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A Practical Approach to Revising Prioritized Knowledge Bases

Abstract. This paper investigates simple syntactic methods for revising prioritized belief bases, that are semantically meaningful in the frameworks of possibility theory and of Spohn's ordinal conditional functions. Here, revising prioritized belief bases amounts to conditioning a distribution function on interpretations. The input information leading to the revision of a knowledge base can be sure or uncertain. Different types of scales for priorities are allowed: finite vs. infinite, numerical vs. ordinal. Syntactic revision is envisaged here as a process which transforms a prioritized belief bases into a new prioritized belief base, and thus allows a subsequent iteration.

Keywords: belief revision, Spohn ordinal conditional functions, possibility theory.

1. Introduction

During the last past fifteen years, the problem of belief change has become a major issue in Artificial Intelligence and information systems for the purpose of managing the dynamics of knowledge bases. The revision of a database consists of the insertion of some input information while preserving consistency. This problem has been studied axiomatically (see e.g., (Gärdenfors, 1988)) leading to the so-called AGM axioms; however revision is computationally expensive to implement in practice (see (Nebel, 1994)). Whether theoretically or practically oriented, all revision tools reveal the necessity of prioritizing belief bases or their deductive closures, belief sets. A prioritized belief base can be cast in the framework of possibilistic logic (Dubois et al., 1994) or in Ordinal Conditional Functions (OCF for short) frameworks (Spohn, 1988). It induces a complete pre-order on interpretations that can be encoded by means of possibility distributions (Zadeh, 1978), or equivalently by means of Spohn's ordinal conditional functions. The possibilistic setting does not necessarily require a numerical scale such as $[0,1]$ but can also be used with finite linearly ordered scales (Dubois and Prade, 1998). Spohn's OCFs use classes of ordinals but are usually defined with the integer scale. At the semantical level, the revision of a possibility distribution or an OCF by a sure input can be achieved by conditioning (Dubois and Prade, 1992; Spohn, 1988), and satisfies the AGM axioms (Gärdenfors, 1988). More generally, the revision of a possibility distribution can be viewed as a so-called "transmutation" (Williams, 1994) that modifies the ranking of

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interpretations so as to give priority to the input information. Clearly, this allows the iteration of the revision process contrary to classical AGM belief revision. In OCF framework, two basic belief revisions have been investigated: Adjustment (Williams, 1994) and conditionalisation (Spohn, 1988).

Revision operations with uncertain input have been studied by Spohn (Spohn, 1988) who has shown their close relationship with Jeffrey's rule of revision in probability theory. Possibilistic counterparts to the revision by uncertain inputs have been discussed in (Dubois and Prade, 1997). At the syntactical level such a form of revision comes down to adding a formula to a belief base at a certain prescribed level. The problem is made difficult because the belief base must be modified so that the added formula maintains its prescribed priority, that is, it is neither implicitly inhibited by higher priority formulas that contradict it, nor pushed to higher priority levels by formulas that imply it. An efficient way for doing this is proposed in this paper.

This paper, which is a revised version of a conference paper (Benferhat et al., 1999), pursues the study of revision with uncertain input in possibility theory framework started in (Dubois and Prade, 1997) by investigating efficient syntactic implementation schemes for both belief revision and contraction in possibilistic logic and shows their full agreement with semantics. Moreover, taking advantage of the connections of OCF with possibility theory (Dubois and Prade, 1991), we also offer syntactic couterparts for revision methods developed in OCF frameworks and we show for instance that the syntactic counterpart of adjustment given in this paper extends the one proposed by Williams (1995). Possibilistic (or OCF-based) revision can be naturally iterated, and is more generally discussed with respect to the Darwiche and Pearl's postulates (1997).

Sections 3 and 4 restate the necessary background on possibilistic logic and on Spohn's ordinal conditional functions, and recalls the connection between these two frameworks. Sections 4 and 5 discuss semantic and syntactic iterated belief revision in the possibilistic logic framework. A comparative study with some existing works on iterated belief revision is also provided. Section 6 presents conditioning and adjustment for OCFs and gives their relationships with possibilistic belief revision. Section 7 is devoted to semantic and syntactic contraction for both possibilistic logic and OCFs.

2. Possibilistic representations of epistemic states

Let L be a finite propositional language. $\vdash$ denotes the classical consequence relation. Ω is the set of classical interpretations or worlds, and $[\phi]$ is the set of classical models of ϕ. Often epistemic states (or cognitive states), viewed as a set of beliefs about the real world (based on the available information), are represented by a total pre-order either on Ω, or on the set of formulas. The latter is called an epistemic entrenchment relation (Gärdenfors, 1988). These orderings reflect the strength of the various knowledge maintained by an agent. A priority ordering over knowledge can be encoded using different types of scale: a finite linearly ordered scale, the integers (possibly completed by $+\infty$), the unit interval $[0, 1]$, etc. In this section, we describe the representation of epistemic states in possibilistic logic both at the syntactic and semantic level, where priorities are encoded by reals in the interval $[0, 1]$.

2.1. Semantic representation of epistemic states

At the semantic level, an epistemic state is represented by a possibility distribution π, which is a mapping from Ω to the interval $[0, 1]$. $\pi(\omega)$ represents the degree of compatibility of ω with the available information (or beliefs) about the real world. $\pi(\omega) = 0$ means that the interpretation ω is impossible, and $\pi(\omega) = 1$ means that nothing prevents ω from being the real world. The interpretations such that $\pi(\omega) = 1$ are considered as normal, expected. When $\pi(\omega) > \pi(\omega')$, ω is a preferred candidate to ω' for being the real state of the world. The less $\pi(\omega)$ the more abnormal ω is. A possibility distribution π is said to be normal if $\exists \omega \in \Omega$, such that $\pi(\omega) = 1$.

Given a possibility distribution π, we can define two different measures on formulas of the language:

- the possibility degree $\Pi_\pi(\phi) = \max\{\pi(\omega) : \omega \in [\phi]\}$ which evaluates the extent to which ϕ is consistent with the available information expressed by π.
- the necessity degree $N_\pi(\phi) = 1 - \Pi(\neg\phi)$ which evaluates the extent to which ϕ is entailed by the available information.

When there is no ambiguity, we simply write $\Pi(\phi)$ (resp. $N(\phi)$) instead of $\Pi_\pi(\phi)$ (resp. $N_\pi(\phi)$). Note that $\Pi(\phi)$ is evaluated from the assumption that the situation where ϕ is true is as normal as can be. The duality equation $N(\phi) = 1 - \Pi(\neg\phi)$ extends the one existing in classical logic, where a formula is entailed from a set of classical formulas if and only if its negation is inconsistent with this set.

Lastly, given a possibility distribution π, the semantic determination of the belief set (corresponding to the agent's current beliefs) denoted by $BS(\pi)$, is obtained by considering all formulas which are more plausible than their negation, namely:

$$BS(\pi) = \{\phi : \Pi(\phi) > \Pi(\neg\phi)\}.$$

Namely, $\mathrm{BS}(\pi)$ is a classical base whose models are the interpretations having the highest degrees in π. When π is normalized, models of $\mathrm{BS}(\pi)$ are interpretations which are completely possible, namely $[\mathrm{BS}(\pi)] = \{\omega : \pi(\omega) = 1\}$. The formula ϕ belongs to $\mathrm{BS}(\pi)$ when ϕ holds in all the most normal situations (hence ϕ is expected, or accepted as being true).

2.2. Syntactic representation of epistemic states

An epistemic state can also be represented syntactically by means of possibilistic knowledge bases which are made of a finite set of weighted formulas

$$\Sigma = \{(\phi_i, a_i) : i = 1, n\},$$

where a_i is understood as a lower bound of the degree of necessity $\mathrm{N}(\phi_i)$ (namely $\mathrm{N}(\phi_i) \geq a_i$). Formulas with null degree are not explicitly represented in the knowledge base (only beliefs which are somewhat accepted by the agent are explicitly represented). The higher the weight, the more certain the formula.

