# Metric Spaces in Synthetic Topology

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**Abstract**

We investigate the relationship between the synthetic approach to topology, in which every set is equipped with an intrinsic topology, and constructive theory of metric spaces. We relate the synthetic notion of compactness of Cantor space to Brouwer's Fan Principle. We show that the intrinsic and metric topologies of complete separable metric spaces coincide if they do so for Baire space. In Russian Constructivism the match between synthetic and metric topology breaks down, as even a very simple complete totally bounded space fails to be compact, and its topology is strictly finer than the metric topology. In contrast, in Brouwer's intuitionism synthetic and metric notions of topology and compactness agree.

## 1 Introduction

By *synthetic topology* we broadly understand a formulation of topology which emphasizes representability of the lattice of open subsets. More precisely, rather than axiomatizing lattices of open sets, we axiomatize the Sierpiński space $\Sigma$ and identify the topology $\mathcal{O}(X)$ of $X$ with the exponent $\Sigma^X$. Topology is thus an intrinsic structure of a space, and not something that we add to a set at will. This changes the spirit of the subject by linking topology more closely with intuitionistic logic, $\lambda$-calculus, and computation in general.

Synthetic topology is not a finished subject. Among possible approaches [19, 6] we chose the one that is most similar to synthetic domain theory [18, 10], in which spaces are treated as objects of a suitable topos. We do not emphasize toposes as categories, however, but rather view them as models of constructive mathematics and develop our theory constructively in the style of Errett Bishop.

Metric spaces are prime examples of spaces, both in classical and constructive mathematics [2]. A metric $d : X \times X \to \mathbb{R}$ induces a topology on the underlying set $X$. How this topology is related to the intrinsic topology $\Sigma^X$ is one of the concerns of the present paper. We also investigate the relationship between basic (synthetic) topological notions and their metric analogues: continuity vs. $\epsilon$-$\delta$ continuity, compactness vs. totally bounded completeness, and overtness vs. separability.

As is customary in constructive mathematics, we use intuitionistic logic and restrict the use of the axiom of choice to the countable version. In fact, in the proofs we only apply the following form of Number Choice:

$$(\forall\, n \in \mathbb{N} \,.\, \exists\, m \in \mathbb{N} \,.\, \phi(n,m)) \implies \exists\, f \in \mathbb{N}^{\mathbb{N}} \,.\, \forall\, n \in \mathbb{N} \,.\, \phi(n, f(n)).$$

This principle is also known as $\mathrm{AC}_{0,0}$ and number-number choice. The pedantic readers will notice that we only apply Number Choice to the case when $\phi$ is an open relation. Nevertheless, we note that the background theory of metric spaces contains other uses of countable choice, for instance in the construction of the Cauchy complete field of real numbers $\mathbb{R}$ as the Cauchy completion of the rational numbers $\mathbb{Q}$.

The paper is organized as follows. In Section 2 we review basic notions of synthetic topology. Section 3 makes initial observations about metric spaces in synthetic topology. Section 4 relates synthetic and metric compactness. We prove that synthetic compactness of Cantor space is equivalent to a variant of Brouwer's Fan Principle, provided that the metric and synthetic topologies on Cantor space match. Section 5 is devoted to showing that the intrinsic and metric topologies of complete separable metric spaces match if they do so for the Baire space. Section 6 derives basic consequences of such a desirable match, as well as a way of transferring topological bases along open surjections. This is applied to the computation of topology of the non-Hausdorff space $\Sigma^{\mathbb{N}}$. The last section presents topology in three models of constructive mathematics: classical mathematics as a trivial example, Russian constructivism in which the match between synthetic and metric topology is poor, and Brouwer's intuitionism with a perfect match.

## 2 Synthetic Topology

We briefly review the basic notions of synthetic topology, as developed by Martín Escardó [7]. A *dominance* is a subset $\Sigma \subseteq \Omega$ of the set of truth values $\Omega$ such that[1] $\bot, \top \in \Sigma$, and the *dominance axiom* holds: for every

[1]The original definition by Rosolini [18] does not require $\bot \in \Sigma$ and one can indeed develop some amount of topology without this assumption.

$p \in \Omega$ and $u \in \Sigma$, if $u \implies (p \in \Sigma)$ then $(u \wedge p) \in \Sigma$. From the axiom it follows easily that $\Sigma$ is closed under $\wedge$.

We call the elements of $\Sigma$ *open* truth values. Similarly, a subset $U \subseteq X$ is *(intrinsically) open* when $x \in U$ is an open truth value for all $x \in X$, i.e., $U$ is classified by a map $X \to \Sigma$. The *(intrinsic) topology* $\mathcal{O}(X)$ on $X$ is the set of all open subsets of $X$. The sets $\mathcal{O}(X)$ and $\Sigma^X$ are isomorphic, and in fact we identify them.

The dominance axiom implies that an open subset $U \subseteq V$ of an open subset $V \subseteq X$ is open in $X$. Because the decidable truth values $2 = \{\bot, \top\} = \{p \in \Omega \mid p \vee \neg p\}$ are contained in $\Sigma$, decidable subsets are open, special cases of which are the empty subset and the whole set.

In our setting all maps are automatically continuous: given any $f : X \to Y$ and an open $U \subseteq Y$, $f^{-1}(U)$ is open in $X$ because $x \in f^{-1}(U)$ is equivalent to $f(x) \in U$, which is an open truth value.

In the usual accounts of topology arbitrary unions of open sets are open. This is not so in synthetic topology. Actually, if all unions of opens are open then $\Sigma = \Omega$ because each $U \subseteq 1 = \{\star\}$ is the union of the family $\{1 \mid \star \in U\}$. If $I$ is a set such that every $I$-indexed union of open sets is open, then we say that $I$ is *overt*. A set $I$ is overt if, and only if, for every $U \in \mathcal{O}(I)$, the truth value of $\exists\, x \in I \,.\, x \in U$ is open.

The empty set and singletons are overt. To get other overt sets we must make further assumptions. We postulate:

*The set of natural numbers* $\mathbb{N}$ *is overt.*

This also makes the integers $\mathbb{Z}$ and the rational numbers $\mathbb{Q}$ overt because they are isomorphic (as sets) to $\mathbb{N}$. Because $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ have decidable equality they are not only overt but also *discrete*, by which we mean that equality $=$ is an open relation, and *Hausdorff*, by which we mean that negation of equality $\neq$ is an open relation. Similarly, the order relations $<$ and $\leq$ on $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are open because they are decidable.

In contrast, we cannot show in general that the reals $\mathbb{R}$ are overt, although $<$ is still open.

**Proposition 2.1** *For all* $x, y \in \mathbb{R}$, $x < y$ *is open.*

*Proof.* Because $\mathbb{Q}$ is overt and $x < y$ is equivalent to $\exists\, q \in \mathbb{Q} \,.\, x < q \wedge q < y$, it suffices to show that $x < q$ and $q < y$ are open. There is a Cauchy sequence $(r_n)_n$ in $\mathbb{Q}$ such that $|x - r_n| < 2^{-n}$ for all $n \in \mathbb{N}$. The statement $x < q$ is open because it is equivalent to $\exists\, n \in \mathbb{N} \,.\, r_n + 2^{-n} < q$. The statement $q < y$ is open for a similar reason. ∎

The corollary is that "open intervals are open", i.e., the intervals $(a, b)$, $(a, \infty)$ and $(-\infty, a)$ are intrinsically open.

The following proposition describes basic properties of overt sets.

**Proposition 2.2**

1. *Overtness is preserved under open subsets, images, and (binary) cartesian products.*

2. *A set is overt if it contains an overt dense subset.*

3. *Countable, and more generally, separable sets are overt.*

*Proof.* If $U \subseteq X$ is open and $X$ is overt then $U$ is overt because $\exists\, x \in U \,.\, x \in V$ is equivalent to $\exists\, x \in X \,.\, x \in V$, where we used the fact that if $V$ is open in $U$ then it is open in $X$.

If $Y$ is the image of $f : X \to Y$ and $X$ is overt then $Y$ is overt because $\exists\, y \in Y \,.\, y \in U$ is equivalent to $\exists\, x \in X \,.\, f(x) \in U$.

The cartesian product $X \times Y$ of overt spaces $X$ and $Y$ is overt because $\exists\, p \in X \times Y \,.\, p \in U$ is equivalent to $\exists\, x \in X \,.\, \exists\, y \in Y \,.\, \langle x, y \rangle \in U$.

Recall that $D \subseteq X$ is *dense* if, for every $U \in \mathcal{O}(X)$, $\exists\, x \in X \,.\, x \in U$ is equivalent to $\exists\, x \in D \,.\, x \in U$. Thus $X$ is overt if $D$ is.

We say that a set $S$ is *countable* if there exists a surjection[2] $e : \mathbb{N} \to 1 + S$, which is the same as saying that $S$ is the image of a decidable subset of $\mathbb{N}$. But since decidable subsets of $\mathbb{N}$ are overt, $S$ is the image of an overt set, therefore overt.

Lastly, a separable set is one that contains a dense countable set. As such, it is overt. ∎

In Section 4 we discuss (synthetic) compactness, which is the dual notion of overtness: a set $X$ is *compact* when $\forall\, x \in X \,.\, x \in U$ is open for all $U \in \mathcal{O}(X)$. Equivalently, $X$ is compact if, and only if, intersections of $X$-indexed families of open sets are open. Kuratowski finite sets are always compact. For other examples of compact spaces we would need further axioms. The only property of compactness we use is preservation by images: if $X$ is compact and $f : X \to Y$ surjective then $Y$ is compact because $\forall\, y \in Y \,.\, y \in U$ is equivalent to $\forall\, x \in X \,.\, f(x) \in U$.

