CONSTRUCTING AN ALMOST HYPERDEFINABLE GROUP

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Abstract. This paper completes the proof of the group configuration theorem for simple theories started in [BY00]. We introduce the notion of an almost hyperdefinable (poly-)structure, and show that it has a reasonable model theory. We then construct an almost hyperdefinable group from a polygroup chunk.

The group configuration theorem is one of the cornerstones of geometric stability theory. It has many variants, stating more or less that in a stable theory, if some dependence/independence situation exists, then there is a non-trivial group behind it. In a one-based theory, any non-trivial dependence/independence situation gives rise to a group. One should consult [Pil96] for these results.

The obvious question from a simplicity theorist's point of view would be how much of this can we prove in a simple theory? In the stable case, the proof could be decomposed into two main steps:

1. Obtain a generic group chunk whose elements are germs of generic functions, and whose product is the composition.
2. Apply the Weil-Hrushovski generic group chunk theorem.

The second step was generalised to simple theories by the third author in [Wag01]. In [BY00], the first author tried to generalise the first, with limited success. As it turned out, the generic chunk obtained is a generic *poly*group chunk, meaning that the generic product is defined up to a bounded (non-zero) number of possibilities. As far as we know, this is still the best we can get if we are not ready to go beyond hyperimaginaries. So there was a gap, and this paper suggests how to bridge over it.

One of the first attempts in [Tom00a] was to quotient out the multiple values by dividing by an invariant (non-type-definable) equivalence relation, thus obtaining a group chunk and eventually a group. It was noted that the group chunk theorem was still applicable regardless of the fact that we were no longer in a hyperimaginary sort. However, there was no proof that the resulting group was not trivial, as there was no bound on the coarseness of the equivalence relation. More or less in the same time, polygroups were introduced into the picture and the model theory of polygroups in simple theories was studied; in particular, the basic theory of generic elements was generalised to polygroups. This was done first in [Tom00b] for supersimple theories, and later extended by the first author to general simple theories. One of the basic examples of polygroups is the double coset space (see 2.3): if $G > H$, and H is commensurate with all its G-conjugates, then $G/\!\!/H = \{HaH : a \in G\}$ has a natural polygroup structure (with bounded products). One has the impression that this situation is analogous to the polygroup chunk. For example, the construction in [Tom00a] would correspond to descending to the group

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$G/ncl(H)$ (ncl denotes normal closure), displaying the problems of non-triviality clearly. On the other hand, if we could recover G/N for some $N \triangleleft G$ commensurate with H, no triviality problems arise, and one could hope to deduce an analogous construction given a polygroup chunk.

The breakthrough came when the third author showed how to do precisely this, if G is connected. The idea was to obtain N from H directly using Schlichting's theorem (see [Wag97]). Dividing by N, one reduces to the case where H is finite, and then a blow-up argument shows that G/N is definable in $G/\!\!/NH$, and both $G/\!\!/H \to G/\!\!/NH$ and $G/N \to G/\!\!/NH$ are surjective finite-to-one maps.

This gave the strategy for the generic chunk case: start with a polygroup chunk, divide by something finite (or bounded), and then give a blow-up argument to extend the elements finitely (or boundedly), hoping to obtain a group chunk and apply the group chunk theorem. It turned out to work quite well.

Two more hurdles were to be passed: first one had to point out the relation by which a generic polygroup chunk had to be divided before blowing up. This relation, eventually named the core equivalence, has unfortunately nothing to do with Schlichting's theorem. Second, the core equivalence is not type-definable. Luckily, it satisfies some nice properties by which it merits the name almost type-definable. So some theory of almost hyperimaginaries had to be developed, and it was shown that reasonable model theoretical tools still generalise to them. In particular, generic elements were shown to exist for almost hyperdefinable polygroups and polyspaces in a simple theory, using suitable stratified local ranks. This was done by the first author, resulting in a variety of blow-up arguments by the first and then the second author, transforming a coreless almost hyperdefinable polygroup chunk into an almost hyperdefinable group chunk (a posteriori, it is interesting to notice a familiar algebraic-geometric flavour of the blow-up construction (see 3.9) and a similarity to the classical reconstruction of the division ring from a projective geometry, expounded in 3.8). The first author showed that the group chunk theorem preserves almost hyperdefinability, so the construction was complete. As both the quotient and the blow-up are bounded-to-one, there is no triviality problem. The generic polygroup chunk can be obtained from a group configuration as in [BY00], or just as the set of generic elements of an almost hyperdefinable polygroup. In the latter case, Theorem 4.4 (proved by the first author) is used in [BY01a] to prove that if P is coreless then $P \cong G/\!\!/H$ for an almost hyperdefinable group G and a bounded $H < G$.

More applications of the group configuration theorem in simple theories, including recovering the space (and not just the group), obtaining the space interpretable in case of ω-categoricity, finding a vector space over a finite field in a one-based (nontrivial) regular type in an ω-categorical simple theory, and a proof that pseudo-linearity implies linearity in an ω-categorical simple theory, can be found in [Tom01] and [TW01]. In [BYW01] the binding group problem is reduced to the group configuration, thus characterizing almost orthogonality of almost internal types in simple theories.

Notation is mostly standard. In particular, the concatenation of two tuples a and b will be denoted by ab. Occasionally, we write $A \approx B$ to express '$A \cap B \neq \varnothing$'. For simplicity-theoretic background, see [Wag00].

1. Ultraimaginaries and almost hyperimaginaries

This section consists mostly of definitions and useful observations concerning ultraimaginaries.

Definition 1.1. Let $(I, \leq)$ be a directed partial order, and X a sort (real, imaginary, or hyperimaginary).

1. An equivalence relation on X is *invariant* if it is automorphism-invariant.
2. A *graded equivalence relation (g.e.r.)* R on X is the direct limit of reflexive symmetric type-definable relations $(R_i : i \in I)$ on X, such that:
   (a) If $i \leq j$ then R_j is coarser than R_i.
   (b) For every i, j there is k (which can be taken to be $\geq i, j$) such that $xR_iyR_jz \Longrightarrow xR_kz$.
   We then note $R = R_I = \bigvee_{i \in I} R_i$, which is an invariant equivalence relation, and say that the R_i give a *grading* of R. If we want to emphasize I, we say *I-graded* and *I-grading.*
3. The class of a modulo R is noted a_R. Even when R is just a reflexive symmetric relation we note $a_R = \{x : xRa\}$ and call this the R-class of a. For a set A we may also note $A_R = \bigcup_{a \in A} a_R$. We also write $x \in_i A$ instead of $x \in A_{R_i}$, and $\pi(x_{R_i})$ for $\exists y\, [xR_iy \wedge \pi(y)]$, where π is a partial type. If there are too many indices, we may occasionally use a/R instead.
4. An invariant equivalence relation R is *almost type-definable* if there is a type-definable symmetric and reflexive relation R' finer than R such that any R-class can be covered by boundedly many R'-classes. If in addition R is graded and R' is finer than some R_i, then we say that it is *gradedly almost type-definable (above i).*
5. A class modulo a (graded) invariant equivalence relation is called a *(graded) ultraimaginary.* A class modulo a (gradedly) almost type-definable equivalence relation is called a (graded) almost hyperimaginary.
6. Let $R = R_I$ and $R' = R'_J$ be g.e.r.'s on sorts X and Y respectively, $f(x, y)$ a type-definable relation on $X \times Y$, and put $f(x) = \{y \in Y :\models f(x, y)\}$.
   (a) Then f defines a *gradedly type-definable partial multi-map* $\bar{f} : X/R \to Y/R'$, if
      (i) There is some R'_0 such that for every $x \in X$ there is a bounded set of elements $y_\alpha \in f(x)$ with $f(x) \subseteq \bigcup_\alpha y_{\alpha R'_0}$.
      (ii) For every $i \in I$ there is $j \in J$ such that $f(x_{R_i}) \subseteq f(x)_{R'_j}$ for every $x \in X$.
   (b) If in the above we need at most a single y_α, then f defines a *gradedly type-definable partial map.*
   (c) If in addition $f(x) \neq \varnothing$ for every $x \in X$, then f defines a *gradedly type-definable total multi-map* or *map*, as the case may be.
7. An n-ary gradedly type-definable map or multi-map $\bar{f} : \prod_{m<n} \bar{X}_m \to Y/R'_J$ is *full in the k-th argument* if there is $0 \in J$ and for every $j \in J$ there is $i \in I$, such that for every $\bar{a} \in \prod_{m<n} X_m$
$$f(a_0, \ldots a_{k-1}, a_k/R_i, a_{k+1}, \ldots, a_{n-1})_{R_0} \supseteq f(\bar{a})_{R_j},$$
   where $\bar{X}_m = X_m/R_I$. It is *full* if it is full in every argument.
8. Two gradedly type-definable multi-maps $\bar{f}$ and $\bar{f}'$ are gradedly equal if there is i such that $f(x) \subseteq f'(x)_{R_i}$ and $f'(x) \subseteq f(x)_{R_i}$ for every $x \in X$.

Remark 1.2.
1. Let $R(x,y)$ be an invariant equivalence relation, and $\mathcal{R}$ a collection of type-definable relations $r \vdash R$ such that for every complete type $p \vdash R$ there is $r \in \mathcal{R}$ with $p \vdash r$ (we may take, for example, the collection of all types in R, or the collection of all type-definable relations $r \vdash R$). Let I be the set of all finite subsets of $\mathcal{R}^{<\omega}$, ordered by inclusion. For $\bar{r} \in \mathcal{R}^n$, we write $\bar{r}(x,y) = \exists z_0 \dots z_n\ x = z_0 \wedge y = z_n \wedge \bigwedge_{i<n} r_i(z_i, z_{i+1})$. For $i \in I$, put $R_i(x,y) = \bigvee_{\bar{r}\in i}(\bar{r}(x,y) \vee \bar{r}(y,x))$, a type-definable reflexive symmetric relation. Then $(R_i : i \in I)$ is a g.e.r., equivalent to R as an invariant relation. Thus every ultraimaginary can be graded.
2. If $R = R_I$ is a g.e.r. which is almost type-definable via some R', we may assume that $R' \in \mathcal{R}$. Thus R is equivalent, as an invariant relation, to a gradedly almost type-definable equivalence relation.
3. The juxtaposition of almost type-definable equivalence relations is almost type-definable, and this is witnessed by the juxtaposition of the witnesses.
4. If one has a graded equivalence relation R defined on a hyperimaginary sort X/E (rather than on a tuple of reals), one can always incorporate E into the R_i. Therefore, we may assume without loss of generality that all the g.e.r.'s we consider are defined on real tuples (possibly infinite). On the other hand, a reader who is used to reasoning directly on hyperimaginary sorts, without resorting to the original real tuples, will see that everything we do here goes through for hyperimaginary sorts as well. Thus, our results still apply in the context described in [BY01b], where there are no "reals".

Convention 1.3. In this section we shall abuse somewhat the notation a_R: We may be dealing with several equivalence relations, but we assume that for every given tuple a the applicable relation R (which may be a juxtaposition of relations on sub-tuples) is clear from the context, thereby giving meaning to a_R. We may thus write a_R and b_R when a and b are not even in the same sort, without confusion.

