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Mathematical Infinity, Its Inventors, Discoverers, Detractors, Defenders, Masters, Victims, Users, and Spectators

The definitive clarification of the nature of the infinite *has become necessary, not merely for the special interests of the individual sciences, but rather for the* honour of the human understanding *itself. The infinite has always stirred the* emotions *of mankind more deeply than any other question; the infinite has stimulated and fertilized reason as few other* ideas *have ; but also the infinite, more than other* notion, *is in need of* clarification.

[Hilbert 1925], pp. 370-371.

Had I been present at the creation, I would have given some useful hints for the better ordering of the universe.

Alfonso X the Wise, King of Castile (1252-84),
cited in [*ODQ* 1980], p. 3.

§1. Preamble.

No one shall be able to drive us from the paradise that Cantor created for us.

[Hilbert 1925], p. 376.

Let us start by the following trivial observation of a pure phenomenological character : before Georg Cantor has entered the scene of mathematical infinity, two types of infinite totalities were already known experimentally to the mathematical community, and at least as early as at the time of Euclid, - we are talking here about denumerable and continuum-like totalities.

Technically, Cantor's legacy includes : first, a remarkable clarification of the notions of both the denumerable ω (or $\aleph_0$) [1] and the continuum $\mathfrak{c}$; second, the discovery of the huge gap between them, $\aleph_0 < \mathfrak{c}$; third, the invention of an uncountable host of new infinite

[1] Whereas ω notations correspond to *ordinal* infinite totalities, their $\aleph$ counterpants correspond to *cardinal* infinite totalities ; see below §3 and, e. g., [Hrbacek, Jech 1984].

totalities supposed either to fill the gap between $\aleph_0$ and $\mathfrak{c}$, or to go far beyond $\mathfrak{c}$. Cantor's followers invented even more formidable infinite totalities; they also have tried hard to fill the hiatus $\{\aleph_0, \mathfrak{c}\}$, albeit without much success.

Of course, such modest phenomenological technicalities cannot be held accountable for what really happened in mathematics after Cantor. The truth is that mathematical infinity has become both an attractive and perilous mathematical «Klondike» of sorts : a gold-mine for some infinity prospectors, a moral and psychological ruin for others, with the founder [Dauben 1979], [Meschkowski 1964] and his most perspicacious follower [Feferman 1986], belonging, sadly, to the last category. Many mathematical «fortunes» were made there, and even more hopes were dashed, provoking from time to time sudden and powerful mathematical and philosophical «quakes» which could be felt far away from the «infinite epicentre». Fratricidal wars were waged, won and lost [van Dalen 1990].

And, similarly to what has happened to the American gold-rush, the risky «infinite» adventure has led to a tremendous expansion of mathematics : first, into logic, philosophy, then into computer science, physics, and back into mathematics. (It is outside the scope of the present paper to dwell upon these developments.)

However, the «infinite» dust is still very far from being settled!

A leading researcher into infinite has still to write papers with titles and preliminaries sounding both defensive and soothing [Shelah 1992] (on another occasion, the same author wonders without a shadow of irony, why so *«many of my colleagues, including the best minds in the field of set theory, feel apologetic about their subject»* [Shelah 1993], p. 2; cf. also [Jensen 1995], p. 407). Meanwhile, his well-known contemporary flatly dismisses his and his colleagues efforts of *«setting up new axioms in the never-never land of large cardinals»* [Mac Lane 1983]. To consummate the disunion, on the «infinite» side, dramatic announcements abounds, both apocalyptic [Friedman 1986] and exuberant [Fremlin 1993], whereas the other side remains unconvinced, indifferent, unaware, if not outright hostile, — just have a look at the never-ending dialogue of the deaf [Mathias 1992] - [Mac Lane 1992], [Mathias 2000] - [Mac Lane 2000].

This painful discord continues to be accompanied by no less painful conflicts of foundational philosophies of mathematics [Hersh 1979] (cf. also an almost pathetic dialogue [Henle 1991], [Paris 1992], [Henle 1992], in this magazine), as well as of educational methodologies [Bishop 1985], [Bishop, Bridges 1985], and of policies of funding mathematical research [Smorynski 1988], [Mathias 1992]. In fact, the integrity of mathematics [Simpson 1988], if not its very existence [Arnold 1995], are at stake.

The question is now : *Why?* Why it always happens to «us», people searching (and, in fact, so successfully!) for, and into, the infinite? Why not to «them», to «others», working in other fields of mathematics? Or less humorously, and more responsibly :

What is the meaning of this «*foundational crisis of nearly unprecedented magnitude*» (paraphrasing [Friedman 1986], p. 93), and what are the good lessons we can learn from it ?

To address these, as well as a couple of others, naive and yet pertinent questions (called below *Frequently Asked Questions, FAQ*) concerning modern set theoretical and foundational research, one needs to look closely, and in a broader cultural and intellectual context, at both the multifaceted *Mathematical Infinity* and the century-long attempts, called *Set-Theoretical Infinity*, of its scientific appropriation and customization.

However, to venture into *the unsafe ground of set theory* ([Cohen 1971] p. 15), with its *surrealist landscape* ([Mathias 1979] p. 109), in search for genuine samples of *Mathematical Infinity*, one needs to pay as much attention to the glamorous pictures of official travel guides as to from-sober-to-bitter assessments of experienced, occasionally disgruntled infinity prospectors, or just to friendly warnings and testimonies of often incredulous, never malicious *compagnons de route*. The considerable attention which these contradicting insights are enjoying in this study might be, in the final analysis, its only (if any) merit and novelty.

The present paper is a very personal tribute to both the exceptional beauty of the subject and the wealth and depth of mathematical and philosophical contributions of many contemporary mathematicians, starting with Georg Cantor. The abundant quotes, these pearls borrowed from many authors on the occasion of our friendly *Get-Together at Mathematical Infinity*, are acknowledged here with the author's deep gratitude and self-effacing admiration.

An Apology. For the good understanding we must appeal to the reader's patience and indulgence : not everything can be said at once, and many important issues simply cannot put in an appearance, at least explicitly, in such a short article. Thus, axiomatic aspects of set-theoretical investigations will come to light much later, and in much more modest form, than it might be expected by a knowledgeable reader. This will be even more true with respect to formal philosophical deliberations. Also, we beg pardon for a couple of, possibly too overt, smiles (could they be compared to a laughter in *Paradise*?) which were intended to extenuate the occasional embarrassment of our official and not-so-official set-theoretical guides.

§2. Answering the First Frequently Asked Question.

> *Infinity in Mathematics : Is Cantor Necessary ?*
>
> [Feferman 1987], the title.

> *A team of Hollywood techno-wizards set out to "bring 'em back alive" ... So they took a little artistic license ... [and] decided to make them half again as large. Anyway, what did books know? Then a surprising thing happened. In Utah, paleontologists found bones of a real raptor, and it was the size of the movie's beast. "We were cutting edge", says the film's chief modelmaker with a pathfinder's pride. "After we created it, they discovered it."*
>
> [Dorfman 1993], p. 53.

As many visionaries and prophets before him, Georg Cantor has been not granted the grace to see good fruits of his set-theoretical revelations; typically, quite the opposite happened, and the immediately ensuing set-theoretical controversies have had disastrous consequences for his scientific activity, as well as for his moral and mental health [Dauben 1979]).

And yet, after all, elegant and powerful extra-set-theoretical applications have completely vindicated at least some of the crucial features of Cantor's vision of Mathematical Infinity.

A recent research paper on termination proof techniques for *Term Rewriting Systems* (*TRS* play an important role in *Theoretical Computer Science*, in particularly, in automated deduction and abstract data type specifications) starts as follows [Dershowitz 1993], p. 243:

«*Cantor invented the ordinal numbers*

$$0, 1, 2, 3, \ldots, n, n+1, \ldots \omega, \omega+1, \ldots \uparrow$$

$$\omega 2, \ldots \omega n, \ldots \omega^2, \ldots \omega^n, \ldots \omega^\omega, \ldots \omega \;\; n,$$

$$\ldots \varepsilon_0, \ldots \varepsilon_0^{\varepsilon_0}, \ldots \varepsilon_1, \ldots \varepsilon_{\varepsilon_0}, \ldots, \textit{and so on}.$$

Each ordinal is larger than all preceding ones, and is typically defined as the set of them all :

ω = *the set of all natural numbers* ;

$\omega 2 = \omega \cup \{\omega + n \mid n \in \omega\}$;

$\omega n = \cup_{i<n} \omega i$;

$\omega^2 = \cup_{n \in \omega} \omega n$;

$\omega \uparrow n = \cup_{i<n} \omega \uparrow i$;

$\varepsilon_0 = \omega^{\varepsilon_0} = \cup_{n \in \omega} \omega \uparrow n$;

$\varepsilon_0^{\varepsilon_0} = \omega^{\varepsilon_0^2}$;

$\varepsilon_1 = \cup_{n \in \omega} \varepsilon_0 \uparrow n$.

The notation $\alpha \uparrow n$ *represents a tower of n* α*s.*»

After this most succinct and transparent introduction to Cantor's transfinite numbers, the author demonstrates, and all this on just 6 pages, how the *ordinal descent* can be used to prove termination for specific *TRSs*. The general *TRS* termination problem (which is, of course, a specialization of the halting problem for Turing machines) is undecidable.

Ordinal descent is an important special case of descent along partially ordered sets (say, along trees). One of Cantor's most fruitful ideas has been the notion of a *well-ordering, WO*, i. e., of a *linearly ordered* set fulfilling the conditionof *finite descent*, *FD*, i. e., of termination after a finite number of steps of any descending subsequence (ordinals are, of course, special *WOs*). The principal merit of the *FD* condition is the extendibility of the mechanism of *Mathematical Induction* beyond natural numbers to any *WO* and, in particular, to any ordinal.

