Logic of Ruler and Compass Constructions

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Euclidean Constructive Geometry **ECG**

Does *constructive* refer to the use of intuitionistic logic?

Or does it refer to geometrical constructions with ruler and compass?

What is the relation between these two?

In our constructive geometry, they are closely related: things proved to exist can be constructed with ruler and compass.

Continuous geometry

Constructive proofs yield continuity in parameters.

In practice, constructive proofs *require* continuity in parameters.

There are thus *three* ways of looking at this subject: geometry with intuitionistic logic, geometry of ruler-and-compass constructions, geometry with continuous dependence on parameters.

Is Euclid's reasoning constructive?

Yes, Euclid's reasoning is generally constructive; indeed the only irreparably non-constructive proposition is Book I, Prop. 2, which shows that a rigid compass can be simulated by a collapsible compass. We just take Euclid I.2 as an axiom, thus requiring a rigid compass in $\mathbf{ECG}$. Only one other repair is needed, in the formulation of the parallel axiom, as we shall see below.
We also take as an axiom $\neg\neg\mathbf{B}(x,y,z) \rightarrow \mathbf{B}(x,y,z)$, or "Markov's principle for betweenness", enabling us to drop double negations on atomic sentences. Here betweenness is strict.

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- *Circle* (A,B) (center A, passes through B) collapsible compass
- *Circle* (A,B,C) (center A, radius BC) rigid compass

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- *IntersectCircles1* (c_1,c_2)
- *IntersectCircles2* (c_1,c_2)

Models of the Elementary Constructions

- The "standard plane" $\mathbb{R}^2$
- The "recursive plane". Points are given by recursive functions giving rational approximations to within $1/n$.
- The minimal model, the points constructible by ruler and compass
- The algebraic plane, points with algebraic coordinates
- The Poincaré model. These constructions work in non-Euclidean geometry too.

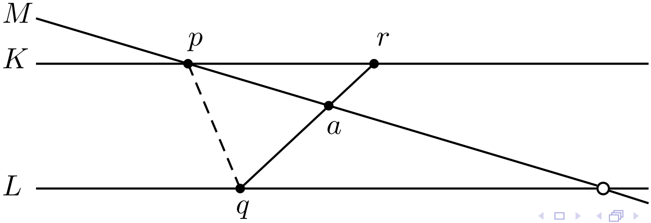
What ECG proves to exist, can be constructed with ruler and compass

In earlier work I proved that if **ECG** proves an existential statement $\exists y A(x, y)$, with A negative, then there is a term t of **ECG** such that **ECG** proves $A(x, t(x))$. In words: things that **ECG** can prove to exist, can be constructed with ruler and compass. Of course, the converse is immediate: things that can be constructed with ruler and compass can be proved to exist in **ECG**. Hence the two meanings of "constructive" coincide for **ECG**: it could mean "proved to exist with intuitionistic logic" or it could mean "constructed with ruler and compass."

Euclid's Parallel Postulate

If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

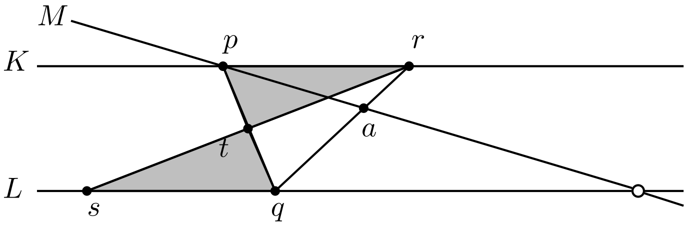
That is, M and L must meet on the right side, provided $\mathbf{B}(q, a, r)$ and pq makes alternate interior angles equal with K and L. The point at the open circle is asserted to exist.



Euclid 5 without mentioning angles

We need to eliminate mention of "alternate interior angles", because angles are not directly treated in **ECG**, but instead are treated as triples of points.

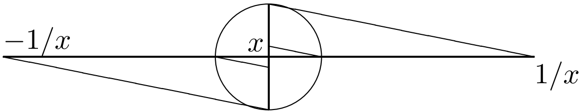
M and L must meet on the right side, provided $\mathbf{B}(q, a, r)$ and $pt = qt$ and $rt = st$.



Inadequacy of Euclid 5

Although we have finally arrived at a satisfactory formulation of Euclid 5, that formulation is satisfactory only in the sense that it accurately expresses what Euclid said. It turns out that this axiom is not satisfactory as a parallel postulate for **ECG**. The most obvious reason is that it is inadequate to define division geometrically. But it turns out to also be inadequate for continuous (constructive) addition and multiplication!

Division and parallels



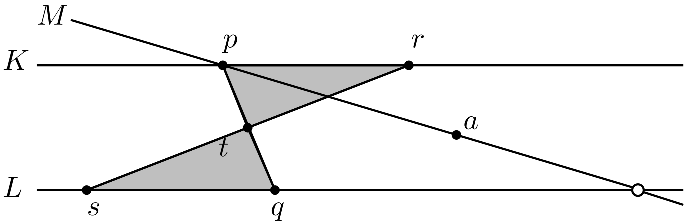
The circle has radius 1. The slanted lines are parallel. $1/x$ is defined if and only if the horizontal line intersects the long slanted line. If we know the sign of x then Euclid 5 suffices; the vertical line is a transversal and on one side the interior angles are less than two right angles.

Division and parallels

Without knowing the sign of x, we will not know on which side of the transversal pq the two adjacent interior angles will make less than two right angles. In other words, with Euclid 5, we will only be able to divide by a number whose sign we know; and the principle $x \neq 0 \ \rightarrow \ x < 0 \vee x > 0$ is not an axiom (or theorem) of **ECG**. The conclusion is that if we want to divide by nonzero numbers, we need to strengthen Euclid's parallel axiom.