We conclude this section with a couple of examples of $\Sigma$. The largest dominance is the subobject classifier itself, $\Sigma = \Omega$. With this choice of $\Sigma$, topology collapses: all sets are open and all spaces are discrete, overt, and compact. The smallest dominance (containing $\bot$ and $\top$) for which $\mathbb{N}$ is overt is Rosolini's dominance [18] of *semidecidable truth values*

$$\Sigma^0_1 = \left\{ p \in \Omega \mid \exists\, q \in 2^{\mathbb{N}} \,.\, (p \iff \exists\, n \in \mathbb{N} \,.\, q(n)) \right\} .$$

In this case the open and countable subsets of $\mathbb{N}$ coincide.

Because $\Sigma^0_1$ is the smallest dominance, it induces the smallest intrinsic topology, which is more likely to be in good agreement with other structures that a set may possess, such as a metric or a topological basis. Thus $\Sigma = \Sigma^0_1$ implies desirable properties, for instance Corollary 4.5 and Proposition 6.8.

[2] If $S$ is inhabited, $e$ may be replaced by a surjection $\mathbb{N} \to S$.

# 3 Topology of Metric Spaces

We use standard notions from the constructive theory of metric spaces [2, 20]. For a metric space $(X, d)$ the *(metric) ball* centered at $x \in X$ with radius $r > 0$ is the set $B(x, r) = \{y \in X \mid d(x, y) < r\}$. Balls are intrinsically open because $<$ is open on $\mathbb{R}$.

Most topological notions in synthetic topology have their corresponding parts in metric topology. We distinguish them by using the adjective *metric* for the metric notions, and when necessary the adjective *intrinsic* for the synthetic notions.

For instance, a metric space $(X, d)$ is *metric separable* if there exists a countable subset of $X$ which meets every ball. Clearly, an (intrinsically) separable space is metric separable. For the converse we need an additional assumption.

**Proposition 3.1** *Suppose every intrinsically open subset of a metric space $X$ is a union of balls. If $X$ is metric separable then it is separable.*

*Proof.* If the countable subset meets every ball then it meets every inhabited open set because such a set is an inhabited union of open balls. ∎

While the previous proposition indicates that it is desirable for open sets to be unions of balls (which is the classical definition of metric topology), this is not enough in our setting because an arbitrary union of balls need not be open. Instead, it makes sense to consider only *overt* unions of balls.

**Definition 3.2** A subset $U \subseteq X$ of a metric space $(X, d)$ is *metric open* when it is an overt union of metric balls in the sense that there exists an overt index set $I$ with families of centers $(x_i)_{i \in I}$ and radii $(r_i)_{i \in I}$ such that $U = \bigcup_{i \in I} B(x_i, r_i)$. The *metric topology* on $X$ induced by $d$ consists of all metric open subsets of $X$.

Because metric balls are open the metric topology is coarser than the intrinsic one. We would like to know when they coincide.

**Definition 3.3** We say that a set $X$ is *metrized* by a metric $d : X \times X \to \mathbb{R}$ when the metric topology on $X$ induced by $d$ coincides with the intrinsic topology $\mathcal{O}(X)$.

An example of a metrized space is an overt set $X$ with decidable equality, such as $\mathbb{N}$, $\mathbb{Z}$, or $\mathbb{Q}$. It is metrized by the *discrete metric*

$$d_D(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

Indeed, every $U \in \mathcal{O}(X)$ is overt because $X$ is overt, hence $U$ is an overt union of unit balls $U = \bigcup_{x \in U} B(x, 1)$.

Metrization of one space may imply metrization of another if there is a good connection between them, such as in the following proposition.

**Proposition 3.4** *A retract of an overt metrized space is overt and metrized.*

*Proof.* Suppose $(X, d)$ is overt and metrized, and $r : X \to A$ is a retraction to a subset $A \subseteq X$. Then $A$ is overt because it is the image of an overt set. For any open $V \in \mathcal{O}(A)$ the preimage $r^{-1}(V)$ is open in $X$, so we may write it as $r^{-1}(V) = \bigcup_{i \in I} B_X(x_i, r_i)$ for an overt $I$. The set

$$J = \{\langle i, a \rangle \in I \times A \mid a \in B_X(x_i, r_i)\}$$

is overt by Proposition 2.2. We claim that

$$V = \bigcup_{\langle i,a \rangle \in J} B_A(a, r_i - d(x_i, a)).$$

For one direction, take any $a \in V$ and observe that $r(a) = a$ implies $a \in r^{-1}(V)$, hence $\langle i, a \rangle \in J$ for some $i \in I$, which gives us $a \in B_A(a, r_i - d(x_i, a))$. For the other direction, if $x \in B_A(a, r_i - d(x_i, a))$ and $\langle i, a \rangle \in J$, then $x \in B_X(x_i, r_i) \subseteq r^{-1}(V)$, from which we conclude that $x = r(x) \in V$. ∎

If metrics on a set induce the same metric topologies, we call them *(topologically) equivalent.* Every metric $d : X \times X \to \mathbb{R}$ on an overt set $X$ is equivalent to a bounded one $d'$, defined by $d'(x, y) = \min(d(x, y), 1)$. To see this, observe that an overt union of open balls $U = \bigcup_{i \in I} B(x_i, r_i)$ may be written as an overt union of balls of radius at most 1,

$$U = \bigcup_{(i,x) \in J} B(x, \min(r_i - d(x_i, x), 1)),$$

where $J = \{(i, x) \in I \times X \mid x \in B(x_i, r_i)\}$ is overt by Proposition 2.2.

We can now easily find metric spaces which are not metrized simply because there may be several non-equivalent metrics on a given set. For instance, in addition to the discrete metric $d_D$, the rationals $\mathbb{Q}$ may also be endowed with the Euclidean metric $d_E(p, q) = |p-q|$. The former metrizes $\mathbb{Q}$ because it is overt and discrete, while the latter is too coarse. We shall see that the relevant distinguishing characteristic between $d_D$ and $d_E$ is completeness. This example also tells us that maps between metric spaces need not be continuous in the usual $\epsilon$-$\delta$ sense, e.g., the identity on $\mathbb{Q}$ as a map from $(\mathbb{Q}, d_E)$ to $(\mathbb{Q}, d_D)$, even though they are always intrinsically continuous.

However, even if we choose a reasonable metric on a space, its intrinsic topology might still be strictly finer than the metric one. For example, if we take $\Sigma = \Omega$, then all subsets of $\mathbb{R}$ are open, but of course not all of them are unions of open intervals. We could blame this anomaly on a poor choice of $\Sigma$, but in the Russian constructivism a counter-example of Friedberg's [8], see Corollary 7.3, shows that even for $\Sigma = \Sigma^0_1$ the intrinsic topology of *Baire space* $\mathbb{N}^{\mathbb{N}}$ is strictly finer than its metric topology. Recall that the Baire space is a complete separable[3] metric space (CSM) with the *comparison metric*

$$d_C(\alpha, \beta) = 2^{-\min\{k \in \mathbb{N} \mid \alpha_k \neq \beta_k\}}.$$

This is even an *ultrametric*, i.e., it satisfies the inequality

$$d_C(\alpha, \gamma) \leq \max(d_C(\alpha, \beta), d_C(\beta, \gamma)).$$

In an ultrametric space, every point of a ball is its center.

# 4 Synthetic and Metric Compactness

Recall that a *complete totally bounded space (CTB)* is a metric space in which every Cauchy sequence has a limit and for every $\epsilon > 0$ the space is covered by a finite family of balls with radius $\epsilon$. Every CTB is a CSM. The prototypical CTB is the *Cantor space* $2^{\mathbb{N}}$ with the comparison metric. In fact, inhabited CTB's are precisely the $\epsilon$-$\delta$ uniformly continuous images of Cantor space, see e.g. [20, 7.4.4].

We cannot expect to get for free a good relationship between (synthetic) compactness and the CTB property. In the trivial case $\Sigma = \Omega$ every set is compact but not every metric space is CTB, while in Section 7.2 we show that even a very simple CTB need not be compact.

Let us first analyze compactness of the simplest interesting CTB, namely the space

$$\mathbb{N}^+ = \left\{\alpha \in 2^{\mathbb{N}} \mid \forall\, n \in \mathbb{N}\,.\, \alpha_n \leq \alpha_{n+1}\right\}$$

with the comparison metric. A good way to think of $\mathbb{N}^+$ is as one-point compactification of $\mathbb{N}$, where a number $n \in \mathbb{N}$ is represented by the sequence

$$\underbrace{0, \ldots, 0}_{n}, 1, 1, \ldots,$$

and the point at infinity $\infty$ by the zero sequence $0, 0, \ldots$ In view of this we think of $\mathbb{N}$ as a subset of $\mathbb{N}^+$. The usual order relation $<$ may be extended from $\mathbb{N}$ to $\mathbb{N}^+$ by

$$s < t \iff \exists\, k \in \mathbb{N}\,.\, (s_k = 1 \land t_k = 0)\,.$$

Clearly, $<$ is an open relation, and if $s \in \mathbb{N}$ or $t \in \mathbb{N}$ then it is even decidable. We define $s \leq t$ to mean $\neg(t < s)$. We compute the minimum $u = \min(s, t)$

[3]We mean metric separability.

as $u_k = \max(s_k, t_k)$ and analogously the maximum $u = \max(s, t)$ as $u_k = \min(s_k, t_k)$.

