Local Homogeneity

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Abstract

We study the expansion of stable structures by adding predicates for *arbitrary* subsets. Generalizing work of Poizat-Bouscaren on the one hand and Baldwin-Benedikt-Casanovas-Ziegler on the other we provide a sufficient condition (Theorem 4.7) for such an expansion to be stable. This generalization weakens the original definitions in two ways: dealing with arbitrary subsets rather than just submodels and removing the 'small' or 'belles paires' hypothesis. We use this generalization to characterize in terms of pairs, the 'triviality' of the geometry on a strongly minimal set (Theorem 2.5). Call a set A *benign* if any type over A in the expanded language is determined by its restriction to the base language. We characterize the notion of benign as a kind of local homogenity (Theorem 1.7). Answering a question of [8] we characterize the property that M has the finite cover property over A (Theorem 3.9).

Let M be a stable structure in a language L and form an $L(P) = L^*$-structure (M, A) by interpreting a new predicate P as the set A. When is the new structure stable? Clearly the structure induced on A (called A_{ind}) must be stable and so it is natural to assume A_{ind} is stable. But stability of A_{ind} is not sufficient [12]. Poizat [12] and Bouscaren [6] restrict the question by assuming the set A is the universe of a submodel. Baldwin and Benedikt [3] restrict the question by assuming the set A is a set of L-indiscernibles. Casanovas and Ziegler [8] generalize 'model' and 'indiscernible' to 'nfcp over A'. (nfcp is short for 'does

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not have the finite cover property'; we spell out the technical definitions below.) All of this earlier work also makes a 'smallness' or 'belles paires' assumption on (M, A). In the spirit of [3], we call (M, A) *pseudosmall* if (M, A) is elementarily equivalent to a structure (N, B) which is *small*: every L-type over $B\mathbf{m}$ (for finite $\mathbf{m}$) is realized in N.

This article springs from two intuitions of the first author. 1) The smallness hypothesis can be replaced by a weaker notion, which we call *benign*, (in a sense explained in Section 3 replacing 'saturation' with 'homogeneity'). 2) Almost all subsets of a stable model are benign. Most intuitively, A is benign if for any $\alpha \in M - A$, $\mathrm{tp}(\alpha/A) \models \mathrm{tp}_*(\alpha/A)$. (Here, tp denotes the L-type and tp_* the type in L^*.) We see below that the first intuition is largely correct. In [1] with Shelah we provide an example of a subset of a superstable theory which is not benign. We then modify the notion of benign to get a notion (weakly benign) that, as shown below, has the useful consequences of benign and such that ([1]), every subset of a superstable theory is weakly benign.

In Section 1 we rephrase 'benign' as a homogeneity condition which is local in two senses: we work over a fixed set A and we work with respect to a finite set Δ of formulas. We pass to the uniform (true in all L^*-elementarily equivalent structures) version of benign and locally homogeneous and show uniform locally homogeneous is equivalent to uniformly benign. There are several reasons beyond curiosity for generalizing the study of expansions from naming submodels to naming arbitrary sets. Indiscernible sets formed the natural subject in [3]; such expansions arise, for example, in the study of bicolored fields [4]; in Section 2, we will use these expansions to characterize geometric properties of strongly minimal sets. Section 3 describes three properties of an $L(P)$ theory: pseudosmall, bounded, uniformly benign which are increasingly weaker. As in Section 1 pseudosmall is derived from a notion, small, defined for a single structure (M, A) and pseudosmall means all elementarily equivalent $L(P)$-structures are small. Answering a question of [8], we show that small and pseudosmall are equivalent if and only if M does not have the fcp over A. Further we show that for theories without fcp, pseudosmall is equivalent to 'all models are *locally saturated*' (defined in Section 3). In Section 4 we show that for uniformly benign structures, (M, A) is stable if M is stable and the induced (properly construed) structure on A is stable.

The stability of (M, A) depends on two inputs: a 'smallness' hypothesis on A and the stability of the 'induced' structure on A. The smallness hypothesis might be given either on the single model (M, A) or on its theory; these alternatives coalesce in the presence of the nfcp over A (Section 3). There are a number of alternatives for the stability hypothesis.

Definition 0.1 *The basic formulas induced on A can be:*

1. *the traces on A of parameter free L-formulas (*induced structure*);*
2. *the traces on A of L-formulas with parameters;*
3. *the traces on A of parameter free $L(P)$-formulas (#-*induced structure*, $A^{\#}$);*

4. the traces on A of $L(P)$-formulas with parameters.

If the ambient theory is stable, the first two are same so we name only one. Although 3) and 4) do define different classes, we use only 3). The following example of Benedikt uses the idea of the third example in Example 3.19 to show the need to consider 3) instead of 1).

Example 0.2 *Form a structure M with a two sorted universe; one sort contains the complex field, a binary relation E links the two with each field element indexing one member of a partition of the second sort into infinite sets. Now let N extend M by putting one new point in the set indexed by a if and only if a is a real number. Now M and N are isomorphic and are ω stable nfcp. But the structure (N, M) is unstable. The induced structure on M is stable since in fact no new sets are definable. In the #-induced structure the formula $(\exists x)E(x, y) \wedge x \notin P$ defines the reals so the #-induced structure is unstable.*

It may not be easy to check either that A_{ind} or $A^{\#}$ is stable. In A_{ind} while the quantifier-free formulas are just induced from L, in order to verify the stability one must do an induction on quantifiers which is nontrivial. For example, Baldwin and Holland [4] constructed a bicolored field by a variant of the Hrushovski construction that is not ω-stable but whose stability class is unknown. It fairly easy to check that the structure imposed on the black points is minimal (every definable set is finite or cofinite) at the R_ϕ-level but the stability of even A_{ind} remains open.

Remark 0.3 Although we deal in this paper exclusively with expansions by unary predicates, this is not an important restriction; the case of arbitrary n-ary relations reduces to the unary case. Let M be a stable L-structure and for some $1 <\leq n < \omega$, let R be an n-place predicate on M, which to be interesting is not L-definable. Denote by M^{eq} a model of T^{eq} constructed from M. Let $\epsilon(\overline{x}, \overline{y})$ be the equivalence relation on M^n defined by

$$[M \models \epsilon(\overline{x}, \overline{y}) \leftrightarrow \wedge_i x_i = y_i].$$

Let M_ϵ be the quotient structure by ϵ. For any $0 < m < \omega$, denote by $Def_{\emptyset,m}(M_\epsilon)$ a set of all $\emptyset$-definable subset of M_ϵ^m and by $Def_\emptyset(M\epsilon) := \bigcup_m Def_{\emptyset,m}(M_\epsilon)$. Define a model M' with universe M_ϵ in the language $\{=\} \cup \{P_X | X \in Def_\emptyset(M_\epsilon)\}$ such that $\forall m < \omega, \forall X \in Def_{\emptyset,m}(M_\epsilon), \forall a_, \ldots, a_m \in M_\epsilon$

$$[M' \models P_X(a_1, \ldots, a_m) \leftrightarrow (a_1, \ldots, a_m) \in X].$$

So, $M' = (M_\epsilon, P_X)_{X \in Def_\emptyset(M_\epsilon)}$. Denote by R' the set of "names" of n-tuples from M^n satisfying R. Then, (M, R) and (M', R') are mutually interpretable so (M, R) is stable iff $(M,' R')$ is stable.

1 Benign and Locally Homogeneous Pairs

In this section we introduce the notions of benign and locally homogeneous pairs. These concepts represent a weakening of the notion 'small' from [8], which provided a common framework for Poizat's [12] notion of 'belles paires' and the Baldwin-Benedikt [3] notion of pseudosmall. We describe the relations among these notions in Section 3. In the following definitions, each formula in Δ (Δ') has the same partition of its variables among 'true variables' and parameters. Throughout the paper we deal with a language L and an expansion $L(P)$ by a unary predicate P. We often use $*$ for $L(P)$-for brevity. If no language is specified we mean L; but sometimes we write the L for emphasis. And if we speak of (M, A) this implies the language is $L(P)$.

We are speaking of types of finite tuples unless we explicitly say otherwise.

Definition 1.1 *1. The set A is* benign *in M if for every $\alpha, \beta \in M$ if $p = \mathrm{tp}(\alpha/A) = \mathrm{tp}(\beta/A)$ then $\mathrm{tp}_*(\alpha/A) = \mathrm{tp}_*(\beta/A)$ where the $*$-type is the type in the language with a new predicate P denoting A.*

2. *(M, A) is* uniformly benign *if every (N, B) which is $L(P)$-elementarily equivalent to (M, A) is benign.*

3. *The set A is* weakly benign *in M if for every $\alpha, \beta \in M$ if $\mathrm{stp}(\alpha/A) = \mathrm{stp}(\beta/A)$ implies $\mathrm{tp}_*(\alpha/A) = \mathrm{tp}_*(\beta/A)$ where the $*$-type is the type in the language with a new predicate P denoting A.*

4. *(M, A) is* uniformly weakly benign *if every (N, B) which is $L(P)$-elementarily equivalent to (M, A) is weakly benign.*

These notions are closely related to the following homogeneity conditions on M.