First, we define the type of an ultraimaginary. We usually consider sets of ultraimaginaries as a single one, modulo the juxtaposed equivalence relation. As the types of ultraimaginaries do not satisfy finite character, considering the type of an infinite set of ultraimaginaries may be misleading and will be avoided. We also only consider the types of ultraimaginaries over hyperimaginaries, and never the type of anything over an ultraimaginary — for our purposes this will not be needed.

Lemma 1.4. *For two ultraimaginaries a_R and b_R and a hyperimaginary c, the following are equivalent:*
1. *There are $a' \in a_R$ and $b' \in b_R$, such that $a' \equiv_c b'$ in the usual sense.*
2. *There is an automorphism fixing c sending a_R to b_R.*
3. *For every $a' \in a_R$ there is $b' \in b_R$ such that $a' \equiv_c b'$.*

And the following are also equivalent:
1. *There are $a' \in a_R$ and $b' \in b_R$ such that $a' \equiv^{\mathrm{Ls}}_c b'$.*
2. *a_R and b_R are equivalent modulo any bounded c-invariant equivalence relation.*
3. *There are $n < \omega$ and c-indiscernible sequences $(a^j_i : i < \omega)$ for $j \leq n$ such that aRa^0_0, $a^j_1 R a^{j+1}_0$ and $a^n_1 Rb$.*
4. *For every $a' \in a_R$ there are $n < \omega$ and c-indiscernible sequences $(a^j_i : i < \omega)$ for $j \leq n$ such that $a' = a^0_0$, $a^j_1 = a^{j+1}_0$ and $a^n_1 Rb$.*

5. *For every $a' \in a_R$ there is $b' \in b_R$ such that $a' \equiv_c^{\mathrm{Ls}} b'$.*

Proof. The first equivalence is easy. For the second (compare with [Pil, Lemma 3.15]):

(1) $\Longrightarrow$ (2) Clear.

(2) $\Longrightarrow$ (3) The relation in (3) is clearly a c-invariant equivalence relation, and one has to show it is bounded. If not, one can obtain arbitrarily long sequences of representatives of inequivalent ultraimaginaries. By a standard argument we obtain an indiscernible sequence that contradicts the assumption that the represented ultraimaginaries were inequivalent.

(3) $\Longrightarrow$ (4) Take the given a' and "propagate" indiscernible copies of it ${a'}_i^j$ along the indiscernible sequences. So at every index we have ${a'}_i^j R a_i^j$, and finally $a_1^n R b$.

(4) $\Longrightarrow$ (5) Clear.

(5) $\Longrightarrow$ (1) Clear.

QED

So it makes sense to define:

Definition 1.5. 1. Two ultraimaginaries a_R and b_R have the same type over a hyperimaginary c, denoted $a_R \equiv_c b_R$, if there are $a' \in a_R$ and $b' \in b_R$ such that $a' \equiv_c b'$ in the usual sense.

2. Two ultraimaginaries a_R and b_R have the same Lascar strong type over a hyperimaginary c, denoted $a_R \equiv_c^{\mathrm{Ls}} b_R$, if there are $a' \in a_R$ and $b' \in b_R$ such that $a' \equiv_c^{\mathrm{Ls}} b'$ in the usual sense.

Clearly, this coincides with the definitions for hyperimaginaries.

Remark 1.6. 1. We *could* have used one of the above equivalent conditions to define types *over* an ultraimaginary. However, as we said above, we will not do so.

2. In a simple (or G-compact) theory, for ultraimaginaries a_R, b_R and a hyperimaginary c, we have $a_R \equiv_c^{\mathrm{Ls}} b_R$ if and only if $a_R \equiv_{\mathrm{bdd}(c)} b_R$.

We now assume that the theory is simple. Continuing by analogy with hyperimaginaries, we define:

Definition 1.7. We say that $a_R \mathop{\smile\!\!\!\!\!\!|}_c b_R$ if there are $a' \in a_R$ and $b' \in b_R$ such that $a' \mathop{\smile\!\!\!\!\!\!|}_c b'$.

The independence of ultraimaginaries takes a nicer form if they are almost hyperimaginaries. More specifically, we have:

Lemma 1.8. *The following are equivalent:*

1. *R is an (I-gradedly) almost type-definable equivalent relation.*
2. *There is a type-definable reflexive symmetric relation R' finer than R (finer than some R_i), such that whenever $a_R \mathop{\smile\!\!\!\!\!\!|}_c b$ for some hyperimaginaries b, c, then there is $a'R'a$ such that $a' \mathop{\smile\!\!\!\!\!\!|}_c b$.*
3. *There are a cardinal κ and a type-definable reflexive symmetric relation R' finer than R (finer than some R_i), such that within an R-class there are no κ disjoint R'-classes.*

4. *There are a cardinal κ and a type-definable reflexive symmetric relation R'' finer than R (finer than some R_i), such that among any κ R-equivalent elements there are necessarily two which satisfy R''.*

Proof. (1) $\Longrightarrow$ (2) Let R' witness almost type-definability, and choose $a' \in a_R$ with $a' \mathop{\smile\!\!\!\!\!\!\vert\,}_c b$. Take $\{a_j : j \in J\}$ such that $a_R = a'_R = \bigcup_J a_{j\,R'}$; we may assume $\{a_j : j \in J\} \mathop{\smile\!\!\!\!\!\!\vert\,}_{a'c} b$, whence $\{a_j : j \in J\} \mathop{\smile\!\!\!\!\!\!\vert\,}_c b$. As $a \in a_{i\,R'}$ for some $j \in J$, the result follows.

(2) $\Longrightarrow$ (3) If there is no such κ for the R' given, we can obtain an indiscernible sequence $(a_j : j < \omega)$ of R-equivalent elements such that the classes $a_{j\,R'}$ are disjoint. Let a' realise $\operatorname{tp}(a_1/a_0)$ with $a' \mathop{\smile\!\!\!\!\!\!\vert\,}_{a_0} a_1$. Then $a_1 R a'$, so there should be $a'' R' a_1$ such that $a'' \mathop{\smile\!\!\!\!\!\!\vert\,}_{a_0} a_1$. The a_0-indiscernible sequence $(a_i : i > 0)$ contradicts this.

(3) $\Longrightarrow$ (4) $R'' = R'^2$ will do.

(4) $\Longrightarrow$ (1) By the assumption and Zorn's lemma, in any R-class there is a maximal set of representatives, no two of which satisfy R'. Its maximality means that the corresponding R'-classes cover the class.

QED

We thus get a "first-order" characterisation of independence for almost hyperimaginaries:

Lemma 1.9. *Assume that R' witnesses that R is almost type-definable. Write $p(x,y) = \operatorname{tp}(ab/c)$, $p'(x,y) = p(x_{R'}, y_{(R'^2)})$. Then $a_R \mathop{\smile\!\!\!\!\!\!\vert\,}_c b_R$ if and only if $p'(x,b)$ does not divide over c.*

Proof. $\Longrightarrow$ Take $a' \in a_R$ and $b' \in b_R$ with $a' \mathop{\smile\!\!\!\!\!\!\vert\,}_c b'$. Then $a_R \mathop{\smile\!\!\!\!\!\!\vert\,}_c b'$ and we may choose $a' \in a_{R'}$, and then similarly $b' \in b_{R'}$. Take now $b'' \equiv_{cb'} b$ such that $a' \mathop{\smile\!\!\!\!\!\!\vert\,}_c b'b''$. Then $p'(a', b'')$, so $p'(x, b'')$ does not divide over c; as $b \equiv_c b''$, neither does $p'(x,b)$.

$\Longleftarrow$ There is $a' \mathop{\smile\!\!\!\!\!\!\vert\,}_c b$ with $p'(a', b)$, meaning that there are $a'' \in a'_{R'}$ and $b'' \in b_{R'^2}$ with $p(a'', b'')$. Hence $a''_R \mathop{\smile\!\!\!\!\!\!\vert\,}_c b''_R$; sending $a''b''$ to ab by a c-automorphism shows that $a_R \mathop{\smile\!\!\!\!\!\!\vert\,}_c b_R$.

QED

In general, we do not claim that non-dividing has finite character (again, infinite tuples of ultraimaginaries are problematic). However, when R is almost type-definable, we obtain finite character for R-classes.

Some of the other ordinary properties of independence hold for ultraimaginaries. Here are some trivial but handy results:

Symmetry: Clear from the definition.

Transitivity: Some caution may be needed, as we do not allow independence over an ultraimaginary. Instead we have:

Lemma 1.10. 1. *Assume that $a_R \mathop{\smile\!\!\!\!\!\!\vert\,}_c b_R d_R$ and $b_R \mathop{\smile\!\!\!\!\!\!\vert\,}_c d_R$. Then $a_R b_R \mathop{\smile\!\!\!\!\!\!\vert\,}_c d_R$ and $a_R \mathop{\smile\!\!\!\!\!\!\vert\,}_c b_R$.*

2. *$a_R \mathop{\smile\!\!\!\!\!\!\vert\,}_c b d_R$ if and only if $a_R \mathop{\smile\!\!\!\!\!\!\vert\,}_c b$ and $a_R \mathop{\smile\!\!\!\!\!\!\vert\,}_{bc} d_R$.*

Proof. 1. We may assume that $a \mathop{\smile\!\!\!\!\!\!\vert\,}_c bd$, and then there are $b' \in b_R$ and $d' \in d_R$ with $b' \mathop{\smile\!\!\!\!\!\!\vert\,}_c d'$. We may further assume that $b'd' \mathop{\smile\!\!\!\!\!\!\vert\,}_{bdc} a$, whereby $b'd' \mathop{\smile\!\!\!\!\!\!\vert\,}_c a$, and $\{a, b', d'\}$ is a c-independent set. The statement follows.

2. Similarly.

QED

This gives reasonable sense to finite c-independent sets of ultraimaginaries. When R is almost type-definable, this generalises to infinite sets as well, by finite character.

Local character: Follows from local character for any representative (over a set of hyperimaginaries).

Finite character: As stated above, this holds for almost hyperimaginaries.

Extension: Is implied by extension for the representatives.

Morley sequences: Staying loyal to our oath not to consider infinite sets of ultraimaginaries, we let indiscernible sequences and Morley sequences in by the back door:

Definition 1.11. An *indiscernible (Morley) sequence* for a_R over c is a sequence of the form $(a_{iR} : i \in I)$ where $(a_i : i \in I)$ is an indiscernible (Morley) sequence over c for some $a' \in a_R$.

So Morley sequences clearly exist, and we have:

Lemma 1.12. *The following are equivalent:*

1. $a_R \mathop{\smile\!\!\!\!\!\!|}_c b_R$.
2. *For every indiscernible sequence* $(b_{iR} : i \in I)$ *for* b_R *over* c *there is* a' *such that* $a'_R b_{iR} \equiv_c a_R b_R$ *for all* $i \in I$.
3. *For some Morley sequence for* b_R *over* c*, for every similar (over* c*) sequence* $(b_{iR} : i \in I)$ *of arbitrary length, there is* a' *such that* $a'_R b_{iR} \equiv_c a_R b_R$.