DEFINITION 1. Let Σ be a possibilistic knowledge base, and $a \in [0, 1]$. We call the a-cut of Σ (resp. strict a-cut), denoted by $\Sigma_{\geq a}$ (resp. $\Sigma_{>a}$), the set of classical formulas in Σ having a certainty degree at least equal (resp. strictly greater than) a.

A possibilistic knowledge base Σ is said to be consistent if the classical knowledge base, obtained by forgetting the weights, is classically consistent. Each inconsistent possibilistic base is associated with a level of inconsistency in the following way:

DEFINITION 2. Let Σ be a possibilistic knowledge base. The inconsistency degree of Σ is:

$$\mathrm{Inc}(\Sigma) = \mathrm{Max}\{a : \Sigma_{\geq a} \text{ is inconsistent}\},$$

with $\mathrm{Max}(\emptyset) = 0$.

Lastly, the syntactic computation of the belief set induced by Σ, denoted by $\mathrm{BS}(\Sigma)$, is obtained by classical deduction from the set of formulas with certainty levels higher than $\mathrm{Inc}(\Sigma)$, namely:

$$\mathrm{BS}(\Sigma) = \{\phi : \mathrm{Cons}(\Sigma) \vdash \phi\},$$

where $\mathrm{Cons}(\Sigma) = \{\phi_i : (\phi_i, a_i) \in \Sigma \text{ and } a_i > \mathrm{Inc}(\Sigma)\}$. Clearly, possibilistic reasoning copes with partial inconsistency. It yields non-trivial conclusions by using a consistent sub-part of Σ, which contains formulas belonging to the layers having sufficiently high levels of certainty. Moreover, checking if a formula belongs to BS(Σ) can be done with a complexity very close to that of classical logic.

2.3. From the syntactic to the semantic representation

Given a possibilistic belief base Σ, we can generate a possibility distribution from Σ by associating to each interpretation, its level of compatibility with agent's beliefs, namely with Σ, as explained now.

When a possibilistic belief base is only made of one formula $\{(\phi, a)\}$, then each interpretation ω which satisfies ϕ gets a possibility degree $\pi(\omega) = 1$ (since it is completely consistent with ϕ) and each interpretation ω which falsifies ϕ gets a possibility degree $\pi(\omega)$ such that the higher a is (i.e., the more certain ϕ is), the lower $\pi(\omega)$ is. In particular, if $a = 1$ (i.e., ϕ is completely certain), then $\pi(\omega) = 0$, namely ω is impossible if ω falsifies ϕ. One way to represent this constraint is to assign to $\pi(\omega)$ the degree $1 - a$ with a numerical encoding. More generally if π takes its value on a linearly ordered scale, $1 - (\cdot)$ is to be understood as an order-reversing map of the scale. Therefore, the possibility distribution associated to $\Sigma = \{(\phi, a)\}$ is: for any $\omega \in \Omega$,

$$\pi_{\{(\phi,a)\}}(\omega) = \begin{cases} 1 & \text{if } \omega \models \phi \\ 1-a & \text{otherwise.} \end{cases}$$

When $\Sigma = \{(\phi_i, a_i) : i = 1, n\}$ is a general possibilistic belief base then all the interpretations satisfying all the beliefs in Σ will have the highest possibility degree, namely 1, and the other interpretations will be ranked w.r.t. the highest belief that they falsify, namely we get (Dubois et al., 1994):

DEFINITION 3. The possibility distribution associated with a knowledge base Σ is defined by: for any $\omega \in \Omega$,

$$\pi_\Sigma(\omega) = \begin{cases} 1 & \text{if } \forall(\phi_i, a_i) \in \Sigma, \omega \in [\phi_i] \\ 1 - \max\{a_i : (\phi_i, a_i) \in \Sigma \text{ and } \omega \notin [\phi_i]\} & \text{otherwise.} \end{cases}$$

An equivalent relationship set in the traditional AGM framework for infinite first order logic can be found in (Peppas and Williams, 1995).

EXAMPLE 1. *Let* $\Sigma = \{(q, .3), (q \vee r, .5)\}$. *Then:*

ω	$\pi_\Sigma(\omega)$
qr	1
$q\neg r$	1
$\neg qr$	.7
$\neg q\neg r$	.5

The two interpretations qr and $q\neg r$ are the preferred ones since they are the only ones which are consistent with Σ, and $\neg qr$ is preferred to $\neg q\neg r$, since the highest belief falsified by $\neg qr$ (i.e. $(q, .3)$) is less certain than the highest belief falsified by $\neg q\neg r$ (i.e. $(q \vee r, .5)$).

The possibility distribution π_Σ is not necessarily normal. However π_Σ is normalized iff Σ is consistent. Moreover, the following correspondences between syntactic and semantic representations can be verified (Dubois et al., 1994):

$$\mathrm{Inc}(\Sigma) = 1 - \max_\omega \pi_\Sigma(\omega), \qquad \text{and } [\mathrm{BS}(\Sigma)] = [\mathrm{BS}(\pi_\Sigma)].$$

EXAMPLE 1 (continued). *In this case:* $[\mathrm{BS}(\pi_\Sigma)] = \{qr, q\neg r\}$. *Syntactically, we have* $\mathrm{Inc}(\Sigma) = 0$. *Then:* $\mathrm{BS}(\Sigma) = \{q, q \vee r\} \equiv q$. *It is clear that:* $[\mathrm{BS}(\Sigma)] = [\mathrm{BS}(\pi_\Sigma)]$.

We just see that semantic and syntactic computation of belief sets coincide, however note that there exist several syntactic forms which are semantically equivalent.

3. Ordinal conditional functions and possibilistic logic

3.1. Semantics representation

Spohn (1988) has proposed a theory for the representation of epistemic states that bears strong similarities with possibility theory and possibilistic logic, as a tool for ordering a set of possible worlds. More precisely, the ordinal conditional functions (OCF), introduced by Spohn for updating purposes, are similar to possibility (and necessity) measures. An OCF is a function κ from L into the class of ordinals rather than in [0,1]. Here, for simplicity we consider the set of natural integers N. An OCF, κ, should satisfy: for all ϕ,

$$\kappa(\phi) = \min\{\kappa(\omega) : \omega \in [\phi]\},$$

and $\kappa(\bot) = +\infty$. K(ϕ) is a degree of impossibility of ϕ (with the convention that 0 corresponds to the minimal impossibility, i.e., maximal plausibility). We have: $\kappa(\phi \vee \psi) = \min\{\kappa(\phi), \kappa(\psi)\}$. Moreover, OCF are called admissible by Spohn when $\kappa^{-1}(0) \neq \emptyset$. Clearly, this condition is similar to the normalisation condition used in possibilistic logic.

In OCF's, the lower $\kappa(\omega)$ is, the more preferred ω is (which is the converse of possibilistic logic convention). When $\kappa(\omega) < \kappa(\omega')$, ω is preferred to ω'. $\kappa(\omega) = +\infty$ means that ω is impossible, while $\kappa(\omega) = 0$ means that absolutely nothing prevents ω from being the real world. Note that in Spohn's paper, the notion of impossible world is not explicitely stated, and every interpretation is considered somewhat possible.

