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Some non-classical logics seen from a variety of perspectives*

Nuel Belnap

Logicians have worked with so many different logical systems that it is not possible even to estimate the number. Of these, many are best seen as extensions of classical logic, including both those of interest to mathematics and those of interest to philosophy and computer science. (Henceforth I will use the term "intelligent systems theory" for the common ground of philosophical logic and that part of computer science that concerns itself with activities plausibly taken to embody intelligence in some degree.) On the mathematical side are, of course, higher order logics, set theories, systems of arithmetic, and so forth. On the intelligentsystems side, useful extensions of classical logic include modal logic, deontic logic, epistemic logic, tense logic, indexical logic, and so forth.

This essay, however, does not deal with those logics; instead, it concerns itself with non-classical logics of interest to intelligent－systems theory. There are doubtless hundreds of non-classical logics, and I consider only a few[①]. Chiefly I will talk about "relevance" logics and some close cousins. Sometimes these are called "substructural logics" for reasons that will emerge. Concerning these logics, I wish to emphasize the very large number of approaches to them that have proved enlightening and useful.

1 Outline

1. Proof theory. I go over two basic forms: (1) the Fitch "method of subordinate proofs" form of Gentzen's and Jaskowski's natural deduction and (2) the "display logic" form of Gentzen's "sequenzen kalkül." Both are elegant tools for investigating substructural logics of all kinds. I give very brief examples of each, emphasizing the light they can shed on some non-classical logics. The natural deduction framework for relevance logic gives a very clear way in which to appreciate the intuitive idea of a premiss being relevant to a conclusion insofar as it can be *used* in obtaining that conclusion. The display-logic framework gives a clear overview of what is going on in so-called "substructural logics"

*收稿日期：2003－03－09
作者简介：Nuel Belnap，美国人，匹兹堡大学哲学系教授。
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in general, of which many relevance logics are a species.

2. Relational and operational semantics. I discuss three forms, all of which can be seen as generalizations and deepenings of the Kripke approach to modelling modal logics: Urquhart' s operational semantics, Fine' s operationplus-binary-relation semantics, and the Routley-Meyer three-termedrelation semantics.

3. The theory of theories. The structures mentioned above are abstract. They take concrete form in the theory of theories. This discussion explains the motivation for the move from so-called "worlds" in the semantics of modal logic to "theories" in connection with a wide variety of non-classical logics.

4. Non-classical algebras. Concrete non-classical algebras emerge directly from the theory of theories: The algebraic operations and relations are operations and relations on theories. Also given mention are the ideas of formulas as types and of types as formulas, as well as the category-theory approach.

5. Finally, I skip to an entirely different topic: Gupta' definitely non-classical theory of circular definitions.

2 Natural deduction

I begin with proof theory, and in particular with the *natural deduction* approach to relevance logic. This approach is especially well-suited for creating an appreciation of the underlying concept of "relevance" that relevance logic is trying to clarify. Here I give just a hint. The idea is to think of conditional $A \rightarrow B$ of relevance logic as meaning something like "that A *relevantly* implies that B , so that for $A \rightarrow B$ to be true, there is to be a relevance of A to B , a way, that is, in which B depends on A . The natural deduction system **R** of relevance logic clarifies this idea of "dependence" by not allowing the derivation of $A \rightarrow B$ unless in passing from A to B one can actually find a use for A. The technical device, first suggested by Anderson, was to modify Fitch' s "method of subordinate proofs" by marking each hypothesis or assumption with an index. Then the system detects whether or not a hypothesis has been used in getting to a certain step by looking to see whether or not the index appears on that step. If it does, that is a signal that the hypothesis has been used; but if the index is missing, that indicates irrelevance of that hypothesis to that step. The natural-deduction rule of "conditional proof" standardly permits deduction of $A \rightarrow B$ from a Fitch subproof with hypothesis A and last step B . For relevance logic, this rule is modified to take account of the "relevance indices" as follows: Conditional proof cannot be used unless the index of the hypothesis actually appears on the last step of the subproof. I give two brief illustrations, with the hope that the reader can fill in the details. ①

①The idea is presented in most detail in Anderson and Belnap 1975, but also in Anderson, Belnap and Dunn 1992 and publications there cited. The calculus **R** that we mention from time to time dates back to early work by Orlov 1928, Moh 1950, and Church 1951.