Proof. (1) $\Longrightarrow$ (2) Consider an indiscernible sequence $(b_{iR} : i \in I)$ over c for b_R, and take $b' \in b_R$ with $a \mathop{\smile\!\!\!\!\!\!|}_c b'$. Then there is an indiscernible sequence $b_i b'_i$ for bb' over c, and therefore some a' such that $a' b'_i \equiv_c ab'$. Then $a'_R b_{iR} = a'_R b'_{iR} \equiv_c a_R b'_R = a_R b_R$.

(2) $\Longrightarrow$ (3) Clear.

(3) $\Longrightarrow$ (1) Take a sequence $(a' b_i : i \in I)$ which is long enough. By local character there is some i such that $a' \mathop{\smile\!\!\!\!\!\!|}_c b_i$. Then $a'_R \mathop{\smile\!\!\!\!\!\!|}_c b_{iR}$ and therefore $a_R \mathop{\smile\!\!\!\!\!\!|}_c b_R$.

QED

The independence theorem: Assume that $a_{0R} \mathop{\smile\!\!\!\!\!\!|}_c a_{1R}$, $b_{iR} \mathop{\smile\!\!\!\!\!\!|}_c a_{iR}$, and $b_{0R} \equiv_c^{\mathrm{Ls}} b_{1R}$. We may assume that c is boundedly closed, and that in fact $a_0 \mathop{\smile\!\!\!\!\!\!|}_c a_1$. For $i < 2$, we may assume that $b_i \mathop{\smile\!\!\!\!\!\!|}_c a'_i$ for some $a'_i \in a_{iR}$. There is $a''_i \equiv_{ca'_i} a_i$ with $b_i \mathop{\smile\!\!\!\!\!\!|}_c a''_i a'_i$, so in fact we may assume that we had $a'_i \equiv_c a_i$ to begin with. Sending a'_i to a_i over c we obtain b'_i such that $a_i b'_i \equiv_c a'_i b_i$. Then there is $b''_1 \in b'_1/R$ such that $b''_1 \equiv_c b'_0$, and we may assume that $b''_1 \mathop{\smile\!\!\!\!\!\!|}_c a_1$. Apply the independence theorem to get $b \mathop{\smile\!\!\!\!\!\!|}_c a_0 a_1$ such that

$$b_R a_{0R} \equiv_c b'_{0R} a_{0R} \equiv_c b_{0R} a'_{0R} = b_{0R} a_{0R}$$

$$b_R a_{1R} \equiv_c b''_{1R} a_{1R} = b'_{1R} a_{1R} \equiv_c b_{1R} a'_{1R} = b_{1R} a_{1R}.$$

Bounded elements: Assume that $a_R \mathop{\smile\!\!\!\!\!\!|}_c b_R$, and a'_R is bounded over $a_R c$, i.e. there are boundedly many images of a'_R by automorphisms fixing $a_R c$. Then $a'_R \mathop{\smile\!\!\!\!\!\!|}_c b_R$. Moreover, if $a_R \mathop{\smile\!\!\!\!\!\!|}_{b'} a_R$ for every $b' \in b_R$, then a_R is bounded over b_R.

Indeed, let $\bar{a}'$ be a set of representatives for all $a_R c$-conjugates of a'_R. We may assume that $a \mathop{\smile\!\!\!\!\!\!\,|}_c b$, and then we may assume that $a\bar{a}' \mathop{\smile\!\!\!\!\!\!\,|}_c b$. Then $a'' R a'$ for some $a'' \in \bar{a}'$, and $a'' \mathop{\smile\!\!\!\!\!\!\,|}_c b$. For the moreover part, assume this were not the case. Then there would be a b-indiscernible sequence $(a_i b_i : i \in I)$ in $\mathrm{tp}(ab)$ such that $a_{iR} \neq a_{jR}$ for $i \neq j$, and $b_\alpha R b$. By hypothesis $a_{iR} \mathop{\smile\!\!\!\!\!\!\,|}_b a_{iR}$, so there is a' with $a'_R a_{iR} \equiv_b a_{iR} a_{iR}$, whence $a_{0R} = a'_R = a_{1R}$, a contradiction.

2. Almost hyperdefinable polygroups

The notion of a graded equivalence relation is suitable for a study of yet more general a class of equational (that is, algebraic) (poly-)structures than the hyperdefinable ones. We shall first give a formal definition of the definability of such a structure modulo a graded equivalence relation. For the classical theory of multivalued algebraic structures, we refer the reader to [Cor93].

2.1. **Gradedly definable (poly-)structures.** An equational structure is given by a positive universal theory in a purely functional language (so the only relation is equation; constants are 0-ary functions). An equational polystructure is given by a theory whose language is purely functional aside for a binary relation $\in$, without equality, and whose axioms are universal quantifications over formulas of the form $\bigwedge x_i \in \tau_i \to \bigvee y_j \in \sigma_j$ where τ_i and σ_j are terms. The interpretation, however, is such that every function can take multiple values (at least one) for any parameters, and everything on the right-hand side of a $\in$ symbol is considered as sets (variables being singletons). Thus we can define satisfaction of the formulas of the form $x \in \tau$, and therefore of the theory (note that $x \in y$ is interpreted as $x = y$). For notational convenience, we allow ourselves to write axioms that can be transformed in an obvious manner to equivalent conjunctions of axioms of the given form. We leave to the reader to verify that this is indeed a generalisation of the notion of an equational structure. (For instance, the fact that τ is a singleton is axiomatized by $(x \in \tau \wedge y \in \tau) \to x \in y$.)

Example 2.1. A group is axiomatised in the language $\{\cdot, e, ^{-1}\}$ by the following axioms:

1. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
2. $x \cdot e = x$
3. $x \cdot x^{-1} = e$

Example 2.2. A polygroup is axiomatised in the language $\{\cdot, ^{-1}\}$ by the following axioms:

1. $t \in (x \cdot y) \cdot z \leftrightarrow t \in x \cdot (y \cdot z)$,
2. $t \in x \cdot y^{-1} \leftrightarrow x \in t \cdot y$,
3. $t \in x^{-1} \cdot y \leftrightarrow y \in x \cdot t$.

A polygroup with identity carries in addition a 0-ary function e satisfying:

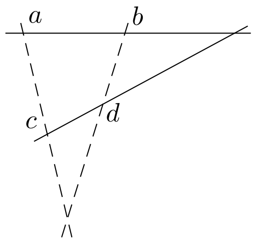
4. $t \in x \cdot e \leftrightarrow t \in x$,
5. $t \in e \cdot x \leftrightarrow t \in x$.

A map $\phi : P \to P'$ between polygroups P and P' is called a homomorphism, if $\phi(a * b) \subseteq \phi(a) * \phi(b)$. It is of type 3, if for all $a, b \in P$, $\phi^{-1}(\phi(a) * \phi(b)) = a_\phi * b_\phi$, where $x_\phi := \phi^{-1}(\phi(x))$. An isomorphism is a bijective map ϕ satisfying $\phi(a * b) = \phi(a) * \phi(b)$.

Two principal examples of polygroups (with identity) that have played an important role in the development of the blowup procedure are given next.

Example 2.3. Let G be a group, and H a (not necessarily normal) subgroup. The double coset space $G/\!\!/H$ is a polygroup with the multioperation $HaH * HbH := \{HahbH : h \in H\}$.

Example 2.4. A projective geometry is an incidence system (P, L, I) consisting of a set of points P, a set of lines L and an incidence relation $I \subseteq P \times L$ satisfying the following axioms:



1. any line contains at least three points;
2. two distinct points a, b are contained in a unique line denoted by $L(a, b)$;
3. if a, b, c, d are distinct points and $L(a, b)$ intersects $L(c, d)$, then $L(a, c)$ must intersect $L(b, d)$ (Pasch axiom), as shown in the figure.

Let $P' := P \cup \{e\}$, where e is not in P, and define:

- for $a \neq b \in P$, $a \circ b := L(a, b) \setminus \{a, b\}$;
- for $a \in P$, if any line contains exactly three points, put $a \circ a := \{e\}$, otherwise $a \circ a := \{a, e\}$;
- for $a \in P'$, $e \circ a = a \circ e := \{a\}$.

Then it is easily verified that $(P', \circ)$ is a polygroup.

Example 2.5. A polyspace is a two-sorted polystructure $\langle P, X\rangle$ in the language $\{\cdot, ^{-1}\}$, where P is a polygroup as above, and $\cdot$ is also a function symbol $P \times X \to X$, satisfying the additional axioms (we use $g, h, \dots$ for elements of P, and $x, y, z, \dots$ for elements of X):

1. $y \in (g \cdot h) \cdot x \leftrightarrow y \in g \cdot (h \cdot x)$
2. $y \in g^{-1} \cdot x \leftrightarrow x \in g \cdot y$

A polyspace with identity is such that P is a polygroup with identity, and in addition:

3. $t \in e \cdot x \leftrightarrow t \in x$.

When ambiguity might arise, we note $\cdot_P : P \times P \to P$ and $\cdot_X : P \times X \to X$.

Remark 2.6. A polygroup P gives rise to natural polyspaces:

1. $\langle P, P\rangle_L$, where $g \cdot_X h = g \cdot_P h$.
2. $\langle P, P\rangle_R$, where $g \cdot_X h = h \cdot_P g^{-1}$.

Remark 2.7.
1. In a polyspace associativity is equivalent to:
   Whenever $k \in g \cdot h$ and $y \in h \cdot x$, there is $z \in k \cdot x \cap g \cdot y$, and whenever $y \in g \cdot x$ and $z \in h \cdot x$, there is $k \in g \cdot h^{-1}$ such that $y \in k \cdot z$.
2. In a polygroup associativity is equivalent to either of the following:
   (a) Whenever $x' \in x \cdot y$ and $y' \in y \cdot z$, there is $z' \in x' \cdot z \cap x \cdot y'$.
   (b) $x \cdot y \approx z \cdot w$ if and only if $z^{-1} \cdot x \approx w \cdot y^{-1}$ (transposition property).
   Notice the similarity of (b) to the Pasch axiom of 2.4.

Proof. 1. Assume associativity. For the first assertion, there is $g' \in g^{-1}$ with $h \in g' \cdot k$, whence $y \in h \cdot x \subseteq (g' \cdot k) \cdot x = g' \cdot (k \cdot x)$ and we can find $z \in k \cdot x$ with $y \in g' \cdot z \subseteq g^{-1} \cdot z$, that is $z \in g \cdot y$.
For the second assertion, $y \in g \cdot x \subseteq g \cdot (h^{-1} \cdot z) = (g \cdot h^{-1}) \cdot z$.

Conversely, assume the two conditions hold. If $z \in (g \cdot h) \cdot x$, say $z \in k \cdot x$ for some $k \in g \cdot h$, then $h \in g' \cdot k$ for some $g' \in g^{-1}$, so there is $y \in h \cdot x \cap g' \cdot z \subseteq h \cdot x \cap g^{-1} \cdot z$, and $z \in g \cdot y \subseteq g \cdot (h \cdot x)$.
On the other hand, if $z \in g \cdot (h \cdot x)$, say $z \in g \cdot y$ for some $y \in h \cdot x$, then $x \in h^{-1} \cdot y$, so there is $k \in g \cdot (h^{-1})^{-1}$ such that $z \in k \cdot x$, and $g \in k \cdot h^{-1}$, whence $k \in g \cdot h$.