3.2. Syntactic representation

The syntactic representation in the OCF framework, largely developed in (Williams, 1994; Williams, 1995), slightly differs from the one of possibilistic logic. Williams starts with sets of integer-valued formulas $K = \{(\phi_i, k_i) : k_i \in N\}$ which are (partial) epistemic entrenchment rankings, namely those which satisfy $\forall(\phi_i, k_i) \in K$:

1. if $\vdash \phi_i$ then $k_i = +\infty$ (tautology).
2. $\{\phi_j : k_j > k_i\} \nvdash \phi_i$ (non redundancy).
3. if K is inconsistent, then $\forall(\phi_i, k_i) \in \min(K), k_i = 0$.

where $\min(K)$ contains all formulas in K having the lowest rank. Intuitively, partial epistemic entrenchment rankings represent agent's explicit beliefs, where the higher the rank assigned to formulas in K the more firmly held they are. The first condition simply means that tautologies are given the highest rank. The second condition means that (ϕ_i, k_i) should not be entailed by formulas of rank higher than k_i and the last condition means that when K is inconsistent the lowest rank of formulas in K should be equal to 0, the rank of inconsistent formulas.

Given a partial epistemic entrenchment ranking K, Willams (1994; 1995) gives a way to extend it to a full epistemic entrenchment ranking. This is done by first defining:

$$\mathrm{Exp}(K) = \{\phi_i : (\phi_i, k_i) \in K \text{ and } k_i > 0\},$$

the set of explicit beliefs of an agent. $\mathrm{Exp}(K)$ is always consistent. The full epistemic entrenchment associated with K is obtained by associating a unique rank to each belief ϕ denoted by $z_K(\phi)$, and defined in the following way:

$$z_K(\phi) = \begin{cases} 0 & \text{if } \mathrm{Exp}(K) \nvdash \phi, \\ \max\{k_i : \{\psi_j : (\psi_j, k_j) \in K, k_j \geq k_i\} \vdash \phi\} & \text{otherwise;} \end{cases}$$

$$z_K(\top) = +\infty\,.$$

3.3. Correspondences between OCF and Possibilistic logic

It is clear that possibilistic logic and OCF's are similar frameworks. The basic difference is that OCF's are defined on set of integers while possibilistic logic measures are defined on [0,1]. The close relationship between these two frameworks has been first pointed out in (Dubois and Prade, 1991) showing that the set function N_κ defined by $\mathrm{N}_\kappa(\phi) = 1 - e^{-\kappa(\neg\phi)}$ is a necessity measure, with values in a subset of the unit interval. Namely, it satisfies the characteristic property of necessity measures. Indeed,

$$\mathrm{N}_\kappa(\phi \wedge \psi) = 1 - e^{-\kappa(\neg\phi \vee \neg\psi)} = \min(\mathrm{N}_\kappa(\phi), \mathrm{N}_\kappa(\psi)).$$

Moreover, letting $\mathrm{N}_\kappa(\phi) = 1 - e^{-\kappa(\neg\phi)}$, it is easy to check that $\pi_\kappa(\omega)$ is equal to $e^{-\kappa(\omega)}$. Indeed: $\pi_\kappa(\omega) = 1 - \mathrm{N}_\kappa(\neg\phi_\omega) = 1 - (1 - e^{-\kappa(\omega)}) = e^{-\kappa(\omega)}$, where ϕ_ω is a formula having exactly one model which is ω.

At the syntactic level, it is clear that possibilistic knowledge bases are not necessarily partial epistemic entrenchment rankings. However if we remove tautologies and the so-called subsumed beliefs then the resulting base satisfies conditions i) and ii) of partial epistemic entrenchment. A formula (ϕ, a) of Σ is said to be subsumed if $\Sigma_{>a}$ classically entails ϕ. It can be easily verified that removing tautologies and subsumed formulas leads to an equivalent knowledge base, namely it generates the same possibility distribution (in the sense of Definition 3). Lastly, if Σ is consistent then condition iii) of partial epistemic entrenchment is also satisfied. When Σ is inconsistent, beliefs with lowest weight are not necessarily assigned 0.

The direct approach for generating a full epistemic entrenchment in the OCF framework is basically the same as the one in possibilistic logic. However usually in possibilistic logic, given a possibilistic knowledge base, a possibility distribution is generated instead of a necessity measure while in the OCF framework a full epistemic entrenchment is generated rather than a ranking on the set of interpretations. This is a matter of convenience, since each full epistemic entrenchment (or a necessity measure) uniquely determines a total pre-order on the set of interpretations (or a possibility distribution) and conversely. From K one can first generate an OCF κ_K in the following way: for all ω,

$$\kappa_K(\omega) = \max\{k_i : (\psi_i, k_i) \in K \text{ and } \omega \notin [\psi_i]\}$$

with $\max(\emptyset) = 0$.

It can be verified that the full epistemic entrenchment can be recovered from κ_K, and conversely. Indeed, we have $z_K(\phi) = \kappa_K(\neg\phi)$.

Table 1 summarizes the translation from the OCF framework to possibilistic logic.

OCF	Possibility theory
$\kappa(\omega)$	$\pi_K(\omega) = e^{-\kappa(\omega)}$
$\kappa(\phi)$	$\Pi_K(\phi) = e^{-\kappa(\phi)}$
K	$\Sigma_K = \{(\phi_i, 1 - e^{-k_i}) : (\phi_i, k_i) \in K\}$

Table 1. From OCF to possibility theory

The converse transformation is only possible when $\kappa(\omega) = -\ln(\pi(\omega))$ takes its value in the set of integers. However, with a finite language leading to a finite set Ω, it is always possible to re-encode the ordering induced by $-\ln(\pi(\omega))$ with integers. Clearly, the advantage of the scale $[0, 1]$ is its capacity to accommodate as many intermediary levels as is necessary for expressing the ranking between beliefs.

4. Iterated semantic revision in possibilistic logic

Belief revision results from the effect of accepting a new piece of information called the input information. In this paper, it is assumed that the current epistemic state (represented by a possibility distribution), and the input information, do not play the same role. The input must be incorporated in the epistemic state. In other words, it takes priority over information in the epistemic state. This assymmetry is expressed by the way the belief change problem is stated, namely the new information alters the epistemic state and not conversely. This assymmetry will appear clearly at the level of belief change operations. This situation is different from the one of information fusion from several sources, where no epistemic state dominates. In this context, the use of symmetrical rules is natural especially when the sources are equally reliable (Cholvy, 1998; Benferhat et al., 1999).

The choice of a revision method partially depends on the status of the input information. Here, we first consider revising with a totally reliable input, then we discuss the revision with an uncertain input. In the case of uncertain information, the input is of the form (ϕ, a) which means that the classical formula ϕ should be believed to a degree of certainty a exactly.

4.1. Revision with a totally reliable input

In the case of revision with a totally reliable (or certain, sure) input p, it is assumed that all interpretations that falsify p are declared impossible. This is performed by means of a conditioning device which transforms a possibility distribution π and a new and totally reliable information p into a new possibility distribution denoted by $\pi' = \pi(.|p)$. We assume that p is not a contradiction and that π is positive. Natural properties for π' are:

$\mathbf{A_1}$ π' should be normalized,

$\mathbf{A_2}$ $\forall\omega \notin [p]$ then: $\pi'(\omega) = 0$,

$\mathbf{A_3}$ $\forall\omega, \omega' \in [p]$ then: $\pi(\omega) > \pi(\omega')$ iff $\pi'(\omega) > \pi'(\omega')$,

$\mathbf{A_4}$ if $\mathrm{N}(p) > 0$ then $\forall\omega \in [p] : \pi(\omega) = \pi'(\omega)$,

$\mathbf{A_5}$ if $\pi(\omega) = 0$ then $\pi'(\omega) = 0$.

$\mathbf{A_1}$ means that the new epistemic state is consistent. $\mathbf{A_2}$ confirms that p is a sure piece information. $\mathbf{A_3}$ means that the new possibility distribution should not alter the previous relative order between models of p. $\mathbf{A_4}$ means that when $\mathrm{N}(p) > 0$ (p is a priori accepted) then revision does not affect π. $\mathbf{A_5}$ stipulates that impossible worlds remain impossible after conditioning. Then it can be verified that any revision of the belief set $\mathrm{BS}(\pi)$ by p, leading to $\mathrm{BS}(\pi(. \mid p))$ with $\pi(. \mid p)$ obeying $\mathbf{A_1}$–$\mathbf{A_5}$, satisfies all AGM postulates.