The space $\mathbb{N}^+$ is a retract of $2^{\mathbb{N}}$ by the retraction

$$r(\alpha)(k) = \begin{cases} 1 & \text{if } \exists\, j \leq k \,.\, \alpha_k \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The comparison metric makes $\mathbb{N}^+$ into a CTB. The $\epsilon$-$\delta$ continuous maps from $\mathbb{N}^+$ to a complete metric space $(X, d)$ correspond precisely to convergent sequences with their limits. Indeed, if $a : \mathbb{N}^+ \to X$ is $\epsilon$-$\delta$ continuous then $\lim_k a(k) = a(\infty)$, and every convergent sequence $(x_n)_{n\in\mathbb{N}}$ appears as such a map $a : \mathbb{N}^+ \to X$, defined by

$$a(t) = \lim_k x_{\min(k,t)} \ .$$

The following principle, which we dub WSO for "weakly sequentially open", is related to metrization of $\mathbb{N}^+$:

$$\forall\, U \in \mathcal{O}(\mathbb{N}^+) \,.\, (\infty \in U \implies \exists\, n \in \mathbb{N} \,.\, n \in U) \ . \tag{WSO}$$

The principle has generally useful consequences such as the following.

**Proposition 4.1** *If WSO holds then CSM's are overt.*

*Proof.* Let $X$ be a CSM with a countable dense subset $D \subseteq X$ and $U \in \mathcal{O}(X)$. Suppose $U$ is inhabited by $x$. By Number Choice there is a Cauchy sequence $(a_n)_n$ in $D$ such that $\lim_n a_n = x$, hence there is a map $f : \mathbb{N}^+ \to X$ such that $f(n) = a_n$ for $n \in \mathbb{N}$ and $f(\infty) = x$. Because $f(\infty) = x \in U$ there is $n \in \mathbb{N}$ such that $a_n = f(n) \in U$. Therefore, every inhabited open set intersects $D$. Because $D$ is overt, $X$ is overt too. ∎

Before characterizing metrization of $\mathbb{N}^+$, we need a little preparation. For a metric space $(X, d)$, we define the *closed (metric) ball* in the usual way,

$$\overline{B}(x, r) = \{y \in X \mid d(x, y) \leq r\} \,.$$

It is convenient to allow the radius $r$ to be zero. In particular, we consider radii of the form $2^{-u}$ for $u \in \mathbb{N}^+$, where $2^{-u} = \lim_k 2^{\min(u,k)}$. This definition agrees with the standard one for $u < \infty$, while $2^{-\infty} = 0$. In any case, $2^{-u} \geq 0$, so $x \in \overline{B}(x, 2^{-u})$.

**Lemma 4.2** *For any $t, u \in \mathbb{N}^+$, the following holds:*

1. $\overline{B}(t, 2^{-u}) = \{x \in \mathbb{N}^+ \mid \forall\, k \in \mathbb{N} \,.\, (u_k = 0 \implies x_k = t_k)\}$.
2. $\overline{B}(t, 2^{-u})$ *has the minimal element, namely* $\min(t, u)$*, and the maximal one, which we denote by* $m(t, u)$.

3. $\overline{B}(t, 2^{-u})$ *is a retract of* $\mathbb{N}^+$.

*Proof.* The first statement follows from equivalences $a \leq 2^{-u} \iff \forall\, k \in \mathbb{N}\,.\, a \leq 2^{-\min(u,k)}$ and $(u_k = 0 \implies x_k = t_k) \iff (u \geq k+1 \implies d_C(x,t) \leq 2^{-k-1})$.

We proceed to the second statement. The first statement implies that $\min(t, u) \in \overline{B}(t, 2^{-u})$ and that for any $x \in \overline{B}(t, 2^{-u})$, we have, for all $k \in \mathbb{N}$, $u_k = 0 \implies x_k = t_k$. From $\min(t,u)_k = \max(t_k, u_k) = 0$ we obtain $u_k = 0$ and $x_k = t_k = 0$ and conclude $\min(t, u) \leq x$. Define

$$m(t,u)_k = \begin{cases} t_k & \text{if } u_k = 0 \text{ or } t_k = 0, \\ 0 & \text{if } u_k = t_k = 1 \text{ and } u \leq t, \\ 1 & \text{if } u_k = t_k = 1 \text{ and } u > t. \end{cases}$$

By the first statement $m(t,u) \in \mathbb{N}^+$ is clearly in $\overline{B}(t, 2^{-u})$. Take any $x \in \overline{B}(t, 2^{-u})$ and $k \in \mathbb{N}$ such that $x_k = 0$; to obtain $x \leq m(t,u)$, we need to show $m(t,u)_k = 0$. If $t_k = 0$, then $m(t,u)_k = t_k = 0$, and we are done. If $t_k = 1$, then $t \leq k < x$. So $2^{-t} = d(t,x) \leq 2^{-u}$, whence $u \leq t \leq k$, in particular also $u_k = t_k = 1$. Again we conclude $m(t,u)_k = 0$.

Lastly, we prove that $\overline{B}(t, 2^{-u})$ is a retract of $\mathbb{N}^+$. By the second statement, $\overline{B}(t, 2^{-u}) = \{x \in \mathbb{N}^+ \mid \min(t,u) \leq x \leq m(t,u)\}$. The desired retraction $r : \mathbb{N}^+ \to \overline{B}(t, 2^{-u})$ may be defined as

$$r(x) = \min\left(\max(\min(t,u), x), m(t,u)\right).$$

∎

**Proposition 4.3** *The following are equivalent:*

1. $\mathbb{N}^+$ *is metrized by the comparison metric.*
2. *For every* $U \in \mathcal{O}(\mathbb{N}^+)$ *if* $\infty \in U$ *then* $B(\infty, r) \subseteq U$ *for some* $r > 0$.
3. $\mathbb{N}^+$ *is compact and WSO holds.*

*Proof.* Suppose $\mathbb{N}^+$ is metrized by the comparison metric and $U \in \mathcal{O}(\mathbb{N}^+)$ contains $\infty$. Because $U$ is a union of open balls there exists an open ball $B(t,r)$ such that $\infty \in B(t,r) \subseteq U$. Because $\mathbb{N}^+$ is ultrametric, $B(t,r) = B(\infty, r)$, therefore the second statement is implied by the first one.

Suppose the second statement holds. Evidently, it implies WSO so we just have to show that $\mathbb{N}^+$ is compact. Consider any $U \in \mathcal{O}(\mathbb{N}^+)$. Because

$$(\forall\, t \in \mathbb{N}^+ \,.\, t \in U) \iff \infty \in U \wedge (\forall\, t \in \mathbb{N}^+ \,.\, t \in U)$$

it suffices to show that the right-hand side of the equivalence is open, which we derive from the dominance axiom. If $\infty \in U$ then there exists $k \in \mathbb{N}$

such that $B(\infty, 2^{-k}) \subseteq U$, whence $\forall\, t \in \mathbb{N}^+ \,.\, t \in U$ is equivalent to the open statement $\forall\, t \leq k \,.\, t \in U$.

Lastly, suppose the third statement holds. A closed ball $\overline{B}(t, 2^{-k})$ is compact because it is the image of $\mathbb{N}^+$ by Lemma 4.2. For any $U \in \mathcal{O}(\mathbb{N}^+)$ the set

$$I = \left\{ \langle t, k \rangle \in \mathbb{N}^+ \times \mathbb{N} \mid \forall\, u \in \overline{B}(t, 2^{-k}) \,.\, u \in U \right\}$$

is overt because $\mathbb{N}^+$ is overt by Proposition 4.1. Moreover,

$$U = \bigcup_{\langle t,k \rangle \in I} B(t, 2^{-k}).$$

Only the inclusion of $U$ in the right-hand side needs proof. For $t \in U$ the map $f : \mathbb{N}^+ \to \Sigma$ defined by $f(u) = (\forall\, x \in \overline{B}(t, 2^{-u}) \,.\, x \in U)$ classifies an open subset of $\mathbb{N}^+$. Because $f(\infty) = (t \in U) = \top$ there exists $k \in \mathbb{N}$ such that $f(k) = \top$, i.e., $t \in B(t, 2^{-k}) \subseteq \overline{B}(t, 2^{-k}) \subseteq U$ and $\langle t, k \rangle \in I$. ∎

An obvious question to ask is how compactness of $2^{\mathbb{N}}$ and Brouwer's Fan Principle are related. Let $2^*$ be the set of finite sequences of 0's and 1's. The length of a finite sequence $a = (a_0, \ldots, a_{n-1})$ is $|a| = n$. If $\alpha \in 2^{\mathbb{N}}$ let $\overline{\alpha}(n) = (\alpha_0, \ldots, \alpha_{n-1})$ and write $a \sqsubseteq \alpha$ if $a$ is a prefix of $\alpha$, i.e., $a = \overline{\alpha}(|a|)$. Denote the concatenation of $a \in 2^*$ and $\beta \in 2^{\mathbb{N}}$ by $a{::}\beta$. For $t \in \mathbb{N}^+$ and $\alpha, \beta \in 2^{\mathbb{N}}$, let $\alpha {\restriction_t} \beta \in 2^{\mathbb{N}}$ be

$$(\alpha {\restriction_t} \beta)_k = \begin{cases} \alpha_k & \text{if } k < t, \\ \beta_{k-t} & \text{if } t \leq k. \end{cases}$$

If $t < \infty$ then $\alpha {\restriction_t} \beta = \overline{\alpha}(t){::}\beta$, while $\alpha {\restriction_\infty} \beta = \alpha$. A *bar* is a subset $S \subseteq 2^*$ such that $\forall\, \alpha \in 2^{\mathbb{N}} \,.\, \exists\, a \in S \,.\, a \sqsubseteq \alpha$. A bar is *uniform* if there exists a bound $n \in \mathbb{N}$ such that $\forall\, \alpha \in 2^{\mathbb{N}} \,.\, \exists\, a \in S \,.\, (|a| \leq n \land a \sqsubseteq \alpha)$. Brouwer's Fan Principle states that every decidable bar is uniform.