1	$A \to B_{\{a\}}$	hypothesis
2	$B \to C_{\{b\}}$	hypothesis
3	$A_{\{c\}}$	hypothesis
4	$B_{\{a,c\}}$	Modus ponens, 1, 3
5	$C_{\{a,b,c\}}$	Modus ponens, 2, 4
6	$A \to C_{\{a,b\}}$	Rel. Cond. Proof, 3–5
7	$(B \to C) \to (A \to C)_{\{a\}}$	Rel. Cond. Proof, 2–6
8	$(A \to B) \to ((B \to C) \to (A \to C))_{\varnothing}$	Rel. Cond. Proof, 1–7

This is the principle called "suffixing". You can see by inspection how the subscripted indices keep track of relevance. Next I exhibit a *failed* try at a proof, with relevance indices, of one of the "paradoxes of implication":

1	$A_{\{a\}}$	hypothesis
2	$B_{\{b\}}$	hypothesis
3	$A_{\{?\}}$	?
4	$A \to B_{\{a\}}$	Rel. Cond. Proof, 2–3
5	$A \to (B \to A\ \varnothing$	Rel. Cond. Proof, 1–4

The failure comes to this. For the use of relevant conditional proof at line 4 to be correct, the question mark at line 3 must contain a "*b*". But although you can certainly copy the "$A_{\{a\}}$" at line 1, that won't provide the *b* needed as a sign that you have used line 2 in obtaining line 3. Indeed, there is no way that line 2 can be used in obtaining *A* on line 4, so that this proof-sketch of a famous "paradox" cannot be completed.

Let me repeat that although the natural deduction formulations of **R** and related non-classical logics is of mathematical interest in its own right, the chief benefit is that they provide insight into the underlying concepts. One can *see* the idea of relevance at work in proofs.

3 Gentzen's calculus

A second form of proof theory that has proved instructive in the analysis of nonclassical logics is due to Gentzen 1969. I start with a version of Gentzen's prooftheoretical analysis of the logical connectives in terms of the roles they play in inference, either as premises or as conclusions. ① At the very bottom of Gentzen's analysis sits a *consecution* having the form

$$X \vdash Z,$$

where X is a finite (perhaps empty) sequence of sentences, and where Z is either a single sentence or empty. ② X is called the antecedent, and Z the consequent. The empty symbol is interpreted as

①For expository reasons I am going to take liberties with the exact characterization of Gentzen's system.

②Gentzen showed that when Z is permitted to consist of multiple sentences, and when corresponding adjustments are made to the various rules, the result is a formulation of classical logic. In this important case, the comma on the left represents conjunction, while the comma on the rightrepresents disjunction. When, as here, Z must be either empty of a single sentence, the result is intuitionist logic. I myself think of intuitionist logic as "nearly" classical since, like classical logic, it cannot tolerate contradictions.

"Truth" when an antecedent, and as "Falsehood" when a consequent.

The central relation-symbol is called the turnstile. This turnstile statement or consecution is intended to say that Y is a logical consequence of a conjunction of the members of X. Gentzen's idea was to characterize the logical connectives in terms of consecutions and their relations. His formal system began with a single axiom, the axiom of identity,

Identity: $A \vdash A$,

which can and should be restricted to atomic sentences. To this thin beginning Gentzen added two families of rules. One family, the structural rules, characterized the very idea of logical consequence itself. There were for Gentzen three "proper" structural rules, and then one more. The first three——the proper ones——were Weakening ($K \vdash$), which permitted adding extra sentences as premisses; Permutation ($C \vdash$), which allowed one to interchange adjacent sentences; and Contraction ($W \vdash$), which permitted two adjacent occurrences of the same sentence to be contracted to a single occurrence. The fourth structural rule, cut, was a kind of generalized transitivity that permitted "cutting out" (or better, "replacing") a sentence when the sentence stood as a sort of "middle term," being implied by certain sentences and itself implying (with the help of other sentences) some sentence. Let us use X, Y, and Z for arbitrary (perhaps empty) sequences of sentences. Then the structural rules of Gentzen are as follows:

$K\vdash$: From $X \vdash B$ to infer $X, A \vdash B$. Also, when negation is present,

one needs in addition $\vdash K$: From $X \vdash$ to infer $X \vdash Z$

$C \vdash$: From $X, A_1, A_2, Y \vdash B$ to infer $X, A_2, A_1, Y \vdash B$

$W \vdash$: From $X, A, A, Y \vdash B$ to infer $X, A, Y \vdash B$

Cut: From $X_1 \vdash A$ and $X_2, A, X_3 \vdash Z$ to infer $X_2, X_1, X_3 \vdash Z$.

This fourth structural rule, cut, occupies a very special place. I return to it after offering a sample of Gentzen's rules for the various logical connectives. I illustrate with the following examples: a binary connective for each of implication (or the conditional) and conjunction; a unary connective for negation; and a 0-ary connective for truth. For each there is a rule treating the appearance of the connective in the conclusion and a rule that takes care of its appearance in a premiss.