Notice that, typically, «*Cantor invented*» or «*created*», not «*discovered*», the ordinal numbers. Later, it was Gerhard Gentzen [Gentzen 1936] who has *discovered* that, assuming the validity of the law of mathematical induction along Cantor's ordinal segment

$$[\ 0, 1, 2, 3, \ldots\ \omega,\ \omega+1, \ldots, \ldots\ \varepsilon_0\ ]\ ,$$

one can prove the consistency of Peano arithmetic. Then, a remarkable general interpretation of (explicitly defined denumerable) ordinals as succinct symbolic notations for algorithmic structures with multiple loops has been given by Alan Turing [Turing 1950], and his approach has substantially contributed to the development of the modern theory of program verification.

Even before Turing, and building on [Gentzen 1936], Reuben L. Goldstein has constructed a truly elementary function $n \to g(n)$ whose arithmetic structure mimics Cantor's transfinite hierarchy up to ε_0, and whose iterates $g^k(n)$ ultimately terminate at0 for any n. However, with n growing, it takes them very long indeed to arrive at 0, which means that the function

$$K(n) = min\ (k,\ g^k(n) = 0)$$

is growing so fast, that any proof of this fact necessary uses a mathematical induction through transfinite numbers up to ε_0 [Goodstein 1944]. The case has become a paradigm of an *independent confirmation* of the existence of an infinite totality through its *necessary use* in a proof of an elementary theorem.

Taking the lead, Harvey Friedman discovered a remarkably transparent, *finitis* version, called *FFF*, of Kruskal's theorem concerning *infinite* sequences of finite trees. The proof of *FFF demonstrably* requires mathematical induction up to the first *impredicative denumerable ordinal* Γ_0 [Gallier 1991]. (The *impredicativity* of Γ_0 signifies, in particular, that no explicit transfinitely recursive formula for it, similar to those for ε_0, ε_1, and $\varepsilon_{\varepsilon_0}$, could be displayed.)

Verily, «*after Cantor created ordinals, they have discovered them*» ! Moreover, Friedman proposed that any newly *invented* infinite totality might be *rediscovered* through its *necessary uses* in a «natural» solution of a «natural» finitistic, i. e., number-theoretical or combinatorial problem :

> *«For at least twenty years, a principal issue in set theory has been the extent to which abstract set theory is necessary for proofs in normal mathematical contexts where abstract set theory plays little or no role in the formulation of the results.»* [Friedman 1986], p. 192.

§3. Flying over Cantor's Paradise with One's Cup of Tea.

> *Let us say that the assertion of a large cardinal property is a* strong axiom of infinity. *The adaptation of strong axioms of infinity is thus a theological venture, involving basic questions of belief concerning what is true about the universe. ... There is here a pleasant analogy: In order for a true believer to really know Mount Everest, he must slowly and painfully trudge up its forbidding side, climbing the rocks amid the snow and the slush, with his confidence waning and his skepticism growing as to the possibility of ever scaling the height. But in these days of great forward leaps in technology, why not get into a helicopter, fly up to the summit, and quickly survey the rarefied realm - all while having a nice cup of tea?*
>
> [Kanamory, Magidor 1978], pp. 103-104.

Emboldened by the outstanding confirmations of Gentzen, Goodstein, Turing, Friedman (to mention just those four leading researchers), a common and shy mathematical fellow is finally ready to follow the friendly invitation and to contemplate in peace the awe-inspiring beauty of the transfinite universe :

> *«This appears to me to be the most admirable flower of the mathematical intellect and in general one of the highest achievements of pure rational human activity.»* [Hilbert 1925], p. 373

Starting from the *0* level, our «helicopter» passes the natural numbers and enters the region of infinite (denumerable) ordinals described above, §2 :

$$\omega,\ \omega+1,\ \dots\ \omega^2,\ \dots\ \omega^n,\ \dots\ \omega^\omega,\ \dots\ \dots\ \varepsilon_0,\ \dots\ \dots\ \varepsilon_{\varepsilon_0}\ ,\ \dots\ \textit{and so on}\ .$$

To ascend further, one asserts that all countable (i. e., finite or denumerable) ordinals a followed by the *first uncountable* ordinal, ω_1, which, in its turn, is followed by ω_2, the first ordinal beyond ω_1 and not equipotent with ω_1, *etc.* :

$$0,\ 1,\ 2,\ 3,\ \dots\ \omega\ =\ \omega_0,\ \dots\ \omega_1,\ \dots\ \omega_2,\ \dots\ ,\ \textit{and so on}\ .$$

To accelerate ascent, one introduces «*absolute set-theoretical values*» of ordinals, o *cardinals*, with, say, $|\omega_0| = \aleph_0$. The notion is based on the equivalence relationship introduced by Cantor and called *one-to-one correspondence,* or *equipotency*. Different but *equipotent ordinals* correspond to one *cardinal*, as, say, $\aleph_0$ corresponds to ω, ε_0 , $\varepsilon_{\varepsilon_0}$, etc., - in short, to all *denumerable* ordinals :

$$\begin{array}{l} 0,\ 1,\ \ldots\ \omega_0,\ \ldots\ \omega_1,\ \ldots\ \omega_2,\ \ldots,\ \textit{and so on} \\ \qquad\quad\ \ \| \qquad\ \ \| \qquad\ \| \\ \qquad\quad\ \aleph_0, \quad\ \aleph_1, \quad \aleph_2, \ldots \end{array}$$

Those are already very strong assumptions of existence of new infinite totalities. Any step behind the last «*and so on*» should involve a new notion, a new construction, a new «*theological venture*» [Kanamory, Magidor 1978], p. 104.

Here is the most modern and very-large-scale map (borrowed, with minor aesthetical modifications, from [Jech 1995], p. 414) of Cantor's mountainous *Paradise* :

____ *????????????????????????????*

|| ____ *inconsistency*

|| || ____ $j : V_\lambda \longrightarrow V_\lambda$

|| || || ____ *huge*

|| || || || ____ *supercompact*

|| || || || || ____ *Woodin*

|| || || || || || ____ $o(\kappa) = \kappa^{++}$

|| || || || || || || ____ $o(\kappa) = 2$

|| || || || || || || || ____ *measurable*

|| || || || || || || || || ____ *Ramsey*

|| || || || || || || || || || ____ $0^{\#}$

|| || || || || || || || || || || ____ *weakly compact*

|| || || || || || || || || || || || ____ *Mahlo*

|| || || || || || || || || || || || || ____ *inaccessible*

|| || || || || || || || || || || || || || ____ $\aleph_1$

|| || || || || || || || || || || || || || || ____ $\aleph_0$

|| || || || || || || || || || || || || || || || ________ 0

The experienced guides direct the attention of a flying-by spectator to two remarkable features of this splendid transfinite landscape. First, the infinite universe has a «roof» called the *inconsistency ceiling*. (Which means that the axiomatic foundations of Cantor's Paradise would «crumble» under any new «storey» built on its top. Those foundations are *ZFC*, Zermelo-Fraenkel's axioms with the axiom of choice.) The second impressive feature of Cantor's Paradise, - its linearly ordered structure :

«As our edifice grew, we saw how one by one the large cardinals fell into place

> *in a* linear *hierarchy. This is especially remarkable in view of the ostensibly disparate ideas that motivate their formulation. As remarked by H. Friedman, this hierarchical aspect of the theory of large cardinals is somewhat a mystery ... In other words, is there a hierarchy of set-theoretical principles in another galaxy above ZFC, disjoint and incomparable to our large cardinals ?»* [Kanamory, Magidor 1978], p. 104.

(In what follows, the spectator turned *Devil's set-theoretical advocate*, will be compelled to submit a less glamourous assessment, as well as to search for a quite different interpretation, of the same infinite phenomena, §§6, 7, 12-14.)

Our «*quick survey of the rarefied realm*», and with it, the «*nice cup of tea*», being drawn to the end, we are leaving the friendly «*helicopter*» with a mixed feeling. It has been nice, of course, and very reassuring indeed, to rub shoulders on this breath-taking adventure with such luminaries as Saunders Mac Lane [Mac Lane 1992], and to meet there our old friends, natural numbers, resting nicely between the first two levels, from 0 to $\aleph_0$, of the transfinite mountain. After all, it's nothing to be surprised about : Mac Lane has been always frank about both his foundational preferences [Mac Lane 1986] and his interest in a good pastime [Mac Lane 1994], and the linear transfinite ascent has been somehow modelled on natural numbers!

But our guides assume (or are they just begging the question ?) that *all* existing or imaginable infinite totalities are somewhere on the steep slope, out to pasture. Then :

3.1. Frequently Asked Question. *What about the continuum* $\mathfrak{c}$*, where is it to be located on this transfinite surrealistic landscape ?*

Everybody knows that Cantor has strongly believed to unmask $\mathfrak{c}$ as the covert $\aleph_1$ (this is the *ordinal version* of Cantor's Continuum Hypothesis, *CH* : cf. FAQ 7.3). During the trip, our mathematical yokel has somehow overheard that Kurt Gödel has been inclined to believe that the continuum size should be $\aleph_2$, the second uncountable cardinal [Moore 1990], p. 175. As they tell us, a recent paper, referring to «*the actual evidences accumulated by 30 years of forcing considerations*» [Judah, Rosłanowski 1995], p. 375, tends to confirm Gödel's intuition and, building on the previous work, develops a sophisticated machinery toward the eventual proof of Gödel's conjecture : see for details [Woodin 2001].

Unfortunately, official travel guides are either silent about this, or worse still, are optimistically elusive :

> *«Despite efforts of Cantor himself and others, the question ... remained unanswered until the emergence of methods of modern logic.»* [Jech 1995], p. 409.