2. The first equivalence follows from part 1. For (b), assume associativity and let $x \cdot y \approx z \cdot w$. This implies that $x \in (z \cdot w) \cdot y^{-1} = z \cdot (w \cdot y^{-1})$, so there must be an $v \in w \cdot y^{-1}$ with $x \in z \cdot v$, so $v \in z^{-1} \cdot x$.
Conversely, if transposition holds, suppose $v \in (x \cdot y) \cdot z$, and let $w \in x \cdot y$ such that $v \in w \cdot z$. Then $w \in v \cdot z^{-1} \cap x \cdot y$, so $x^{-1} \cdot v \approx y \cdot z$, implying that $v \in x \cdot (y \cdot z)$.

QED

An ultradefinable equational structure S in a given theory is given by a definable set S_0, some I-g.e.r. R on S_0 such that $S = S_0/R$, and for each n-ary function symbol f a gradedly definable map $\bar{f}^S : (S_0/R)^n \to S_0/R$, such that:

> For every axiom (a disjunction of equations) there is $i \in I$, such that if we substitute elements from S_0 for the variables and interpret the function symbols as set operations on S_0 as above, then for one of the equations, there are elements on the right-hand and left-hand side sets that satisfy R_i.

Remark 2.8. This is equivalent to the "stronger" version:

> For every axiom (a disjunction of equations) and every $i \in I$ there is $j \in I$, such that if we substitute elements from S_0 for the variables, allowing the substitution of different elements for different occurrences of the same variable provided they satisfy R_i between them (in simpler words: we substitute elements for variables, up to R_i), and then interpret the right-hand and left-hand sides of each equation as sets as above, then for one of the equations, for every choice of elements from both sides, those elements satisfy R_j.

Similarly, an ultradefinable equational polystructure is given by similar information, only each $\bar{f}^S$ is a gradedly definable multi-map, and we require:

> For every axiom $(\bigwedge x_n \in \tau_n \to \bigvee y_m \in \sigma_m)$ and every $i \in I$ there is $j \in I$, such that for every substitution of elements from S for the variables in the axiom, if the conditions hold up to R_i (that is, for every n there is $x'_n \in_i x_n$ such that $x'_n \in \tau_n$), then one of the conclusions holds up to R_j.

Remark 2.9. We have formulated the definitions for one-sorted structures, the adaptations needed for many-sorted ones, such as polyspaces, being obvious.

Definition 2.10. 1. A gradedly ultradefinable structure or polystructure S is *full* if every functions $\bar{f}^S$ is full.

2. A gradedly ultradefinable polyspace $\langle P_0/R_I, X_0/R'_J\rangle$ is *weakly full* if $\bar{\cdot}_X$ is full in the first argument (that is, there is $0 \in J$ such that for every $j \in J$ there is $i \in I$ with $(g_{R_i} \cdot x)_{R'_0} \supseteq (g \cdot x)_{R_j}$ for every $g \in P_0$ and $x \in X_0$).

Lemma 2.11. *A gradedly ultradefinable polygroup is full as a polygroup or as a left (or right) polyspace.*

Proof. As $y_R \in g_R \bar{\cdot}\, x_R \leftrightarrow g_R \in y_R \bar{\cdot}\, x_R^{-1}$, there is $0 \in I$ such that $g \in y \cdot x^{-1}$ implies $y \in_0 g \cdot x$, and for all $i \in I$ there is $j \in I$ such that $y \in_i g \cdot x$ implies $g \in_j y \cdot x^{-1}$, for all $g, x, y \in P_0$. It follows that for any $y \in_i g \cdot x$ there is $g' \in g_{R_j}$ with $g' \in y \cdot x^{-1}$, whence $y \in_0 g' \cdot x$, and $(g \cdot x)_{R_i} \subseteq (g_{R_j} \cdot x)_{R_0}$. QED

Lemma 2.12. *A weakly full gradedly ultradefinable polyspace with identity is full.*

Proof. We only need to show that $\bar{\cdot}_X$ is full in the second variable. Indeed, assume that $y \in_j g \cdot x$. Then, by weak fullness, there is $g' \in g_{R_i}$ such that $y \in_0 g' \cdot x$. As a polygroup is always full, we find $e' \in e_{R_{i'}}$ with $g' \in_0 g \cdot e'$. We get $y \in_1 g \cdot e' \cdot x$; since $e' \cdot x \subseteq x_{R'_{j'}}$ for some j' depending only on i, we have $y \in_1 g \cdot x_{R_{j'}}$ QED

Lemma 2.13. *Let $\langle P_0/R_I, X_0/R'_J\rangle$ be a weakly full gradedly ultradefinable polyspace, as witnessed by R'_0. Then for every $j \in J$ there is $i \in I$ such that, whenever $g, g' \in P_0$, $x, y, y' \in X_0$ and $y \in_j g \cdot x$, $y' \in_j g' \cdot x$, there exists $h \in_i g' \cdot g^{-1}$ with $y' \in_0 h \cdot y$.*

Proof. From associativity we obtain $i' \in I$ and $j' \in J$ such that, under the assumption of the lemma, there is $h' \in_{i'} g' \cdot g^{-1}$ such that $y' \in_{j'} h' \cdot y$. By weak fullness $(h' \cdot y)_{R'_{j'}} \subseteq (h'_{R_{i''}} \cdot y)_{R'_0}$ for some $i'' \in I$. We thus find $h \in h'_{R_{i''}} \subseteq (g' \cdot g^{-1})_{R_{i'} R_{i''}}$ with $y' \in_0 h \cdot y$. Take $i \in I$ such that $h \in_i g' \cdot g^{-1}$; we can choose it independently of g, g', x, y, y'. QED

2.2. **Generic elements and stratified ranks.** We now define generic elements. For generics in hyperdefinable groups and homogeneous spaces, one may wish to compare with [Pil96], [Pil98], or [Wag01]. As a group is a private case of a polygroup, we only consider polygroups and polyspaces:

Definition 2.14.
1. Let $\langle P, X\rangle = \langle P_0/R, X_0/R'\rangle$ be a gradedly almost hyperdefinable (or, for the time being, even invariant) polyspace in a simple theory. A *generic* element of X is an element $x_{R'}$ such that whenever $x_{R'} \mathop{\smile\!\!\!\!\!\!\perp} g$ for $g \in P_0$ and $y \in g \cdot x$, then $y_{R'} \mathop{\smile\!\!\!\!\!\!\perp} g$.
2. A generic element of P is a generic element in the sense of $\langle P, P\rangle_L$.

Note that when R is type-definable, this coincides with the usual definition of a generic element of a hyperdefinable (poly-)group. The basic theory of generics holds with this definition. In particular we have:

Lemma 2.15.
1. *Let $x_{R'}$ be a generic element of X, and assume that $g_R \mathop{\smile\!\!\!\!\!\!\perp} x_{R'}$ and $y_{R'} \in g_R \cdot x_{R'}$. Then $y_{R'}$ is generic.*
2. *If $g_R \in P$ is generic, then so is g_R^{-1}.*
3. *If $g_R \in P$ is generic, then it is generic in the sense of $\langle P, P\rangle_R$ as well. In other words, whenever $h \mathop{\smile\!\!\!\!\!\!\perp} g_R$ for $h \in P_0$ and $f \in g \cdot h$, then $f_R \mathop{\smile\!\!\!\!\!\!\perp} h$.*

Proof.
1. We may assume that $g \mathop{\smile\!\!\!\!\!\!\perp} x$ and $y \mathop{\smile\!\!\!\!\!\!\perp} g$. Consider some $h \mathop{\smile\!\!\!\!\!\!\perp} y_{R'}$, say $h \mathop{\smile\!\!\!\!\!\!\perp} y'$ for some $y' \in y_{R'}$, and any $z_{R'} \in h_R \cdot y_{R'}$. Applying a $y'h$-automorphism we may assume that $gxy \mathop{\smile\!\!\!\!\!\!\perp}_{y'} h$. We have $z_{R'} \in h_R \cdot (g_R \cdot x_{R'}) = (h_R \cdot g_R) \cdot x_{R'}$, so $z_{R'} \in f_R \cdot x_{R'}$ for some $f_R \in h_R \cdot g_R$. As we have $x \mathop{\smile\!\!\!\!\!\!\perp} gh$, we may further assume that $x \mathop{\smile\!\!\!\!\!\!\perp} ghf$, and by genericity $f \mathop{\smile\!\!\!\!\!\!\perp} z_{R'}$. We have $x \mathop{\smile\!\!\!\!\!\!\perp}_f h$ and $z_{R'} \in \mathrm{bdd}(xf)$, so $z_{R'} \mathop{\smile\!\!\!\!\!\!\perp}_f h$, whereby $z_{R'} \mathop{\smile\!\!\!\!\!\!\perp} h$, as required.
2. Let g_R be generic, and let $g' \mathop{\smile\!\!\!\!\!\!\perp} g$ realise the same type as g. Choose some $h_R \in g'_R \cdot g_R$; we may assume that $h \mathop{\smile\!\!\!\!\!\!\perp} g'$. Then there is $h' \in h_R^{-1}$, and we may assume $g' \mathop{\smile\!\!\!\!\!\!\perp} hh'$. Finally, $g_R^{-1} \in h'_R \cdot g'_R$, so by the previous item it is generic (g'_R is generic, of course).

3. By the previous item.

QED

We now pass to the definition of stratified ranks and the existence of generics.

Convention 2.16. 1. We fix $\langle P, X\rangle = \langle P_0/R_I, X_0/R'_J\rangle$, a gradedly almost hyperdefinable weakly full polyspace.

2. For convenience, we give names to certain elements of P_0, I and J:
   (a) We fix $0 \in I$, $0 \in J$ and an infinite cardinal ν such that every R-class can be covered by ν R_0-classes, every R'-class can be covered by ν R'_0-classes, and every operation ($\cdot_P$, $\cdot_X$ or $^{-1}$) has at most ν values.
   (b) For an arbitrary $e \in P_0$ let $(e_\alpha : \alpha < \nu)$ be such that $(e \cdot e^{-1})_R = \bigcup_\alpha e_\alpha/R_0$. We choose $1 \in J$ such that $(g_{R_0} \cdot x)_{R'_0} \subseteq (g \cdot x)_{R'_1}$, and $x \in_1 \bigcup_\alpha e_\alpha \cdot x$ for all $g \in P_0$ and $x \in X_0$.

Local division ranks were defined on hyperimaginary sorts in [Wag01] and in [BY01b], with somewhat different approches. Here we adapt the latter to the (stratified) almost hyperimaginary case, although the former would have done just as well.

Definition 2.17. Let $k < \omega$, and $\Phi(x, y)$, $\Psi(y_0, \dots , y_k)$ be pure partial types. We say that Ψ is a *k-inconsistency witness* for Φ if $\Psi(\bar y) \wedge \bigwedge_{j<k} \Phi(x, y_j)$ is contradictory.