$\vdash \rightarrow$: From $X, A \vdash B$ to infer $X \vdash A \rightarrow B$.

$\rightarrow \vdash$: From $X_1 \vdash A$ and $X_2, B \vdash Z$ to infer $X_2, A \rightarrow B, X_1 \vdash Z$.

$\vdash \&$: From $X_1 \vdash A_1$ and $X_2 \vdash A_2$ to infer $X_1, X_2 \vdash A_1 \& A_2$.

$\& \vdash$: From $X, A_1, A_2, Y \vdash Z$ to infer $X, A_1 \& A_2, Y \vdash Z$.

$\sim \vdash$: From $X \vdash A$ to infer $X, \sim A \vdash$.

$\vdash \sim$: From $X, A \vdash$ to infer $X \vdash \sim A$.

$t \vdash$: From $X \vdash Z$ to infer $X, t \vdash Z$.

$\vdash t$: To infer $\vdash t$.

That concludes the presentation of (a slight adaptation of) Gentzen's system: one axiom (Identity), three proper structural rules ($K \vdash$, $C \vdash$, and $W \vdash$), rules for implication, conjunction, negation, and Truth as premisses and conclusions; and, finally cut.

Gentzen's profound result was that this rule, cut, is provably redundant. The result is calle "cut elimination." A principle consequence of cut elimination, which Curry justly calls one of th most important results in modern logic, give a "normal form" to proofs of statements of logical conse quence (consecutions): As one proceeds down the proof, no sentence ever disappears. In othe words, if one follows the proof upwards, no sentence ever pops into play "from nowhere," as it were. Therefore, in attempting to prove a certain consecution, it always suffices to consider proofs that involve only subformulas of the candidate in question. The proof of Gentzen' s cut-elimination theory is in one sense not so very difficult, but it is always a delicate matter, since the redundancy of cut depends on exactly how the various rules are formulated, and one must routinely examine a large number of cases.

4 Display logic

So far we have said nothing at all about non-classical logics. Display logic is a special case of a Gentzen systematization of statements of logical consequence that permits us to throw a great deal of light on a large variety of such logics.

Display logic puts severe restrictions on the grammar of these statements. By so doing it tremendously increases our control over the structural rules, a feature that is essential to treating non—classical logics, and at the same time the restrictions reduce proof of cut elimination to the relatively easy verification of a few properties of the rules. The chief insistence of display logic is that everything structural be made explicit:

> Gentzen's comma, which, as Gentzen says, is to be interpreted as a conjunction when on the left [disjunction when on the right; see note 3], is unlike conjunction [disjunction] in its grammar, since conjunction [disjunction] is always binary, whereas the comma is of no fixed polyadicity. In display logic, the element of structure which is to be interpreted conjunction-like on the left [disjunction-like on the right] must be binary; we call it a structure connective. To remind us of binariness, we will for this purpose replace Gentzen' s comma by "$\circ$," as in $(X \circ Y)$. This alone gives rise to much more delicate control over structural rules. For one thing, it permits us to envisage more than one structure connective with conjunction—like features. As illustration we will assume two, one "extensional" and one "intensional," using e and i to mark the difference. In each case we shall be careful to use parentheses in order to force binariness. Thus, when X and Y are structures, so is each of $(X \circ^e Y)$ and $(X \circ^i Y)$.
>
> Display logic adds a unary element of structure. Negation is based on this structure connective in just the way that conjunction is based on the comma. Again we shall have two, one related to $\circ^e$, and the other to $\circ^i$. To keep track of which is which, for this purpose we will use a prefixed symbol #, annotated with one of e or i, so that if X is a structure, so are $\#^e X$ and $\#^i X$.
>
> Display logic adds a 0-ary structure connective. Gentzen uses "the empty symbol" for this purpose, which is conceptually confusing. The empty symbol has a definite structural meaning, and it is much easier to be clear about this if one uses in its place a symbol to which to attach this meaning. We will use I for this purpose, and again we duplicate by using markings e or i, so that each of I^e and I^i are structures.

The triple $\{\circ^e, \#^e, I^e\}$ forms what we call a "family" (the e-family or extensional family) as does $\{\circ^i, \#^i, I^i\}$ (the i-family or intensional family).

The proof-theory for display logic comes in pieces. Much of it is identical for the two families, so that often we shall indicate generality by omitting e and i. From now on we use W, X, Y, and Z as variables ranging over structures, which are built inductively from sentences by means of the three structure-connectives from either family. The fundamental form of a consecution is now $X \vdash Y$. Given such a consecution, each substructure is an antecedent part [consequent part] if it can be reached in X through an even [odd] number of occurrences of # or reached in Y through an odd [even] number of occurrences of #.