**Theorem 4.4** *The following are equivalent:*

1. $2^{\mathbb{N}}$ *is compact and WSO holds.*
2. $2^{\mathbb{N}}$ *is metrized by the comparison metric and every overt bar is uniform.*

*Proof.* Suppose $2^{\mathbb{N}}$ is compact and let $S \subseteq 2^*$ be an overt bar. The set

$$U = \left\{ t \in \mathbb{N}^+ \mid \forall\, \alpha \in 2^{\mathbb{N}} \,.\, \exists\, a \in S \,.\, (|a| \leq t \land a \sqsubseteq \alpha) \right\}$$

is open and $\infty \in U$ because $S$ is a bar. By WSO there exists $m \in \mathbb{N}$ such that $m \in U$, which means that $m$ is a bound for $S$. Take any $U \in \mathcal{O}(2^{\mathbb{N}})$ and define

$$I = \left\{ a \in 2^* \mid \forall\, \beta \in 2^{\mathbb{N}} \,.\, a{::}\beta \in U \right\}.$$

This is an overt set because $2^*$ is isomorphic to $\mathbb{N}$ and $2^{\mathbb{N}}$ is compact. Let $o \in 2^{\mathbb{N}}$ be the zero sequence. We claim that

$$U = \bigcup_{a \in I} B(a{::}o, 2^{-|a|}).$$

The right-to-left inclusion is clear. For the opposite inclusion, take $\alpha \in U$ and consider the set

$$V = \left\{ t \in \mathbb{N}^+ \mid \forall \beta \in 2^{\mathbb{N}} \, . \, \alpha{\upharpoonright_t}\beta \in U \right\} .$$

This is an open subset of $\mathbb{N}^+$ and since $\alpha{\upharpoonright_\infty}\beta = \alpha \in U$, we have $\infty \in V$. By WSO there exists $n \in \mathbb{N}$ such that $B(\alpha{\upharpoonright_n}\beta, 2^{-n}) \subseteq U$ for all $\beta \in 2^{\mathbb{N}}$, which finishes the first part of the proof because then $\overline{\alpha}(n) \in I$ and we have $\alpha \in B(\overline{\alpha}(n){::}o, 2^{-n})$.

To prove the converse, suppose $2^{\mathbb{N}}$ is metrized. Then it is overt by Propositions 3.1 and 2.2. Furthermore, by Proposition 3.4 $\mathbb{N}^+$ is metrized, so by Proposition 4.3 the principle WSO holds. Suppose additionally that every overt bar is uniform and consider any $U \in \mathcal{O}(2^{\mathbb{N}})$. We borrow Proposition 6.5 from a later section to write $U$ as an overt union $U = \bigcup_{i \in I} B(\beta_i, 2^{-k_i})$ of balls whose radii are powers of $1/2$. The set

$$S = \left\{ b \in 2^* \mid \exists \, i \in I \, . \, b = \overline{\beta_i}(k_i) \right\}$$

is overt and $\alpha \in U$ is equivalent to $\exists \, b \in S \, . \, b \sqsubseteq \alpha$. From this we get the equivalences

$$\begin{aligned}
\forall \alpha \in 2^{\mathbb{N}} \, . \, \alpha \in U \iff \forall \alpha \in 2^{\mathbb{N}} \, . \, \exists \, b \in S \, . \, b \sqsubseteq \alpha \iff \\
\exists \, m \in \mathbb{N} \, . \, \forall \alpha \in 2^{\mathbb{N}} \, . \, \exists \, b \in S \, . \, (|b| \leq m \wedge b \sqsubseteq \alpha) \iff \\
\exists \, m \in \mathbb{N} \, . \, \forall a \in 2^m \, . \, \exists \, b \in S \, . \, (|b| \leq m \wedge b \sqsubseteq a) \, .
\end{aligned}$$

The second equivalence holds because every overt fan is uniform. The last statement is open, therefore $2^{\mathbb{N}}$ is compact. ∎

A more transparent connection between compactness of $2^{\mathbb{N}}$ and the Fan Principle is expressed by the following corollary.

**Corollary 4.5** *If $2^{\mathbb{N}}$ is metrized then the following are equivalent:*

1. *$2^{\mathbb{N}}$ is compact.*
2. *Every overt bar is uniform.*

*In the special case $\Sigma = \Sigma^0_1$, the second statement may be replaced by Brouwer's Fan Principle "every* decidable *bar is uniform".*

*Proof.* In the proof of Theorem 4.4 we saw that metrization of $2^{\mathbb{N}}$ implies WSO, which is enough to establish the equivalence.

If $\Sigma = \Sigma^0_1$ then the overt bars are the semidecidable bars. By a result of Ishihara's [11] the Fan Principles for decidable and semidecidable bars imply each other. ∎

## 5 Transfer of Metrization

Having examined connections between metric and intrinsic topology for special spaces, we now focus on the general case. To this end we consider how metrization of one space affects metrization of another that is related to it by a map. Call a map $f : X \to Y$ *metric continuous* if preimages of metric open subsets of $Y$ are metric open in $X$. This is a strengthening of the usual $\epsilon$-$\delta$ continuity because we require the unions of balls to be overt. The following lemma provides simple examples of metric continuous maps.

**Lemma 5.1**

1. *The inclusion of an overt subspace into a metric space is metric continuous.*
2. *A map from a metrized space to a metric space is metric continuous.*
3. *Let $f : X \to Y$ be a map from an overt metric space $(X, d_X)$ to a metric space $(Y, d_Y)$, and suppose there exist $\epsilon > 0$ and $c > 0$ such that $d_Y(f(x), f(y)) \leq c \cdot d_X(x, y)$ for all $x, y \in X$ satisfying $d_X(x, y) < \epsilon$. Then $f$ is metric continuous. In particular, a Lipschitz map is metric continuous.*

*Proof.* The first statement is a special case of the last one.

For the second statement, recall that if $V$ is metric open in $Y$, then it is also intrinsically open, so $f^{-1}(V)$ is open in $X$. Because $X$ is metrized, $f^{-1}(V)$ is metric open.

Let us prove the third statement. Suppose $V = \bigcup_{i \in I} B_Y(y_i, r_i)$ is an overt union of balls in $Y$. The set $U = f^{-1}(V)$ is overt because it is an open subset of $X$. By Proposition 2.2 the set

$$J = \{\langle i, x\rangle \in I \times U \mid f(x) \in B_Y(y_i, r_i)\}$$

is overt. We then have

$$U = \bigcup_{\langle i,x\rangle \in J} B_X\left(x, \min\left(\epsilon, \frac{r_i - d_Y(y_i, f(x))}{c}\right)\right).$$

■

**Definition 5.2** A map $f \colon X \to Y$ between metric spaces is a *metric quotient map* if it is surjective and, for every $V \subseteq Y$, $V$ is metric open in $Y$ if, and only if, $f^{-1}(V)$ is metric open in $X$ (the "only if" part is metric continuity).

**Proposition 5.3** *If $X$ is metrized by its metric and $f : X \to Y$ is a metric quotient map then $Y$ is metrized by its metric.*

*Proof.* For every open $V \subseteq Y$, $f^{-1}(V)$ is open in $X$, hence metric open in $X$, therefore $V$ is metric open in $Y$. Notice that we do not need the metric continuity of $f$. ∎

The previous proposition is a generalization of Proposition 3.4 because a retraction of an overt metrized space is a metric quotient map. Indeed, a retraction is surjective, and if $X$ is overt, an examination of the proof of Proposition 3.4 reveals that $r : X \to A$ satisfies the "if" part of the condition for metric quotient map. Finally, if $X$ is metrized[4] then $r$ is metric continuous by Lemma 5.1.

It is well known that every inhabited CSM is an image of Baire space $\mathbb{N}^{\mathbb{N}}$ by an $\epsilon$-$\delta$ continuous map. If we can show that the map is metric quotient then we can transfer metrization of Baire space to all CSM's.

**Theorem 5.4** *For every inhabited CSM $(X, d)$ there is a retract $T$ of $\mathbb{N}^{\mathbb{N}}$ and a surjection $q : T \to X$ which is a metric quotient map when $\mathbb{N}^{\mathbb{N}}$ is overt.*

*Proof.* The first part is a well-known result. Nevertheless, we spell out the proof, following [20, 7.2.4], to verify the second part of the theorem.

