the standard interpretation it is guaranteed that expressions of arithmetic such as "3 + 0 = 3" o "∀x (x + 0 = x)" speak of natural numbers. Of course, there are no categoricity results forbidding the existence of non-standard models. But the important thing is that *PA*'s axioms and theorems come out to be true in those structures whose domains are exclusively composed by natural numbers.

Now, part of the reasons we have to introduce a truth-predicate applicable to *PA*'s axioms and theorems is that we wish that "∀x (x + 0 = x)" would come out as true exactly when this formula spoke of those numbers. That is, the enlargement of *PA* that allows the application of an alleged truth-predicate to *PA*'s formulas obtaining in turn formulas as "*Tr* (∀x (x + 0 = x))", shouldn't alter *PA*'s intended ontology. Formally, what is required if we wish to preserve this intuition is that the enlarged theory should be ω-conservative: if *PA has ω-models,* the extension of *PA* that includes occurrences of an alleged truth-predicate *must also have ω-models.* Notwithstanding, as we have seen, Yablo's sequences along with arithmetic, won't comply with this result. They can indeed have a model, but they have to contain, besides standard elements, non-standard numbers. For this reason, the introduction of the alleged truth-predicate doesn't maintain the standard ontology and therefore, the expression ends up being unable to express a legitimate truth.

The unfulfilling of ω-conservativeness causes that the truth of "*Tr* (∀x (x + 0 = x))" inside a theory including Yablo's biconditionals *cannot depend on* the standard natural numbers. Given that the theory including them does not have a model whose domain is that of natural numbers, -since it is ω-inconsistent- there is no way of guaranteeing that the truth of the sentences of the theory will supervene on the theory's pretended ontology. Even so, it seems that the distinction itself between the standard and non-standard model seems to assume that the semantic notions, and particularly language's truth-predicate, has a standard interpretation. For this, assuring that there is no standard model for Yablo's sequence would be conceptually enough for ensuring that a truth predicate that is part of the expressive resources of a first order language allowing to express infinite ordered series of sentences does not represent a legitimate concept of truth. What the ω-conservativeness guarantees is that the arithmetic truth supervenes on the arithmetic ontology. That is, since they end up being ω-inconsistent, Yablo's sequences represent an alleged truth-predicate whose application conditions cannot make it to depend on natural numbers. Otherwise, whatever the principles are that allow us to establish that a monadic predicate expresses the truth-predicate, I consider that they should guarantee that the mentioned predicate applies to certain expressions of language, assuring that the result offers *the* right extension for the concept of *truth*. But if, following Ketland, we concede that the components of Yablo's sequence are interpreted by means of non-standard models (models whose domains include non-standard numbers), the interpretation of the alleged truth-predicate does not seem to offer an appropriate interpretation for that language. In any case, it seems to offer a characterization of a *non-standard* concept of truth.

In sum, conservativeness over ω-models seems obviously to be a highly desirable feature of a theory of truth [32], and Ketland's result proves that the theory $PA \cup \{Y_n \leftrightarrow$ for all $k > n$, Y_k is not true: $n \in \omega\}$, because it doesn't comply with this feature, end up being unable of expressing correctly the semantic properties of the sentences of *PA.* A theory of truth should be ω-consistent because, if it is not, this theory may not be interpreted as speaking (only) about the intended ontology of the theory to which it applies. For this reason, the enlargement of Yablo's sequence to *PA* should be ω-conservative. Only in this way we would have certain warrant that the sequence expresses legitimate truth. No monadic predicate that we add to language will express legitimate truth if its introduction to the language of arithmetic introduces in turn *a dramatic deviation in the theory's intended ontology*: in order to be able to speak of the concept of *arithmetic truth*, the arithmetic has to abandon the possibility of speaking about standard natural numbers. For this reason, the ω-conservativeness will be an additional requirement that should be satisfied every time we are concerned with expressing truth: not only do we want that the adding of Yablo's sequence to *PA* ends up being consistent, we also want that the result will be capable of conserving the ontology of the pretended interpretation of *PA,* for it seems plausible to