Clearly, Ψ is a k-inconsistency witness for Φ if and only if there are $\varphi(x, y) \in \Phi$ and $\psi(\bar y) \in \Psi$ such that ψ is a k-inconsistency witness for φ.

Definition 2.18. Let ψ be a k-inconsistency witness for $\varphi(x_{R'_0}, y)$. We define a local rank $D_P(-, \varphi, \psi)$ with values in $\omega + 1$ on (consistent) partial types (with parameters) extending $x \in X_0$:

$D_P(\pi, \varphi, \psi) \geq n + 1$ if there are a sequence $(c_\ell : \ell < \omega)$, any k-subsequence of which satisfies ψ, and $g \in P_0$, such that $D_P(\pi(x) \wedge \varphi((g \cdot x)_{R'_1}, c_\ell), \varphi, \psi) \geq n$ for all $\ell < \omega$.

Remark 2.19. The above defines local ranks with respect to ψ rather than k, as it is classically done.

We now proceed with a series of claims:

1. The statement "$D_P(\pi(x), \varphi, \psi) \geq n$" is type-definable in the parameters of π, as it states the consistency of a certain tree. By compactness, we can replace ω in the definition of $D_P(-, \varphi, \psi)$ by any infinite cardinal.
2. Ultrametric property: If $\pi(x) \vdash \bigvee_{\alpha<\mu} \pi_\alpha(x)$, where the disjunction may be infinite, then $D_P(\pi, \varphi, \psi) = \sup_\alpha D_P(\pi \wedge \pi_\alpha, \varphi, \psi)$:
   We show by induction on n that if $D_P(\pi, \varphi, \psi) \geq n$, then there is α such that $D_P(\pi \wedge \pi_\alpha, \varphi, \psi) \geq n$. For 0 this is clear. For $n + 1$, consider a sequence $(c_\ell : \ell < \mu^+ + \omega)$ and apply the induction hypothesis for each $D_P(\pi(x) \wedge \varphi((g \cdot x)_{R'_1}, c_\ell), \varphi, \psi)$. Now choose some $\alpha < \mu$ which appears infinitely often.
3. In a simple theory, every $D_P(-, \varphi, \psi) < \omega$:
   We claim that for some k' depending on φ and ψ the set $\{\varphi((z \cdot x)_{R'_1}, y_\ell) : \ell < k'\}$ is inconsistent with $\bigwedge_{\ell_0<\dots<\ell_{k-1}<k'} \psi(y_{\ell_0}, \dots , y_{\ell_{k-1}})$. Indeed if not, by compactness we could find g, a and $(c_\ell : \ell < \nu^+)$ with

$$\models \bigwedge_{\ell_0<\dots<\ell_{k-1}<\nu^+} \psi(c_{\ell_0}, \dots , c_{\ell_{k-1}}) \wedge \bigwedge_{\ell<\nu^+} \varphi((g \cdot a)_{R'_1}, c_\ell)$$

and by the choice of ν there are $(a_\alpha : \alpha < \nu)$ with $(g \cdot a)_{R'} = \bigcup_{\alpha<\nu} a_{\alpha R'_0}$. For $\ell < \nu^+$ let $a'_\ell \in_1 g \cdot a$ be such that $\varphi(a'_\ell, c_\ell)$ holds. Then there are $\alpha < \nu$ and $\ell_0 < \ldots < \ell_{k-1} < \nu^+$ such that $\bigwedge_{i<k} R'_0(a'_\alpha, a_{\ell_i})$, contradicting the choice of ψ.

If $D_P(\pi, \varphi, \psi) \geq n$ for all $n < \omega$, by compactness we obtain a tree of height $|T|^+$, with the root labeled p and the ω successors of a node η labeled $\varphi((g_\eta \cdot x)_{R'_1}, c_{\eta\frown\ell})$ for $\ell < \omega$, where $(c_{\eta\frown\ell} : \ell < \omega)$ is an indiscernible sequence over the previous parameters whose k-subsequences satisfy ψ, and such that the tree has consistent branches. It follows that a branch divides over any subset of size $|T|$ of its domain, contradicting simplicity.

4. Translation-invariance: $D_P(\pi(x), \varphi, \psi) \geq D_P(\pi((g \cdot x)_{R'_j}), \varphi, \psi)$ for all $g \in P_0$ and $j \in J$. Then in particular

$$D_P(\pi(x), \varphi, \psi) \geq D_P(\pi((g^{-1} \cdot (g \cdot x)_{R'_j})_{R'_j}), \varphi, \psi),$$

so there is equality all the way and we obtain translation-invariance. Assume then that $D_P(\pi((g \cdot x)_{R'_j}), \varphi, \psi) \geq n + 1$. Then there are $g' \in P_0$ and a sequence $(c_\ell : \ell < \nu^+)$, whose k-subsequences satisfy ψ, such that

$$\begin{aligned} n &\leq D_P(\pi((g \cdot x)_{R'_j}) \wedge \varphi((g' \cdot x)_{R'_1}, c_\ell), \varphi, \psi) \\ &= D_P(\exists y\ [y \in_j g \cdot x \wedge \pi(y)] \wedge \exists y'\ [y' \in_1 g' \cdot x \wedge \varphi(y', c_\ell)], \varphi, \psi). \end{aligned}$$

Consider some realisation of the above. Then $y \in_j g \cdot x$ and $y' \in_1 g' \cdot x$; by Lemma 2.13 for some $i \in I$ depending only on j there is $h \in_i g' \cdot g^{-1}$ with $y' \in_0 h \cdot y$. However, one can find elements $(h_\alpha : \alpha < \nu)$ such that $(g' \cdot g^{-1})_R \subseteq \bigcup_{\alpha<\nu} h_{\alpha R_0}$. Then for one of them, say h, we have $y' \in (h_{R_0} \cdot y)_{R'_0}$, and by the ultrametric property

$$\begin{aligned} n &\leq D_P(\pi((g \cdot x)_{R'_j}) \wedge \varphi((g' \cdot x)_{R'_1}, c_\ell), \varphi, \psi) \\ &= D_P(\exists y\ [y \in_j g \cdot x \wedge \pi(y) \wedge \varphi((h_{R_0} \cdot y)_{R'_0}, c_\ell)], \varphi, \psi) \\ &\leq D_P(\exists y\ [y \in_j g \cdot x \wedge \pi(y) \wedge \varphi((h \cdot y)_{R'_1}, c_\ell)], \varphi, \psi) \end{aligned}$$

for infinitely many c_ℓ. By the induction hypothesis we obtain

$$D_P(\pi(x) \wedge \varphi((h \cdot x)_{R'_1}, c_\ell), \varphi, \psi) \geq n.$$

So $D_P(\pi(x), \varphi, \psi) \geq n + 1$.

5. Let $p(x) \in S(A)$ imply $x \in X_0$. Then $D_P(p, \varphi, \psi) = D_P(p(x_{R'_j}), \varphi, \psi)$ for all $j \in J$. If $p' \in S(A)$ extends $p(x_{R'_j})$, then $D_P(p, \varphi, \psi) = D_P(p', \varphi, \psi)$ for all φ and ψ.

The first assertion is an immediate corollary of the previous claim. For the second, we see first that $D_P(p, \varphi, \psi) \geq D_P(p', \varphi, \psi)$. For the other direction let $a \models p'$, so there is $b \models p$ with $aR'_i b$. But then we see that $p = \mathrm{tp}(b/A) \vdash p'(x_{R'_i})$, and we conclude by symmetry.

6. Local ranks witness dividing: Assume that $p \subseteq q$ is an extension of complete types over $A \subseteq B$. Then the partial type $q(x_{R'_0})$ does not divide over A if and only if $D_P(p, \varphi, \psi) = D_P(q, \varphi, \psi)$ for every φ and ψ.

Equivalently, if $x \in X_0$ and $A \subseteq B$, then $x_{R'} \mathop{\smile\!\!\!\!\!\!\mid}_A B$ if and only if $D_P(x/A, \varphi, \psi) = D_P(x/B, \varphi, \psi)$ for every φ, ψ.

$\Longrightarrow$ Assume that $q(x_{R'_0})$ does not divide over A. Then there is a complete type $q' \in S(B)$ extending it which does not divide over A; put $p' = q' \restriction A$.

By standard arguments $D_P(q', \varphi, \psi) = D_P(p', \varphi, \psi)$, and we finish using the previous claim.

$\Longleftarrow$ If $q(x_{R'_0}, B)$ divides over A, then there is $\varphi(x, b) \in q(x)$ such that $\varphi(x_{R'_0}, b)$ divides over A. So we find a formula ψ such that $\psi(\bar{y}) \wedge \bigwedge_{\ell<k} \varphi(x_{R'_1}, y_\ell)$ is contradictory, and an A-indiscernible sequence $(c_\ell : \ell < \nu^+)$ in $\mathrm{tp}(b/A)$ whose k-subsequences satisfy ψ. Then for some $\alpha < \nu$ we have for infinitely many ℓ:

$$\begin{aligned} D_P(q, \varphi, \psi) &\le D_P(p \wedge \varphi(x, b), \varphi, \psi) = D_P(p \wedge \varphi(x, c_\ell), \varphi, \psi) \\ &\le D_P(p \wedge \varphi((e_\alpha \cdot x)_{R'_1}, c_\ell), \varphi, \psi) \le D_P(p, \varphi, \psi) - 1. \end{aligned}$$

7. Generic elements exist in every orbit (over the parameters needed to define this orbit).
The usual proof holds: for $a_0 \in X_0$, the orbit of a_{0R} is precisely $(P_0 \cdot a_0)/R$. As we always work over $\varnothing$ we add constants such that $P_0 \cdot a_0$ is type-definable over $\varnothing$. Enumerate the possible pairs (φ, ψ) and choose a type $p \vdash x \in P_0 \cdot a_0$ of maximal $D_P(-, \varphi, \psi)$ ranks in lexicographic order according to this enumeration. Let $a \models p$, and assume that $g \mathop{\smile\!\!\!\!\!\mid} a_R$ and $b_R \in g_R \cdot a_R$. By the claims above we obtain

$$D_P(a, \varphi, \psi) = D_P(a/g, \varphi, \psi) = D_P(b/g, \varphi, \psi) \le D_P(b, \varphi, \psi).$$

By the maximality property we have equality; putting $q = \mathrm{tp}(b/g)$ we see that $q(x_{R'_0})$ does not divide over $\varnothing$. Therefore $b_R \mathop{\smile\!\!\!\!\!\mid} g$.

QED

Remark 2.20. 1. One may have noticed that we defined stratified ranks straight away, without "bothering" to define the unstratified analogues, which is after all common practice. In fact, as far as we know, we need the stratification in order to bridge over the non-type-definable bit of almost hyperimaginary types. In order to get some intuition, consider that ordinary local ranks on a hyperimaginary sort X_0/E are stratified ranks of the full polyspace over the trivial polygroup $P = \{e\}$, with multiplication $y \in e \cdot x$ given by $E(x, y)$. Conversely, if $\langle P_0/R, X_0/R' \rangle$ is an almost hyperdefinable polyspace and P_0/R acts on X_0/R' trivially and weakly fully, then R' is type-definable. In the strictly almost hyperdefinable case, a non-trivial (weakly) full action is needed.