Axioms $A \vdash A$, for atomic A.

The one-step **display equivalences** as follows: $(X \circ Y) \vdash Z$ is display equivalent to $X \vdash \# Y \circ Z$; $X \vdash Y \circ Z$ is display-equivalent to each of $X \circ \# Y \vdash Z$ and $X \vdash Z \circ Y$; and finally, $X \vdash Y$ is display-equivalent to each of $\# Y \vdash \# X$ and $\#\# X \vdash Y$.

Consecutions are display equivalent iff they are mutually inferable by a series of applications of the one-step display equivalences. The point of these so-called display equivalences is exactly this: Take a consecution $X \vdash Y$ and any substructure Z of it. If Z is an antecedent part [consequent part], then $X \vdash Y$ is display-equivalent to some consecution having the form $Z \vdash Y_1$ in which Z is the entire antecedent [having the form $X_1 \vdash Z$ in which Z is the entire consequent]. That is what is called the display property: Any antecedent part [consequent part] can be displayed in an equivalent consecution as the entire antecedent [consequent]. This simplifies every other part of the proof theory.

The connective rules. Because of the display property, the connective rules can be severely restricted in form. Each connective is duplicated in each family and is to be thought of as marked with e or i, but we suppress this in the interest of reducing complexity of exposition.

$\vdash\rightarrow$: From $(X \circ A) \vdash B$ to infer $X \vdash A \rightarrow B$

$\rightarrow\vdash$: From $X \vdash A$ and $B \vdash Z$ to infer $A \rightarrow B \vdash (\# X \circ Z)$

$\vdash \&$: From $X_1 \vdash A_1$ and $X_2 \vdash A_2$ to infer $(X_1 \circ X_2) \vdash A_1 \& A_2$

$\& \vdash$: From $(A_1 \circ A_2) \vdash Z$ to infer $A_1 \& A_2 \vdash Z$

$\sim\vdash$: From $X \vdash A$ to infer $\sim A \vdash \# X$

$\vdash\sim$: From $A \vdash Z$ to infer $\# Z \vdash \sim A$.

$t\vdash$: From $I \vdash Z$ to infer $t \vdash Z$

$\vdash t$: to infer $I \vdash t$

Cut. The cut rule has the form of a simple transitivity:

$X \vdash A$ and $A \vdash Z$ to infer $X \vdash Z$

Cut-elimination theorem. Already the cut elimination theorem (i. e., the redundancy of the cut rule) can be proved without any structural rules whatsoever, and regardless of which families are present or absent.

Structural rules. Furthermore, the proof of cut elimination is so straightforward that it can be seen by inspection that it will go through for any of a vast array of structural rules. Here are a few examples. (There are literally dozens of additional structural rules that can play a role in characterizing non-classical logics.) The chief point here illustrated is that different families can be characterized by different structural rules, without in any way disturbing cut elimination. Again I will reduce complexity by stating the rules and their names without using e and i.

I: From $X \vdash Y$ to infer $I \circ X \vdash Y$.

I−: From $I \circ X \vdash Y$ to infer $X \vdash Y$.

B: From $X \circ (Y \circ Z) \vdash W$ to infer $(X \circ Y) \circ Z \vdash W$.

B′: From Y ∘ (X ∘ Z) ⊢ to infer (X ∘ Y) ∘ Z ⊢ W.

W: From (X ∘ Y) ∘ Y) ⊢ W to infer X ∘ Y ⊢ W.

C: From (X ∘ Y) ∘ Z ⊢ W to infer (X ∘ Z) ∘ Y ⊢ W.

K: From X ⊢ W to infer X ∘ Y ⊢ W.

We are immediately led to the very heart of "substructural logic" in the sense of Schröder-Heister and Došen 1993.① A logic is said to be substructural if it can be defined in terms of any number of families, each of which is characterized by including certain structural rules (and omitting others). Among substructural logics are a large sample of non-classical logics. In order to convey the basic idea without too much complexity, I shall assume that the logical vocabulary is restricted by omitting any form of negation.

For two-valued logic, assume all structural rules above.

For the calculus R of relevant implication, for the intensional structureconnectives (marked with i), assume all but K.

For the calculus T of "ticket entailment" (Anderson and Belnap 1975, 6), assume just I, I—, B, B′, and W.

The point is this. For all of these logics, and for many more, display logic permits one easily to find a transparent cut-free formulation. Occasionally such a formulation may lead to a decision procedure; for example, this happens for Gentzen's original cut-free formulation of intuitionist logic, and for Kripke's cutfree formulation of the pure implicational fragment of the relevance logic called R. With or without a decision procedure, however, always one can rely on the existence of a cut-free formulation to guarantee that the logic in question makes sense, and is at least to some degree more than an idle collection of axioms: The meaning of its connectives will uniformly be found to fall out of their respective roles in inference.