They are just forgetting to add that it remains unanswered ever after : it has been shown that the Continuum Hypothesis can be neither proved (the famous forcing method of Paul Cohen [Cohen 1966]), nor disproved [Gödel 1964] in Zermelo-Fraenkel's set theory. Worse still, «*the generalized continuum hypothesis can fail everywhere*» [Foreman, Woodin 1991] (the issues of the continuum and the Generalised Continuum Hypothesis, *GCH*, will be raised anew, and in a more serious vein, in §§7, 11) ... Tell me another, fumes bombastically Mac Lane :

> «*I admire Gödel's accomplishments, but I suspect that it is futile to wonder now what he imagined to be the «real» cardinal of the continuum. Those earnest specialists who still search for that cardinal may call to mind that infamous image of the philosopher - a blind man in a dark cellar looking for a black cat that is not there.* » [Mac Lane 1992], p. 121.

The following cartoon (Fig. 1) [2] will hopefully help to dissipate the unpleasant aspects of the ongoing, and very important, discussion of the nature and future of *Mathematical Infinity* :

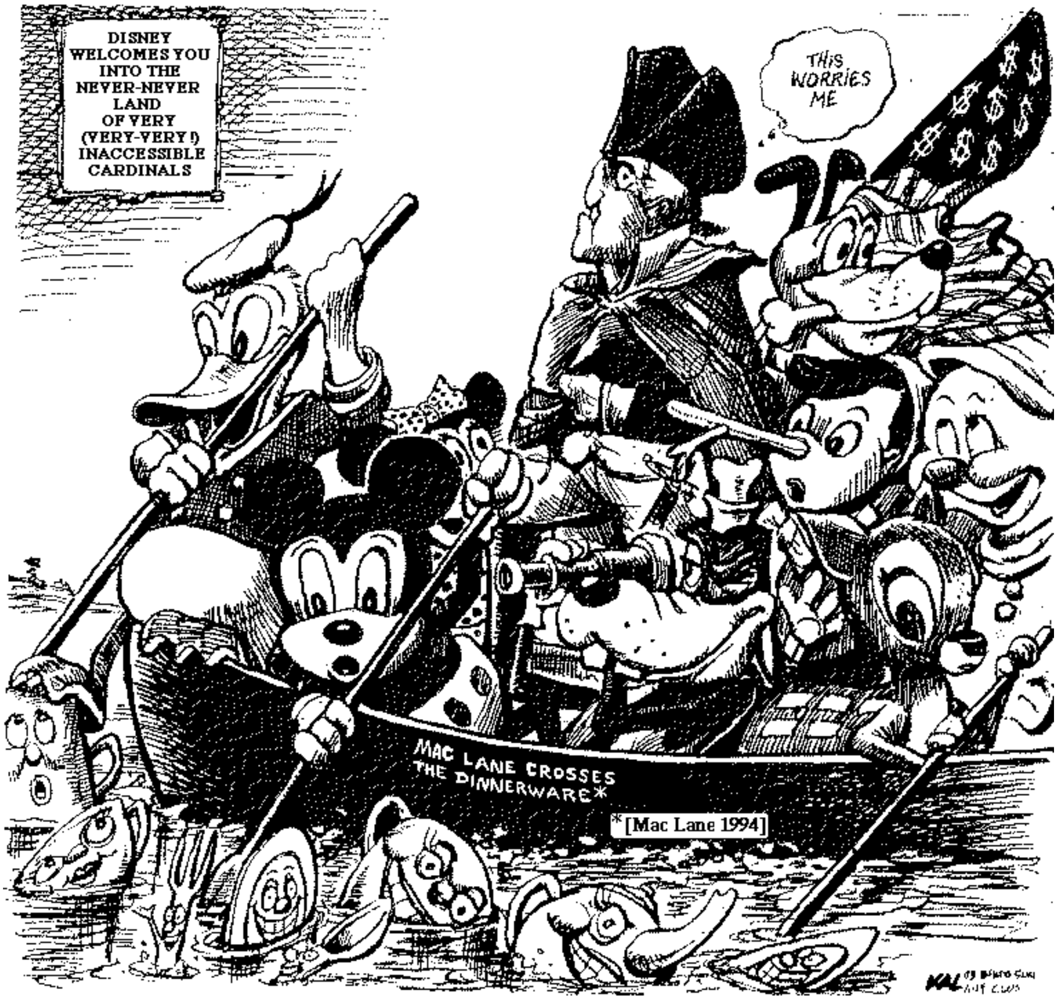

Figure 1

§4. But How Do We Know Indeed That All These New Infinite Totalities Really Exist ?

> *Just think : in mathematics, this paragon of reliability and truth, the very notions and inferences, as everyone learns, teaches, and uses them, lead to absurdities. And where else would reliability and truth be found if even mathematical thinking fails?*
>
> [Hilbert 1925], p. 375.

Back home from the splendid transfinite trip, with his confidence deep shaken, one

[2] Borrowed from *International Herald Tribune, November 18, 1993*, and slightly modified, with the permission which is here gratefully acknowledged. The original cartoon, created by KAL, represents «Washington crossing the Dinnerware» into a «Theme Park based on US History to be build by Disney». The rejoinder «This worries me» belongs to KAL.

has to confront the bitter truth : the Cantorian and post-Cantorian dreams about infinity have become a nightmare. Let's face it : the first crisis provoked by set theory, that of logical paradoxes (in Hilbert's words, «*absurdities*»), has given place to the modern crisis of the *arbitrariness* of both transfinite insights and of extremely elaborated formal notions and constructions inspired by those insights. Nowhere in mathematics (or, for that matter, in natural sciences) does one need to *blindly believe* in so many conceptual inventions and infinite artifacts without *any* benefit of illumination and/or confirmation [Maddy 1988].

His usual naive mathematical sobriety intact, our mathematical country cousin timidly but distinctly utters :

4.1. Frequently Asked Question. *Emotions and travel guides aside, but do they really exist ?!*

Have all new infinite totalities discovered by Cantor and after him «*the same strong claim to existence*» (a paraphrase of [Barwise 1975], p. 113) as the denumerable and the continuum known already to the Greeks ? In other words, what are our reasons to be committed to their existence, as we are committed to the existence of the natural numbers ω and of the continuum $\mathfrak{c}$?

Poor yokel : how could he expect that such simple and natural *mathematical* questions are invariably provoking a terrible storm ...

For Georg Cantor [Dauben 1979], pp. 132-133, and David Hilbert [Hilbert 1925], pp. 375-376, the answer was straightforward and generous *in extremis* : all mathematical objects, whose definitions do not contradict the formal framework of a theory, exist. In other words, *consistency* is the only condition for existence.

On the other hand, for Luitzen Brouwer and Henri Poincaré, neither of new (uncountable) infinite totalities exists *as a matter of principle*, because neither has been ever properly defined : the advanced definitions did not satisfy some *a priori* criteria of philosophical correctness, for example, they employ *the law of the excluded middle*, or lack *predicativity*. Here is a more modern brand of a violent denial of Cantor's and Hilbert's existential generosity :

> «*At the beginning of this century a self-destructive democratic principle was advanced in mathematics (especially by Hilbert), according to which all axiom systems have equal right to be analyzed, and the value of mathematical achievement is determined, not by its significance and usefulness as in other sciences, but by its difficulty alone, as in mountaineering. This principle quickly led mathematicians to break from physics and to separate from all other sciences. In the eyes of all normal people, they were transformed into a sinister priestly caste of a dying religion, like Druids.* » [Arnold 1995], pp. 7-8.

The persistence of such extreme, mutually (and violently) incompatible attitudes explain how the issue has become a hostage in the war of mathematical habits and philosophical tastes.

The atmosphere surrounding, from its very beginning [Dauben 1979], [Moore 1982], [van Dalen 1990] this extremely difficult problem has been, and still is, so opinionated, the arguments have been, and still remain, so personal, arbitrary [Jensen 1995], p. 401 (note 18), and even violent [Mac Lane 1992], p. 121, that the people who prefer to stick to their set-theoretical interests have become somewhat cynical about it. Some are just going after their formal kills, having freed themselves from any ontological fetters; as Craig Smorynski has uncharitably put it:

> *«The subject attracted careerists, who were trained to solve problems, to belittle anything that wasn't hard, and who were not taught anything about the history or philosophy of their subject and quickly learned that such knowledge did not help their careers.»* [Smorynski 1988], p. 13.

Others are acknowledging the legitimacy of the problem, only to address it straightaway in the «didn't ask, wouldn't tell» manner :

> «*The question "what large cardinals are there?" is, although undecidable (unless there are none) surely a natural one. Not that these strong inaccessibles obviously exist; but if caution was to be exercised it should have been exercised a long way earlier. Anyone who is happy about unlimited application of the power set operation can feel few qualms about an inaccessible*». [Dodd 1982], p. xxii.

(Incidentally, and in anticipation of the ensuing deliberations, §§8, 11, the present author has been never «*happy about unlimited application of the power set operation*» [Belaga 1988], and thus, according to [Dodd 1982], he is somehow entitled to feel qualms about new infinite totalities).

Clearly, at stake is so much that one cannot but understand and deeply respect the indignation of Kurt Gödel, who has written more than thirty years ago :

> *«Brouwer's intuitionism is utterly destructive in its results. The whole theory of $\aleph$'s greater than $\aleph_1$ is rejected as meaningless.»* [Gödel 1964], p. 257.

Ours is not a destructive attitude, and we are not rejecting anything. And yet, risking to offend the sensitivity of high-handed dwellers of Cantor's Paradise, we feel relieved to be at this fateful juncture in the company of such good mathematicians and serious thinkers as Luitzen Brouwer, Henri Lebesgue, and Harvey Friedman : people's infinite fantasies have to be somehow *independently verified and confirmed*. Moreover, we even have a few ideas how it could be done in the spirit of, and with all due respect to the achievements of our transfinite colleagues, §6. But the storm we have provoked not only continues unabated, it grows even more bizarre and destructive ...