The Strong Parallel Postulate of **ECG**

Figure: Strong Parallel Postulate: M and L must meet (somewhere) provided a is not on K and $pt = qt$ and $rt = st$.



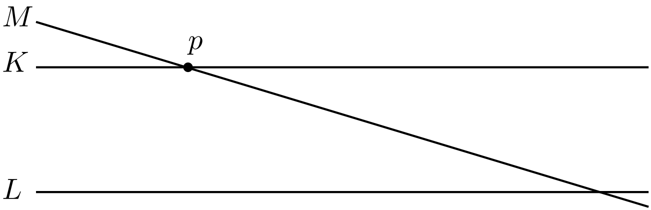
Is the "strong" parallel axiom stronger than Euclid 5?

The strong parallel axiom differs from Euclid's version in that we are not required to know in what *direction* M passes through P; but also the conclusion is weaker, in that it does not specify *where* M must meet L. In other words, the betweenness hypothesis of Euclid 5 is removed, and so is the betweenness conclusion. Since both the hypothesis and conclusion have been changed, it is not immediate whether this new postulate is stronger than Euclid 5, or equivalent, or possibly even weaker, but it turns out to be stronger–hence the name.

Playfair's Axiom

Let P be a point not on line L. We consider lines through P that do not meet L (i.e., are parallel to L). Playfair's version of the parallel postulate says that two parallels to L through P are equal. That is, in the picture not both M and K are parallel to L.

But no point is asserted to exist.



Euclidean rings

Classical Euclidean geometry has models $K^2 = K \times K$ where K is a Euclidean field, i.e. an ordered field in which nonnegative elements have square roots.

We define a Euclidean ring to be an ordered ring in which nonnegative elements have square roots. We use a language with symbols $+$ for addition and $\cdot$ for multiplication, and a unary predicate $P(x)$ for "x is positive".

Euclidean fields

Euclidean fields: nonzero elements have reciprocals.

Weakly Euclidean fields: positive elements have reciprocals.

Playfair rings: elements without reciprocals are zero, and if x is greater than a positive invertible element, then x is invertible.

ECG corresponds to Euclidean fields

The models of $\mathbf{ECG}$ are of the form F^2, where F is a Euclidean field. More specifically, given such a field, we can define betweenness, incidence, and equidistance by analytic geometry and verify the axioms of **ECG**. Conversely, and this is the hard part, we can define multiplication, addition, and division of points on a line (having chosen one point as zero), in **ECG**. It turns out that we need the strong parallel axiom to do that.

We also need new constructions. Those of Descartes and Hilbert require case distinctions and do not provide continuous dependence on parameters.

Weakly Euclidean fields correspond to Euclid 5

If we replace the parallel axiom of **ECG** by Euclid's parallel postulate, we get instead models of the form F^2, where F is a weakly Euclidean field (that is, a Euclidean ring in which positive elements have reciprocals).

We cannot go the other way by defining multiplication and addition geometrically without the strong parallel axiom. (That is, if we only had Euclid 5, we would need case distinctions, as Hilbert and Descartes did.)

Playfair rings correspond to Playfair's axiom

We now work out the field-theoretic version of Playfair's axiom. Playfair says, if P is not on L and K is parallel to L through P, that if line M through P does not meet L then $M = K$. Since $\neg\neg M = K \;\rightarrow\; M = K$, Playfair is just the contrapositive of the parallel axiom of **ECG**, which says that if $M \neq K$ then M meets L. Hence it corresponds to the contrapositive of $x \neq 0 \;\rightarrow\; 1/x \downarrow$; that contrapositive says that if x has no multiplicative inverse, then $x = 0$. Thus Playfair geometries have models F^2 where F is a Playfair ring (as defined above). (We cannot prove the converse because we need the strong parallel axiom to verify multiplication.)

Strong Parallel Axiom implies Euclid 5

The reduction of geometry to field theory described above shows that (relative to a base theory), the strong parallel axiom implies Euclid's postulate 5 (since if reciprocals of non-zero elements exist, then of course reciprocals of positive elements exist). (A direct proof has also been given.)
And Euclid 5 easily implies Playfair's postulate.

Euclid 5 does not imply the strong parallel axiom

That is, relative to the base theory **ECG** minus its parallel axiom. Since non-constructively, the implications *are* reversible, we cannot hope to give counterexamples. In terms of field theory, we won't be able to construct a Euclidean ring in which positive elements have reciprocals but nonzero elements do not. The proof proceeds by constructing appropriate Kripke models.

The field elements (coordinates of points) in those models are real-valued functions belonging to carefully-constructed rings.

Summary

Euclid needs only two modifications to be completely constructive: we have to postulate a rigid compass, rather than relying on Prop. I.2 to simulate it, and we have to take the strong parallel axiom instead of Euclid 5. With those changes Euclid is entirely constructive, and **ECG** formalizes Euclid nicely.

The classical constructions used to define addition and multiplication involve non-constructive case distinctions, but these can be replaced by more elaborate constructions that are continuous (and constructive), so geometry can still be shown equivalent to the theory of Euclidean fields, and different versions of the parallel axiom correspond to weakenings of the field axiom about reciprocals.

Summary

ECG has the nice property that things it can prove to exist can be constructed with ruler and compass, and hence depend continuously on parameters.

ECG permits us to distinguish between versions of the parallel axiom with different constructive content, even though non-constructively they are equivalent, and using Kripke models whose "points" are real-valued functions, we proved formal independence results to make those distinctions sharp.

To read more

Google *Michael Beeson Cambridge.pdf*
for the 10-page version.

Google *Michael Beeson ConstructiveGeometryLong.pdf*
for the unfinished 260 page version (it is unfinished, but it has complete proofs).