Completeness of Public Announcement Logic in Topological Spaces

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Outlook of the Talk

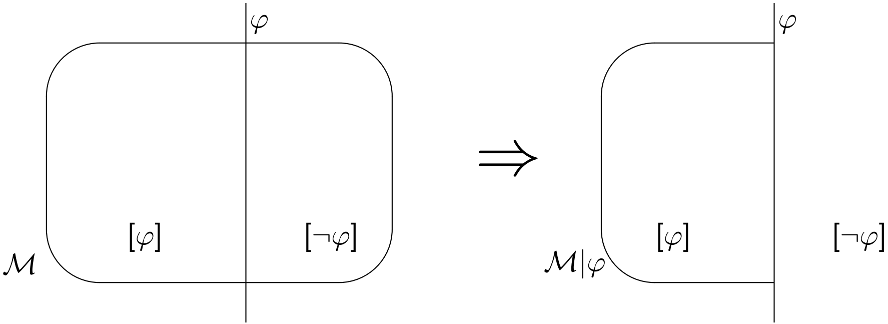
- Public Announcement Logic
- Topological Semantics
- Completeness
- Conclusion

What is Public Announcement Logic?

A paradigm for state-elimination based dynamic epistemology and communication! (Plaza, 1989; van Ditmarsch *et al.*, 2007)

1. A truthful announcement φ is made (by an external agent) to the "public", i.e. to all of the agents/knowers,
2. The announcement φ becomes **common knowledge** among the agents,
3. The agents "update" their epistemic status by state elimination,
4. The agent eliminate the states that do not agree with the announcement

A Simple Illustration



where $[\varphi]$ is the extension of φ, i.e. the points where φ is true.

Language

The language of public announcement logic (PAL) is that of epistemic logic extended with an additional announcement operator.

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_i \varphi \mid [\varphi]\varphi$$

here $\Box_i \varphi$ and $[\varphi]\psi$ will read "the agent i knows φ" and "after the public announcement of φ, the formula ψ is true" respectively.

Semantics

Let $\mathcal{M} = \langle W, \{R_i\}_{i \in I}, V \rangle$ be a model where W is a non-empty set, $\{R_i\}_{i \in I}$ is a collection of binary relations defined on W for each agent i, and V is a valuation sending propositional variables to subsets of W, and i is an agent from the set of agents I.
For atomic propositions, negations and conjunctions, the semantic definition is as usual.

Semantics

For modal operators, we have the following semantics:

$$\mathcal{M}, w \models \Box_i \varphi \quad \text{iff} \quad \mathcal{M}, v \models \varphi \text{ for each } v \text{ such that } (w, v) \in R_i$$
$$\mathcal{M}, w \models [\varphi]\psi \quad \text{iff} \quad \mathcal{M}, w \models \varphi \text{ implies } \mathcal{M}|\varphi, w \models \psi$$

Here the updated model $\mathcal{M}|\varphi = \langle W', R', V' \rangle$ is defined by restricting $\mathcal{M}$ to those states where φ holds.
(Plaza, 1989)

Reduction Axioms

The axiom system of PAL is that of multi-modal (multi-agent) S5 epistemic logic with the following additional ones.

Atoms	$[\varphi]p \leftrightarrow (\varphi \to p)$
Partial Functionality	$[\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\psi)$
Distribution	$[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
Knowledge Announcement	$[\varphi]\Box_i\psi \leftrightarrow (\varphi \to \Box_i[\varphi]\psi)$

The additional rule of inference for $[\cdot]$ is called the *announcement generalization* and is described as follows.

From $\vdash \psi$, *derive* $\vdash [\varphi]\psi$.

Soundness and Completeness

Soundness

Soundness of the axioms is a simple and fun exercise.

Completeness

Completeness is easy.
Axioms show that any formula in the new language is reducible to the basic modal language. Therefore, PAL is *equi-expressible* as the basic modal logic (Plaza, 1989).
Thus, the completeness follows immediately.

Some Properties

- $(\varphi \to [\varphi]\psi) \leftrightarrow [\varphi]\psi$
- $[\varphi \wedge [\varphi]\psi]\chi \leftrightarrow [\varphi][\psi]\chi$

(van Ditmarsch *et al.*, 2007)

Topological Definitions

A topological space $\mathcal{S} = \langle S, \sigma \rangle$ is a structure with a set S and a collection σ of subsets of S satisfying the following axioms:

1. The empty set and S are in σ.
2. The collection σ is closed under arbitrary union.
3. The collection σ is closed under finite intersection.

Interior Operator

Recall now that the topological interior operator $\mathbb{I}$ satisfies the following properties for each open $X, Y \in \sigma$:

1. $\mathbb{I}(X) = X$
2. $\mathbb{I}(X \cap Y) = \mathbb{I}(X) \cap \mathbb{I}(Y)$
3. $\mathbb{I}(\mathbb{I}(X)) = \mathbb{I}(X)$

In topological models, we will use I operator for modality instead of the usual operator $\Box$.

Logical Definitions

A topological model $\mathcal{M}$ is a triple $\langle S, \sigma, v \rangle$ where $\mathcal{S} = \langle S, \sigma \rangle$ is a topological space, and v is a valuation function sending propositional letters to the subsets of S, i.e. $v : P \to \wp(S)$.