§5. Worse Still : Does Mathematical Infinity Exist at All ?

> *To be sure, the discussion of the paradoxes of set theory led research in the foundation of mathematics a long way from the classical view of the nature of mathematics so passionately defended by Cantor. Intuitionists and formalists are united in their effort to eliminate all metaphysical elements from the foundations of exact sciences. .... Georg Cantor, schooled in Plato and scholastics, thought differently about the matter. .... It is part of the tragedy of our investigator's life, so full of disappointments, that his own theory gave rise to a new concept of mathematics which, for good reasons, put an end to basing the exact sciences on metaphysics.*
>
> [Meschkowski 1964], pp. 94, 95.

Thus, before even attempting to reflect on the above existential problem, one is confronted with a much more formidable one :

5.1. Frequently Asked Question. *Does Mathematical Infinity exist at all ? Or, in other words : Can one «really know» anything about infinity ?*

The answers of two leading modern schools of thought, *formalism* and *constructivism*, which split between them the majority of votes of philosophically affiliated members of the mathematical community, vary from a mild «Not very much indeed» to the unapologetic «Nothing, and do not make a fool of yourself». (We apologize to the reader of a *Platonist* or any other *idealistic* persuasion for classifying him as an ideological minority, and we implore him to wait for a while patiently in line. As to *nominalists* and other *pragmatists*, they do not belong here anyway.)

The ultimate *intuitionist*, or *constructivist*, reason is philosophical, even a religious one : the understanding of man as a purely finite being having no reliable access to infinite. Pushing Brouwer's original and deep vision to its almost absurd limits, Errett Bishop claims :

> «*Our point of view is to describe the mathematical operations that can be carried out by finite beings, man's mathematics for short. In contrast, classical mathematics concerns itself with operations that can be carried out by God.*» [Bishop 1985], p. 9.

And

> «*If God has mathematics of his own that needs to be done, let him do it himself.*» [Bishop, Bridges 1985], p. 5.

Of course, not every constructivist could easily swallow such a brutal brand of the intuitionist philosophy ; Hermann Weyl, for once, has been of another opinion :

> «*Mathematics has been called the science of the infinite. Indeed, the mathematician invents finite constructions by which questions are decided that by their very nature refer to the infinite. That is his glory.*» [Weyl 1985], p. 12.

As to the *formalist* school, whose historical *raison d'être* has been the urgent need to defend the mathematical *Vaterland* from the onslaught of intuitionism, and judging by what we have heard from David Hilbert in the first epigraph to the present paper, one might expect that it would defend the infinite with at least as much good will as Weyl did ... Surprisingly, Hilbert's defence against Brouwer's foundational critique of classical mathematics has been based on not less sweeping a denial of the «real existence» of the infinite than Bishop's [Hilbert 1925], p. 392.

Yet, the dubious honour to unambiguously and terminally affirm the *formalist* death of the infinite, and to do this confessedly, on the grounds of the absence of any inspiring philosophical convictions, fell finally to Abraham Robinson (who, too, has been just pushing the founder's original vision to its clearly absurd limits) :

> «*My position concerning the foundations of Mathematics is based on the following two main points or principles.* ***(i)*** *Infinite totalities do not exist in any sense of the word (i. e., either really or ideally). More precisely, any mention, or purported mention, of infinite totalities is, literally,* meaningless. ***(ii)*** *Nevertheless, we should continue the business of Mathematics "as usual", i. e., we should act* as if *infinite totalities really existed.*» [Robinson 1965], p. 230.

Two «merits» of this famous doctrine bear on the subject of the present study.

First, it plagiarizes, with minor adjustments, another famous maxim : that, of all things ... of Aristotelian *Physics* [3] !! The Greeks strike again ...

[3] As a matter of fact (to the best knowledge of the present author), neither Abraham Robinson, no any other source mention the striking similarity between the spirit and letter of Robinson's dictum and the following passage from *Physics* of Aristotle (we following the translation of [Hintikka 1996], p. 201) :

> «*Our account does not rob the mathematicians of their study, by disproving the actual existence of the infinite ... In point of fact they do not need the infinite and do not use it. They postulate only that the finite straight line may be produced as far as they wish. It is possible to have divided in the same ration as the largest quantity another magnitude of any size you like. Hence, for the purposes of proof, it will make no difference to them to have such an infinite instead, while its existence will be in the sphere of real magnitudes* ». [*Phys*. III, 7, 207b27-34]

Of course Aristotle's is a more consistent and, from the modern point of view, more radical assertion : in the up-to-date parlance it would be dubbed something like «ultra-intuitionistic criticism» [Yessenin-Volpin 1970].

Second, Robinson's dictum has been the final affirmation that Hilbert's *formalism à la Robinson*, and the inherent in it schizophrenic vision of the mathematical theory and practice [Cohen 1971], [Hersh 1979], [Bishop 1985], have become a normative mathematical thinking.

§6. The Greeks Strike Again and Again : Problems **versus** *Paradoxes.*

> *Yes, I once gave a lecture with the flamboyant title, «Set theory is obsolete.» In this and few others contentious articles, I have violated one of the cardinal principles of mathematical activity : Mathematicians do not make pronouncements; they prove theorems. My apologies.*
>
> [Mac Lane 1992], p. 119.

Although the ills of modern set theory, as the reader might have already noticed, are numerous, the present author is convinced that it is neither obsolete, not terminally sick. Still, in the light of all important (and only partially mentioned in the present paper) *pronouncements* concerning the past [Dauben 1997], [Hallett 1984], [Moore 1982], present [Jech 1995], [Jensen 1995], [Mac Lane 1992], and future [Shelah 1993] of set theory, a simple and down-to-earth set-theoretical diagnosis would be no luxury. The rest of this study, §§6-14, represents such a diagnosis.

Our first observation : all above *Frequently Asked Questions* have about them an air of somehow touching a *mystery*, and not just an *unknown*. This is a typical phenomenon :

6.1. Frequently Asked Question. *What makes the problems concerning Mathematical Infinity more akin to logical paradoxes than to open problems of other mathematical domains* ?

Here are a few words of explanation. Open problems excite the imagination of a mathematician, some of them for years, others for decades, still others for centuries (as, for example, Fermat's Last Problem), even millennia (Euclid's Fifth Postulate). Yet, impenetrable and deep as an *open problem* might be, it represents an exact question raised in an exact mathematical context. *An open problem can be compared to a clearly marked trail leading into as yet inaccessible but absolutely real* terra incognita

By contrast, paradoxes do not have such a privilege : a deep mathematical or logical *paradox* is a double-edged question concerning both the object and the subject of study, - the subject being our intellectual ability to decently handle the object. *An unresolved paradox is similar to a mirage, with its clear but deceptive image, deprived of any certitude of reality, not speaking about possible ways to eventually reach it.* In particular, to answer a good paradox, one needs to invent from scratch a proper conceptual (mathematical, logical, or even cultural) context in which the hidden in the paradox question becomes explicit and exact, in other words, becomes an *open problem*. Also, if a problem is solved, then it is definitely solved; by contrast, a good paradox tends to remain open and attractive in every generation, after it has been «successfully resolved» as many times as many philosophers have addressed it.

6.2. Meta-Paradox of Mathematical Infinity. The most salient feature of all inquiries into *Mathematical Infinity*, starting with the Greeks [4], has been, and still remains, their «*disconcerting tendency to produce*» (a paraphrase from [Shelah 1992], p. 197)

(i) more new paradoxes than new open problems,

(ii) and more new (often, extremely sophisticated) refinements of known paradoxes than new solutions of known problems.

Let us illustrate these statements by one of the most marvellous set-theoretical stories concerning the *existence* of the countable and the continuum, as well as of the relationship between them :

6.3. Example of an Apparently Resolved But in the Last Analysis, Aggravated Paradox. *The confrontation between the countable and the continuum, from Zeno to Cantor, and beyond.*

The Greeks have been the first to «colonize» two basic (and fundamental for us as well) *infinite mathematical habitats*, the natural numbers and the continuum, ω and $\mathfrak{c}$ in modern notations. They existed for the Greeks (as they exist for us, present-day mathematical yokels) simply because of :

6.4. The Criterion of «Real Existence» of an As Yet Only Intuitively Perceived Mathematical Notion. *Beautiful mathematical theories about and around it, rich in fruitful applications.*

Notice that this criterion is neither *platonist*, nor *constructivist*, nor *formalist* : it just doesn't claim anything about the «object behind the notion». For the *Platonist*, the existence is related to an «*object*» :

> «*But, despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true.* » [Gödel 1964], p. 268.

For the *Constructivist*, the existence of an object is *sine qua non* condition for a theory to be mathematics, and there exist only constructively defined objects. For the *Formalist*, nothing existential matters.

Become acquainted with ω and $\mathfrak{c}$ *experimentally*, the Greeks have been absolutely fascinated by the obvious to them differences in the «*origins*» and «*natures*» of these two infinities, as it is clear from the paradoxes advanced by Zeno of Elea (C. 464/460 B.C.) [Bochenski 1970], p. 26, [Anglin, Lambek 1995], pp. 54-57.

In particular, Zeno's paradox «*Achilles and Tortoise*» clearly demonstrates the conceptual confrontation between two different types of experiences which led to two different models of infinity. One type is best encapsulated by the counting experience (through observations of hearth-beatings, walking as a step-by-step movement, building of towers, etc.), - the only humanly available «*accumulation of infinity*» by *finite and discrete* portions. The second type can be observed in the external world as a *continuous* infinity (points on the horizon, the flight of an arrow, etc.). Zeno clearly doubted that the two infinities could be reconciled : one can run, but one cannot «understand» this phenomenon, because *our understanding is finite and discrete*, whereas our movement (a

[4] «*A "foundational crisis" occurred already in Greek mathematics, brought about by the Pythagorean discovery of incommensurable quantities. It was Eudoxos who provided new foundations, and since then Greek mathematics has been unshakeable. If one reads modern mathematical textbooks, one is normally told that something very similar occurred in modern mathematics.* » [Lorenzen 1958], p. 241.

mystery in itself) is, as the sky itself, *continuous*. [5] This does not mean, of course, that Zeno doubted the *existence* of the continuum.