Definition 1.2 *1. The pair (M, A) is* locally homogeneous *if $\mathrm{tp}_L(\alpha/A) = \mathrm{tp}_L(\beta/A)$ implies for every finite $\Delta \subseteq L$ and any $\mathbf{m} \in M$, with $\mathrm{tp}_\Delta(\mathbf{m}/\alpha, A) = q(\mathbf{x}, \alpha, A) \in S_\Delta(A\alpha)$, $q(\mathbf{x}, \beta, A)$ (the result of replacing α with β in $q_\Delta(\mathbf{x}, \alpha)$) is realized in M.*

2. *The pair (M, A) is* uniformly locally homogeneous *if for every finite Δ there is a finite $\Delta' \supseteq \Delta$ such that for any $\mathbf{m} \in M$, with $\mathrm{tp}_\Delta(\mathbf{m}/\alpha, A) = q(\mathbf{x}, \alpha, A) \in S_\Delta(A\alpha)$ and $\mathrm{tp}_{\Delta'}(\alpha/A) = \mathrm{tp}_{\Delta'}(\beta/A)$ then $q(\mathbf{x}, \beta, A)$ (the result of replacing α with β in $q_\Delta(\mathbf{x}, \alpha)$) is realized in M.*

Note that the concept of local homogeneity is orthogonal to saturation notions. Unlike saturation it deals with finite Δ rather than all of L, but since $|A|$ may be the same as $|M|$ it is not implied by saturation either. The following lemma characterizes uniform local homogeneity. For convenience, we assume Δ is closed under negation. Recall that a structure is *weakly saturated* if it realizes every consistent type over the empty set.

Lemma 1.3 *1. Local homogeneity is preserved by $L(P)$-elementary equivalence.*

2. If (M, A) is locally homogeneous and weakly saturated then (M, A) is uniformly locally homogeneous.

Proof. For any finite $\Delta \subset L(P)$, let $\mu_{\Delta,\Delta'}$ denote

$$(\forall \mathbf{t}_1)(\forall \mathbf{t}_2)((\forall \mathbf{u} \in P) \bigwedge_{\chi \in \Delta'} [\chi(\mathbf{t}_1, \mathbf{u}) \leftrightarrow \chi(\mathbf{t}_2, \mathbf{u})] \rightarrow (\forall \mathbf{x})(\exists \mathbf{y}) \bigvee_{\theta \in \Delta} (\forall \mathbf{u} \in P)\theta(\mathbf{x}, \mathbf{t}_1, \mathbf{u}) \leftrightarrow \theta(\mathbf{y}, \mathbf{t}_2, \mathbf{u})).$$

To see i) note that (M, A) is uniformly locally homogeneous if and only if for any finite $\Delta \subset L(P)$, there is a finite $\Delta' \supseteq \Delta$ such that $(M, A) \models \mu_{\Delta,\Delta'}$.

Using i), if (M, A) is locally homogeneous but not locally homogeneous, there is a finite Δ such that for every $\Delta' \supseteq \Delta$, the following set of formulas $\Gamma_{\Delta,\Delta'}(\mathbf{t}_1, \mathbf{t}_2)$ is consistent.

$$\{(\forall \mathbf{u} \in P) \bigwedge_{\chi \in \Delta'} [\chi(\mathbf{t}_1, \mathbf{u}) \leftrightarrow \chi(\mathbf{t}_2, \mathbf{u})] : \chi \in L\} \cup \{\neg(\forall \mathbf{x})(\exists \mathbf{y}) \bigvee_{\theta \in \Delta} (\forall \mathbf{u} \in P)\theta(\mathbf{x}, \mathbf{t}_1, \mathbf{u}) \wedge \neg\theta(\mathbf{y}, \mathbf{t}_2, \mathbf{u})).$$

But if (M, A) is weakly saturated, $\bigcup_{\Delta' \subset L} \Gamma_{\Delta,\Delta'}(\mathbf{t}_1, \mathbf{t}_2)$ is realized and then (M, A) is not locally homogeneous. $\square_{1.3}$

With this in hand we easily conclude:

Corollary 1.4 *The following are equivalent.*

1. *The pair (M, A) is uniformly locally homogeneous.*
2. *Every elementary extension of the $L(P)$-structure (M, A) is uniformly locally homogeneous.*
3. *Every elementary extension of the $L(P)$-structure (M, A) is locally homogeneous.*
4. *Every weakly saturated elementary extension of the $L(P)$-structure (M, A) is locally homogeneous.*

Lemma 1.5 *Suppose T is stable and the $L(P)$-structure (M, A) is benign, then (M, A) is locally homogeneous.*

Proof. Fix a type $q(\mathbf{x}, \alpha, A)$ which is realized in M and a finite $\Delta \subseteq L$. Let the L-formula over A $d_q\theta(\alpha, \boldsymbol{a})$ define the restriction of q to Δ. That is, $d_q\theta(\alpha, \boldsymbol{a})$ holds if and only if $\theta(\mathbf{x}, \alpha, \boldsymbol{a}) \in q$. Then the formula

$$(\exists \mathbf{x} \notin P)(\forall \mathbf{u} \in P) \bigwedge_{\theta(\mathbf{x},\mathbf{y},\mathbf{z}) \in \Delta} \theta(\mathbf{x}, \mathbf{y}, \mathbf{u}) \leftrightarrow d_q\theta(\mathbf{y}, \mathbf{u})$$

is in $\mathrm{tp}_*(\alpha/A)$ and thus in $\mathrm{tp}_*(\beta/A)$, so $q_\Delta(\mathbf{x}, \beta, A)$ is realized. $\square_{1.5}$

But, this does not yield *uniform* local homogeneity. The following example is derived from Shelah's example ([1]) of a pair (M, A) where A is not benign.

Example 1.6 *We build a pair (M, A) which is benign but not uniformly benign (and consequently locally homogeneous but not uniformly locally homogeneous). We start by describing the prime model M of the theory T^*. The language L^* contains relation symbols $S_i(x), E(x, y), U(x), R(x, y, z)$ and $P(x)$. The elements not in U are α_i, β_i for $i < \omega$ and S_i holds exactly of α_i. Each of $R(x, y, \alpha_i)$, $R(x, y, \beta_i)$, define an equivalence relation refining the same equivalence class of E. The equivalence relations $R(x, y, \alpha_i)$ and $R(x, y, \beta_i)$ cross cut each other. In fact $R(x, y, \alpha) \wedge R(x, y, \beta)$ implies $x = y$. We say 'subdiagonal of the checkerboard on an E-class' for a set which contains exactly one representative of each equivalence class of $R(x, y, \alpha_i)$ and all but one class of $R(x, y, \beta_i)$. For each n, $P(M)$ intersects the nth equivalence class of E in a subdiagonal of the checkerboard determined by $R(x, y, \alpha_n)$ and $R(x, y, \beta_n)$. Then T^* satisfies both*

$$(\exists^\infty w)(\forall v)(\forall x)E(x, w) \rightarrow (\exists y)P(y) \wedge R(y, x, v)$$

and

$$(\exists^\infty w)(\exists v)(\exists x)E(x, w) \rightarrow (\forall y)P(y) \wedge \neg R(y, x, v).$$

Thus both (M, A) and $N, B)$ satisfy T^* where $A = P(M)$ as described, while (N, B) adds an infinite E class D and $P(N) \cap D$ is a subdiagonal for the checkerboard on D given by new elements of $\neg U$, α and β. But then (M, A) is benign (since S_i distinguishes α_i from β_i) but (N, B) is not.

The following requires no stability hypothesis.

Theorem 1.7 *Suppose (M, A) is uniformly locally homogeneous. Then, (M, A) is benign.*

Proof. Suppose $\alpha, \beta \in M - A$ and $\mathrm{tp}(\alpha/A) = \mathrm{tp}(\beta/A)$. For each $\phi \in L(P)(A)$, we want to show that

$$(M, A) \models \phi(\alpha) \text{ iff } (M, A) \models \phi(\beta)$$

Let n be the number of quantifiers in ϕ and Δ_0 the set of atomic formulas from $L(P)$ which occur in ϕ. Let Δ_{i+1} be Δ_i', where ' is the operator from the uniform homogeneity of (M, A) which assigns to Δ the set of formulas Δ' which are sufficient to guarantee Δ-equivalence. Now we claim (M, A, α) is n-game equivalent to (M, A, β) in the language with atomic formulas Δ_0. At the ith play of the game, duplicator plays the same point γ as spoiler if the point is in A and plays the δ_i realizing $\mathrm{tp}_{\Delta_i}(\gamma_i, \beta A)$ if not. (The formula exhibited in the last paragraph of Lemma 1.4, guaranteed by the uniform local homogenity, shows δ_i exists.) $\square_{1.7}$ Now the main result of this analysis is:

Theorem 1.8 *If M is stable then (M, A) is uniformly benign if and only if it is uniformly locally homogenous.*

2 Expansions of Strongly Minimal Sets

The next few results illustrate the distinction between the situation here, where the predicate P is allowed to name an arbitrary subset of the universe and the earlier restriction that only submodels be named. When the ambient structure is strongly minimal, naming a submodel always preserves ω-stability. In fact, [7], the rank of the pair theory can be used to classify the geometry of the strongly minimal set. The pair theory has finite rank if and only if the strongly minimal set is locally modular. By allowing arbitrary subsets, we can extend this classification to distinguish trivial geometries. (On reading a preliminary version of this paper, Yevgeniy Vasilev pointed out that a variant on Buechler's argument allows one to characterize trivial strongly minimal sets by: the pair theory has rank one if and only if the strongly minimal set is trivial.)