2. For almost hyperimaginaries modulo R (as witnessed by R_0), one might be tempted to define unstratified ranks with respect to a formula φ and a k-inconsistency witness ψ by
$D(\pi, \varphi, \psi) \ge n + 1$ if there is a sequence $(c_\ell : \ell < \omega)$, any k-subsequence of which satisfies ψ, such that $D(\pi(x) \wedge \varphi(x_{R_0}, c_\ell), \varphi, \psi) \ge n$ for all $\ell < \omega$,
and $D(\pi(x_R), \varphi, \psi) = \max_{i \in I} D(\pi(x_{R_i}), \varphi, \psi)$; since $D(\pi(x_R), \varphi, \psi) \le D(x{=}x, \varphi, \psi) < \omega$, these ranks are finite and R-invariant. For almost hyperimaginaries, they witness dividing: $a_R \mathop{\smile\!\!\!\!\!\mid}_A B$ if and only if $D(a_R/A, \varphi, \psi) = D(a_R/AB, \varphi, \psi)$ for all φ and inconsistency witnesses ψ.
Indeed suppose first that $a_R \mathop{\smile\!\!\!\!\!\mid}_A B$, and fix φ, ψ; we may clearly assume that $a \mathop{\smile\!\!\!\!\!\mid}_A B$. There is $i \in I$ and $a' \in a_{R_i}$ with $D(a'/A, \varphi, \psi) = D(a_R/A, \varphi, \psi)$. Put $p(x) = \mathrm{tp}(a/A)$ and $q(x) = \mathrm{tp}(a/AB)$. Then $\bigcup_{\ell<\omega} q(x, B_\ell)$ has a realization $a_0 \models p$ for any indiscernible sequence $(B_\ell : \ell < \omega)$ in $\mathrm{tp}(B/A)$, and

there is $a_0' \models p'$ with $a_0 R_i a_0'$. Hence $p'(x) \cup \bigcup_{\ell<\omega} q(x_{R_i}, B_\ell)$ is realized by a_0' and thus consistent: $q(x_{R_i}, B) \cup p'(x)$ does not divide over A, and can be extended to a non-dividing extension $q'(x) \in S(AB)$ of p'. Then $a \models q'(x_{R_i})$, and

$$\begin{aligned} D(a_R/AB, \varphi, \psi) &\geq D(q'(x_{R_i}), \varphi, \psi) \geq D(q'(x), \varphi, \psi) \\ &= D(p'(x), \varphi, \psi) = D(a_R/A, \varphi, \psi) \end{aligned}$$

The converse is similar to the proof for stratified ranks.
However, since the condition "$D(\pi(x_R), \varphi, \psi) \geq n$" is not type-definable in the parameters of π, it is not clear how useful this rank is.

Generic elements in a (poly-)group lead naturally to generic chunks. Note that this is precisely what the object given in [BY00, Theorem 4.9] satisfies.

Definition 2.21. Let $S = S_0/R$ be an I-gradedly almost hyperdefinable set, $\bar{\cdot} : S^2 \to S$ a gradedly type-definable partial (multi-)map (induced by a partial (multi-)map $\cdot : S_0^2 \to S_0$), defined for every pair of independent elements (so $a \cdot b \neq \varnothing$ for $a_R \mathop{\smile\!\!\!\!\!\!\mid\,} b_R$), and $\overline{^{-1}} : S \to S$ a gradedly type-definable (multi-)map. Then $\langle S, \bar{\cdot}, \overline{^{-1}} \rangle$ is a *gradedly almost hyperdefinable generic (poly-)group chunk* (or simply, a *generic chunk*) if for some $0 \in I$

1. *Generic independence:* If $a \mathop{\smile\!\!\!\!\!\!\mid\,} b$ and $c \in a \cdot b$, then $c_R \mathop{\smile\!\!\!\!\!\!\mid\,} a$ and $c_R \mathop{\smile\!\!\!\!\!\!\mid\,} b$.
2. *Generic associativity:* Whenever a, b, c are independent and $d \in a \cdot (b \cdot c)$, then there is $d' \in (a \cdot b) \cdot c$ such that $d \in_0 d'$, and vice versa.
3. *Inverse:* $a_R \in b_R \mathbin{\bar{\cdot}} c_R$ if and only if $b_R \in a_R \mathbin{\bar{\cdot}} c_R^{\overline{-1}}$ if and only if $c_R \in b_R^{\overline{-1}} \mathbin{\bar{\cdot}} a_R$, gradedly (i.e. in the sense of remark 2.8).

Remark 2.22.
1. For every $i \in I$ there is $j \in I$ such that for independent $a, b, c \in S_0$, whenever $d \in_i a \cdot (b \cdot c)$, there is $d' \in (a \cdot b) \cdot c$ with $d' \in_j d$.
2. Axiom 3. implies that we may choose 0 big enough such that in addition $a \in b \cdot c$ implies $b \in_0 a \cdot c^{-1}$ and $c \in_0 b^{-1} \cdot a$, and similarly for the other implications.
3. In a gradedly almost hyperdefinable polygroup chunk, the set $\{(a, b) \in S_0^2 : a_R \mathop{\smile\!\!\!\!\!\!\mid\,} b_R\}$ is type-definable, by Lemma 2.24 below.

The stratified ranks introduced above along with standard techniques and results (see [Wag97] or [Wag01]) allow us to prove:

Fact 2.23. *Let $P = P_0/R$ be a gradedly almost hyperdefinable (poly-)group. Then the generic elements of P are precisely those who have the same stratified ranks as P (over $\varnothing$), and thus the set $g(P)$ of all the representatives of generic elements is a gradedly almost hyperdefinable generic (poly-)group chunk.*

This also means that the independence of generic elements is type-definable, using stratified ranks. For generic chunks, we have:

Lemma 2.24. *Let $S = S_0/R_I$ be a generic chunk, and let w be some sort. Then there is a partial type $\Phi(x, w)$ such that $\Phi(a, d)$ if and only if $a \in S_0$ and $a_R \mathop{\smile\!\!\!\!\!\!\mid\,} d$. Moreover, if R' is an almost type-definable equivalence relation on w, then there is $\Phi'(x, y)$ such that $\Phi(a, d)$ if and only if $a \in S_0$ and $a_R \mathop{\smile\!\!\!\!\!\!\mid\,} d_{R'}$.*

Proof. Fix a complete type p_0 in S_0, let R_0 witness that R_I is almost type-definable, and choose $1 \in I$ such that $c \in_0 b^{-1} \cdot a_{R_0}$ implies $a \in_1 b \cdot c$. We claim that the partial type

$$\Phi(x,w) := \exists y,z\ [p_0(y) \wedge y \mathop{\smile\!\!\!\!\!\!|}\, zw \wedge x \in_1 y \cdot z]$$

(such a partial type exists, as the independence of a complete type is type-definable) will work:

$\Longrightarrow$ Assume that $a' \in_0 a$ with $a' \mathop{\smile\!\!\!\!\!\!|}\, d$. Take $b \models p_0$ independently of both, and obtain $a' \mathop{\smile\!\!\!\!\!\!|}\, bd$. Pick $c \in b^{-1} \cdot a'$; as $c_R \mathop{\smile\!\!\!\!\!\!|}\, b$ there is $c' \in c_{R_0}$ with $c' \mathop{\smile\!\!\!\!\!\!|}\, b$; we choose it such that $c' \mathop{\smile\!\!\!\!\!\!|}_{a'b} d$, which eventually gives $b \mathop{\smile\!\!\!\!\!\!|}\, c'd$. Now $c' \in_0 b^{-1} \cdot a_{R_0}$ implies $a \in_1 b \cdot c'$, whence $\Phi(a,d)$ holds.

$\Longleftarrow$ Assume that $\Phi(a,d)$ holds, with b,c witnessing this as y,z respectively. Pick $a' \in b \cdot c$ with $a' \mathop{\smile\!\!\!\!\!\!|}_{bc} d$; we get $a' \mathop{\smile\!\!\!\!\!\!|}_c d$. As $a'_R \mathop{\smile\!\!\!\!\!\!|}\, c$ there is $a'' \in_0 a'$ with $a'' \mathop{\smile\!\!\!\!\!\!|}\, c$; we choose it such that $a'' \mathop{\smile\!\!\!\!\!\!|}_{a'c} d$, whence $a'' \mathop{\smile\!\!\!\!\!\!|}\, d$. Finally, as $a'' R a$, we get $a_R \mathop{\smile\!\!\!\!\!\!|}\, d$.

For the moreover part, assume that R'_0 witnesses that R' is almost type-definable. Then clearly $\exists w'\ [\Phi(x,w') \wedge w' \in_0 w]$ will do. QED

2.3. **The core equivalence.** A polygroup is in a sense a group with some added background noise. The core is an essential part of this noise, which can be eliminated while doing minimal changes (we divide by a bounded normal sub-polygroup, or by an equivalence relation with bounded classes). However, even on a hyperdefinable polygroup, the core equivalence may well be only gradedly almost type-definable, whence the need to consider almost hyperimaginaries. In the special case where $P = G/\!\!/H$, the double quotient of a $\varnothing$-connected type-definable group by a non-normal relatively-definable subgroup commensurable with all its conjugates, we have an alternative construction which keeps everything type-definable (or even relatively definable). In this case we divide by a finite normal sub-polygroup whose existence is proved using Schlichting's theorem, but this is not discussed here.

Definition 2.25. Let $P = P_0/R_I$ be a gradedly almost hyperdefinable polygroup.

1. For $a,b \in P_0$ and $i \in I$, we say that $a \sim_{i1} b$ if there is a generic $g \mathop{\smile\!\!\!\!\!\!|}\, ab$ such that $a,b \in_i g \cdot h$ for some h (which must also be generic). $\sim_{in}$ is the n-closure of $\sim_{i1}$, and $\sim$ is $\bigvee_{in} \sim_{in}$. We shall show that $\sim$ is an $(I\times\omega)$-gradedly almost type-definable equivalence relation, which we call the *core equivalence*.
2. We define the *core* N of P as follows: $N_{i1} \subseteq P_0$ is the set of all a such that $a \in_i g \cdot g^{-1}$ for some generic $g \mathop{\smile\!\!\!\!\!\!|}\, a$, and $N_{in} = N_{i1}^n$. One verifies that $\bigcup_i N_{in}$ is a union of R-classes closed under inverse for all $n < \omega$, so we can put $N_n = \bigcup_i N_{in}/R = N_1^n$, and $N = \bigcup_n N_n \leq P$, the sub-polygroup generated by N_1.
3. P is *coreless* if the core equivalence is the same as R, that is for every $(i,n) \in I \times \omega$ there is $j \in I$ such that R_j is coarser than $\sim_{in}$.