*(1) Construction of the retraction $r : \mathbb{N}^{\mathbb{N}} \to T$.*

Let $(s_i)_{i \in \mathbb{N}}$ be a dense sequence in $X$. The simplest idea for the construction of $T$ is to take those $\alpha \in \mathbb{N}^{\mathbb{N}}$ for which $(s_{\alpha(i)})_{i \in \mathbb{N}}$ is a Cauchy sequence, but this does not work as we would get stuck when trying to show that $T$ is a retract of $\mathbb{N}^{\mathbb{N}}$. A slightly more refined idea works, though. By Number Choice, there exists a map $\delta : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ such that $|d(s_i, s_j) - \delta(i, j, k)| < 2^{-k}$. That is, $\delta(i, j, k)$ is a rational $2^{-k}$-approximation of the distance between $s_i$ and $s_j$. Now let

$$T = \left\{ \alpha \in \mathbb{N}^{\mathbb{N}} \mid \forall\, k \in \mathbb{N} \,.\, \delta(\alpha(k), \alpha(k+1), k) < 2^{-k+2} \right\} .$$

To construct a retraction $r : \mathbb{N}^{\mathbb{N}} \to T$, first define, for $\alpha \in \mathbb{N}^{\mathbb{N}}$ and $k \in \mathbb{N}$,

$$f(\alpha, k) = \min \left\{ j \in \mathbb{N} \mid j = k \lor \delta(\alpha(j), \alpha(j+1), j) \geq 2^{-k+2} \right\} ,$$

and then $r(\alpha)(k) = \alpha(f(\alpha, k))$. Since $r(\alpha)(k)$ depends only on the first $k$ terms of $\alpha$, $r$ is Lipschitz with coefficient 1. Furthermore, if $\alpha \in T$ then $f(\alpha, k) = k$ for all $k \in \mathbb{N}$, hence $r(\alpha) = \alpha$, which proves that $r$ is a retraction.

*(2) Construction of the map $q : T \to X$.*

For every $\alpha \in T$ and $k \in \mathbb{N}$ we have $d(s_{\alpha(k)}, s_{\alpha(k+1)}) < \delta(\alpha(k), \alpha(k+1), k) + 2^{-k} < 5 \cdot 2^{-k}$, therefore $(s_{\alpha(k)})_{k \in \mathbb{N}}$ is a Cauchy sequence and we may define

[4] We require the assumption that $X$ is metrized: for an example of a retraction which is not metric continuous, take $\mathbb{Q}$ with Euclidean metric and $r : \mathbb{Q} \to \mathbb{Q} \cap (0, \infty)$, defined by $r(q) = 1$ for $q \leq 0$, and $r(q) = q$ for $q > 0$.

a map $q : T \to X$ by $q(\alpha) = \lim_k s_{\alpha(k)}$. Observe that, for all $\alpha, \beta \in T$ with $d_C(\alpha, \beta) < 2^{-m}$,

$$d(q(\alpha), q(\beta)) \leq d(q(\alpha), s_{\alpha(m)}) + d(s_{\beta(m)}, q(\beta)) < 5 \cdot 2^{-m+2},$$

where we used the facts that $\alpha(m) = \beta(m)$ and $d(q(\alpha), s_{\alpha(m)}) < 5 \cdot 2^{-m+1}$. This tells us $q$ is Lipschitz when restricted to balls with radius 1: if $d_C(\alpha, \beta) < 1$ then there is $m \in \mathbb{N}$ such that $d_C(\alpha, \beta) < 2^{-m} < 3 \cdot d_C(\alpha, \beta)$, which gives us the admittedly rough Lipschitz coefficent

$$d(q(\alpha), q(\beta)) < 5 \cdot 2^{-m+2} = 20 \cdot 2^{-m} < 20 \cdot 3 \cdot d(\alpha, \beta) = 60 \cdot d(\alpha, \beta).$$

Next, given any $x \in X$, by Number Choice there exists $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $d(x, s_{\alpha(k)}) < 2^{-k}$ for all $k \in \mathbb{N}$. Then $\delta(\alpha(k), \alpha(k+1), k) < d(s_{\alpha(k)}, s_{\alpha(k+1)}) + 2^{-k} \leq 2^{-k} + 2^{-k-1} + 2^{-k} < 2^{-k+2}$, hence $\alpha \in T$ and $q(\alpha) = x$. We showed $q$ is surjective.

*(3) A property of $q : T \to X$.*

Before proving that $q$ is a metric quotient map when $\mathbb{N}^{\mathbb{N}}$ is overt, we verify the following claim:

> Given any $\alpha \in T$, let $\beta = \lambda n.\, \alpha(n+4)$. Then $\beta \in T$, $q(\alpha) = q(\beta)$, and $B_X(q(\beta), r/8) \subseteq q(B_T(\beta, r))$ for all $0 < r \leq 1$.

First, $\beta \in T$ because

$$\begin{aligned}\delta(\beta(k), \beta(k+1), k) = \delta(\alpha(k+4), \alpha(k+5), k) <\\ d(\alpha(k+4), \alpha(k+5)) + 2^{-k} <\\ \delta(\alpha(k+4), \alpha(k+5), k+4) + 2^{-k-4} + 2^{-k} <\\ 2^{-k-2} + 2^{-k-4} + 2^{-k} < 2^{-k+2}.\end{aligned}$$

Clearly, $q(\alpha) = q(\beta)$. Suppose $x \in X$ such that $d(q(\beta), x) < r/8$. There is $m \in \mathbb{N}$ such that $r/8 < 2^{-m-1} < 2^{-m} < r$. By Number Choice there exists $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $d(x, s_{\gamma(k)}) < 2^{-k}$ for all $k \in \mathbb{N}$. Let $\epsilon \in \mathbb{N}^{\mathbb{N}}$ be defined by

$$\epsilon(k) = \begin{cases} \beta(k) & \text{if } k \leq m, \\ \gamma(k) & \text{if } k > m. \end{cases}$$

Then $\epsilon \in T$ because for all $k \in \mathbb{N}$,

$$\begin{aligned}\delta(\gamma(k), \gamma(k+1), k) < d(s_{\gamma(k)}, s_{\gamma(k+1)}) + 2^{-k} \leq\\ d(s_{\gamma(k)}, x) + d(x, s_{\gamma(k+1)}) + 2^{-k} < 2^{-k} + 2^{-k-1} + 2^{-k} < 2^{-k+2},\end{aligned}$$

and

$$
\begin{aligned}
\delta(\beta(m), \gamma(m+1), m) < d(s_{\beta(m)}, s_{\gamma(m+1)}) + 2^{-m} \leq \\
d(s_{\beta(m)}, q(\beta)) + d(q(\beta), x) + d(x, s_{\gamma(m+1)}) + 2^{-m} < \\
5 \cdot 2^{-m-3} + 2^{-m-1} + 2^{-m-1} + 2^{-m} = 21 \cdot 2^{-m-3} < 2^{-m+2} \ ,
\end{aligned}
$$

where we used the fact that $d(s_{\beta(m)}, q(\beta)) = d(s_{\alpha(m+4)}, q(\alpha)) < 5 \cdot 2^{-m-3}$. Furthermore, $q(\epsilon) = x$ and $\epsilon \in B_T(\beta, 2^{-m}) \subseteq B_T(\beta, r)$. This proves the claim.

*(4) If $\mathbb{N}^{\mathbb{N}}$ is overt then $q$ is metric quotient.*

To conclude the proof, suppose $\mathbb{N}^{\mathbb{N}}$ is overt. Then $T$ is overt because it is an image of $\mathbb{N}^{\mathbb{N}}$. We already showed $q$ is surjective, and it is metric continuous by Lemma 5.1. We verify that $V \subseteq X$ is metric open if $U = q^{-1}(V)$ is metric open. Suppose $U = \bigcup_{i \in I} B_T(\alpha_i, r_i)$ for an overt $I$. The set

$$
J = \{\langle \alpha, i \rangle \in T \times I \mid \alpha \in B_T(\alpha_i, r_i)\}
$$

is overt and $U = \bigcup_{\langle \alpha, i \rangle \in J} B_T(\alpha, r_i)$ because $B_T(\alpha_i, r_i) = B_T(\alpha, r_i)$ whenever $\alpha \in B_T(\alpha_i, r_i)$. Let $\sigma : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be the shift map $\sigma(\beta) = \lambda n.\, \beta(n+4)$. Define the overt set

$$
K = \{\langle \beta, \alpha, i \rangle \in T \times J \mid \sigma(\beta) \in B_T(\alpha, r_i)\} \, .
$$

We claim that

$$
V = \bigcup_{\langle \beta, \alpha, i \rangle \in K} B_X(q(\sigma(\beta)), \min(r_i, 1)/8) \ .
$$

The set $V$ contains the right-hand side because by the claim from *(3)* above, for all $\langle \beta, \alpha, i \rangle \in K$,

$$
\begin{aligned}
B_X(q(\sigma(\beta)), \min(r_i, 1)/8) \subseteq q(B_T(\sigma(\beta), \min(r_i, 1))) \subseteq \\
q(B_T(\sigma(\beta), r_i)) = q(B_T(\alpha, r_i)) \subseteq V.
\end{aligned}
$$

To establish the opposite inclusion, suppose $x \in V$. There is $j \in I$ and $\beta \in B_T(\alpha_j, r_j)$ such that $x = q(\beta)$. Because $\sigma(\beta) \in T$ and $q(\sigma(\beta)) = q(\beta) = x$, there exists $\langle \alpha, i \rangle \in J$ such that $\sigma(\beta) \in B_T(\alpha, r_i)$. Therefore $\langle \beta, \alpha, i \rangle \in K$ and $x = q(\sigma(\beta)) \in B_X(q(\sigma(\beta)), \min(r_i, 1)/8)$, which proves that $x$ is a member of the union on the right-hand side. ∎

We are now able to transfer metrization of Baire space to that of any CSM.