For any set A contained in a strongly minimal model M, if α, β are in $\mathrm{acl}(A)$ and $\mathrm{tp}_L(\alpha/A) = \mathrm{tp}_L(\beta/A)$ then there is an elementary permutation of $\mathrm{acl}(A)$ taking α to β. If $A \subseteq M$ *and* M is strongly minimal this permutation extends to an automorphism of M. Again, when M is strongly minimal any two points not in the algebraic closure of a set A are automorphic over A. This establishes the following well-known fact:

Fact 2.1 *If M is strongly minimal every subset of M is benign. By Lemma 1.4 and Theorem 1.7, this means every subset of M is uniformly benign.*

We included the argument for the previous result to emphasize that strong minimality was used twice in the proof. In fact there are sets that are not benign because of algebraic types. Now we see the role of the geometry.

Fact 2.2 *Let M be a trivial strongly minimal model in a countable language L and $A \subset M$. Then the pair (M, A) is superstable.*

Proof. Let (N, B) be an $|M|^+$-saturated elementary extension of (M, A) in language $L(P)$. It suffices to show that the group G of automorphisms of N which fix M pointwise and B setwise has only $2^{\aleph_0}$ orbits on $N - (M \cup B)$. If not there exist $\langle a_i : i < (2^{\aleph_0})^+ \rangle \subset N - (M \cup B)$ such that no $g \in G$ maps a_i to a_j if $i \neq j$. By strong minimality, for each i there exists an L-elementary monomorphism $f_i : M \cup \mathrm{acl}(a_0) \to M \cup \mathrm{acl}(a_i)$ and is the identity on M . If for each $i \neq j$, $f_i^{-1}(\mathrm{acl}(a_i) \cap B) \neq f_j^{-1}(\mathrm{acl}(a_j) \cap B)$, there are more than continuum many subsets of $\mathrm{acl}(a_0)$. So for some i, j, $h_{ij} = f_j \cdot f_i^{-1} : \mathrm{acl}(a_i) \to \mathrm{acl}(a_j)$ is an L-elementary map which preserves B. Moreover, $h = h_{ij} \cup h_{ji}$ is an elementary permutation of $M \cup \mathrm{acl}(a_i) \cup \mathrm{acl}(a_j)$. By triviality, h extends to an L-automorphism $\hat{h}$ of N which fixes $N \setminus (\mathrm{acl}(a_1) \cup \mathrm{acl}(a_2))$ pointwise. Since h also preserves B on $(\mathrm{acl}(a_i) \cup \mathrm{acl}(a_j))$, $\hat{h} \in G$. This contradiction yields the result. $\square_{2.2}$

Note that the restriction to a countable language here is purely for convenience. In fact, the triviality is necessary for this result. Modifying an argument from [5], we show:

Fact 2.3 *If G is a group which is not finitely generated, there is an expansion of G by a unary predicate which is unstable.*

Proof. Choose inductively $\langle a_i, b_i \rangle$ so that no a_i (or b_i) is in the subgroup generated by the set of $a_j, b_j, j < i$ ($a_j : j \leq i, b_j, j < i$). Let $c_{ij} = a_i \cdot b_j$. Now we define the order property by the formula $P(x \cdot y)$ by letting P name $\{c_{ij} : i < j\}$. $\square_{2.3}$

This shows that no locally modular strongly minimal set remains stable under all unary expansions. To include strongly minimal sets which have no group structure we have the following variant.

Fact 2.4 *Let M be a strongly minimal set such that algebraic closure is not trivial. There is an expansion of M by a unary predicate which is unstable.*

Proof. First consider $\langle a_i, b_i : i < \omega \rangle$ a family of algebraically independent elements from a saturated elementary extension of M. Choose $c_{00} \in \mathrm{acl}(a_0, b_0) \setminus (\mathrm{acl}(a_0) \cup \mathrm{acl}(b_0))$ witnessed by a formula $\phi(x, y, v)$. Since all pairs of elements from $\langle a_i, b_i \rangle$ realize the same type there exists $c_{ij} \in \mathrm{acl}(a_i, b_j) \setminus (\mathrm{acl}(a_i) \cup \mathrm{acl}(b_j))$ satisfying $\phi(a_i, b_j, c_{ij})$. Moreover, unless $k = i$ and $\ell = j$, we have $\neg\phi(a_k, b_\ell, c_{ij})$. Now for each $n < \omega$, choose in M by elementary submodel a set $\langle a_i^n, b_i^n : i < n \rangle \cup \langle c_{i,j}^n : i, j < n \rangle$ which satisfy the $\phi(x, y, v)$-type of $\langle a_i, b_i : i < n \rangle \cup \langle c_{i,j} : i, j < n \rangle$. Make these sets disjoint. Now let P denote the set $\langle c_{i,j}^n : i < j < n \rangle$. The formula

$$(\exists v)(P(v) \wedge \phi(x, y, v))$$

defines the order property for arbitrarily long sequences so (M, P) is unstable. $\square_{2.4}$

Thus by passing from model-pairs to arbitrary subsets we are able to describe triviality. In summary:

Theorem 2.5 *Let M be a strongly minimal model and $A \subset M$. Then the geometry on M is trivial if and only if for every subset A of M, the pair (M, A) is superstable.*

3 The role of the local fcp

Casanovas and Ziegler introduced useful localizations to sets of two important model theoretic notions: the finite cover property over A and stability over A. We show here that these notions form a bridge between small and pseudosmall. We then discuss the notion of bounded, introduced in [3] but fully explored in [8]. We show that in the presence of nfcp over A, pseudosmall implies bounded implies uniformly benign and these implications are strict. In this section, the only stability assumptions are made explicitly.

Definition 3.1 *We say M has the* finite cover property over A *if there is a formula $\phi(\mathbf{x}, \mathbf{y}, \mathbf{z})$ such that for each $k < \omega$ there are a tuple $\mathbf{m} \in M$ and a family $(\boldsymbol{a}_i)_{i \in I}$ of tuples from A such that the set*

$$\{\phi(\mathbf{x}, \boldsymbol{a}_i, \mathbf{m}) : i \in I\}$$

is k-consistent but not consistent.

Note that 'fcp over A' is preserved by $L(P)$-elementary equivalence. This notion is key to relating the following concepts.

Definition 3.2 *1. The pair (M, A) is* small *if for every finite sequence $\beta \in M$, every L-type over $A\beta$ is realized in M.*

2. The pair (M, A) is pseudosmall *if the $L(P)$-structure (M, A) is $L(P)$-elementarily equivalent to a small pair.*

3. The $L(P)$-theory T^* is pseudosmall if some model (M, A) of T^* is small; whence every model of T^* is pseudosmall.

4. (M, A) is locally saturated *if for any $\mathbf{b} \in M$, for any L-formula $\phi(\mathbf{x}, \mathbf{y}, \mathbf{u})$, any $\phi(\mathbf{x}, \mathbf{y}, \mathbf{b})$-type over A is realized in M.*

Our definition of pseudosmall is the Casanovas-Ziegler [8] definition of small; we want to preserve the distinction between small and pseudosmall made in [3], although both of the notions in [3] were marginally stronger than these. Small is a natural weakening of Poizat's concept [12] of 'belles paires' by dropping the requirement of $|L|^+$-saturation (and in our context working over subsets rather than models). We will show that T^* is pseudosmall is the same as every model of T^* is locally saturated. For this we need first a little more notation.

Note that in the definition of local saturation, it makes little difference whether we think of a ϕ-type as containing only instances of ϕ or both positive and negative instances since we can always code ϕ by a ϕ' so that positive instance of ϕ' determine both positive and negative instances of ϕ.