As I made plain above, in order to get started with display logic,② one needs structure connectives, and in particular one needs a "family" of three structure connectives: one binary, one unary, and one zero-place. There are two further things to emphasize.

In the first place, one may need more than one family in order to cast some particular logic as a display logic. Relevance logics, for example, typically require two families: one "intensional" family for the distinctively relevant connectives of the logic, and another "extensional" family for its two-valued connectives. In this case, one marks all intensional formula-connectives and structure-connectives with i, and all the two-valued ones with e. In addition, one may fashion combined relevance logics, featuring, perhaps in addition to the two-valued connectives, connectives from two or more distinct relevance families. In this case, one would need distinct markings for the various relevance logics. The beauty of display logic is that it just doesn't care about this. As long as the connective rules and the structural rules satisfy easily-verified conditions, as indicated above, exactly the same proof of cut-elimination will deliver exactly the same satisfying results.

The second thing is that one may include a variety of other structure connectives in addition to those that fall into "families" as described above. Of particular interest is the addition of specific mo-

①For a powerful unified treatment of substructural logics, see Restall 2000.

②There are many forms of display logic; it is only one form that starts out in exactly the way described, that of Belnap 1982. Similar remarks, however, govern other forms as well.

dal structural connectives by Wansing 1998, other structural connectives added by Restall 2000 for the "split negations" of Dunn 1991 and Dunn 1993, and still others besides. ① All of these can be added either to some form of classical logic, or to a relevance logic or to any other non-classical logic such as linear logic or intuitionist logic; the display-logic formulation keeps everything in order, as guaranteed by the cut-elimination theorem. That is how display logic helps give us insight into non-classical logics.

5 Relational and operational semantics

A vast array of classical and non-classical logics have been illuminated semantically by relativizing the truth-predicate in various ways. When the truth of A is to be relativized to an entity a in any of the ways that we shall be considering, for maximum brevity I will write

A_a,

trusting the reader to read this as

A is true at (or in or with respect to) a.

Of the various semantics based on relativizing the truth-predicate in unexpected ways, doubtless the best-known are those of Kripke for various classical modal logics. Let us begin with these, and see what can usefully be changed in order to deal with non-classical logics akin to relevance logics. In the first place, Kripke relativizes the truth of A to worlds, and on the set of these worlds there is imposed a binary relation, R, to be considered as a relation of relative possibility. Then the semantic clause for e. g. the modal operator of necessity, written $\Box$ A, says that $\Box$ A is true at a world a just in case A is true in every world b that is possible relative to a:

$\Box A_a$ iff $\forall$ b (if Rab then A_b).

In adapting Kripke's semantics to relevance logics and other similar non-classical logics, there are two big changes to be made. First of all, as I have said, Kripke relativizes truth to "worlds," so that the semantic primitive is something like "A is true at world a." A more general notion is needed for nonclassical logics. As Urquhart 1992 points out, worlds have two features neither of which is always wanted: consistency (the property that not both A and ~A can be true at one and the same world) and completeness (the property that at least one of A and ~A must be true at each world). When one gives up only completeness and keeps consistency, Urquhart suggests calling the truth-relatum an "evidential situation" rather than a "world," and he points out the connection of this idea to intuitionism. When in addition one gives up consistency, so that one has neither completeness nor consistency, Urquhart suggests that what we have as a relatum is a "piece of information." As an alternative to "piece of information," one may with Meyer think of a "theory," which may of course be neither complete nor consistent. I shall come back to theories.

That is the first departure from the Kripke semantics for modal logics. Second, as I noted above, Kripke gives his various semantics in terms of a binary relation of "worlds" to be thought of as a relation of "relative possibility." Giving useful semantics for non-classical logics in the relevance family requires more than just moving from "worlds" to "pieces of information" or "theories." In addition, it

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①Modal logics, including modal non-classical logics, can be treated within the framework described above, but it is much better to formulate their display logic by adding the special modal structural elements of Wansing 1998.

will no longer do to have a Kripke binary relation be the only element of semantic structure. There are a number of options here. Three of them are spelled out in Anderson et al. 1992 (see pp. 161 – 162 for some history).

5.1 Operational semantics

One option is to do the semantic work with a binary operation $\circ^r$ (with "r" for relevance). This has been pursued e. g. by Urquhart 1992. The idea is that $a \circ^r b$ is the piece of information that results from combining the two pieces of information a and b. Then the clauses for relevant implication look something like this:

$(A \rightarrow B)_a$ iff $\forall$ b [if A_b then $B_{(a \circ^r b)}$].