[32] In the last few years several constructions have emerged that, as Yablo's series, are consistent although ω-inconsistent. All of them lead to results of non-standardness rather than to inconsistency. In this line of thought, McGee has presented an infinite Liar-sentence. This sentence asserts that not every iterated application of the truth predicate to itself will be true. There is a non-standard model for this sentence, even when, of course, it lacks a standard model itself. Visser has presented a construction analogous to McGee's that, unlike the latter, involves a non-wellfounded hierarchy of languages, each level with their respective truth predicates. Leitgeb, instead, has formulated a construction ω-inconsistent assuming the existence of beings with infinite capacities. It can be shown that Uzquiano's paradox on the denotation of certain definite descriptions within *PA* and that of the Gods blocking the road to a man also generate ω-inconsistencies. See Leitgeb, H. [16], McGee, v. [18], Uzquiano, G. [27] and Visser, A. [28], Yablo, S. [30].

maintain that no theory of truth should imply a substantive answer to what numbers are or to what the ontology of arithmetic is.

However, that ω-conservativeness should be treated as an *adequacy condition* for any theory pretending to express legitimate truth is not out of discussion. Recently a number of investigations on truth theories have appeared, all of them lacking of a standard model. [33] In this direction, Halbach and Horsten[34] claim that there are some truth theories that are ω-inconsistent. In the first place, because theories with this characteristic are arithmetically sound, that is, this type of theories does not prove any false arithmetical sentence. For this reason, the ω-inconsistency concerns only that part of the theory that deals with the truth predicate.[35] But, precisely, my point is that ω-inconsistency implies that the theory that includes the truth predicate is not conservative in ω-models. The lack of attractiveness is not the result of permitting to prove false sentences of arithmetics, but that of allowing to preserve arithmetic's pretended ontology. On the face of this reply, Halbach and Horsten could answer that the idea that such theories don't have nice models is at least questionable. "We accept set theory although we cannot prove that there is any nice model for set theory. Because of Gödel's second incompleteness theorem we cannot even prove that set theory has any model." [36] But there seem to be differences between the impossibility of proving the existence of a nice model and even between the impossibility of proving the existence of a model (as it happens in the case of the set theory [37]), and the possibility of proving the non-existence of a nice model (as it happens in the case of the truth theories lacking standard models). This last point seems to show that, unlike what happens in set theory, the truth theory doesn't express legitimate truth, for we have proofs enough showing that there is not any ω-model in which the theory doesn't end up being truth.

In the same way I'm asserting it, they do accept that there are some differences between the case of set theory and the case in which we get a truth theory without a ω-model. Nevertheless, they claim that set theory does not reject the existence of a strong inaccessible cardinal number, but in the case of this type of theory of truth, that ω-inconsistency refutes the

[33] Leitgeb, H. [16].
[34] Halbach, V. and Horsten, L. [12]
[35] Halbach, V. and Horsten, L. [12], p. 213.
[36] Halbach, V. and Horsten, L. [12], p. 214.
[37] However, vann McGee has recently show that the axioms of second-order Zermelo-Fraenkel set theory plus choice with urelements (ZFCU) plus the axiom that urelements form a set are able to characterize the structure of the universe of pure sets up to isomorfism. This result has to assume unrestricted quantification. Then, any two models of ZFCU+ the Urelement Set Axiom in which our quantifiers take unrestricted range over all the objects there are have isomorfic pure sets. Of couse, McGee's point doesn't tell us if there are seven inaccesible cardinals or if there are more or fewer, but it does tell us that the sentence of set theory that states that there are seven inaccesible cardinals is either true in all models of second order of ZFCU+ the Urelement Set Axiom in which our quantifiers take unrestricted range or false in all such models. Anyway, this result is very controversial and I don't need it in order to defend my position. Cfr McGee, v. [19].