Notation 3.3 *1. For a pair of formulas $\phi(\mathbf{x}, \mathbf{y}, \mathbf{u})$ and $\psi(\mathbf{y}, \mathbf{z}, \mathbf{u})$ we emulate Poizat [12] and say $[\phi(\mathbf{x}, \mathbf{y}, \mathbf{u}), \psi(\mathbf{y}, \mathbf{z}, \mathbf{u}), n](\mathbf{z}, \mathbf{u})$ holds if*

$$(\forall \mathbf{y}_0 \in P) \ldots (\forall \mathbf{y}_{n-1} \in P)[\bigwedge_{i<n} \psi(\mathbf{y}_i, \mathbf{z}, \mathbf{u}) \rightarrow (\exists \mathbf{x}) \bigwedge_{i<n} \phi(\mathbf{x}, \mathbf{y}_i, \mathbf{u})].$$

2. Suppose M does not have the finite cover property over A. Let T' be the theory axiomatized by the sentences

$$(\forall \mathbf{z}\mathbf{u})[[\phi(\mathbf{x}, \mathbf{y}, \mathbf{u}), \psi(\mathbf{y}, \mathbf{z}, \mathbf{u}), n_\phi](\mathbf{z}, \mathbf{u}) \wedge (\exists \mathbf{y}_0, \ldots \mathbf{y}_{n_\phi})\psi(\mathbf{y}_i, \mathbf{z}, \mathbf{u})] \\ \rightarrow (\exists \mathbf{x})(\forall \mathbf{y})[P(\mathbf{y}) \rightarrow (\psi(\mathbf{y}, \mathbf{z}, \mathbf{u}) \rightarrow \phi(\mathbf{x}, \mathbf{y}, \mathbf{u}))], \tag{1}$$

where n_ϕ is chosen by nfcp so the every n_ϕ-consistent ϕ-type over A is consistent.

By the remark about ϕ and ϕ' just before the last result, we could as well have written $\psi(\mathbf{y},\mathbf{z},\mathbf{u}) \leftrightarrow \phi(\mathbf{x},\mathbf{y},\mathbf{u})$ in last line of part 2). Note that while nfcp implies stability, nfcp over A does not.

The notion of a bounded formula comes from [3].

Definition 3.4 *An $L(P)$ formula ϕ is bounded if has the form:*

$$(Q_1\mathbf{y}_1 \in P), \ldots (Q_k\mathbf{y}_k \in P)\psi(\mathbf{x},\mathbf{y})$$

where each Q_i is $\exists$ or $\forall$ and ψ is an L-formula.

Casanovas and Ziegler [8] localized stablity to a subset by the following definition.

Definition 3.5 *M is stable over A if in every (N, B) elementarily equivalent to (M, A), every type over B is definable by a bounded formula with parameters from B.*

Equivalently, in every (N, B) elementarily equivalent to (M, A), there are only $|B|$ types over B. They note that ncp over A implies stable over A. It is easy to show:

Fact 3.6 *If M has nfcp over A, $(M, A) \models T'$ if and only if (M, A) is locally saturated.*

Answering a question of [8], we are going to characterize the finite cover property over A. For this we require some special notation.

Definition 3.7 *For any formulas $\phi(\mathbf{x},\mathbf{y},\mathbf{u})$ and $\psi(\mathbf{y},\mathbf{z},\mathbf{u})$, we say $FCP_{\phi,\psi}(M, A)$ holds if for all $n < \omega$, there are $\mathbf{c}_n, \mathbf{m}_n \in M$ such that $\{\phi(x, \boldsymbol{a}, \mathbf{m}_n) : \boldsymbol{a} \in \psi(M, \mathbf{c}_n, \mathbf{m}_n) \bigcap A^{lg(\mathbf{y})}\}$ is n-consistent but not consistent.*

Note that $(M, A) \equiv (N, B)$ and $FCP_{\phi,\psi}(M, A)$ then $FCP_{\phi,\psi}(N, B)$.

Lemma 3.8 *If $FCP_{\phi,\psi}(M, A)$ holds then (M, A) is not both small and $|L|^+$-saturated.*

Proof. Consider the type $q_{\phi,\psi}(\mathbf{z},\mathbf{u})$:

$$\{[\phi(\mathbf{x},\mathbf{y},\mathbf{u}), \psi(\mathbf{y},\mathbf{z},\mathbf{u}), n](\mathbf{z},\mathbf{u}) : n < \omega\} \cup \{\neg(\exists \mathbf{x})(\forall \mathbf{y} \in P)[\psi(\mathbf{y},\mathbf{z},\mathbf{u}) \to \phi(\mathbf{x},\mathbf{y},\mathbf{u})\}$$
$$\cup\{(\exists \mathbf{y}_0 \in P) \ldots (\exists \mathbf{y}_n \in P) \bigwedge_{i<n+1} \psi(\mathbf{y}_i,\mathbf{z},\mathbf{u}) : n < \omega\}.$$

This $L(P)$-type is consistent, since by $FCP_{\phi,\psi}$, $[\phi(\mathbf{x},\mathbf{y},\mathbf{u}), \psi(\mathbf{y},\mathbf{z},\mathbf{u}), k](\mathbf{c}_n, \mathbf{m}_n)$ holds for each $k \leq m$ but $\neg(\exists \mathbf{x})(\forall \mathbf{y} \in P)[\psi(\mathbf{y}, \mathbf{c}_n, \mathbf{m}_n) \to \phi(\mathbf{x},\mathbf{y},\mathbf{m}_n)$ since $\neg(\exists \mathbf{x}) \bigwedge_{i<n} \phi(\mathbf{x}, \boldsymbol{a}_i^n, \mathbf{m}_n)$. As (M, A) is $|L|^+$-saturated, $q_{\phi,\psi}(\mathbf{z},\mathbf{u})$ is realized by some $\mathbf{m}, \mathbf{c} \in M$. For such $\mathbf{m}, \mathbf{c}$, any extension to a complete type over $A \bigcup \mathbf{m}$ of the consistent infinite set of ϕ-formulas: $\{\phi(x, \boldsymbol{a}, \mathbf{m}) : M \models \psi(\boldsymbol{a}, \mathbf{c}, \mathbf{m}), \boldsymbol{a} \in A\}$, is not realized in M. This contradicts that (M, A) is small. $\square_{3.8}$

We rely on a theorem of Casanovas and Ziegler for one direction of the following equivalence. The new direction answers a question raised in [8].

Theorem 3.9 *Suppose $T^* = \mathrm{Th}(M, A)$ is pseudosmall. M does not have fcp over A if and only if every $|L|^+$-saturated $(N, B) \models T^*$ is small and M is stable over A.*

Proof. Deducing pseudosmall implies small from nfcp over A is Theorem 5.2 of [8]. Conversely, suppose M does have fcp over A, witnessed by a formula $\phi(\mathbf{x}, \mathbf{y}, \mathbf{u})$.

Suppose that $\phi(\mathbf{x}, \mathbf{y}, \mathbf{u})$ has the finite cover property over A. Our goal is to find a formula ψ which witnesses this property as in Definition 3.7. Take an infinite $W \subseteq \omega$ such that for any $n \in W$, there are elements $\boldsymbol{a}_i^n$ such that $\{\phi(\mathbf{x}, \boldsymbol{a}_i^n, \mathbf{m}_n) : i \leq n\}$ is n-consistent but inconsistent and so $n+1$-inconsistent. Let $A_0 := \{\boldsymbol{a}_i^n : n \in W, i \leq n\}$. By the choice of A_0, ϕ has the fcp over A_0.

Theorem 3.9 is immediate from Lemma 3.8 and the following Proposition 3.10. For the proof of Proposition 3.10, we will refine W several times. It seems clearest to give the proof as a series of reductions.

Proposition 3.10 *Suppose that $\phi(\mathbf{x}, \mathbf{y}, \mathbf{u})$ has the finite cover property over A. Then for any A' ($A_0 \subseteq A' \subseteq M$) such that M is stable over A' there exists $\psi(\mathbf{y}, \mathbf{z}, \mathbf{u})$ such that $FCP_{\phi,\psi}(M, A')$.*

Proof. For any $n \in W$, $i \leq n$ put

$$B_i^n := \bigcap\{\phi(M, \boldsymbol{a}_j^n, \mathbf{m}_n) : j \neq i,\ j \leq n\}.$$

It follows from the definition that $B_i^n \subset \neg\phi(M, \boldsymbol{a}_i^n, \mathbf{m}_n)$ and $B_i^n \cap B_j^n$ is empty when $i \neq j \leq n$. For any formula $\eta(\mathbf{x})$, let $C_j^n(\eta)$ denote $\eta(M) \cap B_j^n$ and $\ell(n, \eta) := |\{j \leq n : C_j^n(\eta) \neq \emptyset\}|$.