In order to fit a given logic, one needs of course to put some constraints on the operation. For example, the operational semantics fits the calculus R of relevance logic if the operation is characterized as a semi-lattice with zero: 0, which represents "logic," is a left identity, and the operation is commutative, associative, and idempotent (i. e., $(w \circ^r w) = w$).

It is not so easy to adapt the operational approach to the presence of all the usual connectives. If however we restrict the logical vocabulary by omitting extensional disjunction and every negation, one may characterize various logics by (1) basing the semantics on a partial—order relation instead of identity and (2) postulating a selection from among principles similar to the following.

I: $0 \circ a \leqslant a$

W: $(a \circ b) \circ b \leqslant a \circ b$

B: $a \circ (b \circ c) \leqslant (a \circ b) \circ c$

There are at least two ways of modifying the operational semantics in order to permit simultaneous treatment of disjunction and negation. In the interest of simplicity, however, I shall omit further mention of negation.

5.2 Operation plus binary relation

In the version of Fine 1992, to the binary operation (and 0 for "logic") one adds a binary relation that amounts to something like a "subtheory" or "sub-piece-ofinformation" relation, and one adds a special classification of the entities to which the operation applies to pick out those that respect disjunction in the proper way (if a disjunction is true at once of these, then so is at least one disjunct). One also adds something for negation, but I am not going into that, continuing to restrict attention to the positive vocabulary.

5.3 Three-termed relational semantics

The Routley-Meyer three-termed-relation approach replaces the operation and the binary relation entirely by a three-termed relation R (see especially Mares and Meyer 2001 or Restall 2000). The truth clauses for intensional implication and intensional conjunction, and for extensional conjunction and disjunction, are these:

$(A \rightarrow^i B)_a$ iff $\forall b\ \forall c$ [if (R abc and A_b) then B_c].

$(A \,\&^{i}\, B)_c$ iff $\exists a\ \exists b$ [A_a and B_b and Rabc].

$(A \,\&^{e}\, B)_a$ iff A_a and B_b.

$(A \vee^{e} B)_a$ iff A_a or B_b.

Abstract properties of R look like the following samples. (To make them readable, we use the following abbreviations: Rabcd $\leftrightarrow_{df} \exists x$ (Rabx and Rxcd), and Ra (bc) d $\leftrightarrow_{df} \exists y$ [Rayd and Rbcy.)

I: R0aa

B: if Rabcd then Ra (bc) d.

C: if Rabcd then Racbd

W: if Rabc then Rabbc

These correspond, in their own way, to various important logical principles.

6 The theory of theories

When one is doing formal semantics for classical modal logics, one speaks semantically, as we have said, of worlds and of whether or not a sentence is true at a world. In the course of completeness proofs for these modal logics, one invariably represents a "world" by a set of sentences, a theory of, as we might say. In such a case one does not think that a possible world "really is" a theory; rather, the theory plays an important mathematical role in modelling the modal logic in question. The situation for non-classical logics is partly similar, but is different in the following important way: In the modelling of e. g. relevance logics, one does not want "worlds" at all. Instead, one really and truly wants the theories themselves. That is above all because while for modal logics the theory of a "world" should doubtless be complete and consistent (we meant to be speaking only of possible worlds), we wish the target of relevance-like logics to be theories, precisely because theories can be incomplete and theories can be inconsistent.

Therefore, it is good to think in concrete terms when one is building up a model for a non-classical relevance logic L. It is natural and meaningful to take the relata for truth to be theories. For a wide variety of logics L, it is accordingly good to define a L-theory as a set of sentences a that is closed under both extensional conjunction (if $A \in a$ and $B \in a$, then $(A \,\&^{e}\, B) \in a$) and also under L-consequence in the following sense: if $A \in a$ and $(A \rightarrow^{i} B) \in L$ then $B \in a$. It is, as we said, important for the very idea of relevance logics that L-theories need be neitherconsistent nor complete, and we may add that an L-theory a need not contain all the theorems of L. That is one way in which the relevance-logic idea of "theory" corresponds to what is needed for real-world applications, since of course our theories are hardly ever complete, and they can be inconsistent as well. This point is well-underlined by the "theories" approach to relevance logics, and is perhaps one of their chief conceptual (as opposed to purely mathematical) benefits.

Further, although at various stages one must consider all L-theories, in the final mathematical modelling one concentrates on certain prime L-theories, those that respect disjunction: If $(A \vee B) \in a$ then either $A \in a$ or $B \in a$. Let 0 be a particular prime theory that extends L. We now focus on the set of prime 0-theories (they will also be L-theories). One may then canonically recover the three-term relation R on the prime 0-theories as follows:

Rabc iff $\forall A \forall B$ [if $(A \rightarrow B \in a$ and $A \in b)$ then $B \in c$].