**Corollary 5.5** *If $\mathbb{N}^{\mathbb{N}}$ is metrized by the comparison metric then every CSM is metrized by its metric (and the converse holds trivially).*

*Proof.* Suppose $\mathbb{N}^{\mathbb{N}}$ is metrized and consider first an inhabited CSM $X$. Because every open subset of $\mathbb{N}^{\mathbb{N}}$ is a union of open balls, $\mathbb{N}^{\mathbb{N}}$ is overt by Propositions 3.1 and 2.2 as it is metric separable. Let

$$\mathbb{N}^{\mathbb{N}} \to T \to X$$

be a retraction $\mathbb{N}^{\mathbb{N}} \to T$ followed by a metric quotient map $T \to X$, as in Theorem 5.4. By Proposition 3.4 $T$ is metrized and then so is $X$ by Proposition 5.3.

If $X$ is a possibly non-inhabited CSM, we proceed as follows. Because $\mathbb{N}^{+}$ is metrized, $X$ is overt by Propositions 4.3 and 4.1. Replace the metric $d$ on $X$ with an equivalent bounded metric and extend it to a metric $d' : X' \times X' \to \mathbb{R}$ on $X' = 1 + X$ by

$$d'(x, y) = \begin{cases} \min(d(x, y), 1) & \text{if } x, y \in X, \\ 0 & \text{if } x, y \in 1, \\ 1 & \text{otherwise.} \end{cases}$$

Observe that $(X', d')$ is an inhabited CSM, hence metrized. Because $X$ is an open subset of $X'$ it is metrized also, as every open in $X$ is also open in $X'$. ∎

## 6 The Metric Axiom and its Consequences

In view of Corollary 5.5 we consider the following

> ***Metric Axiom:*** *The Baire space is metrized by the comparison metric.*

The axiom ensures a well-behaved theory of complete separable metric spaces.

**Theorem 6.1** *If the Metric Axiom holds then:*

1. *Up to topological equivalence, a set may be equipped with at most one complete separable metric.*
2. *The principle WSO holds.*
3. *Complete separable metric spaces are overt.*
4. *Continuity Principle holds: every map from a CSM to a metric space is metric continuous.*

*Proof.* If $d_1$ and $d_2$ both make the set $X$ complete and separable then by Corollary 5.5 they both induce the same topology, namely the intrinsic one. Hence they are topologically equivalent.

The principle WSO holds by Proposition 4.3 because $\mathbb{N}^+$ is metrized, and complete separable metric spaces are then overt by Proposition 4.1.

The Continuity Principle follows from Lemma 5.1. ∎

The Metric Axiom is non-classical because it implies that the Law of Excluded Middle fails.

**Proposition 6.2** *The Metric Axiom and the Lesser Principle of Omniscience are not both true.*

*Proof.* Recall [2] that the *Lesser Principle of Omniscience* (LPO) is a particular instance of the Law of Excluded Middle,

$$\forall\, f \in \mathbb{N}^{\mathbb{N}} \,.\, (f = o \vee \neg(f = o))\,,$$

where $o$ is the constantly zero map $o(n) = 0$. If LPO holds then the singleton $\{o\}$ is open in $\mathbb{N}^{\mathbb{N}}$, which contradicts the Metric Axiom. ∎

## 6.1 Topological basis and transfer by open surjections

Thus far our attention has been restricted to metric spaces. To deal with non-metric ones, we need to generalize the concept of balls as building blocks of open sets.

**Definition 6.3** A *(topological) basis* for a set $X$ is a family $\{B_j \in \mathcal{O}(X) \mid j \in J\}$ of *basic* open sets, indexed by a (possibly non-overt) set $J$, subject to the following condition: for every $U \in \mathcal{O}(X)$ there exists an overt set $I$ and a map $e : I \to J$ such that $U = \bigcup_{i \in I} B_{e(i)}$.

We explicitly require the indexing set $J$ and the map $e$ in order to avoid the need for choice principles when the same basic open set is indexed several times, e.g., in a metric space balls with different centers and radii may coincide. According to our definition, in such cases specific indices (centers and radii) must be given, rather than just basic opens as sets of points. The following lemma allows us to reindex topological bases if so desired.

**Lemma 6.4** *Suppose $\{B_j \mid j \in J\}$ is a basis for $X$ and $r : J \to K$ is a surjection such that $r(i) = r(j)$ implies $B_i = B_j$ for all $i, j \in J$. Then we may define $C_{r(j)} = B_j$ and the family $\{C_k \mid k \in K\}$ is a basis for $X$.*

*Proof.* If $U = \bigcup_{i \in I} B_{e(i)}$ for an overt $I$ and a map $e : I \to J$, then also $U = \bigcup_{i \in I} C_{(r \circ e)(i)}$. ∎

To say that a metric space is metrized is the same as to say that the family of all balls $\{B(x, r) \mid x \in X, r > 0\}$ is a topological basis for it. Of course, just as in classical theory of metric spaces, we need not take all the balls.

**Proposition 6.5** *Suppose $(X,d)$ is metrized by its metric and $(s_k)_{k\in\mathbb{N}}$ is a dense sequence in $X$. Then $\{B(s_k, 2^{-j}) \mid k, j \in \mathbb{N}\}$ is a basis for $X$.*

*Proof.* Let $U \in \mathcal{O}(X)$; there is an overt $I$ such that $U = \bigcup_{i\in I} B(x_i, r_i)$. Define the overt set

$$J = \left\{\langle k, j\rangle \in \mathbb{N}\times\mathbb{N} \mid \exists\, i \in I \,.\, d(x_i, s_k) + 2^{-j} < r_i\right\}.$$

We claim that

$$U = \bigcup_{\langle k,j\rangle\in J} B(s_k, 2^{-j}).$$

The right-to-left inclusion is straightforward. For the other one, if $x \in U$ then $x \in B(x_i, r_i)$ for some $i \in I$. There exist $k, j \in \mathbb{N}$ such that $2^{-j} < (r_i - d(x_i, x))/2$ and $s_k \in B(x, 2^{-j})$. Then $\langle k, j\rangle \in J$ and $x \in B(s_k, 2^{-j})$. ∎

The previous proposition tells us that a metrized separable metric space has a countable basis. Thus, if the Metric Axiom holds, every CSM has a countable basis.

Call $f : X \to Y$ an *open* map if its images of open subsets of $X$ are open in $Y$. While we could transfer only metrization along metric quotient maps, we can transfer bases along open surjections.

**Proposition 6.6** *Suppose $\{B_j \mid j \in J\}$ is a basis for $X$.*

1. *If $f : X \to Y$ maps basic open sets to open sets then it is an open map.*
2. *If $f : X \to Y$ is an open surjection then $\{f(B_j) \mid j \in J\}$ is a basis for $Y$.*

*Proof.* (1) If $U \in \mathcal{O}(X)$ then $U = \bigcup_{i\in I} B_{e(i)}$ for an overt $I$, hence $f(U) = \bigcup_{i\in I} f(B_{e(i)})$ is open as $I$ is overt and the sets $f(B_{e(i)})$ are open.

(2) If $V \in \mathcal{O}(Y)$ then $f^{-1}(V)$ is open in $X$, therefore $f^{-1}(V) = \bigcup_{i\in I} B_{e(i)}$ for an overt $I$, hence $V = f(f^{-1}(V)) = \bigcup_{i\in I} f(B_{e(i)})$. ∎

With this we can transfer the basis of a CSM to other spaces, even non-metric ones. We demonstrate this by computing the topology of $\Sigma^{\mathbb{N}}$. Let $q : \mathbb{N}^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ be the map

$$q(\alpha) = \{n \in \mathbb{N} \mid \exists\, k \in \mathbb{N} \,.\, \alpha(k) = 1 + n\}. \tag{1}$$

We would like to transfer the basis for $\mathbb{N}^{\mathbb{N}}$ along $q$, for which we need to know first that it is surjective.

**Lemma 6.7** *The map $q : \mathbb{N}^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ is surjective if, and only if, $\Sigma = \Sigma^0_1$.*

*Proof.* If $\Sigma = \Sigma^0_1$ then every $U \in \Sigma^{\mathbb{N}}$ is countable, i.e., there exists a surjective $e : \mathbb{N} \to 1 + U$, hence $U = q(\alpha)$ if we take

$$\alpha(n) = \begin{cases} 1 + e(n) & \text{if } e(n) \in U, \\ 0 & \text{if } e(n) = \star. \end{cases}$$

Conversely, suppose $q$ is surjective. Because $\mathbb{N}$ is overt and $\bot, \top \in \Sigma$, we already know that $\Sigma^0_1 \subseteq \Sigma$. Suppose $p \in \Sigma$. There exists $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that

$$\{n \in \mathbb{N} \mid p\} = \{n \in \mathbb{N} \mid \exists\, k \in \mathbb{N} \,.\, \alpha(k) = 1 + n\}\,.$$

In particular, for $n = 0$ we get $p \iff (\exists\, k \in \mathbb{N} \,.\, \alpha(k) = 1)$, hence $p \in \Sigma^0_1$. ∎

Recall that a subset $\mathcal{U} \subseteq \Sigma^{\mathbb{N}}$ is *Scott open* when $U \in \Sigma^{\mathbb{N}}$ is in $\mathcal{U}$ if, and only if, some finite subset of $U$ is in $\mathcal{U}$.