Definition 3.11 *Let $\theta(\mathbf{x}, \mathbf{v}, \mathbf{u})$ be a formula, W' be an infinite subset of W, $\Gamma := \{\mathbf{e}_n \in M^{lg(\mathbf{v})} : n \in W'\}$ be a family of parameters.*

1. *We say that the triple (θ, W', Γ) is* unbounded *if $\ell(n, \theta(\mathbf{x}, \mathbf{e}_n, \mathbf{m}_n))$ is unbounded on any infinite $W'' \subseteq W'$.*

2. *Let $f < \omega$; we say that the unbounded triple (θ, W', Γ) is* f-minimal, *if $\forall n \in W', \forall \mathbf{b} \in A'$*

$$[\ell(n, \theta(\mathbf{x}, \mathbf{e}_n, \mathbf{m}_n) \wedge \phi(\mathbf{x}, \mathbf{b}, \mathbf{m}_n)) \leq f \textit{ or } \ell(n, \theta(\mathbf{x}, \mathbf{e}_n, \mathbf{m}_n) \wedge \neg\phi(\mathbf{x}, \mathbf{b}, \mathbf{m}_n)) \leq f].$$

Assume momentarily the following lemma.

Lemma 3.12 *There exist a formula $\theta^*(\mathbf{x}, \mathbf{v}, \mathbf{u})$, an infinite subset W^* of W, and a family of parameters $\Gamma^* := \{\mathbf{e}_n \in M^{lg(\mathbf{v})} : n \in W^*\}$ such that the following holds for some $f < \omega$:*

1. *The triple $(\theta^*, W^*, \Gamma^*)$ is unbounded. In particular, for each $n \in W^*$, $\ell(n, \theta^*(\mathbf{x}, \mathbf{e}_n, \mathbf{m}_n) \geq 2f$.*

2. *The triple* $(\theta^*, W^*, \Gamma^*)$ *is* f*-minimal.*

On the basis of Lemma 3.12 we complete the proof of Proposition 3.10 and thus Lemma 3.9.

For each n fix $X_n \subset n+1$ so that if $j \in X_n$, $C^n_j(\theta^*(\mathbf{x}, \mathbf{e}_n, \mathbf{m}_n)) \neq \emptyset$. Then $|X_n| = \ell(n, \theta^*(\mathbf{x}, \mathbf{e}_n, \mathbf{m}_n)) \geq 2f$. Choose $\mathbf{c}^n_j \in C^n_j(\theta^*(\mathbf{x}, \mathbf{e}_n, \mathbf{m}_n))$ for $j \in X_n$. Let $\mathbf{c}_n := \cup_{j \leq 2f} \mathbf{c}^n_j$, $\mathbf{z} := \cup_{j \leq 2f} \mathbf{z}_j$ and $\psi(\mathbf{y}, \mathbf{z}, \mathbf{u})$ be the first order formula expressing:

$$|\phi(M, \mathbf{y}, \mathbf{u}) \cap \{\mathbf{z}_0, \ldots, \mathbf{z}_{2f}\}| > f.$$

For each $n \in W^*$, by f-minimality for any $\boldsymbol{a} \in A'$, $\phi(\mathbf{x}, \boldsymbol{a}, \mathbf{m}_n) \wedge \theta^*(\mathbf{x}, \mathbf{e}_n, \mathbf{m}_n)$ holds either for less than f of the c^n_j (for $j \in X_n$) or for all but f of them. So, for any $\boldsymbol{a} \in A'$ we have $\models \psi(\boldsymbol{a}, \mathbf{c}_n, \mathbf{m}_n)$ if and only if all but f of the $\mathbf{c}^n_j$ are contained in $\phi(M, \boldsymbol{a}, \mathbf{m}_n)$. This implies that if t_n is greatest such that $t_n \cdot f \leq \ell(n, \theta^*(\mathbf{x}, \mathbf{e}_n, \mathbf{m}_n))$ then

$$\{\theta^*(\mathbf{x}, \mathbf{e}_n, \mathbf{m}_n) \wedge \phi(\mathbf{x}, \boldsymbol{a}, \mathbf{m}_n) : \boldsymbol{a} \in \psi(M, \mathbf{c}_n, \mathbf{m}_n) \cap A'\}$$

is t_n-consistent. Clearly then, $\{\phi(\mathbf{x}, \boldsymbol{a}, \mathbf{m}_n) : \boldsymbol{a} \in \psi(M, \mathbf{c}_n, \mathbf{m}_n) \cap A'\}$ is t_n-consistent. By definition, $\phi(\mathbf{c}^n_j, \boldsymbol{a}^n_i, \mathbf{m}_n)$ holds if $i \neq j$ so each $\phi(\mathbf{x}, \boldsymbol{a}^n_i, \mathbf{m}_n)$ is satisfied by more than f of the $\mathbf{c}^n_j$. Thus $\{\boldsymbol{a}^n_0 \ldots, \boldsymbol{a}^n_n\} \subseteq \psi(M, \mathbf{c}_n, \mathbf{m}_n) \cap A'$. By the choice of $\{\boldsymbol{a}^n_0 \ldots, \boldsymbol{a}^n_n\}$, $\{\phi(\mathbf{x}, \boldsymbol{a}, \mathbf{m}_n) : \boldsymbol{a} \in \psi(M, \mathbf{c}_n, \mathbf{m}_n) \cap A'\}$ is not consistent. Since f is fixed and t_n goes to infinity with n, we have $FCP_{\phi,\psi}(M, A')$. $\square_{3.10}$

Proof of 3.12. Let $\Gamma := \{\mathbf{m}_n : n \in W\}$. The triple $(\mathbf{x} = \mathbf{x}, W, \Gamma)$ is unbounded since $\ell(n, \mathbf{x} = \mathbf{x}) = n + 1$. Thus on the basis of Lemma 3.13 (below), we can either find a triple $(\theta^*, W^*, \Gamma^*)$, which is f-minimal for some $f < \omega$, or we can inductively construct a tree of instances of conjunctions of $\phi(\mathbf{x}, \mathbf{w}, \mathbf{u})$ with $\neg\phi(\mathbf{x}, \mathbf{w}, \mathbf{u})$, violating the stability of M over A'; cf. Corollary 4.4 of [8]. The 'in particular' clause is easily satisfied by taking a tail of any W^* which satisfies the other conditions. $\square_{3.12}$

So we are reduced to showing:

Lemma 3.13 *Suppose the triple* $(\theta(\mathbf{x}, \mathbf{y}, \mathbf{u}), W, \Gamma)$*, where* $\Gamma = \langle \mathbf{d}_n \mathbf{m}_n : n \in W \rangle)$*, is unbounded. Either*

1. *for some* $f < \omega$ *and some infinite* $W^* \subseteq W$*, the triple* (θ, W^*, Γ) *is* f*-minimal or*
2. *there is a* $\Gamma' = \langle \mathbf{d}_n \mathbf{e}_n \mathbf{m}_n : n \in W \ \mathbf{e}_n \in A' \rangle$ *such that both* $(\theta(\mathbf{x}, \mathbf{y}, \mathbf{u}) \wedge \phi(\mathbf{x}, \mathbf{w}, \mathbf{u}), W, \Gamma')$ *and* $(\theta(\mathbf{x}, \mathbf{y}, \mathbf{u}) \wedge \neg\phi(\mathbf{x}, \mathbf{w}, \mathbf{u}), W, \Gamma')$ *are unbounded.*

Proof. We claim that for any $f < \omega$ there exists $n_f \in W$ such that for all $n \in W$ with $n \geq n_f$ there exists $\mathbf{e}_n(f) \in A'$ such that:

$$\ell(n, \theta(\mathbf{x}, \mathbf{d}_n, \mathbf{m}_n) \wedge \phi(\mathbf{x}, \mathbf{e}_n(f), \mathbf{m}_n)) > f \text{ and } \ell(n, \theta(\mathbf{x}, \mathbf{d}_n, \mathbf{m}_n) \wedge \neg\phi(\mathbf{x}, \mathbf{e}_n(f), \mathbf{m}_n)) > f. \quad (2)$$

If not, $\exists f<\omega$, $\forall n<\omega$, $\exists k(n,f)\in W$ $(k(n,f)>n)$ such that $\forall \mathbf{b}\in A'$

$$\ell(k(n,f),\theta(\mathbf{x},\mathbf{d}_n,\mathbf{m}_n)\wedge\phi(\mathbf{x},\mathbf{b},\mathbf{m}_n))\leq f \text{ or } \ell(k(n,f),\theta(\mathbf{x},\mathbf{d}_n,\mathbf{m}_n)\wedge\neg\phi(\mathbf{x},\mathbf{b},\mathbf{m}_n))\leq f.$$

This means that the unbounded triple $(\theta, W', \emptyset)$, where $W' := \{k(n,f) : n<\omega\}$, is f-minimal and we are finished. Thus, there exists an infinite family of sets of parameters from M

$$\Gamma^f := \{\mathbf{e}_n(f)\in A' : n\in W, n_f\leq n\},\ f<\omega$$

satisfying property 2. For all $f<\omega$, $(n_f\leq n_{f+1})$. Let

$$\Gamma^* := \{\mathbf{e}_n : n\in W, \forall f<\omega[n_f\leq n<n_{f+1}\rightarrow \mathbf{e}_n=\mathbf{e}_n(f)]\}.$$

Then the triples $(\theta(\mathbf{x},\mathbf{y},\mathbf{u})\wedge\phi(\mathbf{x},\mathbf{w},\mathbf{u}),W,\Gamma')$ and $(\theta(\mathbf{x},\mathbf{y},\mathbf{u})\wedge\neg\phi(\mathbf{x},\mathbf{w},\mathbf{u}),W,\Gamma')$ are unbounded as required. $\square_{3.13}$

Having proved the last reduction we conclude the proof of Theorem 3.9. $\square_{3.9}$

Let us sum up the connections between pseudosmallness and local saturation.