There are numerous details, but they all fit together beautifully into a harmonious whole. Typical properties of the three-termed R needed in order to establish a fit with e. g. the calculus R of relevance logic are those that we listed above as governing an abstract three-termed relation R.

7 Non-classical algebras

A natural way to obtain algebras corresponding to ideas of relevance is by considering operations on 0-theories. Perhaps most illuminating is the operation $\circ^r$ as applied to (not necessarily prime) 0-theories, defined by

$$(a \circ^r b) =_{df} \{A: \exists B\, [A \to B \in a \text{ and } B \in b\}.$$

In connection with the partial ordering $a \leqslant b$ on 0-theories delivered by the subset relation $a \subseteq b$ (equivalently, by R0ab), many properties of various relevance logics can be stated in familiar or nearly familiar forms. For instance, when 0 is taken as a prime extension of the calculus R of relevant implication and o^r is defined as above, o^r turns out to be associative, commutative, and semi-idempotent (i. e., $(ao^ra) \leqslant a$, but not the reverse). Of even more interest is the delicate fashion in which more general algebraic properties correspond to various logical principles.

For example, the principle

$$bo^r\,(ao^rc) \leqslant (ao^rb)\, o^rc$$

corresponds to the logical principle of "suffixing," i. e.

$$(A \to B) \to ((B \to C) \to (A \to C)),$$

of which I gave a natural deduction proof above. Observe that the principle is not an identity such as one might expect. It is a one-way "reduction principle" such as one encounters in combinatory logic. In fact, various systems of non-classical logic correspond in a one-one fashion with various combinatory logics that differ one from the other only in which combinators they feature. For example, in a combinatory logic corresponding to a logic with suffixing, you would need the combinator that Curry calls "B′", and which is characterized by the principle that

$$((B'o^ra)\, o^rb)\, o^rc \leqslant bo^r\,(ao^rc)$$

Following up this line soon makes contact with various calculuses of lambda conversion and with the thought that a non—classical implication $A \to B$ can be understood as representing a function (or set of functions) of a very particular kind that carries the antecedent A into the consequent B. In the case of the relevant implication of R, the idea is that the function must "really depend on its arguments." (The syntactic indication of this in the case of the lambda calculus is that for γx, t to be well-formed, the variable x must actually occur free in the term t, as is required by Church 1941 in the original presentation of the calculus of lambda conversion.) Many of these connections are explored in Anderson et al. 1992, but for up-to-date treatments it is better to consult Restall 2000 and Mares and Meyer 2001. The latter emphasizes that there is a two way street between formulas and types of functions. (1) There is the formulas-as-types idea due essentially to Curry and Howard, where a formula represents a set of higher—level functions of a certain type, a type that can be represented by a combinator such as B′. There is the types-as-formulas idea, due to Meyer, where a combinator such as B′ can be represented as a set of formulas, and in particular as a theory. These connections via non—classical logics are very beautiful.

Going back briefly to the topic of non-classical algebras, it will come as no surprise that such concrete examples as the algebra of theories can be generalized in the direction of abstract algebras, and that in that framework there is much to learn. The algebraic approach to relevance logic, pioneered by Dunn, is very well covered in Restall 2000. The ideas of residuation and of Galois connections come to be important. In that place you will also find a presentation of algebraic non—classical logic in the context of category theory.

Of special interest is the four-element DeMorgan algebra of § 81 of Anderson et al. 1992 whose "told" values correspond to the following four states of a body of information thought of as stored in a computer: (1) told True (but not told False), (2) told False (but not told True), (3) told neither True nor False, and (4) told both True and False. It is argued in that section that automatic reasoning about and-or-not propositions that is based on these four values is well-codified by the theory of "tautological entailments" of § 15 of Anderson and Belnap 1975.

8 Circular definitions

I want to finish by mentioning an entirely different area of non—classical research: the theory of circular definitions. Classical logicians such as Frege and Lesniewski and Tarski considered that to find out that a proposed definition was circular implied that it was entirely wrong-headed, useless, illogical, and a sign of unreason. Only with the recent work of Gupta, reported in part in Gupta and Belnap 1993, have we had available a theory—a non-classical theory—of how circular definitions can be useful, and even sometimes essential. ① A central example is truth. Tarski suggested taking a sentence such as

'Snow is white' is true iff snow is white

as a partial definition of truth. If one applies this in a straightforward way throughout our language, one must consider examples such as the following:

' 'Snow is white' is true' is true iff 'Snow is white' is true.