Lemma 2.26. *Let $P = P_0/R$ be an I-gradedly almost hyperdefinable polygroup.*

1. *$\sim$ is an $(I \times \omega)$-gradedly almost type-definable equivalence relation on P coarser than R, and every $\sim$-class contains boundedly many R-classes (that is, if $a_R \sim b_R$ then a_R and b_R are interbounded as almost hyperimaginaries).*
2. *For every $m,n < \omega$ and $i \in I$ there is $j \in I$ such that for every $a \sim_{in} a'$, $b \sim_{im} b'$ and $c \in a \cdot b$ there is $c' \in a' \cdot b'$ such that $c \sim_{j,n+m} c'$.*
3. *For every $n < \omega$, $\bar m \in \omega^{n+1}$ and $i \in I$ there is $j \in I$ such that whenever $a'_\ell \sim_{i,m_\ell} a_\ell$ for every $\ell \leq n$, where $\{a_{\ell R} : \ell \leq n, \ell \neq k\}$ is a set of independent*

generics for some $k \leq n$, and $b \in \prod_\ell a_\ell$, $b' \in \prod_\ell a'_\ell$ with $bb' \mathop{\smile\!\!\!\!\!\!\;|}\, \{a_{\ell R} : \ell \neq k\}$, then $b \sim_{j,n+\sum_\ell m_\ell} b'$.

4. *$P_0/\!\sim$ is a quotient polygroup of P_0/R.*
5. *For $a, b \in P_0$, we have $a \sim_{in} b$ for some i if and only if $a \cdot b^{-1} \cap N_{jn} \neq \varnothing$ for some j; moreover for any n we can bound j in terms of i and vice versa.*
6. *N is a normal sub-polygroup, i.e. $a_R \cdot N = N \cdot a_R$ for all $a \in P_0$. Define $a \sim'_{in} b$ as $a \cdot b^{-1} \cap N_{in} \neq \varnothing$, and $\sim' = \bigvee_{in} \sim'_{in}$. Then there is a natural bijection between $P_0/\!\sim'$ and P/N as sets, and $P_0/\!\sim'$ and $P_0/\!\sim$ are gradedly isomorphic.*
7. *$P_0/\!\sim$ is coreless. Any almost hyperdefinable group is coreless.*
8. *If P is coreless, then inverses are unique, and a unique identity exists. This is to say that there are $i \in I$ and $e \in P_0$ such that $(a^{-1})^{-1}, e \cdot a, a \cdot e \subseteq a_{R_i}$ for every $a \in P_0$.*

Proof. Let R_0 witness almost type-definability of R.

1. Type definability of $\sim_{i1}$ is shown using stratified ranks, and $\sim$ is clearly a g.e.r. coarser than R. Assume now that $a \sim_{i1} b$; there is generic $g_R \mathop{\smile\!\!\!\!\!\!\;|}\, ab$ and h with $a, b \in_i g \cdot h$. So $g_R \mathop{\smile\!\!\!\!\!\!\;|}_a b$, whereby $g_R h_R \mathop{\smile\!\!\!\!\!\!\;|}_a b$ and finally $b_R \mathop{\smile\!\!\!\!\!\!\;|}_a b_R$, by boundedness of the product. Now for any $a' \in a_R$ we have $b \sim_{j1} a'$ for some j, whence $b_R \mathop{\smile\!\!\!\!\!\!\;|}_{a'} b_R$ by the previous argument. This shows that b_R is bounded over a_R, and every $\sim$-class contains only boundedly many R-classes. Thus $\sim_{01}$ witnesses almost type-definability for $\sim$.
2. By induction and symmetry, it suffices to prove that if $a \sim_{i1} a'$ and $c \in a \cdot b$ then there is $c' \in a' \cdot b$ with $c \sim_{j1} c'$. Take g, h such that g is generic, $a, a' \in_i g \cdot h$, and $aa' \mathop{\smile\!\!\!\!\!\!\;|}\, g$; we may assume that $gh \mathop{\smile\!\!\!\!\!\!\;|}_{aa'} bc$. Then $c \in_{i'} (g \cdot h) \cdot b$ for some i', and we can apply associativity to conclude that there is $h' \in h \cdot b$ such that $c \in_{i''} g \cdot h'$ for some i''. By associativity again there must be some $c' \in_j a' \cdot b \cap g \cdot h'$ for some $j \geq i''$. But we also have $g \mathop{\smile\!\!\!\!\!\!\;|}\, aa'bc$, so $g \mathop{\smile\!\!\!\!\!\!\;|}\, cc'_R$; increasing j we may assume $g \mathop{\smile\!\!\!\!\!\!\;|}\, cc'$, and $c \sim_{j1} c'$.
3. We use induction on n. For $n = 0$ the assertion is trivial, so consider the case $n > 0$. By symmetry we may assume that $k \neq n$. There are $c \in \prod_{\ell<n} a_\ell$ and $c' \in \prod_{\ell<n} a'_\ell$, such that $b \in c \cdot a_n$ and $b' \in c' \cdot a'_n$. As $bb' \mathop{\smile\!\!\!\!\!\!\;|}_{a''_n} \{a_{\ell R} : \ell \neq k, n\}$ for some $a''_n \in_0 a_n$ with $a''_n \mathop{\smile\!\!\!\!\!\!\;|}\, \{a_{\ell R} : \ell \neq k, n\}$, and a'_{nR} is bounded over a_{nR}, whence c_R, c'_R are bounded over $bb'a''_n$, transitivity yields $c_R c'_R \mathop{\smile\!\!\!\!\!\!\;|}\, \{a_{\ell R} : \ell \neq k, n\}$. By inductive hypothesis (for some suitably independent representatives) $c \sim_{i', n-1+\sum_{\ell<n} m_\ell} c'$ for some i'; by the previous item there are $j' \geq 0$ and $b'' \in c \cdot a_n$ with $b'' \sim_{j', n-1+\sum_{\ell \leq n} m_\ell} b'$. Now $bb' \mathop{\smile\!\!\!\!\!\!\;|}\, a_{nR}$ implies $bb''_{R_0} \mathop{\smile\!\!\!\!\!\!\;|}\, a_{nR}$ by part 1., and we conclude that $b \sim_{j, n+\sum m_\alpha} b'$ for some j.
4. Follows from parts 1. and 2.
5. By associativity it suffices to prove the assertion for $n = 1$.
   - $\Longrightarrow$ Let $a \sim_{i1} b$, as witnessed by $g \mathop{\smile\!\!\!\!\!\!\;|}\, ab$ and $a, b \in_i g \cdot h$. By associativity there are i' depending on i, and elements $b' \in b^{-1}$, $g' \in g^{-1}$, such that there exists $c \in a \cdot b' \cap (g \cdot g')_{R_{i'}}$. Then $g \mathop{\smile\!\!\!\!\!\!\;|}\, abc_R$, so there is $c' \in_0 c$ such that $c' \mathop{\smile\!\!\!\!\!\!\;|}\, g$, and $c' \in_{i''} g \cdot g^{-1}$, whence $c' \in N_{i''1}$. Therefore $c \in N_{j1}$ for yet a greater j, that can be bounded in terms of i.
   - $\Longleftarrow$ Assume that $c \in a \cdot b^{-1} \cap N_{j1}$, and pick $g \mathop{\smile\!\!\!\!\!\!\;|}\, c$ with $c \in_j g \cdot g^{-1}$; we may assume that $abc \mathop{\smile\!\!\!\!\!\!\;|}\, g$, and let $g' \in g^{-1}$ be such that $c \in_j g \cdot g'$. Now $a \in (c \cdot b)_R$ and $g' \in (g'' \cdot c)_R$, for another $g'' \in g^{-1}$, so associativity yields

$h \in (g'' \cdot a)_R \cap (g' \cdot b)_R$, whence $a, b \in_i g \cdot h$ for some i that depends only on j.

6. The previous item shows that the identity of P_0 induces a graded isomorphism of $P_0 /\!\sim$ and $P_0 /\!\sim'$, and that $N \neq \varnothing$. This also shows that $\sim'$ is a gradedly almost type-definable equivalence relation, and $P_0 /\!\sim'$ is a quotient polygroup of P. It is clear that $P/N = P_0 /\!\sim'$ as sets, so P/N is a quotient polygroup as well, and N must be normal. (The normality of N can also be proved directly from its definition.)
7. Consider the quotient $P_0 /\!\sim\, = P/N$. Assume that $g_\sim$ is generic, $a, b \in_{in} g \cdot h$, and $g \downarrow ab$. As $\sim$-classes contain boundedly many R-classes, this means that g_R is generic in P. Take $a \sim_{in} a'$ and $b \sim_{in} b'$ such that $a', b' \in g \cdot h$; as $g \downarrow a'_R b'_R$ by boundedness, there are $a'' \in_0 a'$ and $b'' \in_0 b'$ with $g \downarrow a'' b''$, whence $a \sim_{in} a' \sim_{01} b' \sim_{in} b$, and $a \sim_{j,2n+1} b$. The corelessness of groups is clear.
8. Let $a \in P_0$ and $a' \in (a^{-1})^{-1}$. Choose a generic $g \downarrow aa'$, some $g' \in g^{-1}$ and $h \in g' \cdot a'$. Then $h \in g^{-1} \cdot a'$ implies $a' \in_i g \cdot h$, and $h \in g' \cdot (a^{-1})^{-1}$ implies $g' \in_i h \cdot a^{-1}$, for some fixed $i \in I$, whence $h \in_{i'} g' \cdot a \subseteq g^{-1} \cdot a$ and $a \in_{i''} g \cdot h$, for some $i'' \geq i' \geq i$. Thus $a \sim_{i''1} a'$.
   For e just take any element of N (and note that N/R is a singleton in a coreless polygroup).

QED

The same holds more or less for generic chunks.

Definition 2.27. Let $S = S_0/R$ be an I-gradedly almost hyperdefinable generic polygroup chunk.

1. For $a, b \in S_0$ and $i \in I$, we say that $a \sim_{i1} b$ if there is $g \downarrow ab$ such that $a, b \in_i g \cdot h$ for some h. The n-closure of $\sim_{i1}$ is $\sim_{in}$, and $\sim$ is $\bigvee_{in} \sim_{in}$. We shall show that $\sim$ is an $(I \times \omega)$-gradedly almost type-definable equivalence relation, which we call the *core equivalence*.
2. S is *coreless* if the core equivalence is the same as R, that is for every $(i, n) \in I \times \omega$ there is $j \in I$ such that R_j is coarser than $\sim_{in}$.