The family $\mathscr{F}$ of all finite subsets of $\mathbb{N}$ is overt because it is isomorphic to $\mathbb{N}$. For any $F \in \mathscr{F}$, the set $\uparrow F = \left\{U \in \Sigma^{\mathbb{N}} \mid F \subseteq U\right\}$ is open in $\Sigma^{\mathbb{N}}$ because $F$ is compact.

**Proposition 6.8 (Scott's Principle)** *If $\Sigma = \Sigma^0_1$ and the Metric Axiom holds then the family $\{\uparrow F \mid F \in \mathscr{F}\}$ is a basis for $\Sigma^{\mathbb{N}}$. Moreover, every open subset of $\Sigma^{\mathbb{N}}$ is Scott open.*

*Proof.* Let $\mathbb{N}^*$ be the set of all finite sequences in $\mathbb{N}$. Notice that $\mathbb{N}^*$ is isomorphic to $\mathbb{N}$. For every $a \in \mathbb{N}^*$ define $s_a = a{::}o$ where $o$ is the zero sequence. Because $\{s_a \mid a \in \mathbb{N}^*\}$ is metric dense in $\mathbb{N}^{\mathbb{N}}$, the family

$$\left\{B(s_a, 2^{-k}) \mid a \in \mathbb{N}^*, k \in \mathbb{N}\right\}$$

is a basis for $\mathbb{N}^{\mathbb{N}}$ by Proposition 6.5. The map $q : \mathbb{N}^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ defined by (1) maps a basic open ball $B(s_a, 2^{-k})$ to the open subset

$$q(B(s_a, 2^{-k})) = \uparrow r(a, k),$$

where $r : \mathbb{N}^* \times \mathbb{N} \to \mathscr{F}$ is defined by

$$r(a, k) = \{n \in \mathbb{N} \mid \exists\, j \leq k \,.\, s_a(j) = 1 + n\}\,.$$

Therefore, by Lemma 6.7 and Proposition 6.6 the family

$$\left\{q(B(s_a, 2^{-k})) \mid a \in \mathbb{N}^*, k \in \mathbb{N}\right\}$$

is a basis for $\Sigma^{\mathbb{N}}$. We now apply Lemma 6.4 with the reindexing map $r$ to conclude that $\{\uparrow F \mid F \in \mathscr{F}\}$ is a basis, too.

It remains to be shown that every $\mathcal{U} \in \mathcal{O}(\Sigma^{\mathbb{N}})$ is Scott open. There is an overt $I$ and a map $e : I \to \mathscr{F}$ such that $\mathcal{U} = \bigcup_{i \in I} \uparrow e(i)$. Consider any $U \in \Sigma^{\mathbb{N}}$. If there is a finite $F \subseteq U$ such that $F \in \mathcal{U}$ then there is $i \in I$

such that $e(i) \subseteq F$, hence $e(i) \subseteq U$ and $U \in \mathcal{U}$. Conversely, if $U \in \mathcal{U}$ then $e(i) \subseteq U$ for some $i \in I$, but then $e(i) \in {\uparrow} e(i) \subseteq \mathcal{U}$. ■

Scott's Principle furthermore implies that every $\mathcal{U} \in \mathcal{O}(\Sigma^{\mathbb{N}})$ is an overt union of basic open sets in a *canonical* way, namely

$$\mathcal{U} = \bigcup \{ {\uparrow} F \mid F \in \mathscr{F} \cap \mathcal{U} \} .$$

The set $\mathscr{F} \cap \mathcal{U}$ is overt because it is the intersection of an overt and an open set.

Actually, we could have computed the topology of $\Sigma^{\mathbb{N}}$ already from the weaker principle WSO.

**Proposition 6.9** *If $\Sigma = \Sigma^0_1$ and WSO then the family $\{ {\uparrow} F \mid F \in \mathscr{F} \}$ is a basis for $\Sigma^{\mathbb{N}}$. Moreover, every open subset of $\Sigma^{\mathbb{N}}$ is Scott open.*

*Proof.* Let $\mathcal{U} \in \mathcal{O}(\Sigma^{\mathbb{N}})$ be an open set. It suffices to prove

$$\mathcal{U} = \bigcup \{ {\uparrow} F \mid F \in \mathcal{F} \cap \mathcal{U} \} \ ,$$

because $\mathcal{F} \cap \mathcal{U}$ is overt.

If $F \in \mathcal{F} \cap \mathcal{U}$ then ${\uparrow} F \subseteq \mathcal{U}$ because $\mathcal{U}$ is an upper set. To see this, suppose $U \in \mathcal{U}$ and $U \subseteq V \in \Sigma^{\mathbb{N}}$. Define the map $f : \mathbb{N}^+ \to \Sigma^{\mathbb{N}}$ by

$$f(t) = \{ n \in \mathbb{N} \mid n \in U \lor (t < \infty \land n \in V) \} .$$

Because $f(\infty) = U$ we have $f(\infty) \in \mathcal{U}$. By WSO there exists $k \in \mathbb{N}$ such that $f(k) \in \mathcal{U}$, but $f(k) = V$.

For the opposite inclusion, suppose $U \in \mathcal{U}$. Because $\Sigma = \Sigma^0_1$, $U$ is enumerated by some $e : \mathbb{N} \to \mathbf{1} + U$. Define the map $g : \mathbb{N}^+ \to \Sigma^{\mathbb{N}}$ by

$$g(t) = \{ n \in \mathbb{N} \mid \exists\, i \in \mathbb{N} \,.\, (i < t \land e(i) = n) \} .$$

Because $g(\infty) = U \in \mathcal{U}$ by WSO there exists $k \in \mathbb{N}$ such that $g(k) \in \mathcal{U}$. This finishes the proof because $g(k) \subseteq U$ and $g(k) \in \mathcal{F}$. ■

# 7 Varieties of Constructivism

In this section we consider the relationship between synthetic and metric notions in three varieties of constructive mathematics:

1. Classical mathematics (CLASS).
2. Russian constructivism (RUSS), modeled by the effective topos [9], which is based on the original notion of number realizability by Kleene [12]. In computable analysis this setting is known as *Type I* computability.

3. A slightly strengthened version of Brouwer's intuitionism ($\mathrm{INT}^+$), which is modeled by the realizability topos based on Kleene's function realizability [13]. This is also known as *Type II* computability [21].

Figure 1 summarizes validity of various statements in each of these models, where Continuity Principle is as stated in Theorem 6.1 and Scott's Principle as stated in Proposition 6.8. We see that CLASS serves as a trivialization of the theory, in RUSS there is little connection between metric and synthetic topology, while there is a very good match between our theory and $\mathrm{INT}^+$.

| | RUSS | $\mathrm{INT}^+$ | CLASS |
|---|---|---|---|
| WSO | • | • | |
| Metric Axiom | | • | |
| $2^{\mathbb{N}}$ metrized | | • | |
| $\mathbb{N}^+$ metrized | | • | |
| $\mathbb{N}^{\mathbb{N}}$ compact | | | • |
| $2^{\mathbb{N}}$ compact | | • | • |
| $\mathbb{N}^+$ compact | | • | • |
| Scott principle | • | • | |
| Continuity principle | • | • | |

Figure 1: Metrization and compactness in varieties of constructivism

### 7.1 Classical Mathematics

If we assume the Law of Excluded Middle nothing much can be said apart from the fact that synthetic topology collapses because $2 = \Sigma = \Omega$. All spaces are overt, compact, discrete, Hausdorff, and metrized by the discrete metric. By Proposition 6.2 the Metric Axiom fails.

### 7.2 Number realizability or Russian Constructivism

We work within the framework of Russian constructivism in the style of Richman's [17, 3] and synthetic computability [1]. The following two principles are valid in Russian constructivism:

1. *Markov Principle:* If not all terms of a binary sequence are zeros then some of them are ones.

2. *Enumerability Axiom:* There are countably many countable subsets of $\mathbb{N}$.

At first we restrict attention to $\Sigma = \Sigma^0_1$. In this case the set of countable subsets of $\mathbb{N}$ is just $\Sigma^{\mathbb{N}}$. Let $W : \mathbb{N} \to \Sigma^{\mathbb{N}}$ be an enumeration. Markov Principle says that $\neg\neg p \implies p$ for all $p \in \Sigma$.

The Enumerability Axiom implies (and is implied by) Richman's axiom CFP, which states that there is an enumeration $\phi_0, \phi_1, \ldots$ of those partial maps $\mathbb{N} \rightharpoonup \mathbb{N}$ that have countable graphs. A partial function $f : \mathbb{N} \rightharpoonup \mathbb{N}$ has an enumerable graph if, and only if, "$f(n)$ is defined" is semidecidable for all $n \in \mathbb{N}$. We write $f(n)\!\downarrow$ for "$f(n)$ is defined". Because semidecidable truth values are open, this implies that, for all $a = (a_0, \ldots, a_{k-1}) \in \mathbb{N}^k$ and $n \in \mathbb{N}$, the truth value

$$\forall\, i < k\,.\,(\phi_n(i)\!\downarrow \wedge\, \phi_n(i) = a_i) \tag{2}$$

is open. We abbreviate (2) as $\overline{\phi_n}(k) = a$.