Cantor has resolved this particular aspect of Zeno's paradox [6] by inventing an absolutely new mathematical universe, his *Set Theory*, unavailable to Zeno, where the relationship between two, previously incompatible infinite «habitats» can be successfully conceptualized, and then formally studied and understood.

In fact, all what we are now know about the continuum *for sure* (and what remains one of Cantor's most striking and important *discoveries*), can be explained to a schoolgirl or -boy :

6.5. Cantor's Powerset Construction and Proof of the Inequality $\aleph_0 < \mathfrak{c}$. **(i)** Cantor's construction, or definition, of the continuum $\mathfrak{c}$ as the set of all subsets (called the power set) of the set ω of natural numbers,

$$\mathfrak{c} = powerset(\omega) = \mathsf{P}\ (\omega).$$

(ii) The proof, based on Cantor's formidable *diagonal argument*, that this fact implies the *uncountability* of the continuum,

$$\aleph_0 < \mathfrak{c}\ .$$

Notice that this inequality has as yet nothing to do with the above ordinal-cardinal hierarchy. It just means that **(i)** $\mathfrak{c}$ has a subset equipotent with ω, **(ii)** a conjecture that ω can be put into one-to one correspondence with $\mathfrak{c}$ leads to contradiction over an extremely weak subset of *ZF*.

For better or worse, this has not been the end of the story. Falling in an even deeper trap than Zeno, Cantor «*freed himself of all fetters and manipulated the set concept without any restriction*» (as Hermann Weyl puts it disapprovingly in [Weyl 1949], p. 50). Most important, Cantor has invented transfinite ordinals with the explicit purpose to be capable to do exactly what Zeno realized he has been unable to do : namely, to «count up» the continuum ! (The details will be discussed later, §§11, 12).

The resulting state of affairs in set theory far surpasses in its discordancy all known Greek precedents. In particular, the cleavage between the discrete (or *Pythagoreanas*, as Ronald Jensen has chosen to call it [Jensen 1995], p. 401) and continuous (*Newtonian*, according to Jensen) accounts of the world has become even more acute and irreconcilable :

6.6. The Fundamental Problem of the Continuum in Modern Set-Theory. *There is as yet no definitive demonstration of the fact (conjectured by Cantor and proved by Ernest Zermelo in ZFC set theory from even more complicated conjectured properties of sets) that the continuum, as we know it mathematically, can be embedded into the aforementioned ordinal-cardinal hierarchy.*

---

[5] More than two thousand years later, apparently the same intuition has motivated Brouwer's efforts to re-define Cantor's mathematical continuum [Brouwer 1981] ! On the other hand, according to the Bible, a few thousands years before Zeno, the people of Shinar had no such scruples :

> «*And they said: Come, let us build a city and a tower, whose top may reach to heaven.*» (Genesis 11:4).

They have been, of course, mistaken ... Still, as some argue, modern set theory is far from being completely free from a similar imperious insanity.

[6] Other aspects of this paradox are still vividly discussed by philosophers. Thus, for instance, Henri Bergson (1859-1941) is thoroughly discussing and forcefully «explaining» Zeno's paradox anew in his book [Bergson 1959], pp. 1259, 1376, 1377.

§7. *More FAQs Concerning the Continuum and the Ordinal-Cardinal Hierarchy.*

«*For me the essential point is the existence of infinite totalities. The attitude toward infinite sets has traditionally been the great dividing line between mathematicians.* »

[Cohen 1971], p. 10.

Assuming Cantor's Paradise, as it is described above in §3, exists and shelters somehow the continuum, one can ask about such a continuum many (often contradicting) questions and receive many (mostly, mutually incompatible) answers [Judah, Just, Woodin 1992]. Assuming, however, that the problem of the continuum's sojourn in the Paradise is still open (Fundamental Problem 6.6), one is left with at least two open questions, whose merit is their unambiguous and universal mathematical importance.

7.1. Frequently Asked Question. *Could it be that the continuum belongs, in fact, to an «hierarchy of set-theoretical principles in another galaxy above the linearly ordered hierarchy of §3 ?»* (Paraphrazing [Kanamory, Magidor 1978], p. 104, cf. §3.) *In other words, is it possibly that* , in reality, *the continuum cannot be well-ordered ?*

If the answer would be in affirmative, then

7.2. Frequently Asked Question. *Would be all members of the linear ordinal-cardinal hierarchy comparable to the continuum, or would be it «disjoint and incomparable to some or even all of our large cardinals» ?»* (More paraphrazing [Kanamory, Magidor 1978], p. 104, cf. §3.)

Two following questions can be answered affirmatively immediately:

7.3. Frequently Asked Question. *Does the Continuum Hypothesis make sense outside the linear ordinal-cardinal hierarchy ?*

The obvious answer is : yes, it does, if one understands Cantor's inequality in the spirit of his construction 6.5 : one looks for a subset, say, $\mathbb{L}$ of $\mathfrak{c}$, which encloses the set $\mathbb{N}$ of natural numbers, $\mathbb{N} \subset \mathbb{L} \subset \mathfrak{c}$, and such that $\mathbb{N} < \mathbb{L} < \mathfrak{c}$, in the same sense as $\mathbb{N} = \aleph_0 < \mathfrak{c}$. *CH* states that no such $\mathbb{L}$ exists.

Interestingly enough, while Cantor still talks about his conjecture in both of its possible forms, «*no cardinal value* $\mathbb{L}$ *strictly between* $\mathbb{N}$ and $\mathfrak{c}$» and «$\aleph_1 = \mathfrak{c}$», Kurt Gödel's attitude becomes rigid and single-minded : he completely identifies *Mathematical Infinity* with its *ZFC* formalization, *CH* with «$\aleph_1 = \mathfrak{c}$», insisting with the determination of a martyr of the new set-theoretical faith that «*the axioms force themselves upon us as being true*» [Gödel 1964], p. 268.

We will return to Gödel and his tragedy later, in *Cantor's Transfinite Dream, 11.2*; now comes the next

7.4. Frequently Asked Question. *What might be the criterion of «real existence» of members of the linear ordinal-cardinal hierarchy ?*

Our answer follows Kurt Gödel's [Gödel 1964] and Harvey Friedman's [Friedman 1986], p. 192, suggestions (cf. §2) :

7.5. The Criterion of «Real Existence» of a New Infinite Totality Defined in a Fixed Axiomatic Framework : It Mast Be Relevant and Even Necessary (Okham's Razor).

(CRE0) *Its existence is directly confirmed by displaying of a verifiably true* *theorem from number theory, combinatorics, etc., which demonstrably needs in its proof the mathematical induction up to this particular infinite totality.*

The following can be regarded as a supporting evidence :

(CRE1) *The totality is an object of a rich, beautiful, and multifaceted theory.*

(CRE2) *This theory fruitfully interacts with theories from other mathematical domains, and has nontrivial and interesting applications there.*

The following are the instructions for the razor's user :

(CREU) *A most lenient, liberal, and cautious use of the above rc advisable. Still, until the existence of an infinite totality has been indepe confirmed, it would eke out a bare formal notational existence, remaining different interpretations, critique, and even to an outright rejection.*

Of course, the actual absence of such a confirmation does not give to anybody the right to apply Okham's razor as the butcher's axe, proclaiming modern set-theoretical research irrelevant, or to strive for its permanent transfer to mental institutions [Mac Lane 1992], p. 121 (quoted in §3).

After introducing in the next section Cantor's basic mechanisms of fabricating new infinite totalities, we will discuss in §13 what one *really knows* today, beyond the examples of §2, about independent confirmation (the criterion CRE0) of *the real existence* of new infinite totalities. Then we proceed to review the status of infinite totalities whose existential credential are restricted to CRE1-2, or less.

§8. Entering Cantor's Paradise on Foot.

> *For inaccessible or measurable cardinals our intuition is probably not yet sufficiently developed or at least one cannot communicate it. Nevertheless I feel that this is a useful task, to develop our mystical feeling for which axioms should be accepted. Here of course, we must abandon the scientific program entirely and return to an almost instinctual level, somewhat akin to the spirit with which man first began to think about mathematics. I, for one ... feel impelled to resist the great aesthetic temptation to avoid all circumlocutions and to accept set theory as an existing reality ... The reader will undoubtedly sense the heavy note of pessimism which pervades these attitudes. Yet mathematics may be likened to a Promethean labour, full of life, energy and great wonder, yet containing the seed of an overwhelming self-doubt. ... Through all of this, number theory stands as a shining beacon. ... This is our fate, to live with doubts, to pursue a subject whose absoluteness we are not certain of, in short to realize that the only "true" science is itself of the same mortal, perhaps empirical, nature as all other human undertakings.*
>
> [Cohen 1971] p. 15.

Unless one becomes (paraphrazing [Smorynski 1988], p. 13) a «*careerist belittling anything he does not understand*», one needs to go back to §3, and to climb the transfinite slope in person, riveted by the vision of Cantor's Paradise, with its flying-by helicopters carrying our mathematical country cousins [Maddy 1988]. The beautiful Greek icon below, §12, faithfully portrays our dangerous ascent ...