Theorem 3.14 *Suppose M does not have the finite cover property over A. Then the following conditions are equivalent:*

1. *(M,A) is pseudosmall.*
2. *Every $|L|^+$-saturated elementary extension of (M,A) is small.*
3. *$(M,A)\models T'$.*
4. *(M,A) is locally saturated.*

Proof. We have already noted (Fact 3.6)) the equivalence of 3) and 4). 1) implies 2) is immediate from Theorem 3.9. But 2) implies that (M,A) has an $L(P)$-elementary extension which is locally saturated. By Fact 3.6 this extension and *a fortiori* (M,A) model T'. So 2) implies 3).

Now for 4) implies 1), let (M',A') be an $|L|^+$-saturated $L(P)$-elementary extension of (M,A). Let $\mathbf{b}\in M'$. For arbitrary $q\in S(A'\cup\mathbf{b})$ consider the set of restrictions $\{q_\phi:\phi\in L\}$ of q to ϕ-types over $A'\cup\mathbf{b}$. For each q_ϕ there exists an L-formula $d\phi(\mathbf{y},\boldsymbol{a}_\phi,\mathbf{b})$ with parameters from $(A'\cup\mathbf{b})$ such that $d\phi(\mathbf{y},\boldsymbol{a}_\phi,\mathbf{b})$ defines q_ϕ. Denote $A'_q := \cup_{\phi\in L}\boldsymbol{a}_\phi$. Thus, $|A'_q|=|L|$. Then consider the set of $L(P)$-formulas:

$$r(\mathbf{x},A'_q\cup\mathbf{b}) := \{(\forall\mathbf{y})[P(\mathbf{y})\rightarrow(d\phi(\mathbf{y},\boldsymbol{a}_\phi,\mathbf{b})\leftrightarrow\phi(\mathbf{x},\mathbf{y},\mathbf{b}))] : \phi\in L\}.$$

To see $r(\mathbf{x},A'_q\cup\mathbf{b})$ is consistent we invoke the nfcp. Since (M,A) is locally saturated, $(M,A)\models T'$ and for every n and ϕ, the formula (defined in Notation 3.3)

$$[\phi(\mathbf{x},\mathbf{y},\mathbf{u}),\psi(\mathbf{y},\mathbf{z},\mathbf{u}),n](\boldsymbol{a}_\phi,\mathbf{b})\wedge(\exists\mathbf{y}_0,\ldots\mathbf{y}_n)\psi(\mathbf{y}_i,\boldsymbol{a}_\phi,\mathbf{b})]$$

holds in (M, A). (Replace ϕ by $\neg\phi$ if $\phi(x, \boldsymbol{a}, \mathbf{b}) \in q_\phi$ for only finitely many $\boldsymbol{a} \in A$). So both (M, A) and (M', A') satisfy $(\exists \mathbf{x})(\forall \mathbf{y})[P(\mathbf{y}) \rightarrow (\psi(\mathbf{y}, \boldsymbol{a}_\phi, \mathbf{b}) \leftrightarrow \phi(\mathbf{x}, \boldsymbol{a}_\phi, \mathbf{b}))]$. Since this holds for every formula ϕ, $r(\mathbf{x}, A'_q \cup \mathbf{b})$ is consistent. Since (M', A') is $|L|^+$-saturated, $r(\mathbf{x}, A'_q \cup \mathbf{b})$ and thus q is realized in M'. Thus (M', A') is small and consequently (M, A) is pseudosmall. $\square_{3.14}$

Without assuming nfcp one can prove any (M, A) which has a small $L(P)$-elementary extension is locally saturated. But the following example shows that Theorem 3.14 requires the nfcp hypothesis.

Example 3.15 The following (M, A) has fcp over A, and is small (a fortiori pseudosmall and locally saturated) but is not $\aleph_1$-saturated.

L contains one equivalence relation E. There is one class of each finite cardinality. If n is even half the elements of the class with n elements are in A. If n is odd, all elements of the class are in A. There are $\aleph_1$ classes with size $\aleph_1$, each with half its elements in A. There are $\aleph_1$ classes with size $\aleph_1$ which do not intersect A. Note that (M, A) is not $\aleph_1$-saturated as an $L(P)$-structure because the type of an infinite class with all elements in A is consistent but not realized.

Now let (N, B) be elementarily-equivalent in $L(P)$ to (M, A) but have an infinite class which is entirely in P. Then (N, B) is still pseudosmall but no $L(P)$-elementary extension of (N, B) is small and (N, B) is not locally saturated.

We can combine the results of Section 1 with Theorem 3.9 to get a short proof of the following.

Theorem 3.16 *Suppose T^* is pseudosmall. If $(M, A) \models T^*$ and M does not have fcp over A, (M, A) is uniformly benign.*

Proof. By Theorem 3.9, every $|L|^+$-saturated elementary extension (N, B) of (M, A) is small and so obviously locally homogeneous. By Lemma 1.4, (M, A) is uniformly locally homogeneous. Apply Theorem 1.7. $\square_{3.16}$

Now we turn to the analysis of the connections between our notions and those in [3] and [8].

Definition 3.17 (M, A) *is* bounded *if in (M, A) every $L(P)$ formula is equivalent to a bounded formula.*

In [8] 5.3 the authors give two other equivalents of 'bounded', in terms of conditions on extensions of permutations of A. We have three properties of an $L(P)$ theory T^*: pseudosmall, bounded, uniformly benign. The first can be replaced by a property of a single model (M, A) (small) if M does not have fcp over A. The three properties of the theories are related as follows.

Fact 3.18 *1. If (M, A) is small then (M, A) is benign.*

2. M does not have fcp over A and (M, A) is pseudosmall then (M, A) is bounded.

3. If (M, A) is bounded then (M, A) is uniformly benign (= uniformly locally homogeneous).

Fact 3.18 1 is a triviality. Fact 3.18 2 is Proposition 2.1 of [8]; Fact 3.18 3 follows easily from the second equivalent condition to bounded in Lemma 5.3 of [8]. Together, the last two yield a more round-about proof of Theorem 3.16. None of these implications are reversible.

Example 3.19 *1. Let T be the standard ω-stable theory with fcp. (That is, theory whose 'standard' model is an equivalence relation with one class of size n for each finite n and no infinite classes.) Let (N, M) be a pair of models of T with $M \prec N$ such that M contains an entire infinite equivalence class from N. Then (N, M) is not small since the type of an element in that class but not in M is omitted in N. But by Bouscaren [6](or by observation) (N, M) is uniformly benign. Here, N has fcp over M.*

2. Benedikt suggests to modify the previous example by considering (N, A) where A is one entire infinite equivalence class of some nonstandard model N of T. Now, (N, A) is not small but (by observation) (N, A) is uniformly benign and N does not have fcp over A.

3. Let M be a structure with two infinite equivalence classes and let A omit one point from one class and two from another. Then (M, A) is uniformly benign but not bounded. And its theory could not be better behaved.

4 Stable Pairs

Casanovas and Ziegler generalized the arguments of Poizat, Bouscaren, Baldwin and Benedikt to show ' bounded' and A_{ind} stable implies the pair structure is stable. We now show that the hypothesis 'bounded' can be weakened to 'uniformly weakly benign' at the cost of strengthening the stability requirement on A. More precisely, the next few lemmas combine to show that if (M, A) is uniformly weakly benign and both M and $A^{\#}$ (see next paragraph) are stable then (M, A) is stable.

Consider the language $L^{\#}$ whose basic predicates are the relations induced on A in the sense of Definition 0.1.3. We denote by $A^{\#}$ the expansion of A to an $L^{\#}$-structure. It is easy to show by induction on the construction of $L^{\#}$-formulas that for every $\phi^{\#}(\mathbf{x}) \in L^{\#}$, there exists $L(P)$-formula $\phi^*(\mathbf{x})$ such that for every $\boldsymbol{a} \in A$ the following holds:

$$A^{\#} \models \phi^{\#}(\boldsymbol{a}) \leftrightarrow (M, A) \models \phi^*(\boldsymbol{a}). \tag{3}$$

Suppose $(M, A) \prec (N, B)$ and $\mathbf{b}, \mathbf{b}' \in B$. It is immediate that if $\mathbf{b}$ and $\mathbf{b}'$ have the same $L^{\#}$-type over A then $tp_*(\mathbf{b}|A) = tp_*(\mathbf{b}'|A)$. Equation 3 together with Definition 0.1.3. means that for every $\mathbf{b}, \mathbf{b}' \in B$, $\mathbf{b}$ and $\mathbf{b}'$ have the same $L^{\#}$-type over A (in the sense of $B^{\#}$) if and only if they have the same $L(P)$-type over A (in the sense of (N, B).