Definition (Topological Semantics)

$$
\begin{array}{lll}
\mathcal{M}, s \models p & \text{iff} & s \in v(p) \text{ for } p \in P \\
\mathcal{M}, s \models \neg\varphi & \text{iff} & \text{not } \mathcal{M}, s \models \varphi \\
\mathcal{M}, s \models \varphi \wedge \psi & \text{iff} & \mathcal{M}, s \models \varphi \text{ and } \mathcal{M}, s \models \psi \\
\mathcal{M}, s \models \mathsf{I}\varphi & \text{iff} & \exists U \in \sigma (s \in U \wedge \forall t \in U, \mathcal{M}, t \models \varphi)
\end{array}
$$

The C operator can then be defined accordingly:

$$\mathcal{M}, s \models \mathsf{C}\varphi \quad \text{iff} \quad \forall U \in \sigma (s \in U \to \exists t \in U, \mathcal{M}, t \models \varphi)$$

Topological vs Kripkean Semantics

Topological

$\mathcal{M}, s \models \mathsf{I}\varphi$ iff $\exists U \in \sigma(s \in U \wedge \forall t \in U, \mathcal{M}, t \models \varphi)$

Kripkean

$\mathcal{M}, s \models \Box\varphi$ iff $\forall t \in U(sRt \rightarrow \mathcal{M}, t \models \varphi)$

Complexity and Expressivity: Topological Semantics is Σ_2 as opposed to Π_1 Kripke Semantics.

Correspondence: Topological vs Kripke Frames

Every S4 Kripke frame $\langle S, R \rangle$ gives rise to a topological space $\langle S, \sigma_R \rangle$, where σ_R is the set of all upward closed subsets of the given frame. It is easy to see that the empty set and S are in σ_R, and furthermore arbitrary unions and finite intersections of upward closed sets are still upward closed. Hence, σ_R is a (Alexandroff) topology.

Note that Alexandroff spaces are those topological spaces in which intersection of any family of opens is again an open.

For the converse direction, put $sR_\sigma t$ if $s \in Clo(t)$. It is an easy exercise to observe that R_σ is reflexive and transitive.

Some Lemmas

Let σ be a topology on S. For a formula φ, define $S_\varphi = S \cap (\varphi)$ where (φ) is the extension of φ, i.e the points where φ is true. Similarly, define $v_\varphi = v \cap S_\varphi$ for the valuation.

Lemma

Let σ be a topology. Then, $\sigma_\varphi = \{O \cap S_\varphi : O \in \sigma\}$ is a topology, too.

Proof.

An easy exercise.

Definitions

We can now define public announcement operator in topologic setting.
Let $\mathcal{M} = \langle S, \sigma, v \rangle$ be a topological model.
Define $\mathcal{M}_\varphi := \langle S_\varphi, \sigma_\varphi, v_\varphi \rangle$ as before.

Definition (Public Announcements)

$\mathcal{M}, s \models [\varphi]\psi$ *if and only if* $\mathcal{M}, s \models \varphi$ *implies* $\mathcal{M}_\varphi, s \models \psi$

Completeness

Reduction axioms that we have discussed earlier work perfectly in topological spaces. We will only deal with the modal reduction here.

$$[\varphi]\mathsf{I}\psi \leftrightarrow (\varphi \to \mathsf{I}[\varphi]\psi)$$

Theorem

PAL in topological spaces is complete with respect to the earlier axiomatization.

Compeleteness: Proof

Proof.

$$\begin{array}{lll}
\mathcal{M}, s \models [\varphi]\mathsf{I}\psi & \text{iff} & \mathcal{M}, s \models \varphi \rightarrow \mathcal{M}_\varphi, s \models \mathsf{I}\psi \\
& \text{iff} & \mathcal{M}, s \models \varphi \rightarrow \\
& & \exists U_\varphi \in \sigma_\varphi (s \in U_\varphi \wedge \forall t' \in U_\varphi, \mathcal{M}_\varphi, t' \models \psi) \\
& & \text{(so far, definitions)} \\
& \text{iff} & \mathcal{M}, s \models \varphi \rightarrow \\
& & \exists U \in \sigma (s \in U \wedge \forall t \in U (\mathcal{M}, t \models \varphi \rightarrow \mathcal{M}_\varphi, t \models \psi) \\
& & \text{(since } U_\varphi = U \cap (\varphi) \text{ for some } U \in \sigma) \\
& \text{iff} & \mathcal{M}, s \models \varphi \rightarrow \\
& & \exists U \in \sigma (s \in U \wedge \forall t \in U (\mathcal{M}, t \models [\varphi]\psi)) \\
& \text{iff} & \mathcal{M}, s \models \varphi \rightarrow \mathcal{M}, s \models \mathsf{I}[\varphi]\psi \\
& \text{iff} & \mathcal{M}, s \models \varphi \rightarrow \mathsf{I}[\varphi]\psi
\end{array}$$

Future Work

This is a first step to formalize *change* in topological modal logic. However, there is a lot left to do.

- What is the connection between topologies and fixed-points?
- How can we define fixed-point logics in topological settings?
- How can we use continuous functions and homotopies to represent knowledge change?
- Connection with weak topologies (Başkent, 2007)?

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Thanks for your attention!

Talk slides and the paper are available at:

`www.canbaskent.net`