The previous properties $\mathbf{A_1}$–$\mathbf{A_5}$ do not guarantee a unique definition of conditioning. Moreover, the effect of the axiom $\mathbf{A_2}$ may result in a sub-normalized possibility distribution. Restoring the normalisation, so as to satisfy $\mathbf{A_1}$, can be done using two different types of conditioning (Dubois and Prade, 1998) (when $\Pi(p) > 0$):

- In an ordinal setting, we assign maximal possibility to the best models of p, then we get:

$$\pi(\omega \mid_m p) = \begin{cases} 1 & \text{if } \pi(\omega) = \Pi(p) \text{ and } \omega \models p \\ \pi(\omega) & \text{if } \pi(\omega) < \Pi(p) \text{ and } \omega \models p \\ 0 & \text{if } \omega \notin [p]. \end{cases}$$

  This is the definition of *minimum-based conditioning.*

- In a numerical setting, we proportionally rescale all models of p upwards:

$$\pi(\omega \mid_. p) = \begin{cases} \frac{\pi(\omega)}{\Pi(p)} & \text{if } \omega \models p \\ 0 & \text{otherwise.} \end{cases}$$

  This is the definition of *product-based conditioning.*

These two revision methods satisfy an equation of the form for all ω:

$$\pi(\omega) = \pi(\omega \mid p) * \Pi(p)$$

which is similar to Bayesian conditioning, $*$ is min and the product respectively. The rule based on the product is much closer to genuine Bayesian conditioning than the qualitative conditioning defined from the minimum which is purely based on comparing levels; product-based conditioning requires more of the structure of the unit interval. Besides, when $\Pi(p) = 0, \pi(\omega \mid_m p) = \pi(\omega \mid . \, p) = 1, \forall \omega$, by convention.

EXAMPLE 2. *Let us revise the possibility distribution π_Σ given in Example 1 by the information that q is certaintly false. If we use minimum-based conditioning we get:*

ω	$\pi_\Sigma(\omega \mid_m \neg q)$
$\neg q r$	1
$\neg q \neg r$	.5
$q r$	0
$q \neg r$	0

However, if we use the product-based conditioning, we get:

ω	$\pi_\Sigma(\omega \mid . \, \neg q)$
$\neg q r$	1
$\neg q \neg r$	5/7
$q r$	0
$q \neg r$	0

4.2. Revision with an uncertain input

We shall consider the revision of π by some uncertain input information of the form (p, a) into a new epistemic state denoted by $\pi' = \pi(\omega \mid (p, a))$. The input (p, a) is interpreted as a constraint which forces π' to satisfy:

$$\mathrm{N}'(p) = a \text{ (i.e., } \Pi'(p) = 1 \text{ and } \Pi'(\neg p) = 1 - a).$$

Clearly, properties defined for revision are all suitable for revising with uncertain input except $\mathbf{A_2}$ which is no longer appropriate since $\Pi'(\neg p) \neq 0$ for $a < 1$. $\mathbf{A_2}$ is replaced by the following two axioms:

$\mathbf{A'_2}$ $\Pi'(p) = 1, \Pi'(\neg p) = 1 - a$.

$\mathbf{A''_2}$ $\forall \omega, \omega' \notin [p]$ if $\pi(\omega) \geq \pi(\omega')$ then $\pi'(\omega) \geq \pi'(\omega')$.

$\mathbf{A}_2''$ preserves the relative order between countermodels of p, but in a weaker sense than in axiom $\mathbf{A_3}$ for the models of p. Note that there is no further constraints which relate models of p and countermodels of p in the new epistemic state.

$\mathbf{A_3}$ and $\mathbf{A}_2''$ suggest that revising with uncertain input can be achieved using two parallel changes with a sure input: first, a conditioning on p and one on $\neg p$. Then, in order to satisfy $\mathbf{A}_2'$, the distribution $\pi(\cdot \mid \neg p)$ is denormalized, so as to satisfy $\Pi'(\neg p) = 1 - a$. Therefore, revising with uncertain information can be achieved using the following definition:

$$\pi(\omega \mid (p,a)) = \begin{cases} \pi(\omega \mid p) & \text{if } \omega \models p, \\ (1-a) * \pi(\omega \mid \neg p) & \text{otherwise.} \end{cases}$$

where $*$ is either min or the product, depending on whether conditioning is based on the product or the minimum operator. When $*$=product (resp. min) the possibilistic revision is called product-based (resp. minimum-based) conditioning with an uncertain input, denoted $\pi(\omega \mid_. (p,a))$, (resp. $\pi(\omega \mid_m (p,a))$). Table 2 displays the expression of $\pi(\omega \mid (p,a))$ depending if ω is a model or a countermodel of p. From this table, it is clear that the new ranking on models of p is simply obtained using conditioning with a sure input.

	$\omega \models p$	$\omega \models \neg p$
$\pi(\omega \mid_m (p,a))$	$\begin{cases} 1 & \text{if } \pi(\omega) = \Pi(p) \\ \pi(\omega) & \text{otherwise.} \end{cases}$	$\begin{cases} 1-a & \text{if } \pi(\omega) = \Pi(\neg p) \text{ or } \pi(\omega) > 1-a \\ \pi(\omega) & \text{otherwise.} \end{cases}$
$\pi(\omega \mid_. (p,a))$	$\pi(\omega)/\Pi(p)$	$(1-a) \cdot \pi(\omega)/\Pi(\neg p)$

Table 2. Definition of $\pi(\omega \mid (p,a))$

The new ranking of countermodels of p depends on the relative position of the a priori certainty of p, and the prescribed posterior certainty of p:

- If $\mathrm{N}(p) \leq a$ and when $* =$ min, all interpretations that were originally more plausible than $1-a$, are forced to level $1-a$, which means that some strict ordering between countermodels of p may be lost. When $* =$ product, all plausibility levels are proportionally shifted down (to the level $1-a$).
- If $\mathrm{N}(p) > a$ the best countermodels of p are raised to level $1-a$. Moreover, when $* =$ product, the plausibility levels of other countermodels are proportionally shifted up (to level $1-a$).

Note that in Table 2 when $a = 1$, we recover conditioning by a totally reliable input.

When $*$ = product, a stronger version of $\mathbf{A_2''}$ holds whereby the order of countermodels of p is fully preserved, hence it satisfies:

$\mathbf{A_6}$ $\quad \forall \omega_1, \omega_2 \notin [p], \pi(\omega_1) \leq \pi(\omega_2) \quad \text{iff} \quad \pi'(\omega_1) \leq \pi'(\omega_2).$

Moreover if $\mathrm{N}(p) \leq a$, we can check that the following two postulates are also satisfied:

$\mathbf{A_7}$ If $\omega_1 \models p$ and $\omega_2 \models \neg p$ then $\pi(\omega_1) < \pi(\omega_2)$ only if $\pi'(\omega_1) < \pi'(\omega_2)$.

$\mathbf{A_8}$ If $\omega_1 \models p$ and $\omega_2 \models \neg p$ then $\pi(\omega_1) \leq \pi(\omega_2)$ only if $\pi'(\omega_1) \leq \pi'(\omega_2)$.

EXAMPLE 3. *Let us again consider the possibility distribution π_Σ of Example 1. Let $(q \vee r, .2)$ be the uncertain input. Note that $\mathrm{N}_{\pi_\Sigma}(q \vee r) = .5$, and hence taking into account the input should decrease our belief in the information $q \vee r$. Using minimum-based conditioning, we get:*

ω	$\pi_\Sigma(\omega \mid_m (q \vee r, .2))$
qr	1
$q\neg r$	1
$\neg qr$	.7
$\neg q \neg r$	.8

In this example, the product-based conditioning leads to the same result. Note that the main difference with conditioning with sure input is that countermodels of p are no longer impossible.