Entering Cantor's set-theoretical *edifice* on foot, one discovers :

(i) a foundation, with its two powerful primitive concepts, that of set-theoretical *object*, and that of set-theoretical *relationship* between objects; any mathematical object or structure can be, according to Cantor, stripped of all its properties, down to the state of being a structureless collection, or *set*, or ((*in*)*finite*) *totality*, composed of its *elements*, who, in their turn, *belong* to the totality (cf. the *Axiom of Extensionality* below, this section); the only criterion of equality between totalities is that of the *one-to-one correspondence*;

(ii) four «pillars» resting on the foundation, which are four different (and not always mutually independent) methodologies, or meta-procedures, of fabrication of new infinite totalities from the known ones;

(iii) the «dome», Cantor's transfinite theory, or superstructure, of ordinals and cardinals, supported by the four pillars, with Cantor's *Well-Ordering Principle* serving as the «keystone».

Cantor's naive set theory could be axiomatized in many ways. The best known *Zermelo-Fraenkel* axiomatic system, *ZF* [Fraenkel, Bar-Hillel, Lévy 1973] (*ZFC* denotes *ZF* plus the *Axiom of Choice*, *AC*), comprises four types of axioms:

(i) the *Axiom of Extensionality*, *AE*, which, in fact, is a definition of the notion of set with respect to the relationship of membership $\in$: to «know» a set, it is enough to «know» all its elements;

(ii) two axioms of existence, each one postulating the existence of two specific sets, of the empty set and of the (countable) infinite one : *Empty Set Axiom*, *ESA*, and *Axiom of Infinity*, *AI*;

(iii) three axioms of construction, which, given a set x, a pair of sets x, y, or a set z of sets, postulate the existence, respectively, of the power set of x, $\mathsf{P}(x)$, of the pair set (x, y), and of the union $\cup z$ of member-sets of z : *Power Set Axiom*, *PSA*, *Axiom of Pair*, *AP*, *Axiom of Union*, *AU*;

(iv) an axiom schemata (i. e., a recipe to design axioms) of construction, which, given a set x and a «property», or «condition», expressed by a formula, postulate the existence of the image of x under the function (mapping) defined by the given formula : *Axiom Schemata of Replacement*, *ASR*. [7]

Presenting below Cantor's naive set theory, we indicate in brackets the corresponding axiomatic means formalizing Cantor's intuitive notions in the *ZFC* framework.

§9. The Main Principle : Invoking Mathematical Infinity in One Full Swoop : Just Say «And So On !».

> *I am about to introduce a symposium on infinity. I do so, not because I can claim any special intimacy with the infinite, nor yet because I feel myself specially competent to unravel its intricacies, but because I think it all-important that a notion so fundamental should be rescued from the grip of the experts, and should be brought back into general circulation. It is a notion so common and so clear as to lie behind practically every use of the ordinary phrases "and so on" or "and so forth", but it is non the less capable of giving rise to vertiginous bewilderments, which may lead, on the one hand, to the mystical multiplication of contradictions, as also, on the other hand, to that voluntary curtailment of our talk and thought on certain matters, which is as ruinous to our ordered thinking. A notion which is at once so tantalizing and*

[7] To these axioms one usually adds an axiom banning the existence of some «pathological» sets, which are members of themselves, either directly, or through other members, or have any similar property of *infinite descent* through the membership relationship : the *Axiom of Foundation.*

so ordinary plainly deserves the perpetual notice of philosophers. Throughout the history of human reflection the fogs of an interesting, and often interested obscurity have surrounded the infinite; they were dispersed for a brief period by the sense-making genius of Cantor, but have since gathered about it with an added, because wilful, impenetrability. In the growing illiteracy of our time, when the lamp of memory barely sheds its beams beyond the past two decades ...... I must attempt, at any rate, to do what others, better qualified than myself, have so entirely neglected; it is better that someone should discuss this topic with the freedom of philosophy, than that all talk about it should be allowed to flow along those technical channels which, whatever else they may do, never enrich our philosophical understanding.

[Findlay 1953], pp. 146-147.

The dome superstructure being discussed in §11, we are entering now Cantor's factory of infinities.

Among four «pillars» of Cantor's set theory, one is central and distressingly general. In fact, it represents a fundamental *meta-philosophical principle of reaching out* for new infinite totalities, with three others methods being its (meta-) mathematical specifications.

Cantor's «Beyond the Upper-Limits» Principle, BULP (Or Principle of the Ultimate Accessibility of Any Set-Theoretical Inaccessible). *The set-theoretical world you see around you has a limit, and behind this limit a ne world starts. So let's go and take a look at it !*

The spirit of the procedure is aptly captured by the old engraving Fig. 2 below.

One cannot underestimate all importance of this principle for theory of sets. In fact, rarely in the history of science or mathematics can one find a vast and full-fledged theory with such a predominance, both conceptional and formal, of a single, and for that matter, extremely controversial idea !

Starting with Cantor's first and historically unprecedented affirmation that any *potential*, or incomplete, or improper infinity can be viewed, and subsequently dealt with, as an *actual*, or proper, or complete one [Dauben 1979], p. 97, - through *inaccessibility* and *indescribability* [Kanamory 1994], - and up to the most-recent *inconsistency ceiling* for all known *strong hypothesis of infinity*, §3, - everywhere one meets and needs the omnipresent, omnipotent, and, as many are still hoping, omniscient Cantor's «beyond-the-upper-limits» *Symbol of Set-Theoretical Faith*. Notice that in its generality, *BULP* is independent of a specific axiomatic framework, *ZFC* including.

Figure 2

Figure 3

§10. *Three Lesser Principles of Fabrication of New Infinite Totalities.*

10.1. Transfinite Counting. The first offshoot of Cantor's general accessibility principle has been his method of extension of the usual counting procedure, *1, 2, 3, ...*, beyond its infinite «ceiling», as it has been described above, §2. Notice that in the standard set-theoretical expositions, the first (*transfinite counting*) procedure is usually *formally* incorporated into the third (*functional*) scheme, *via* Cantor's concept of *well-ordering*. (In *ZFC*, one needs here, among other things, *AES* and *AI*.)

10.2. Combinatorial Method. The second method extends the aforementioned Powerset Construction 6.5 to any set, including the continuum itself, $\mathfrak{c}_1$ = *powerset*($\mathfrak{c}$) = $\mathsf{P}(\mathfrak{c})$, and beyond. It also employs such basic combinatorial operations over infinite totalities as the sum, product, pairing. (The axioms *PSA*, *AP*, *AU* in *ZFC*.)

10.3. Functional (or Descriptive) Scheme(s). This method designs, or rather *nominates*, new infinite totalities by pure descriptive (or functional) means. Typically, one assembles together *all* already existing or hypothetically available totalities *with a chosen property* into a «basket», and then one declares that the «stuffed» in such a way basket must necessary represent an infinite totality. (The axiom schemata *ASR*.)

10.4. Example of the Application of the Functional Scheme : Cantor's Construction of the First, Second, etc., Uncountable Ordinals. One collects *all countable* ordinals into one «basket», called Cantor's *first number class*. It has to be an ordinal, and it must be (by definition) greater than any one from the first class, - thus, the *smallest uncountable* ordinal, $\omega_1 = \mathsf{L}(\omega)$ (L stands for the *transfinite* L*imit operation*; our notation). All ordinals, which follow ω_1 and are equipotent with it, form Cantor's *second number class* followed by $\omega_2 = \mathsf{L}(\omega_1)$, ... «*and so on*» !

§ 11. *Cantor's Transfinite Superstructure.*

Still, Cantor has been not satisfied with the emerging transfinite universe, and not without reason : after being created according to one (or several) of Cantor's three methods, some of his new infinite totalities bear forever the «marks of their infinite origins», which do not let them to effectively «mix» with other infinities, leaving them «*disjoint and incomparable*» ([Kanamory, Magidor 1978], p. 104; §§2, 7; FAQs 7.1-2).

To overcome these inbred shortcomings, Cantor has conjectured two fundamental and far-reaching properties of the old and new infinite totalities, which forcefully amalgamate disparate infinities into one linearly ordered (in fact, *well-ordered*) transfinite universe, but which strike an attentive observer (see, e. g., [Lebesgue 1905], to mention only one of many prominent critics, past and present) as coming out of nowhere, a sort of a politically motivated, transfinite «affirmative action».

And here it is how it apparently happened. As Cantor has discovered, the power set construction «creates» an uncountable (the continuum) from the countable, $\mathfrak{c} = \mathsf{P}(\omega)$ (Cantor's Powerset Construction 6.5). But so does Cantor's transfinite limit procedure, $\omega_1 = \mathsf{L}(\omega)$ (Example 10.4) ! Also, two operations share an important general property : applied to a set, they increase its cardinal power :

$$\alpha < \mathsf{P}(\alpha),\ \alpha < \mathsf{L}(\alpha)\,.$$

In the case of L, it is true by (the ordinal) definition, for P Cantor has discovered a general, simple, elegant, and influential *diagonal* argument (mentioned in 6.5) which does not depend on Cantor's ordinal construction.

11.1. Cantor's Transfinite Dream. Our guess is, *Cantor believed that both operations have the same cardinal strength and create the same things* .

In other words, for cardinals, $\mathsf{P} = \mathsf{L}$. But, being enacted on ordinals, L is a much more subtle, rich, transparent, and good-behaving operation : **(i)** it defines the *minimal* uncountable totality ω_1; **(ii)** more generally, it increments the cardinal power by the minimal transfinite «quantum». In a word, a perfect assembly line wonderfully explaining away the somewhat obscure and recalcitrant P !

Cantor never claimed that. Instead, he proposed :

(H1) Any set (the continuum including) can be *well-ordered*, or, in other words, is equipotent with some ordinal (Cantor's *Well-Ordering Principle*).

(H2) The Continuum Hypothesis in Its Ordinal Form *(CH)* : the continuum and the first uncountable ordinal are equipotent, $\mathfrak{c} \approx \omega_1$. Later on, the last conjecture has been vastly extended :

(H2*) The Generalized Continuum Hypothesis *(GCH)* : $\mathsf{P}^{\alpha}(\omega) \approx \mathsf{L}^{\alpha}(\omega)$.