The classical nature of Markov Principle and the non-classical nature of Enumerability Axiom combine into a strange mix of consequences. We show that WSO is valid, but $\mathbb{N}^+$ is not metrized or compact.

**Proposition 7.1** *The principle WSO holds.*

*Proof.* A detailed proof can be found in [1, 4.26]. By Lawvere's fixed point theorem [15], every $f : \Sigma \to \Sigma$ has a fixed point, namely $W_n(n) = f(W_n(n))$ where $n \in \mathbb{N}$ is such that $W_n(k) = f(W_k(k))$. By Markov Principle it suffices to show

$$\forall\, U \in \mathcal{O}(\mathbb{N}^+)\,.\,((\forall\, n \in \mathbb{N}\,.\, n \notin U) \implies \infty \notin U)\,.$$

Suppose $U \in \mathcal{O}(\mathbb{N}^+)$ and $n \notin U$ for all $n \in \mathbb{N}$. Let $q : \mathbb{N}^+ \to \Sigma$ be the quotient map $q(t) = (t < \infty)$. The map $U : \mathbb{N}^+ \to \Sigma$ factors through $q$ to give a map $f : \Sigma \to \Sigma$, which has a fixed point $p \in \Sigma$. Because $f(\top) = \bot$ and $f(p) = p$, we see that $p \neq \top$, hence $p = \bot$. Thus we get $U(\infty) = f(q(\infty)) = f(\bot) = \bot$, as required. ∎

**Proposition 7.2** *There exists $V \in \mathcal{O}(\mathbb{N}^+)$ which is not a union of balls.*

*Proof.* Let $r : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^+$ be the retraction

$$r(\alpha)(k) = \begin{cases} 1 & \text{if } \exists\, j \leq k\,.\, \alpha(k) \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

let $\mathsf{Tot} \subseteq \mathbb{N}$ be the set of those $n$ for which $\phi_n$ is a total function, and let $\psi : \mathsf{Tot} \to \mathbb{N}^+$ be the composition $\psi = r \circ \phi$.

We prove that $\mathbb{N}^+$ is not metrized by constructing an open subset $V \subseteq \mathbb{N}^+$ such that $\infty \in V$ but no ball $B(\infty, 2^{-k})$ is contained in $V$. For this purpose, define the map $s : \mathbb{N}^+ \to \mathbb{N}^+$ by $s(\alpha)(k) = \alpha(k^2 + 2k)$. It computes the integer part of square root, i.e., for $n \in \mathbb{N}$, $s(n)$ is the $m \in \mathbb{N}$ such that $m^2 \leq n < (m+1)^2$, and $s(\infty) = \infty$. The set

$$\begin{aligned} U = \{n \in \mathsf{Tot} \mid{}& n < s(\psi_n) \vee \\ & (\psi_n < \infty \wedge \exists\, j < s(\psi_n)\,.\, \overline{\phi_j}(1 + \psi_n) = \overline{\psi_n}(1 + \psi_n))\} \end{aligned}$$

is open. Moreover, we claim that $n \in U$ and $\psi_n = \psi_m$ imply $m \in U$. There are two cases to consider:

1. If $n < s(\psi_n)$ there are two further cases: (1) if $m \leq n$ then $m \in U$ because $m \leq n < s(\psi_n) = s(\psi_m)$, (2) if $n < m$ then either $m < s(\psi_m)$ or $n < s(\psi_n) = s(\psi_m) \leq m$, both of which imply $m \in U$.
2. If $\psi_n < \infty$ and there is $j < s(\psi_n)$ such that $\overline{\phi_j}(1 + \psi_n) = \overline{\psi_n}(1 + \psi_n)$ then me may use the same $j$ to conclude that $m \in U$.

It follows that $U$ induces an open subset $V \subseteq \mathbb{N}^+$, defined by

$$V = \left\{\alpha \in \mathbb{N}^+ \mid \exists\, n \in \mathsf{Tot}\,.\,(\psi_n = \alpha \wedge n \in U)\right\} ,$$

or equivalently,

$$V = \left\{\alpha \in \mathbb{N}^+ \mid \forall\, n \in \mathsf{Tot}\,.\,(\psi_n = \alpha \implies n \in U)\right\} .$$

We prove that, for any $m \in \mathbb{N}$, the set $S_m = s^{-1}(m) = \{m^2, \ldots, m^2 + 2m\}$ is not contained in $V$. For every $k \in S_m$ there is $n_k \in \mathbb{N}$ such that $\psi_{n_k} = k$. Each of the numbers $n_{m^2}, \ldots, n_{m^2+2m}$ is an element of $U$ if it satisfies one or the other disjunct in the definition of $U$. Since $s(\psi_{n_k}) = m$ at most $m$ satisfy the first disjunct, and at most $m$ satisfy the second one because different $n_k$'s receive different witnesses $j$. As there are $2m + 1$ numbers $n_k$, not all of them are in $U$, therefore not all elements of $S_m$ are in $V$.

Clearly, $\infty \in V$. For any $k \in \mathbb{N}$, the open ball $B(\infty, 2^{-k})$ is not contained in $V$ because that would also mean that $S_m$ is contained in $V$ for large enough $m$. ∎

The complement of the set $V$ from previous proof is neither finite nor infinite (does not contain an infinite, strictly increasing sequence). In recursion theory such sets are called *immune*.

By Proposition 4.3 it follows that $\mathbb{N}^+$ is not metrized, and because WSO holds, that it is not compact. This has consequences for $\mathbb{N}^\mathbb{N}$ and $2^\mathbb{N}$.

**Corollary 7.3** *The spaces $\mathbb{N}^\mathbb{N}$ and $2^\mathbb{N}$ are neither metrized nor compact.*

*Proof.* If either were metrized or compact then so would be their retract $\mathbb{N}^+$. ∎

The fact that $\mathbb{N}^\mathbb{N}$ is not metrized is essentially a result of Friedberg's who constructed an effective but not partial recursive operator [8]. If a space is not metrized with respect to $\Sigma^0_1$ then it is not metrized with respect to any larger dominance $\Sigma$. Thus Proposition 7.2 and Corollary 7.3 imply that $\mathbb{N}^+$ and $\mathbb{N}^\mathbb{N}$ are not metrized with respect to any dominance that makes $\mathbb{N}$ overt.

### 7.3 Function Realizability or Brouwer's Intuitionism

In this section we consider a slightly strengthened version of Brouwer's intuitionism. We adopt the following principles:

1. *Function-Function Choice:* for any total relation $R \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ there exists a choice function $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $\langle \alpha, f(\alpha) \rangle \in R$ for all $\alpha \in \mathbb{N}^{\mathbb{N}}$.

2. *Continuity Principle:* for every $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ and $\alpha \in \mathbb{N}^{\mathbb{N}}$ there exists $k \in \mathbb{N}$ such that $\beta \in B(\alpha, 2^{-k})$ implies $f(\alpha) = f(\beta)$.

3. *Fan Principle:* every decidable bar is uniform, cf. Section 4.

In the usual setting INT [3, 5.2] only the weaker *Function-Number* choice is used. Kleene's function realizability and the corresponding realizability topos validate not only Function-Function Choice but even Function Choice: every total relation with domain $\mathbb{N}^{\mathbb{N}}$ has a choice function. In Type II effectivity Function Choice manifests itself as the fact that $\mathbb{N}^{\mathbb{N}}$ has an admissible injective representation.

We show that $\Sigma = \Sigma^0_1$, Function-Function choice, and Continuity Principle together imply the Metric Axiom. It then follows from the Fan Principle and Corollary 4.5 that $2^{\mathbb{N}}$, and more generally any inhabited CTB, is compact.

Let us prove that every open subset $U \subseteq \mathbb{N}^{\mathbb{N}}$ is a union of balls. Because the map $q : 2^{\mathbb{N}} \to \Sigma$, defined by $q(\alpha) = (\exists\, n \in \mathbb{N} \,.\, \alpha_n = 1)$ is surjective, by Function-Function Choice there exists a map $f : \mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that

$$U = \left\{\alpha \in \mathbb{N}^{\mathbb{N}} \mid \exists\, n \in \mathbb{N} \,.\, f(\alpha)(n) = 1\right\}.$$

By Continuity Principle and Function-Number Choice there exists a modulus of continuity $\mu : \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$ such that, for every $\alpha \in \mathbb{N}^{\mathbb{N}}$, $\beta \in B(\alpha, 2^{-\mu(\alpha,n)})$ implies $f(\alpha)(n) = f(\beta)(n)$. Define

$$I = \left\{\langle \alpha, n \rangle \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \mid f(\alpha)(n) = 1\right\}$$

and observe that

$$U = \bigcup_{\langle \alpha, n \rangle \in I} B(\alpha, 2^{-\mu(\alpha,n)}).$$

We now know that open subsets are unions of open balls, therefore $\mathbb{N}^{\mathbb{N}}$ is overt by Propositions 3.1 and 2.2. Consequently, $I$ is overt as well. The Metric Axiom is established.

One may wonder if the Continuity Principle alone implies the Metric Axiom for the case $\Sigma = \Sigma^0_1$. This would make the two equivalent because the Metric Axiom implies the Continuity Principle by Theorem 6.1. However, in Russian constructivism the Continuity Principle is validated by the Kreisel-Lacombe-Shoenfield-Ceitin theorem [14, 5, 4] but the Metric Axiom fails, as we showed in Section 7.2.

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