Denote by $FE(X)$ the family of all L-finite equivalence relations over X and by $FE^*(X)$ the family of all L^*-finite equivalence relations over X. We will write $stp(\mathbf{b}|X) \equiv tp(\mathbf{b}|X)$ if for every $E(\mathbf{x}, \mathbf{y}) \in FE(X)$ there exists $\mathbf{m}_E \in X$ such that $\models E(\mathbf{b}, \mathbf{m}_E)$ and write $stp_*(\mathbf{b}|X) \equiv tp_*(\mathbf{b}|X)$ for the analogous notion for $*$-types. In the stable case, $stp(\mathbf{b}|X) \equiv tp(\mathbf{b}|X)$ implies $tp(\mathbf{b}|X)$ is stationary.

Lemma 4.1 *If $(M, A) \prec (N, B)$ then*

1. *$M \downarrow_A B$ (as L-structures)*
2. *$stp(\mathbf{b}|A) \equiv tp(\mathbf{b}|A)$.*
3. *$stp_*(\mathbf{b}|A) \equiv tp_*(\mathbf{b}|A)$.*

Consequently, for any $\mathbf{b}, \mathbf{b}' \in B$, $tp(\mathbf{b}|A) = tp(\mathbf{b}'|A)$ implies $tp(\mathbf{b}|M) = tp(\mathbf{b}'|M)$.

Proof. It is easy to see using $(M, A) \prec (N, B)$ that any formula $\phi(\mathbf{m}, \boldsymbol{a}, \mathbf{x})$ which is satisfied by an element of B is satisfied by an element of A. This implies the type of B over M does not fork over A. (See for example [2, III.3.12].)

We prove 3) and 2) is a special case. Let $\mathbf{b} \in B$, $E^*(\mathbf{x}, \mathbf{y}) \in FE^*(A)$. Then there exists $\mathbf{m} \in M$ such that $(N, B) \models E^*(\mathbf{m}, \mathbf{b})$ since $E^* \in FE^*(A)$ and each E^*-class has a representative in model (M, A). Then we have $(N, B) \models E^*(\mathbf{m}, \mathbf{b}) \wedge P(\mathbf{b})$. As $(M, A) \prec (N, B)$, for some $\boldsymbol{a} \in A$, $N \models E^*(\mathbf{m}, \boldsymbol{a})$ and by transitivity of E^*, $(N, B) \models E^*(\mathbf{b}, \boldsymbol{a})$. So, $E^*(\mathbf{x}, \boldsymbol{a}) \in tp_*(\mathbf{b}|A)$.

The 'consequently' is immediate from 1) and 2). $\square_{4.1}$

We will say f is a *strong partial automorphism over* X if f preserves all strong types over X. We need the following easy lemma which does not seem to have been remarked.

Fact 4.2 *Suppose $M = \bigcup_{\kappa<\lambda} A_\kappa$ is a special model with $X \subset A_0$. If $b, b' \in M$ realize the same strong type over X there is an automorphism of M which fixes X pointwise, preserves strong types over X, and maps b to b'.*

Proof. Let $M = \bigcup_{\kappa<\lambda} A_\kappa$. If f is a strong partial automorphism over X whose domain contains X and is contained in some $A\kappa$, then for any $c \in M$ there is a $d \in M$ such that $f \cup \langle c, d \rangle$ is a strong partial automorphism over X. (Just realized the type $\{E(v, f(a)) :\models E(c, a), a \in \mathrm{dom}\, f\}$ where E ranges over all finite equivalence relations with parameters in X.) Now fix a *cautious enumeration* of M as in Lemma 10.4.3 of [9]. Copying the proof of

Theorem 10.4.4 of [9] but replacing elementary equivalence with the condition that sending a_i to d_i and b_i to c_i is a strong partial automorphism over X, we get the result. $\square_{4.2}$

We write $S^*(X)$ for the collection of $L(P)$-1-types over X. We also need briefly some quite special notation. Let $S^*_P(X)$ denote the set of $L(P)-1$-types $q(y)$ over X which contain the formula $P(y)$. We first extend from the stability of $A^\#$ to bounding the cardinality of $S^*_P(X)$, for arbitrary X, not just subsets of A.

Lemma 4.3 *Suppose M and $A^\#$ are stable in $|A|$, and (M, A) is uniformly weakly benign. Then, $|S^*_P(M)| \le |A|$.*

Proof. Let (N, B) be an elementary extension of (M, A) which is a special model of $(M, A, x)_{x\in X}$. Then, (N, B) is strongly λ-homogeneous and moreover if two points realize the same $L(P)$-strong type over a set X of size λ there is an automorphism of (N, B) mapping one to the other which fixes X pointwise and preserves $L(P)$-strong types over X. The discussion before Lemma 4.3 shows that it suffices to prove the following:

Claim 4.4 *If $\mathbf{b}, \mathbf{b}' \in B$ realize the same $L(P)$-type over A, then they realize the same $L(P)$-type over M.*

Proof. Notice that by 3) of Lemma 4.1, $\mathbf{b}$ and $\mathbf{b}'$ have the same $L(P)$-strong type over A. The choice of (N, B) implies by Fact 4.2 that there is an $L(P)$-automorphism h of (N, B) which fixes A point-wise, preserves all $L(P)$-strong types over A and takes $\mathbf{b}'$ to $\mathbf{b}$. If the lemma fails there is an $L(P)$-formula $\phi(\mathbf{x}, \mathbf{y})$ such that

$$\phi(\mathbf{m}, \mathbf{b}) \wedge \neg\phi(\mathbf{m}, \mathbf{b}').$$

So

$$\phi(\mathbf{m}, \mathbf{b}) \wedge \neg\phi(h(\mathbf{m}), \mathbf{b}).$$

By Lemma 4.1, M is independent from B over A. In particular, $\mathbf{m} \downarrow_A B$. So $h(\mathbf{m}) \downarrow_{h(A)} h(B)$, i.e. $h(\mathbf{m}) \downarrow_A B$. Since h preserves $L(P)$-strong types over A and consequently L-strong types over A, $stp(\mathbf{m}|A) = stp(h(\mathbf{m})|A)$. The result follows by the following standard result (e.g. Proposition 4.34 of [11]: $stp(h(\mathbf{m})|A) = stp(\mathbf{m}|A)$, $\mathbf{m} \downarrow_A B$ and $h(\mathbf{m}) \downarrow_A B$ imply $stp(h(\mathbf{m})|B) = stp(\mathbf{m}|B)$.

Since (N, B) is weakly benign we conclude $\mathbf{m}$ and $h(\mathbf{m})$ realize the same $L(P)$-type over B. This contradiction yields the conclusion. $\square_{4.4}$

Lemma 4.3 follows since the stability of $A^\#$ implies there are at most $|A|$ $L^\#$-types over A and by Claim 4.4 we have at most $|A|$ $L(P)$-types over M which contain the formula $P(x)$. $\square_{4.3}$

As we see in Lemma 4.5, it would suffice in Lemma 4.3 to show $|S^*_P(M)| \le |M|$. Because B is independent from M over A, we obtained the stronger bound $|S^*_P(M)| \le |A|$, which might be useful in the future. Now we adapt an argument of Bouscaren to get stability of (M, A).

Lemma 4.5 *Let M be stable, (M, A) uniformly weakly benign, and $|S^*_P(M)| \leq |M|$. Then (M, A) is stable.*

Proof. We suppose that $|M| = \lambda$ with $\lambda^{|L|} = \lambda$. Let (N, B) be an $|M|^+$-saturated elementary extension of $(M, A), [(M, A) \prec (N, B)]$. Consider any $\boldsymbol{a} \in N - (B \cup M)$ and let $p := tp(\boldsymbol{a}/B \cup M)$. By what Bouscaren calls the Weak Definability Theorem [6][page 214]; [10]) there is $C(\boldsymbol{a}) \subset B \cup M(|C(\boldsymbol{a}) \cup M| = \lambda)$ such that $tp(\boldsymbol{a}|C(\boldsymbol{a}) \cup M)$ has only one nonforking extension to type over $B \cup M$ and it is p. We use this notation for various choices of $\boldsymbol{a}$ in the following. The following claim is Lemma 2.3 of [6]; we repeat the proof for completeness.