In the above view, the uncertain input is viewed as a constraint which is enforced. However, another view exists (Dubois and Prade, 1997) where the input is taken into account only if it leads to a strengthening of the certainty of π; it corresponds to the following definition:

$$\pi(\omega \mid (p, a)) = \begin{cases} \pi(\omega \mid p) & \text{if } \omega \models p \\ \min(1 - a, \pi(\omega)) & \text{otherwise.} \end{cases}$$

With this definition, in Example 3, no revision would take place.

4.3. Related works

Clearly possibilistic revision operators also deal with iterated belief revision, since the underlying ordering (here a possibility distribution) used in a belief revision process is not lost after a revision step. Several authors have proposed postulates for iterated belief revision which are added to the AGM postulates. Here, we briefly recall the ones proposed by Darwiche and Pearl (1997), which are devoted to iterated belief revision operators which transform a given ordering on interpretations, denoted by $\leq$, in presence of the new information p, into a new ordering, denoted by $\leq'$. These postulates are:

$\mathbf{CR_1}$ If $\omega_1 \models p$ and $\omega_2 \models p$ then $\omega_1 \leq \omega_2$ iff $\omega_1 \leq' \omega_2$.

$\mathbf{CR_2}$ If $\omega_1 \models \neg p$ and $\omega_2 \models \neg p$ then $\omega_1 \leq \omega_2$ iff $\omega_1 \leq' \omega_2$.

$\mathbf{CR_3}$ If $\omega_1 \models p$ and $\omega_2 \models \neg p$ then $\omega_1 < \omega_2$ only if $\omega_1 <' \omega_2$.

$\mathbf{CR_4}$ If $\omega_1 \models p$ and $\omega_2 \models \neg p$ then $\omega_1 \leq \omega_2$ only if $\omega_1 \leq' \omega_2$.

$\mathbf{CR_1}$ (resp. $\mathbf{CR_2}$) simply says that the relative ordering between models (resp. countermodels) of p should be preserved after the revision process. $\mathbf{CR_3}$ and $\mathbf{CR_4}$ say that if some model of p is preferred to some countermodel of p, this preference should be also preserved.

When revising with a completely sure input, none of the two possibilistic revision operators satisfies Darwiche and Pearl's postulates. This is basically due to $\mathbf{A_2}$ which contradicts the axiom $\mathbf{CR_2}$. $\mathbf{A_2}$ stipulates that all countermodels of p should be impossible (in a qualitative setting, this can be done by pushing all countermodels of p in a new lowest rank). Thus, the Darwiche and Pearl's postulates are appropriate in case of uncertain inputs. In case of sure input, the revision process can be iterated, in a non-trivial way, only if the succesive inputs are coherent with the totally sure part of the successive bases (this requires that the inputs are together coherent).

Possibilistic revision with uncertain input is more in the spirit of the Darwiche and Pearl postulates, except that in possibilistic revision there is no limitation on the input (p, a) leading to a revision. In particular, it may happen that in the current agent beliefs, the certainty of p is higher than a, namely $\mathrm{N}(p) > a$. In this case, the uncertain input would require that the agent believes in p less. In such a case, possibilistic revision can reverse a preference between some models of p and some countermodels of p. But this is clearly forbidden by $\mathbf{CR_3}$ and $\mathbf{CR_4}$. However, decreasing the rank of p is close to the idea of contraction rather than to the idea of revision.

Now, when $\mathrm{N}(p) \leq a$, then we can easily check that the product-based possibilistic conditioning, satisfies all Darwiche and Pearl postulates. Indeed, $\mathbf{A_3}$, $\mathbf{A_6}$, $\mathbf{A_7}$, $\mathbf{A_8}$ respectively corresponds to $\mathbf{CR_1}$, $\mathbf{CR_2}$, $\mathbf{CR_3}$ and $\mathbf{CR_4}$.

Natural belief revision (Boutilier, 1993), also hinted by Spohn (1988), can also be viewed as a minimum-based conditioning, with uncertain input (Dubois and Prade, 1997). A natural revision by input p only comes down to assigning to the most plausible interpretations in $[p]$ a degree of possibility higher than those of any other interpretations. This is enough to ensure that $N'(p) > 0$, where N' is a necessity measure obtained after taking into account p. This also retains the same ordering of interpretations as before revision takes place, including *the case of the interpretations outside* of $[p]$. It implies that, after revision, some interpretations where $[p]$ is not true may remain more plausible than interpretations where $[p]$ is true. Let a be such that $1 > a > \max\{\pi(\omega) : \pi(\omega) \neq 1\}$. Then the natural belief revision of a possibility distribution π by an input p, denoted by $\pi_p^n(\omega)$ can be encoded in the following way:

$$\pi_p^n(\omega) = \begin{cases} \pi(\omega) & \text{if } \Pi(p) > \Pi(\neg p), \\ \pi(\omega \mid_m (p, 1-a)) & \text{otherwise.} \end{cases}$$

Lastly, in Darwiche and Pearl postulates, there are only weak constraints which relate models of p and countermodels of p. Papini (Papini, 2001), has considered a stronger constraint (also hinted by Spohn (1988)) by imposing that each model of p should be strictly preferred to each countermodel of $\neg p$, and moreover the relative ordering between models (resp. countermodels) of p should be preserved. This revision operator can be captured by product-based conditioning when possibility distributions are *positive*. A possibility distribution π is said to be positive if $\forall\omega, \pi(\omega) > 0$. Let $\Delta'(p) = \min\{\pi'(\omega) : \omega \vDash p\}$, and a such that $1 - a < \Delta'(p)$. Then we can check that revising a possibility distribution π with an uncertain input (p, a) using product-based possibilistic conditioning leads to Papini's revision operator. Indeed, product-based revision shifts down all countermodels of p below $1 - a$ (which represents a degree smaller than the one of the worst model of p). The *positiveness condition* is necessary, since $\Delta'(p) > 0$ allows to put countermodels of p at levels less than $\Delta'(p)$, which would not be possible otherwise.

5. Syntactic iterated belief revision in possibilistic logic

5.1. Case of totally reliable inputs

A syntactic counterpart of revising with totally reliable information consists of constructing from a possibilistic base Σ and the new information p, a new possibilistic base Σ' such that:

$$\forall\omega, \pi_{\Sigma'}(\omega) = \pi_{\Sigma}(\omega|p)\,,$$

where $|$ can be either the minimum-based conditioning or the product-based conditioning. The construction of Σ' can be performed as follows:

- add the input p to the belief base with highest possible priority (namely 1);
- compute the level of inconsistency $x = \text{Inc}(\Sigma \cup \{(p,1)\})$ of the resulting possibly inconsistent belief base (when $x = 1$ the revision simply acknowledges the input formula $\{(p,1)\}$);
- drop all formulas with priority less than or equal to this level of inconsistency.

This guarantees that the remaining beliefs are consistent with p. Concerning the weights of remaining beliefs, $|_m$ leaves them unchanged, however $|_.$ leads to discounting them by decreasing their weight.

The following proposition gives the formal expression of Σ' for the two possible definitions of conditioning.

PROPOSITION 1. *Let Σ be a possibilistic base and π_Σ be the possibility distribution associated with Σ (in the sense of Definition 3). Let p be new sure information and $x = \text{Inc}(\Sigma \cup \{(p,1)\})$. Then:*

- *the possiblistic base associated with $\pi_\Sigma(\omega|_m p)$ is:*

$$\Sigma' = \{(\phi, b) : (\phi, b) \in \Sigma \textit{ and } b > x\} \cup \{(p,1)\} ;$$

- *the possiblistic base associated with $\pi_\Sigma(\omega|_. p)$ is:*

$$\Sigma'' = \{(\phi, f(b)) : (\phi, b) \in \Sigma \textit{ and } b > x\} \cup \{(p,1)\}$$

with $f(b)=\frac{(b-x)}{(1-x)}$.