The fact is, taken together, *the conjectures H1, H2, H2** amount in Cantor's naive set theory to exactly $\mathsf{P} = \mathsf{L}$! In axiomatic set theory, the constructible universe *L* of Kurt Gödel, together with the conjecture $V = L$ [Devlin 1984], represent the closest and most exquisite formal realization (with necessary adjustments) of Cantor's last set-theoretical will. After all, Gödel has been the most faithful, gifted, subtle, and, inescapably, most pathetic [Feferman 1986] of Cantor's heirs :

11.2. Kurt Gödel's Intellectual Martyr : Believing, first, that all of them really exist, and second, that ZFC represents the heaven-sent (sorry, Platon-sent) axiomatic basis for any adequate formalism capable to eventually capture the «true nature» of Mathematical Infinity.

Never since Hamlet, the famous Prince of Denmark, has lived and acted such a brilliant and relentlessly analytic mind, who would be so puerile and credulous in his fundamental existential beliefs !

Two examples illustrate this intrinsic paradox in an almost tragicomic, if not tragic, way. First, remember how this gentle man [Feferman 1986] would characterize Brouwer («*utterly destructive*», cf. §4; [Gödel 1964], p. 257), only because of the latter's critical (if even intellectually and mathematically perfectly justified, both *a priori* and *a posteriori*) attitude toward Cantor's infinite constructions.

Second, read the hilarious account, in [Feferman 1986], p. 12, of Gödel's, this consummate European and ex-Viennese, acquiring US citizenship. He has been assisted in this endeavour by two faithful ex-European lieutenants (themselves luminaries of sorts : Albert Einstein and Oskar Morgenstern) who have succeeded to save the formal proceedings from Gödel's initiative to correct in the United States Constitution a «*logical-legal possibility by which the U. S. A. could be transformed into a dictatorship* ».

11.3. Historical Aside. As the reader probably already knows, the fates of two conjectures, *H1* and *H2*, turned out to be very different :

(1) Cantor's *Well-Ordering Principle*, or *WOP*, originally conceived by him as

> «a *fundamental law of thought, rich in consequences and particularly remarkable for its general validity* » (cf. [Hallett 1984], p.73),

has become a theorem in *ZFC*. The nontriviality and controversial history of the *Axiom of Choice* are well known [Moore 1982]. The axiom has been invented in 1904 by Ernst Zermelo with the express purpose to prove the *WOP*.

Less known is the fact that Zermelo's proof (or, for that matter, any other proof) of

WOP depends in equal measure on Cantor's powerset construction, or the *Power Set Axiom*, *PSA*, as well. (It is clear from the above remark of [Dodd 1982] in §4, why this particular technical feature is worth to be mentioned here.)

(2) In fact, they are *PSA* and *ASR*, who form together the potent «motor» capable to propel us into Cantor's transfinite Paradise, and beyond [8]. Still new and more powerful motors are needed to even more accelerate our transfinite ascent. Such motors are called *strong axioms of infinity* (cf. the quotation from [Kanamory, Magidor 1978] in §3), and in this «*quest for new axioms of infinity*» [Jensen 1995], p. 401, one has already invented a throng of them [Jech 1995] ...

(3) As to the validity of the *Continuum* and *Generalized Continuum Hypothesis*, *CH* and *GCH*, they remain open problems, in fact, the most famous open problems of modern set theory [Jech 1995].

(4) Still, Cantor's implicit expectation hidden (as we affirm above) behind the combination *WOP* + *CH*, that of the identity of two transfinite operations, $P = L$, has been definitively abandoned : each of the two mutually excluding conjectures $P > L$ and *GCH* are consistent with *ZFC* [Gödel 1964], [Cohen 1964].

§12. Cantor's Dream and the Post-Cantorian Nightmare.

> *And he dreamed, and behold a ladder set up on the earth, and the top of it reached to heaven; and behold the angels of God ascending and descending on it ... And Jacob awoke out of his sleep, and he said: Surely the Lord is in this place; and I knew it not. And he was afraid, and said: How dreadful is this place! This is no other than the house of God, and this is the gate of heaven.*
>
> Genesis 28:12, 16, 17.

Our understanding is : Cantor's transfinite programme has been inspired by his powerful and deep extra-mathematical interests. These interests have shaped Cantor's set theory, and they implicitly continue to shape the bulk of «theological ventures» of modern set theory.

In short : if angels could ascend and descend on the ladder set up on the earth and reaching to heaven, why not men ? Our guess is, such has been the inspiration of Georg Cantor (this was a deeply religious man [Dauben 1979]), who conceived and constructed the transfinite ladder with the express purpose to ascend from the finite («earth» of a mathematician), through the countable (the lower part of the ladder), to the continuum («heaven» of the Greeks), and then beyond (to the «heaven» of scholastics?). The beautiful Greek icon below, this true precursor of the transfinite ladder of §3, perfectly captures his vision.

Has Cantor been directly influenced by this icon, or was it a famous classic of the VIIth century, «*The Ladder of Divine Ascent*» [Climacus 1982], that has enticed him (cf. Fig. 3 above, p. 19) ?... As nowadays, Zen or Tao are enticing some physicists [Capra 1991] who view them as the privileged and most powerful para-spiritual engine of their scientific research.

Cantor himself was quite unapologetic about his motives. Here he is, writing hundred years ago to Father Thomas Esser in Rome:

> «*The establishing of the principles of mathematics and the natural sciences is the responsibility of metaphysics. Hence metaphysics must look on them as*

---

[8] Without *ASR*, one cannot even assemble the ordinal $\omega + \omega = \omega 2$, §8, from the sets $\omega + n$: see, e. g., [Hrbacek, Jech 1984], p. 144.

her children and as her servants and helpers, whom she must not let out of her sight, but must watch over and control, as the queen bee in a hive sends into the garden thousands of industrious bees, to suck nectar from the flowers and then together under her supervision, to turn it into precious honey, and who must bring her, from the wide realm of the material and spiritual world, the building blocks to finish her palace. » [Meschkowski 1964], p. 94.

So far, so good ... The only trouble is, Cantor's haughty metaphysics has been overburdened with silly ideological (in parlance of philosophers, *reductionists*) platitudes which have dominated his (and to some degree, our) age.

Thus, Cantor has shared with Karl Marx, Charles Darwin, and Sigmund Freud the key idea that the *raison d'être* of the word around us can be fully understood as *a linear progress towards an encompassing, assembly-line like universe* (respectively, of sets, societies, organisms, or human beings), *beginning from scratch* (call it the empty set, primitive society, cell, monkey , or baby), *and driven by a single, blind, algorithmic force* (be it the transfinite enumeration or other transfinite mechanisms which will be discussed later, the class struggle, struggle for biological survival, or libido).

This ideological liability has considerably distorted, in the present author's opinion, the future development of set theory.

§13. Another Apocalyptic Scenario : And What if Mahlo Infinite Totalities Really Exist ?!

> *Nevertheless, it will be argued below that* the necessary use of higher set theory in mathematics of the finite has yet to be established. *Furthermore, a case can be made that* higher set theory is dispensable in scientifically applicable mathematics ... P*ut in other terms :* the actual infinite is not required for the mathematics of the physical world.
>
> [Feferman 1987], p. 153.

In particular, the carefree hastiness of Cantor's passage through newly created by him uncountables,

$$\omega_0, \ \dots \ \omega_1, \dots \ \omega_2, \dots, \textit{ and so on !}$$

without any understanding of the necessity of an independent justification of their existence, has provoked a deep crisis of confidence, and this chasm is still hunting us. Sure, would be he right in his (*CH*) conjecture that $\aleph_1 = \mathfrak{c}$, the *real* existence of (at least some of) new uncountables would be assured ...

In the absence of such a proof, one has to look for other justifications. Thus, recently, some people have become convinced that Harvey Friedman's outstanding result (mentioned above, §2) concerning an independent finitistic confirmation of the existence of Γ_0 (the first impredicative denumerable ordinal), shows «*a commitment to* Γ_0 *to entail a commitment to the* [existence of the first] *uncountable* [ordinal ω_1]» [Smorynski 1982], p. 186. As the eloquent Craig Smorynski writes in this magazine:

«Harvey Friedman, who has the most original mind in logic today, has shown a simple finite form of Kruskal's Theorem, FFF, to be independent of a theory much stronger than formal number theory. Through its unprovability in theories of strength greater than Γ_0 *, i. e., the impredicative nature of any proof of it, FFF illustrates beautifully the fallacy of predicativity: FFF is a concrete assertion about finite objects instantly understandable to any predicativist*

(predicatician?); but any proof of it must appeal to impredicative principles. In short, FFF would have been meaningful to Poincaré, but he would not have been able to prove it, disprove it, or accept any proof of it given to him.» [Smorynski 1982], pp. 182, 187.

Convincing, isn't it ? Yet, with all due respect to Friedman's remarkable discovery, let us make it clear that the result in question demonstrates only that predicativity is too restrictive a concept to fully formalize the notion of «finite and elementary»; of course, this is in itself a remarkable achievement ! However, it is still a far (more precisely, uncountably far) cry from a necessary use of even the first uncountable ordinal ω_1.