Claim 4.6 *Let $\mathbf{b} \in N \setminus M$, $C(\mathbf{b}) \subset (B \cup M)$ such that $tp(\boldsymbol{a}C(\boldsymbol{a})/M) = tp(\mathbf{b}C(\mathbf{b})/M))$ and $tp_*(C(\boldsymbol{a})/M) = tp_*(C(\mathbf{b})/M)$. Suppose further that $stp(\mathbf{c}/MC(\mathbf{b})) = stp(\mathbf{b}/MC(\mathbf{b}))$, where $\mathbf{c} \in N$ is the image of $\boldsymbol{a}$ under an $L(P)$-automorphism h fixing M and taking $C(\boldsymbol{a})$ to $C(\mathbf{b})$. Then $tp_*(\boldsymbol{a}/M) = tp_*(\mathbf{b}/M)$.*

Proof. By the choice of $\mathbf{c}$,

$$tp_*(\boldsymbol{a}C(\boldsymbol{a})/M) = tp_*(\mathbf{c}C(\mathbf{b})/M).$$

Since h fixed B setwise, $tp(\mathbf{c}/M \cup B)$ is the unique nonforking extension of its restriction to $C(\mathbf{b}) \cup M$. Now,

$$tp(\boldsymbol{a}C(\boldsymbol{a})/M) = tp(\mathbf{b}C(\mathbf{b})/M) = tp(\mathbf{c}C(\mathbf{b})/M).$$

Since $C(\mathbf{b})$ is chosen as $C(\boldsymbol{a})$ was in the first paragraph, $tp(\mathbf{b}/B \cup M)$ is the unique nonforking extension of its restriction to $C(\mathbf{b}) \cup M$. Since $stp(\mathbf{c}/MC(\mathbf{b})) = stp(\mathbf{b}/MC(\mathbf{b})))$, $stp(\mathbf{b}/M \cup B) = stp(\mathbf{c}/M \cup B)$. This easily implies that for any $\mathbf{d} \in M \setminus A$, $stp(\mathbf{bd}/B) = stp(\mathbf{cd}/B)$. Then, because (N, B) is weakly benign, $tp_*(\mathbf{bd}/B) = tp_*(\mathbf{cd}/B)$. The last implies that $tp_*(\mathbf{b}/M \cup B) = tp_*(\mathbf{c}/M \cup B)$. Then, because $tp_*(\mathbf{c}/M) = tp_*(\boldsymbol{a}/M)$, $tp_*(\boldsymbol{a}/M) = tp_*(\mathbf{b}/M)$. $\square_{4.6}$

Claim 4.6 implies that $tp_*(\boldsymbol{a}/M)$ is characterized by a triple of types: $tp(\boldsymbol{a}C(\boldsymbol{a})/M)$, $tp_*(C(\boldsymbol{a})/M)$, and $stp(\boldsymbol{a}/MC(\boldsymbol{a}))$. Notice, that $tp(\boldsymbol{a}C(\boldsymbol{a})/M) \in S(M)^{|L|}$ and $tp_*(C(\boldsymbol{a})/M) \in S^{**}_P(M)$, where $S^{**}_P(X)$ denotes the set of $L(P) - |L|$-types $q(\mathbf{y})$ containing $P(y_i)$ for each $i < |L|$. For fixed $tp(\boldsymbol{a}/MC(\boldsymbol{a}))$, the finite equivalence relation theorem shows there are at most $2^{|L|}$ choices for $stp(\boldsymbol{a}/MC(\boldsymbol{a}))$. Thus, there are only λ choices for $stp(\boldsymbol{a}/MC(\boldsymbol{a}))$. Since $|S^*_P(M)| \leq |M|$ and $\lambda^{|L|} = \lambda$, $|S^{**}_P(M)| \leq |M| = \lambda$. So,

$$|S^*(M)| \leq |S(M)|^{|L|} \times |S^{**}_P(M)| \times \lambda \leq \lambda \tag{4}$$

and we finish. $\square_{4.5}$

Combining Lemma 4.3 and the argument from Lemma 4.5,we have:

Theorem 4.7 *Let M be (super) stable, (M, A) uniformly weakly benign, and $A^{\#}$ (super) stable. Then (M, A) is (super) stable. And if L is countable and both M and $A^{\#}$ are ω-stable then so is (M, A).*

Proof. We prove the ω-stable case. Superstability implies that each $C(\boldsymbol{a})$ is finite. Thus in the superstable case $S_P^{**}(X)$ reduces to a collection of $L(P)$ k-types for some k and the types of the form $tp(\boldsymbol{a}C(\boldsymbol{a})/M)$ also become finite types over M. So the first two factors in Equation 4 become $|M|$. In the ω-stable case the number of strong types over a set of cardinality X is $|X|$. Thus, the third factor in Equation 4 is also $|M|$ and we finish.

The following example illustrates the role of P^*.

Example 4.8 Let M be the rational numbers under addition and let $p_i : i < \omega$ be an enumeration of the prime numbers. Let $a_0 = 1, b_0 = 1/2$ more generally $a_i = \frac{1}{p_{2i-1}}, b_i = \frac{1}{p_{2i}}$. Let $c_{ij} = a_i \cdot b_j$. Now if P is a subset of the c_{ij}, (M, P) is uniformly benign (by Fact 2.1), but (M, P) is unstable for P chosen as in Fact 2.3. By Theorem 4.7, for such P, P^* is unstable. It is also easy to see directly in this case that $(\forall x)[P(x \cdot a_0^{-1} \cdot v) \rightarrow P(x \cdot a_0^{-1} \cdot w)]$ linearly orders $\{c_{0i} : i < \omega\}$.

5 Context and Further Problems

This study has two somewhat disparate motivations. On the one hand we are trying to understand the arguments arising from the study of 'quantifier collapse' in embedded finite model theory. On the other we look at the problem of finding expansions of models which preserve stability from an unusual perspective. The main aim of such constructions (stemming from Hrushovski) is to find 'sufficiently random expansions' which preserve stability. It is understood that 'most' expansions destroy stability. Our aim is rather to find simple conditions which guarantee that stability is preserved – a possibility which arises from work on embedded finite model theory and 'belles paires'. The following two remarks extend the connections with these two motivations. Theorem 7.3 of [3] still holds when the use of 'pseudosmall/bounded' in the proof is replaced by 'benign'. With the help of [1] this provides a new proof for quantifier collapse in superstable theories. That is the crucial local arguments in [3] or as refined in [8] are replaced by the global analysis of the structure of models using Chapter V of [13]. The material relevant for databases requires only the following (Theorem 5.2) weakening of Theorem 7.3 of [3]. We use the following notation from [3].

Notation 5.1 *Let I be a subset of N and let L^* be the expansion of L^+ by constants $\{c_\alpha : \alpha < |I|\}$. Fix a permutation g of I and an enumeration $\{a_\alpha : \alpha < |I|\}$ of I. Then N_1^g is the expansion of N where c_α is interpreted as a_α and N_2^g is the expansion of N where c_α is interpreted as $g(a_\alpha)$.*

The following result is proved as Theorem 7.3 of [3], just noticing that the weaker hypotheses are all that are used.

Theorem 5.2 *Let N be a model that is stable and I be a set (sequence) of indiscernibles. Suppose that (N, I) is benign and that (N, I) is ω_1-saturated. For any permutation g on I (order-preserving permutation of I),*

$$N_1^g \sim_{L^*,p} N_2^g.$$

Our main question remains:

Question 5.3 *Is every subset of a stable theory weakly benign?*

In view of [1] and Theorem 4.7, a positive response to the following question would yield (M, I) is superstable and so avoid (shift?) much of the difficulty of [3].

Question 5.4 *Let M be superstable and I an infinite set of indiscernibles. Is $I^{\#}$ superstable?*

In fact, the answer is yes in several natural cases: M is strongly minimal or I is based on a regular type. In either case, every permutation of I extends to an automorphism of M and so $I^{\#}$ remains trivial and in particular stable. In the first case this is immediate from strong minimality. The second was remarked in the last paragraph of [3]. So Question 5.4 is solved by a positive response to the problem raised in [3]:

Question 5.5 *Let M be superstable and I an infinite set of indiscernibles. Does every permutation of I extend to an automorphism of M?*

There are analogous versions of Question 5.4 for stable theories if Question 5.3 has a positive answer. In another direction, we ask:

Question 5.6 *None of this touches the work of [3] on theories without the independence property (nip) In that context it is easy to find examples of non-benign sets. E.g. (Bouscaren) Let A be Q without the closed interval $[0, 1]$. Then 0 and 1 realize the same type in L but not in L^+. order. But one could still ask something like, does (M, A) benign, M nip, A_{ind} nip imply (M, A) nip? Or more weakly, and more plausibly, does the order indiscernible version [3] of Theorem 5.2 go through with nip rather than stable as the hypothesis.*

The construction in Section 2 raises the following question.

Question 5.7 *Let M be free abelian on $\langle a_i, b_i \rangle$. (Note M is strictly stable.) Choose P as in Fact 2.3. Is P benign?*

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