The point is that the second example, if taken as a definition, is circular since the defined predicate "is true" occurs as part of the partial definiens. Such circularity can be harmless, as in the given example, or vicious, as in a "paradox" such as the familiar "Paradox of the Liar":

This very sentence is not true.

What Gupta does is to consider not just the example of truth, but the idea of an arbitrary circular definition. He argues for the importance of circular definitions (and for our understanding of truth being circular), and he offers a clean and persuasive theory of these definitions, and of the phenomena to which they give rise. He shows how to extract full meaning from circular definitions, and how to separate off the useful parts from the useless parts. Gupta gives a deep and satisfying account of the paradoxes, an account that recognizes them for what they are, and does not try to pretend that they do not really arise nor to legislate them out of existence. This work of Gupta's is one of the best and most constructive pieces of non-classical logic of which I know.

① Gupta takes the underlying pre-definitional language to be classical. The non-classical part of his logic lies in the logic of definitions, which he treats in an entirely fresh and non-classical way.

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"Truth" when an antecedent, and as "Falsehood" when a consequent.

The central relation-symbol is called the turnstile. This turnstile statement or consecution is intended to say that Y is a logical consequence of a conjunction of the members of X. Gentzen' s idea was to characterize the logical connectives in terms of consecutions and their relations. His formal system began with a single axiom, the axiom of identity,

Identity: $A \vdash A$,

which can and should be restricted to atomic sentences. To this thin beginning Gentzen added two families of rules. One family, the structural rules, characterized the very idea of logical consequence itself. There were for Gentzen three "proper" structural rules, and then one more. The first three——the proper ones——were Weakening ($K \vdash$), which permitted adding extra sentences as premisses; Permutation ($C \vdash$), which allowed one to interchange adjacent sentences; and Contraction ($W \vdash$), which permitted two adjacent occurrences of the same sentence to be contracted to a single occurrence. The fourth structural rule, cut, was a kind of generalized transitivity that permitted "cutting out" (or better, "replacing") a sentence when the sentence stood as a sort of "middle term," being implied by certain sentences and itself implying (with the help of other sentences) some sentence. Let us use X , Y , and Z for arbitrary (perhaps empty) sequences of sentences. Then the structural rules of Gentzen are as follows:

$K \vdash$: From $X \vdash B$ to infer $X , A \vdash B$. Also, when negation is present,

one needs in addition $\vdash K$: From $X \vdash$ to infer $X \vdash Z$

$C \vdash$: From $X , A_1 , A_2 , Y \vdash B$ to infer $X , A_2 , A_1 , Y \vdash B$

$W \vdash$: From $X , A , A , Y \vdash B$ to infer $X , A , Y \vdash B$

Cut: From $X_1 \vdash A$ and $X_2 , A , X_3 \vdash Z$ to infer $X_2 , X_1 , X_3 \vdash Z$.

This fourth structural rule, cut, occupies a very special place. I return to it after offering a sample of Gentzen' s rules for the various logical connectives. I illustrate with the following examples: a binary connective for each of implication (or the conditional) and conjunction; a unary connective for negation; and a 0-ary connective for truth. For each there is a rule treating the appearance of the connective in the conclusion and a rule that takes care of its appearance in a premiss.

$\vdash \rightarrow$: From $X , A \vdash B$ to infer $X \vdash A \rightarrow B$.

$\rightarrow \vdash$: From $X_1 \vdash A$ and $X_2 , B \vdash Z$ to infer $X_2 , A \rightarrow B , X_1 \vdash Z$.

$\vdash \&$: From $X_1 \vdash A_1$ and $X_2 \vdash A_2$ to infer $X_1 , X_2 \vdash A_1 \& A_2$.

$\& \vdash$: From $X , A_1 , A_2 , Y \vdash Z$ to infer $X , A_1 \& A_2 , Y \vdash Z$.

$\sim \vdash$: From $X \vdash A$ to infer $X , \sim A \vdash$.

$\vdash \sim$: From $X , A \vdash$ to infer $X \vdash \sim A$.

$t \vdash$: From $X \vdash Z$ to infer $X , t \vdash Z$.

$\vdash t$: To infer $\vdash t$.

That concludes the presentation of (a slight adaptation of) Gentzen' s system: one axiom (Identity), three proper structural rules ($K \vdash$, $C \vdash$, and $W \vdash$), rules for implication, conjunction, negation, and Truth as premisses and conclusions; and, finally cut.

Gentzen' s profound result was that this rule, cut, is provably redundant. The result is called "cut elimination. " A principle consequence of cut elimination, which Curry justly calls one of the most important results in modern logic, give a "normal form" to proofs of statements of logical consequence (consecutions): As one proceeds down the proof, no sentence ever disappears. In othe