PROOF. First observe that: $\Pi(p) = 1 - x$. Now it is clear, in both cases, we have for a given interpretation ω which falsifies p that $\pi_{\Sigma'}(\omega) = \pi_{\Sigma''}(\omega) = 0$. Now let ω be such that it satisfies p.

If $\pi_\Sigma(\omega) = \Pi(p) = 1 - x$ then this means by definition that ω satisfies all formulas with a weight greater than $1 - x$, hence ω satisfies all formulas of Σ', therefore $\pi_{\Sigma'}(\omega) = 1$. Now assume that $\pi_\Sigma(\omega) < 1 - x$, then by definition:

$$\begin{aligned}
\pi_\Sigma &= \min\{1 - b_i : (\phi_i, b_i) \in \Sigma \text{ and } \omega \not\models \phi_i\} \\
&= \text{Min}\{\{\min\{1 - b_i : (\phi_i, b_i) \in \Sigma \text{ and } \omega \not\models \phi_i \text{ and } b_i > x\}, \\
&\quad \min\{1 - b_i : (\phi_i, b_i) \in \Sigma \text{ and } \omega \not\models \phi_i \text{ and } b_i \leq x\}\} \\
&= \min\{1 - b_i : (\phi_i, b_i) \in \Sigma \text{ and } \omega \not\models \phi_i \text{ and } b_i > x\} \text{ (since } \pi_\Sigma(\omega) < 1 - x) \\
&= \pi_{\Sigma'}(\omega).
\end{aligned}$$

$$\begin{aligned}
\pi_{\Sigma''}(\omega) &= \min\{1 - f(b_i) : (\phi_i, b_i) \in \Sigma \text{ and } \omega \not\models \phi_i \text{ and } b_i > x\} \\
&= \min\{1 - \tfrac{b_i - x}{1-x} : (\phi_i, b_i) \in \Sigma \text{ and } \omega \not\models \phi_i \text{ and } b_i > x\}
\end{aligned}$$

$$= \min\{\tfrac{1-b_i}{1-x} : (\phi_i, b_i) \in \Sigma \text{ and } \omega \not\models \phi_i \text{ and } b_i > x\}$$
$$= \min\{\tfrac{1-b_i}{\Pi(p)} : (\phi_i, b_i) \in \Sigma \text{ and } \omega \not\models \phi_i \text{ and } b_i > x\}$$
$$= \tfrac{\pi_\Sigma(\omega)}{\Pi(p)} \,. \qquad \blacksquare$$

Note that the computation of the resulting base is efficient. Its complexity is the same as the one of computing the inconsistency level of a possibilistic base. This can be done in $log_2 m$ satisfiability tests, where m is the number of layers in Σ. Moreover, the size of the resulting base is at most equal to $|\Sigma|+1$ which is reached when the input is consistent with the possibilistic base.

EXAMPLE 4. *Let us consider Examples 1 and 2 again, where* $\Sigma = \{(q, .3), (q \vee r, .5)\}$, *and* $\neg q$ *be the sure input.*

It can be verified that $x = \text{Inc}(\Sigma \cup \{(\neg q, 1)\}) = .3$. *Using the above proposition, the revision with minimum-based conditioning leads to:*

$$\Sigma' = \{(\neg q, 1), (q \vee r, .5)\}.$$

It can be verified that: $\forall \omega, \pi_{\Sigma'}(\omega) = \pi_\Sigma(\omega \mid_m \neg q)$, *where* $\pi_\Sigma(\omega \mid_m \neg q)$ *is the possibility distribution computed in Example 2.*

Similarly, if we consider the product-based conditioning we get:

$$\Sigma' = \{(\neg q, 1), (q \vee r, 2/7)\}.$$

And again, we can easily check that: $\forall \omega, \pi_{\Sigma'}(\omega) = \pi_\Sigma(\omega \mid_. \neg q)$.

5.2. Case of uncertain inputs

Syntactic revision methods can be provided for efficiently inserting a formula at some prescribed level in a prioritized belief base in accordance with semantic principles given in Section 4.2. Let

$$x = \text{Inc}(\Sigma \cup \{(p, 1)\}), \qquad y = \text{Inc}(\Sigma \cup \{(\neg p, 1)\}),$$
$$f(b) = \frac{b-x}{1-x}, \qquad \text{and} \qquad g(b) = \frac{b-y}{1-y}\,.$$

First, let us start with the case when $x = y = 0$, namely when Σ is consistent with p and $\neg p$, we get $\Sigma' = \Sigma \cup \{(p, a)\}$ (simple addition of $\{(p, a)\}$ in the case min-based conditioning. However, with product-based conditioning,

$$\Sigma' = \Sigma \cup \{(p, a)\} \cup \{(\phi \vee p, a + b - ab) : (\phi, b) \in \Sigma\}$$

which leads to a more informative belief base.

Then the two following propositions give the syntactic counterpart of the two forms of conditioning with uncertain inputs when either p or $\neg p$ is inconsistent with Σ:

PROPOSITION 2. *Let $\pi_\Sigma(\omega \mid_m (p,a))$ be the possibility distribution obtained by revising π_Σ with (p,a) using minimum-based conditioning. Let Σ' be defined in the following way:*

– *if Σ is inconsistent with $\neg p$ then*

$$\Sigma' = \{(p,a)\} \cup \{(\phi,b) : (\phi,b) \in \Sigma \text{ and } b > y\} \\ \cup \{(\phi \vee \neg p, b) : (\phi,b) \in \Sigma \text{ and } b \leq y\};$$

– *if Σ is inconsistent with p then*

$$\Sigma' = \{(p,a)\} \cup \{(\phi,b) : (\phi,b) \in \Sigma \text{ and } b > x\} \\ \cup \{(\phi \vee p, b) : (\phi,b) \in \Sigma \text{ and } a \leq b \leq x\}.$$

Then $\pi_{\Sigma'} = \pi_\Sigma(\omega \mid_m (p,a))$.

PROOF. We only give the proof of the first part of the proposition, the other case has a similar proof. Let us then condider the case where Σ is inconsistent with $\neg p$. This means that: $\Pi(p) = 1$ and $\Pi(\neg p) = 1 - y$.

Let $\omega \vDash p$, then by definition:

$$\begin{aligned} \pi_{\Sigma'}(\omega) &= \mathrm{Min}\{\min\{1 - b_i : (\phi_i, b_i) \in \Sigma \text{ and } , \omega \nvDash \phi_i \text{ and } b_i > y\} \\ &\quad \min\{1 - b_i : (\phi_i, b_i) \in \Sigma \text{ and } \omega \nvDash \phi_i \vee \neg p \text{ and } b_i \leq y\}\} \\ &= \mathrm{Min}\{\min\{1 - b_i : (\phi_i, b_i) \in \Sigma \text{ and } \omega \nvDash \phi_i \text{ and } b_i > y\}, \\ &\quad \min\{1 - b_i : (\phi_i, b_i) \in \Sigma \text{ and } \omega \nvDash \phi_i \text{ and } b_i \leq y\}\} \\ &= \min\{1 - b_i : (\phi_i, b_i) \in \Sigma \text{ and } \omega \nvDash \phi_i\} \\ &= \pi_\Sigma(\omega). \end{aligned}$$

Let $\omega \nvDash p$, then by definition:

$$\pi_{\Sigma'}(\omega) = \mathrm{Min}\{1 - a, \min\{1 - b_i : (\phi_i, b_i) \in \Sigma \text{ and } \omega \nvDash \phi_i \text{ and } b_i > y\}\}.$$

We distinguish two cases,

- either $\pi_\Sigma(\omega) = \Pi(\neg p) = 1 - y$, this means that ω satisfies all the formulas of Σ having a weight strictly greater than y, then:

$$\min\{1 - b_i : (\phi_i, b_i) \in \Sigma \text{ and } \omega \nvDash \phi_i \text{ and } b_i > y\} = 1.$$

  Therefore: $\pi_{\Sigma'}(\omega) = 1 - a$.