existence of a ω-model. And, unlike what I myself think, Halbach and Horsten don't think that this result makes this theory unacceptable. They claim that the semantic functioning of the theory is very natural. For any part of the theory with a finite number of Yablo sentences possesses a nice standard model.[38] Since we can use only finitely resources in any proof, at any step of our reasoning we will have a nice model. The authors seem to think that in the case of Yablo's sequence the ω-inconsistency just reflects the fact that there is no nice limit model at level ω. But, at the same time, the consistency of the sequence ensures that nothing is wrong with the theory. Note, however, that in the same way in which Ketland's result can be interpreted as expressing that for Yablo's sequence there is no nice limit model at level ω, the lack of categoricity of their models could lead us to suspect that there is indeed one. If we accept an ω-inconsistent theory of truth, what we call *infinite sequence* doesn't really mean infinite sequence, and for each thing n that satisfies "natural number" in the model, there are only finitely many objects in the model that satisfy "natural number that precede n". Of course, this results from formalizing Yablo's sequence in a first order language. But we should note here that the sequence turns out to be unsatisfiable under a formalization that includes high order resources. For this reason, I take it that the fact that there is a consistent formulation of the sequence is not enough to prove that there is nothing wrong with the theory. It takes more than that, and I suspect that the non-fulfillment of the ω-conservativeness provide us a good reason to believe that there is indeed something fishy about these theories.

Along with all these, it could be interesting to mention that recently Hartry Field has shown other difficulties in talking about the truth theory for our language being ω-inconsistent.[39] According to Field, such a theory involves a failure analogous to Tarski's schema for "truth of".

For all numbers *n*, <B(x)> is true of *n*, but <(∀n∈Number) B(*n*)> is not true.

Field shows that assuming Tarski'a schema for truth, this sentence is equivalent to

For all numbers *n*, <B(x)> is true of *n*, but it is not the case that all numbers *n*, B(*n*).

what plainly does not fulfill the mentioned schema for "true of". In this sense, if it is agreed that the Tarskian schema of truth constitutes the core of any reflection on truth, Field's point provide us with yet another reason for doubting that truth theories lacking standard models give us a correct axiomatization of arithmetic's truth-predicate.

[38] Halbach, V. and Horsten, L. [12], p. 214.
[39] Cfr. Field, H. [9], p. 93-94.

In a nutshell, I take it that what Yablo's paradox shows us is that a first order theory pretending to express its own truth predicate, if it is capable of expressing infinite ordered series of sentences, not only has to be consistent (it must have a model) but it also has to be ω-consistent. If it wasn't, the theory wouldn't have a ω-model and in that case, it wouldn't have to be interpreted as referring to natural numbers but, instead, to non-standard numbers. In other words: the ω-inconsistency of a theory implies that his models are non-standard, and in the case that a theory pretends to express the truth predicate, its impossibility of having standard models prevents the expressed truth from representing our intuition according to which the truth of Yablo's sequence of biconditionals should depend on standard natural numbers. For that reason, the fact that Yablo's sequence of sentences is ω-inconsistent should be enough to show a new kind of expressive incapacity: the one of representing, within a language, a truth predicate that establishes a close link between truth and the standard ontology. Because of this, the existence of non-standard models, even when it avoids inconsistency, generates a new expressive limitation related to truth: that of allowing an adequate representation of the sequence as talking about the *concept of truth in that language.*

V.-

We have seen that there are several ways of presenting Yablo's sequence of sentences. All of them seem to have significant consequences related to the concept of *truth*. Whether it is because any formulation that enriches the resources of first order languages (such as infinitary languages or those possessing the ω-rule, or directly high order languages) allows us to obtain a contradiction when expressing the sequence, or because its first order formulation is capable of attaining a model (one including standard elements), it will be possible to prove that when expressing a Yablo series within the language of first order arithmetic, a dramatic deviation of the pretended ontology occurs. The resulting truth theory is not ω-consistent, since it is not capable of keeping the capacity of talking about the standard natural numbers. Such a result allows us to find a new expressive limitation: a language capable of expressing infinite series of sentences is unable to talk about its own truth predicate.

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