Now, if one cannot prove as yet that Cantor's «minimal» uncountable «independently» exists (Criterion 7.5), why not to go far ahead and to find some new, huge and «mathematically useful» infinite totality, whose existence would radically justify all «little guys» behind it, including ω_1 ? This is what is hidden, in particular, in the following radical affirmation :

«Here we give necessary uses of the outer reaches of the abstract set theory in a finite mathematical context ... These outer reaches of abstract set theory actually go significantly beyond the commonly accepted axiomatic framework for mathematics (as formalized by ZFC), and are based on the existence of Mahlo cardinals of finite order ... These are among the so-called small large cardinals ... We believe that the example is sufficiently convincing to open up for the first time the realistic possibility, if not probability, that strong abstract set theory will prove to play an essential role in a variety of more standard finite mathematical contexts. Of course this would open up a foundational crisis of nearly unprecedented magnitude since we seem to have no way of convincing ourselves of the correctness of consistency of such set theoretic principles short of faith in our very uneasy intuition about them.» [Friedman 1986], p. 93.

This dramatic pronouncement made ten years ago has remained ever since neither commented on, nor justified or explained, either by Friedman, or by his followers and admirers. The reason is, of course, the far-fetchedness of the claim that the existence of a Mahlo cardinal is necessary to prove Friedman's combinatorial theorem. Our alternative explanation of Friedman's result will appear elsewhere (cf. also [Feferman 1987]).

Note that the radical justification of the existential reliability of new infinite totalities by their *necessary uses* in some well established mathematical domains can be somewhat weakened : the uses could be just *useful*, or even just serve as an *alternative approach* to otherwise discovered mathematical facts [Gödel 1964], [Jensen 1995]. A limited analogy can be drown with the case of elementary and analytical methods in number theory : although analytical methods are dispensable in some cases, they still sometimes provide even in those cases useful alternative insights.

Still, even this weakening of the independent confirmation requirements did not bring us any closer to a proof of the viability of the uncountable part of Cantor's Paradise ...

§14. Philosophical Postlude.

The author ... has come to believe that the debate between various philosophies of mathematics is a particularisation of the debate between various accounts of the world. [*Thus,*] *parallels may be drawn between Platonism and Catholicism, which are both concerned with what is true; between intuitionism and Protestant presentation of*

Christianity, which are concerned with the behaviour of mathematicians and the morality of individuals; between formalism and atheism, which deny any need for postulating external entities; and between category theory and dialectical materialism.

[Mathias 1977], p. 543.

Just compare the aforementioned Cantorian and post-Cantorian conjurations, §§3, 9, 10, of infinite totalities out of nowhere with the following famous lines :

«*And God said, Let there be light : and there was light. And God saw the light, that it was good : and God divided the light from the darkness. And God called the light Day, and the darkness he called Night.* » (Genesis 1:3-5).

We propose to systematically apply this divine methodological scheme to all new infinit totalities : first, invent it, then see if it is good, and only then «divide and name» them. ` also submit that the *Existential Criterion 7.5* might serve as a good approximation to the divine «see that it is good».

To an agnostic (or even atheistic) reader who might be displeased by so many idealistic, if not outright theological references, we express here all our sincere understanding. The fact is, set theoretic research has really become an open «*theological venture*» ([Kanamory, Magidor 1978], p. 104, if not a «*mystical experience*» [Cohen 1971], p. 15.

More precisely, one introduces new axioms of infinity following her or his «theological» beliefs, and then one pretends (according to Robinson's formalist maxim, §5) to just do some formal mathematics :

«*The adaptation of strong axioms of infinity is thus a theological venture, involving basic questions of belief concerning what is true about the universe. However, one can alternatively construe work in the theory of large cardinals as formal mathematics, that is to say the investigation of those formal implications provable in first-order logic.»* [Kanamory, Magidor 1978], p. 104.

It is enough to browse the faithful reporting, called «*Believing the Axioms*» and written during a helicopter flight by an honestly credulous spectator [Maddy 1988], to fully grasp the frivolity of the cohabitation of all this «theology» with mathematics.

And this leaves us with the grave responsibility to answer the following naive but inescapable question :

14.1. (Not So) Frequently Asked Question. *Lacking necessary faith into ZFC and its extensions, how have one to face the challenge of the advanced ZFC research without resorting to one or another form of a purely negative and «utterly destructive» (§4, [Gödel 1964], p. 257) attitude ?*

Let's face it the second time : we do not share the widespread conviction that axiomatic captures adequately the nature of mathematical infinity. Sure, the problem starts not with *ZF* itself but with Cantor's original set-theoretical vision of which *ZF* is the most faithful and best researched formalization.

The opinion that *ZF* is lacking in some, still to discover and to formalize, basic principles of infinity, both «qualitative» and «quantitative» (whatever these qualifiers might mean in a specific context), is widespread. In fact, the notion is forced on us by the discoveries of Kurt Gödel and Paul Cohen of the independence of *CH* over *ZFC*. (See, e. g., [Jech 1995] on the most influential and advanced methods to feel «infinite axiomatic gaps» in *ZF*). However, what makes our pronouncement about the deficiency of *ZF* different is the fact that it is coupled with the claim of its redundancy. Namely, we believe that some axioms of *ZF* are superfluous, because they do not capture any «infinite reality» : they are just instances of our wishful axiomatic thinking about infinity.

The following technological metaphor might help : imagine one wants to build a flying device, say, helicopter, but only knows how to design cars. *ZF*, which has been conceived to «fly», is, as it where, a very powerful and fast «racing car» (much more powerful and fast than drivers usually need [Barwise 1975], [Mathias 1992]). And even after being outfitted with all those brand-new and extra-powerful «engines» (very large cardinal axioms), the overburdened *ZF* still cannot «fly». Just adding two «wings» (principles still to discover) would not help : *ZF* would still remain a heavy «racing car with two wings».

The trouble is hidden exactly where our *ZF* pride resides : in the powerful built-in iterative mechanism of set generation. Here is the crucial *meaningful distinction* (we borrow this expression from several very clever methodological formulas of [Bishop 1985]) : **(i)** one thing is our ability to build recursively some *internal* (or *inside*) mathematical objects in a given axiomatic framework : and here one can be justly proud of the recursive power of the weakest nontrivial subsystem of *ZF*, the *Kripke-Platek* axiomatic system, *KP* [Barwise 1975], [Mansfield, Weitkamp 1985], [Mathias 1992]; **(ii)** quite another thing is, however, to recursively re-create, in *ZFC*, an *external* (or *outside*) set-theoretical universe [Parsons 1977], [Shoenfield 1977] : a modern, but still absolutely illusory attempt to outperform the builders of the Babylonian tower (Genesis 11:4).

It is obvious that *ZF* has gained in its creative power on the expense of its descriptive power; or, as Jon Barwise has put it :

> «*The most obvious advantage of the axiomatic method is lost since ZF has so few recognizable models in which to interpret its theorems*.» [Barwise 1975], p. 8.

In comparison, the axioms of Euclidian geometry are weak in creative power, but extremely strong descriptively. In fact, *ZF* is so powerful that it permits a «user» to create his own infinite totalities, which have nothing to do with real infinity :

14.2. Thesis : ZFC Is an Interactive Programming Language. *(Let Us Call It Tentatively the ZFC-Calculus of Imaginary Infinite-Like Constructions.)* ***(1)*** The advanced *ZFC* set theory is a sophisticated and beautiful structure, which is successfully mimicking some aspects of mathematical infinity, but whose main thrust lies with the providing to advanced «users» sophisticated options of creation of, and manipulation with, artificial infinite totalities (similar to computer graphic images).

(2) The totalities in question are, in fact, pure mathematical notations not related to any «reality» outside the tight structure of their definitions and relationships.

(3) The mathematical beauty of the constructions becomes, thus, a natural outcome of the fascinating interplay between the tight intrinsic recursive structure of the «programming language» *ZFC* (whose «*axioms force themselves upon us as being true*» [Gödel 1964], p. 268) and the wealth of mathematical constructions freely (as some contend, arbitrary) borrowed by set-theorists from the treasury of our science.

In this interpretation finds its proper place the puzzling and «disconcerting» predominance in modern set theory of results on *ZF* consistency and independency:

> «*When modern set theory is applied to conventional mathematical problems, it has a disconcerting tendency to produce independence results rather than theorems in the usual sense. The resulting preoccupation with «consistency» rather than «truth» may be felt to give the subject an air of unreality.*» [Shelah 1992], p. 197.

We claim that those consistency results are just the instances of successful pr verifications.

We understand that our interpretation of *ZF* brings with it the responsibility, bo scientific and moral, to propose a dignified «ontological exit» for *ZF* related research, which has produced, over almost one hundred years, a wealth of beautiful and extremely difficult results and theories. What «mean» those mathematical facts if, as we are arguing, the infinite totalities they describe are «preprogrammed» and exist only «on paper» ? One possible explanation was hinted at by Stephen Simpson :

> «*Only the first few levels of the cumulative hierarchy bear any resemblance to external reality. The rest are a huge extrapolation based on a crude model of abstract thought processes. Gödel himself comes close to admitting as much.*» [Simpson 1988], p. 362.

In other words, *ZF* related research could be viewed as a sophisticated and protracted exercise in perfecting our skills of inductive and iterative imagination. Other interpretations are possible as well, some of them leading to as yet unknown applications to future philosophy of reasoning and computing.

14.3. A Few Farewell Confidential Quips about Mathematical Infinity. ***(1)*** *When invited next time on a transfinite trip, look closely at remarkable results concerning the universe of countable ordinals. We are only beginning to penetrate the fringes of the immense wilderness of the denumerable. There is no doubt that (paraphrazing [Friedman 1986], p. 92) the «outer reaches» of the universe of denumerables will become an important subject of future research in set theory, theory of recursive functions, and mathematical logic [Wainer 1989], [Aczel, Simmons, Wainer 1993].*

(2) *There exists probably nothing well-ordered beyond the (the author conjectures,* proper*) class of denumerable ordinals, in particular, «because» the continuum cannot be well-ordered.*

Bidding good-bye to our good reader, we leave to him the privilege to decide, to what category belongs the author of the present study : is he an *inventor, discoverer, detractor, defender, master, victim, user, or spectator* of mathematical infinity ?

And who are you, my reader ?

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