- or $\pi_\Sigma(\omega) < 1 - y$, then:

$$\begin{aligned}\pi_{\Sigma'}(\omega) &= \mathrm{Min}\{1-a, \min\{1-b_i : (\phi_i, b_i) \in \Sigma \text{ and } \omega \not\models \phi_i \text{ and } b_i > y\}\} \\ &= \mathrm{Min}\{1-a, \min\{1-b_i : (\phi_i, b_i) \in \Sigma \text{ and } \omega \not\models \phi_i \text{ and } b_i > y\}, \\ &\quad \min\{1-b_i : (\phi_i, b_i) \in \Sigma \text{ and } \omega \not\models \phi_i \text{ and } b_i \leq y\}\} \\ &\quad (\text{since } \pi_\Sigma(\omega) < 1-y) \\ &= \mathrm{Min}\{1-a, \pi_\Sigma(\omega)\}.\end{aligned}$$

■

This proposition improves the syntactic revision hinted in (Dubois and Prade, 1997) by guaranteeing the consistency with the semantics. Similarly,

PROPOSITION 3. *Let $\pi_\Sigma(\omega \mid_. (p, a))$ be the possibility distribution obtained by revising π_Σ with (p, a) using product-based conditioning. Let Σ' be defined in the following way:*

– *if Σ is inconsistent with $\neg p$ then*

$$\begin{aligned}\Sigma' = \{(p, a)\} &\cup \{(\phi \vee p, g(b) + a - a.g(b)) : (\phi, b) \in \Sigma \text{ and } b > y\} \\ &\cup \{(\phi \vee \neg p, b) : (\phi, b) \in \Sigma\}\end{aligned}$$

– *and, if Σ is inconsistent with p then*

$$\begin{aligned}\Sigma' = \{(p, a)\} &\cup \{(\phi, f(b)) : (\phi, b) \in \Sigma \text{ and } b > x\} \\ &\cup \{(\phi \vee p, a + b - ab) : (\phi, b) \in \Sigma\}\end{aligned}$$

Then $\pi_{\Sigma'} = \pi_\Sigma(\omega \mid_. (p, a))$.

The size of the revised base is at most $2 \cdot |\Sigma|$, and its computation can be done efficiently since it needs at most $2 \cdot \log_2 m$ satisfiability tests (m is the number of different layers in Σ).

EXAMPLE 5. *Let $\Sigma = \{(q, .3), (q \vee r, .5)\}$, and let the input be $(q \vee r, .2)$. It can be verified that $y = .5$, then the syntactic revision using min-based conditioning leads to*

$$\begin{aligned}\Sigma' &= \{(q \vee r, .2), (q \vee (\neg q \wedge \neg r), .3), ((q \vee r) \vee (\neg q \wedge \neg r), .5)\} \\ &\equiv \{(q \vee r, .2), (q \vee \neg r, .3)\} \quad \textit{(after removing tautologies).}\end{aligned}$$

Note that Σ' still implies q but with a weight .2. So, this revision is respectful of the original belief base. It can be easily verified that:

$$\pi_{\Sigma'}(\omega) = \pi_\Sigma(\omega | (q \vee r, .2))$$

where $\pi_\Sigma(\omega | (q \vee r, .2))$ is the possibility distribution computed in Example 3.

As a corollary of Proposition 2, we can provide a syntactic counterpart of natural belief revision, since we have shown that it corresponds to a particular case of revision with minimum-based conditioning. Let a be such that $a > 0$ and $a < \min\{b_i : (\phi_i, b_i) \in \Sigma\}$. Then, the natural belief revision of Σ, denoted by Σ_p^n, with (p, a) is described as follows:

- Σ remains unchanged if Σ entails p already;
- otherwise if Σ is consistent with p, we simply add p to Σ into a new layer with a priority equal to a,
- and in case of inconsistency with p, natural revision yields $\{(p, a)\} \cup \{(\phi_i, b_i) : b_i > x\} \cup \{(\phi_i \vee p, b_i) : b_i \leq x\}$.

In all cases, natural belief revision can be computed at the syntactic level in an efficient way. Moreover, it can be easily verified that:

$$\pi_{\Sigma_p^n}(\omega) = \pi^n(\omega).$$

6. Syntactic adjustment and conditionalization

6.1. adjustment

Williams (1994; 1995) has defined a general form of belief change she calls "transmutations". Given an uncertain input $(p, i), i \in N \cup \{+\infty\}$ taken as a constraint and an OCF κ describing the agent's a priori epistemic state, a transmutation of κ by (p, i) produces a new OCF κ' such that $\kappa'(\neg p) = i$ and $\kappa'(p) = 0$, i.e., the degree of acceptance of p is enforced to level i. Williams (1994) has introduced a qualitative transmutation called an adjustment. An adjustment of κ by (p, i), is defined as follows:

$$\kappa^*_{(p,i)} = \begin{cases} \kappa_p^- & \text{if } i = 0 \\ (\kappa_p^-)^x_{(p,i)} & \text{if } 0 < i < \kappa(\neg p) \\ \kappa^x_{(p,i)} & \text{otherwise} \end{cases}$$

where

$$\kappa_p^-(\omega) = \begin{cases} 0 & \text{if } \omega \models \neg p \text{ and } \kappa(\omega) = \kappa(\neg p) \\ \kappa(\omega) & \text{otherwise} \end{cases}$$

$$\kappa^x_{(p,i)}(\omega) = \begin{cases} 0 & \text{if } \omega \models p \text{ and } \kappa(\omega) = \kappa(p) \\ \kappa(\omega) & \text{if either } \omega \models p \text{ and } \kappa(\omega) \neq \kappa(p) \text{ or } \omega \models \neg p \text{ and } \kappa(\omega) > i \\ i & \text{otherwise.} \end{cases}$$

κ_p^- is a contraction. Dubois and Prade (1997) have shown that for $i > 0$ the expression of Williams' adjustment can be simplified as follows:

$$\kappa^*_{(p,i)}(\omega) = \min(\kappa(\omega \mid p), \max(i, \kappa(\omega \mid \neg p)))$$

where

$$\kappa(\omega \mid p) = \begin{cases} +\infty & \text{if } \omega \models \neg p \\ \kappa(\omega) & \text{if } \kappa(\omega) > \kappa(p) \\ 0 & \text{if } \kappa(\omega) = \kappa(p) \end{cases}$$

which is the counterpart of the min-based conditioning with uncertain input given in Section 4.2.

6.2. conditionalization

Spohn (1988) also introduces conditioning concepts, which transforms an ordinal conditional function (or kappa function)κ, into a new one. First, he introduces the notion of p-part of κ:

- the p-part of κ is the conditioning by a formula p defined by

$$\forall \omega \in \Omega, \kappa(\omega \mid p) = \kappa(\omega) - \kappa(p).$$

- the (p, i)-conditionalization of κ, say $\kappa(\omega \mid (p, i))$ is a conditioning operation by an uncertain input $\kappa'(p) = i$, defined by

$$\kappa(\omega \mid (p,i)) = \begin{cases} \kappa(\omega \mid p) & \text{if } \omega \models p \\ i + \kappa(\omega \mid \neg p) & \text{if } \omega \models \neg p. \end{cases}$$

Darwiche and Pearl (1997) have proposed an iterated belief revision operator, which satisfies AGM postulates and $\mathbf{CR_1}$–$\mathbf{CR_4}$, based on a dynamic version of Spohn conditionalization. They choose $i = \kappa(p) + 1$. This choice is made in order to guarantee that the degree of belief on p will increase after the revision step.

6.3. relationships with possibilistic conditioning

The following proposition relates the two forms of revision in OCF framework with the two definitions of conditioning with uncertain input in possibilistic logic setting. This is done with the help of Table 1.

PROPOSITION 4. *Let κ be an OCF, (p, i) be the input information and κ_1, κ_2 be the revision of κ using adjustment and conditionalization. Let $\pi = e^{-\kappa}$ (resp. $\pi_1 = e^{-\kappa_1}$, $\pi_2 = e^{-\kappa_2}$) be the possibility distributions associated with κ, κ_1, κ_2 using Table 1. Then:*

- $\forall \omega, \pi_1(\omega) = \pi(\omega \mid_m (p, 1 - e^{-i}))$, *and*
- $\forall \omega, \pi_2(\omega) = \pi(\omega \mid_. (p, 1 - e^{-i}))$.

The above proposition means that adjustment in OCF corresponds to possibilistic minimum-based conditioning with uncertain input, and conditionalization in OCF corresponds to possibilistic product-based conditioning with uncertain input.

As a corollary of this proposition, it is possible to provide a syntactic counterpart of adjustment and conditionalization. Let K be a partial epistemic entrenchment, and κ_K be its associated OCF. Then it is possible to directly construct K' such that:

$$\kappa_{K'} = \kappa^*_{(p,i)} \qquad (\text{resp. } \kappa_{K'} = \kappa(\omega \mid (p, i))).$$

where $\kappa^*_{(p,i)}$ (resp. $\kappa(\omega \mid (p, i))$) is the result of applying adjustment (resp. conditionalization) on κ.

Table 3 summarizes the structure of K'.

1.	K is inconsistent with $\neg p$
Adjustment	$\{(p, i)\} \cup \{(\phi, j) : j > n\} \cup \{(\phi \vee \neg p, j) : j \leq n\}$
Conditionalization	$\{(p, i)\} \cup \{(\phi \vee p, j - n + i) : j > n\} \cup \{(\phi \vee \neg p, j)\}$

2.	K is inconsistent with p
Adjustment	$\{(p, i)\} \cup \{(\phi, j) : j > m\} \cup \{(\phi \vee p, j) : i < j \leq m\}$
Conditionalization	$\{(p, i)\} \cup \{(\phi, j - m) : j > m\} \cup \{(\phi \vee p, j + i) : i < j \leq m\}$

where

$$m = \min\{i : \{\psi : \kappa(\psi) > i\} \text{ is consistent with } p\},$$
$$n = \min\{i : \{\psi : \kappa(\psi) > i\} \text{ is consistent with } \neg p\}.$$

Table 3. Syntactic counterpart of Adjustment and Conditionalization

Besides, from this table we can immediately give the syntactic counterpart of the Darwiche and Pearl (1997) proposal, where they set semantically $i = \kappa_K(p) + 1$ (which is equivalent syntactically to setting $i = n + 1$), we get:

- if K is inconsistent with $\neg p$ then:

$$K' = \{(p,i)\} \cup \{(\phi \vee p, j+1) : (\phi, j) \in K \text{ and } j > n\}$$
$$\cup \{(\phi \vee \neg p, j) : (\phi, j) \in \mathrm{K}\};$$

- if K is inconsistent with p then:

$$K' = \{(p,i)\} \cup \{(\phi, j-m) : (\phi, j) \in K \text{ and } j > m\}$$
$$\cup \{(\phi \vee p, j+i) : (\phi, j) \in \mathrm{K}\}.$$

7. Semantic and syntactic possibilistic contraction

Contraction is the process of forgetting some old beliefs. Hence, one may define a contraction as a particular case of revising with uncertain input (p, a) with $a = 0$. Clearly, using this interpretation, contracting p also leads to contract $\neg p$. Indeed, when $a = 0$, the definition of $\pi(. \mid (p, a))$ (where $\mid$ can be either minimum-based or product-based conditioning) from Section 4.2 yields a result which is symmetric with respect to p and $\neg p$:

$$\pi(\omega \mid (p, 0)) = \begin{cases} \pi(\omega \mid p) & \text{if } \omega \models p \\ \pi(\omega \mid \neg p) & \text{if } \omega \models \neg p\,, \end{cases}$$

such that $\Pi(p) = \Pi(\neg p) = 1$. This is stronger than the definition of contraction used in (Gärdenfors, 1988) since when $\neg p$ is believed, then contracting with p does not modify π. In this section, we propose to see how contraction in the sense of Gärdenfors can be handled in the possibility theory framework.

The contraction of a possibility distribution with respect to p corresponds to forgetting that p is believed if p was previously in the belief set (i.e., if $p \in \mathrm{BS}(\pi)$). In such a case, the result π_p^- of the contraction must lead to a possibility measure Π_p^- such that $\Pi_p^-(p) = \Pi_p^-(\neg p) = 1$, i.e. complete ignorance about p. Intuitively if $\Pi(p) = \Pi(\neg p) = 1$ already exists, then we should have $\pi_p^- = \pi$. Besides if $p \in \mathrm{BS}(\pi)$ then we should have $\pi_p^-(\omega) = 1$ for some $\omega \models \neg p$, and especially for those ω such that $\Pi(\neg p) = \pi(\omega)$. If $\Pi(\neg p) = 1 > \Pi(p)$, i.e. $\neg p$ represents an accepted belief, π should be unchanged. It leads to (Dubois and Prade, 1992):

$$\pi_p^-(\omega) = \begin{cases} 1 & \text{if } \pi(\omega) = \Pi(\neg p) \text{ and } \omega \models \neg p \\ \pi(\omega) & \text{otherwise.} \end{cases}$$

This contraction rule will be called minimum-based contraction.
If $\Pi(\neg p) = 0$, what is obtained is the fullmeet contraction (Gärdenfors, 1988). By construction, π_p^- again corresponds to the idea of minimally changing π so as to forget p, when there is a unique $\omega \models \neg p$ such that $1 > \Pi(\neg p) = \pi(\omega)$. When there are several elements in $\{\omega \models \neg p, \pi(\omega) = \Pi(\neg p)\}$, minimal change contractions correspond to letting $\pi_p^-(\omega) = 1$ for any selection of such situation, and π_p^- corresponds to considering the envelope of the minimal change solutions. This contraction coincides exactly with a natural contraction in the sense of Boutilier and Goldzsmidt (1993).

An alternative contraction rule, called product-based contraction is:

$$\pi_p^-(\omega) = \begin{cases} \frac{\pi(\omega)}{\Pi(\neg p)} & \text{if } \omega \models \neg p \\ \pi(\omega) & \text{otherwise,} \end{cases}$$

that is the companion to the numerical product-based possibilistic revision rule. Table 4 gives the syntactic counterpart of contraction for both definitions of contraction.

	Σ'
min-based contraction	$\{(\phi, b) : b > y\} \cup \{(\phi \vee \neg p, b) : b \leq y\}$
product-based contraction	$\{(\phi, g(b)) : b > y\} \cup \{(\phi \vee \neg p, b)\}$

where $y = \text{Inc}(\Sigma \cup \{(\neg p, 1)\})$ and $g(b) = \frac{(b-y)}{(1-y)}$.

Table 4. Syntactic contraction

8. Conclusion

This paper has presented a simple generic tool for modifying prioritized belief bases. A large class of elementary changes in a prioritized belief base can be captured in terms of conditioning, and/or rescaling step. These operations can be easily implemented in practice at the syntactic level, in agreement with the semantics. Revision operators can be efficiently computed, e.g., for instance if the prioritized belief base is composed of Horn clauses, revising it (with totally reliable or uncertain input) is polynomial. Similar results have been provided for the contraction process.

Revision and contraction operations are not only useful in the handling of knowledge bases describing what is known or believed about the state of the world at some certainty degrees. It should be also of interest for modifying bases representing preference profiles under the form of a set of goals to be reached with different levels of priority, when new requirements are introduced. This second type of problem may lead to still more fruitful applications.

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