Lemma 2.28. *Let $S = S_0/R$ be an I-gradedly almost hyperdefinable polygroup chunk.*

1. *$\sim$ is an $(I \times \omega)$-gradedly almost type-definable equivalence relation on S coarser than R, and every $\sim$-class contains boundedly many R-classes.*
2. *For every $m, n < \omega$ and $i \in I$ there is $j \in I$ such that for every $a \sim_{in} a'$, $b \sim_{im} b'$ and $c \in a \cdot b$ there is $c' \in a' \cdot b'$ such that $c \sim_{j,n+m} c'$.*
3. *For every $n < \omega$, $\bar{m} \in \omega^{n+1}$ and $i \in I$ there is $j \in I$ such that whenever $a'_\ell \sim_{i,m_\ell} a_\ell$ for every $\ell \leq n$, where $\{a_{\ell R} : \ell \leq n\}$ is independent, and $b \in \prod_\ell a_\ell$, $b' \in \prod_\ell a'_\ell$ with $bb' \downarrow \{a_{\ell R} : \ell \neq k\}$ for some $k \leq n$, then $b \sim_{j,n+\sum_\ell m_\ell} b'$.*
4. *$S_0 /\!\sim$ is coreless. Any almost hyperdefinable group chunk is coreless.*
5. *In a coreless polygroup chunk there is some i such that $(a^{-1})^{-1} \subseteq a_{R_i}$ for every a.*

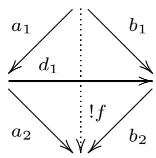
Proof. Similar to 2.26. Use 2.24 to get the definability of independence, and interboundedness of group chunk elements in a $\sim$-class to obtain the necessary independencies for the product to be defined. QED

3. Blowing up generic chunks

The variant of the blow-up construction described in this section is due to the first author. For alternative constructions, see the remarks after Theorem 3.6 and [Tom01].

Convention 3.1. We fix a I-gradedly almost hyperdefinable coreless generic polygroup chunk $S = \langle S_0/R, \cdot, ^{-1} \rangle$, as well as some R_0 which witnesses that R is almost type-definable. We shall no longer distinguish between the multiplication and inverse on S_0, and the maps induced on S_0/R.

Lemma 3.2. *For every $i \in I$ there is $j \in I$ such that whenever $a_1, a_2, b_1, b_2, d_1 \in S_0$, the triplet $\{a_{1R}, b_{1R}, b_{2R}\}$ is independent, and $d_1 \in (a_1^{-1} \cdot b_1)_{R_i} \cap (a_2 \cdot b_2^{-1})_{R_i}$, then there is $f \in a_1 \cdot a_2 \cap (b_1 \cdot b_2)_{R_j}$.*
Moreover, if we have also $c_1, c_2, d_2 \in S_0$ such that $c_{1R} \mathop{\smile\hskip-0.9em\downarrow} c_{2R}$, $a_{1R} \mathop{\smile\hskip-0.9em\downarrow} b_{1R} b_{2R} c_{1R} c_{2R}$ and $d_2 \in (a_1^{-1} \cdot c_1)_{R_i} \cap (a_2 \cdot c_2^{-1})_{R_i}$, and we take $f' \in a_1 \cdot a_2 \cap (c_1 \cdot c_2)_{R_j}$, then $f \in_1 f'$ for some $1 \in I$ dependent only on i. In particular, f is unique up to R_1:

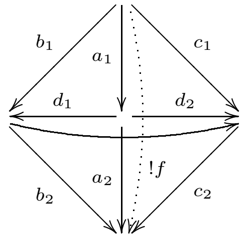


Proof. By genericity of the product, independence of $\{a_{1R}, b_{1R}, b_{2R}\}$ implies independence of $\{b_{1R}, d_{1R}, b_{2R}\}$. As for polygroups, there is an equivalent statement of associativity for polygroup chunks (with the same proof), which applied to the product $b_{1R} \cdot d_{1R}^{-1} \cdot a_{2R}$ asserts that since $a_{1R} \in b_{1R} \cdot d_{1R}^{-1}$ and $b_{2R} \in d_{1R}^{-1} \cdot a_{2R}$, there is $f_R \in a_{1R} \cdot a_{2R} \cap b_{1R} \cdot b_{2R}$. As the associativity axiom is graded, we can find j such that in fact $f \in (a_1 \cdot a_2)_{R_j} \cap (b_1 \cdot b_2)_{R_j}$, and j depends only on i. Re-choosing f and j, we may assume that $f \in a_1 \cdot a_2 \cap (b_1 \cdot b_2)_{R_j}$.
Now let c_1, c_2, d_2, f' be as stated in the moreover clause. Then $a_{1R} \mathop{\smile\hskip-0.9em\downarrow} b_{1R} b_{2R} c_{1R} c_{2R}$ implies $a_{1R} \mathop{\smile\hskip-0.9em\downarrow} f_R f'_R$. As $f, f' \in a_1 \cdot a_2$, they are core-equivalent, and we finish by corelessness.
Finally, to see uniqueness of f up to R_1, we just take $c_1 = b_1$, $c_2 = b_2$ and $d_2 = d_1$. QED

Corollary 3.3. *Let $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$ satisfy:*

1. *$\{a_{1R}, a_{2R}, d_{1R}, d_{2R}\}$ is independent.*
2. *$d_1 \in (a_1^{-1} \cdot b_1)_{R_i} \cap (a_2 \cdot b_2^{-1})_{R_i}$ and $d_2 \in (a_1^{-1} \cdot c_1)_{R_i} \cap (a_2 \cdot c_2^{-1})_{R_i}$.*
3. *$b_{1R}^{-1} \cdot c_{1R} \cap d_{1R}^{-1} \cdot d_{2R} \cap b_{2R} \cdot c_{2R}^{-1} \neq \varnothing$.*

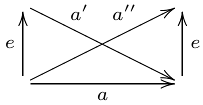
Then the hypotheses of the moreover clause of 3.2 are satisfied:



Proof. Let $d_{3R} \in {b_1}_R^{-1} \cdot c_{1R} \cap {d_1}_R^{-1} \cdot d_{2R} \cap b_{2R} \cdot {c_2}_R^{-1}$. Then we obtain that $\{a_{1R}, b_{1R}, b_{2R}, d_{3R}\}$ is independent. First, this gives us $a_{1R} \perp\!\!\!\perp b_{1R}b_{2R}d_{3R}$ which implies $a_{1R} \perp\!\!\!\perp b_{1R}b_{2R}c_{1R}c_{2R}$. But then, we also have $b_{1R} \perp\!\!\!\perp b_{2R}$, and $c_{1R} \perp\!\!\!\perp c_{2R}$ is obtained similarly. QED

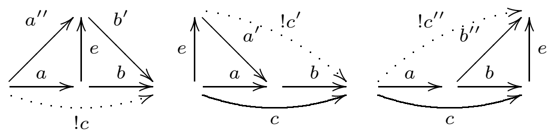
Definition 3.4. 1. We fix some $e \in S_0$, and set $S_0' = \{a \in S_0 : a_R \perp\!\!\!\perp e_R\}$.

2. Define $\tilde{S}_0 = \{(a, a', a'') : a \in S_0', a' \in e^{-1} \cdot a \text{ and } a'' \in a \cdot e\}$ and $\tilde{S} = \tilde{S}_0/R$.



(We follow a tacit understanding that R may also stand for $R \times R \times R$, where this is clear from the context.)

3. A triplet $\tilde{a} = (a, a', a'') \in \tilde{S}_0$ is called a *blow-up* of a. Conversely, we define the *blow-up* map $\pi : \tilde{S}_0 \to S_0'$ by $\pi(a, a', a'') = a$, where a is sometimes referred to as the *axis* of (a, a', a'').
4. Given $\tilde{a}_R \perp\!\!\!\perp_e \tilde{b}_R$, we wish to define $\tilde{a} \cdot \tilde{b}$. First, we know that $e \in (a^{-1} \cdot a'')_{R_1} \cap (b \cdot b'^{-1})_{R_1}$ for some $1 \in I$. By Lemma 3.2 there is $c \in a \cdot b \cap (a'' \cdot b')_{R_2}$, for some $2 \in I$. Again by Lemma 3.2 we find $c' \in e^{-1} \cdot c \cap (a' \cdot b)_{R_2}$ and $c'' \in c \cdot e \cap (a \cdot b'')_{R_2}$. Set $\tilde{a} \cdot \tilde{b}$ to be the set of all $\tilde{c} = (c, c', c'')$ obtained in this manner.



5. Recall that the inverse is a gradedly definable map, so it is only defined up to some R_i. Thus, for $\tilde{a} = (a, a', a'') \in \tilde{S}_0$, we can define its inverse as:

$$\tilde{a}^{-1} = \{(b, b', b'') \in \tilde{S}_0 : b \in a^{-1}, b' \in_j a''^{-1}, b'' \in_j a'^{-1}\}$$

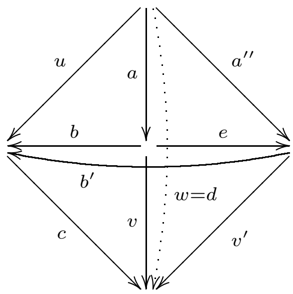
for $j \in I$ big enough to make sure that $\tilde{a}^{-1}$ cannot be empty; re-arranging previous choices we may assume that $j \leq 0$.

Proposition 3.5. 1. *The sets S_0' and $\tilde{S}_0$ are type-definable, $\tilde{S}_0 \subseteq (S_0')^3$, and S_0'/R is a generic polygroup chunk over e.*

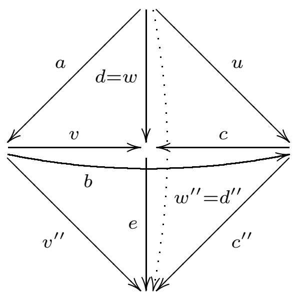
2. *The product operation $\cdot : (\tilde{S})^2 \to \tilde{S}$ is a gradedly type-definable partial map, defined on the (type-definable) set of e-independent pairs of elements of $\tilde{S}$.*
3. *If $\tilde{a}_R, \tilde{b}_R, \tilde{c}_R$ are independent over e, then $\tilde{a}_R \cdot (\tilde{b}_R \cdot \tilde{c}_R) = (\tilde{a}_R \cdot \tilde{b}_R) \cdot \tilde{c}_R$ gradedly.*
4. *Inversion $^{-1} : \tilde{S} \to \tilde{S}$ is a gradedly type-definable map, and $(\tilde{a}_R^{-1})^{-1} = \tilde{a}_R$ gradedly.*
5. *$\tilde{c}_R = \tilde{a}_R \cdot \tilde{b}_R$ if and only if $\tilde{a}_R = \tilde{c}_R \cdot \tilde{b}_R^{-1}$ if and only if $\tilde{b}_R = \tilde{a}_R^{-1} \cdot \tilde{c}_R$, gradedly.*
6. *The blow-up map induces a gradedly type-definable surjective bounded-to-one map $\bar{\pi} : \tilde{S} \to S_0'/R$.*
7. *The blow-up map is (generically) onto of type 3, i.e.*
   - *If $\tilde{a}_R \perp\!\!\!\perp_e \tilde{b}_R$ and $\tilde{c} \in \tilde{a} \cdot \tilde{b}$ then $\pi(\tilde{c}) \in \pi(\tilde{a}) \cdot \pi(\tilde{b})$ (homomorphism);*
   - *If $a_R \perp\!\!\!\perp_e b_R$, $c \in a \cdot b$, and a blow-up $\tilde{c}$ is given, then there are blow-ups $\tilde{a}_R \perp\!\!\!\perp_e \tilde{b}_R$ such that $\tilde{c}_R = \tilde{a}_R \cdot \tilde{b}_R$ (type 3).*

Proof. 1. By 2.24, and easy verifications.

2. Clear by the construction and Lemma 3.2, including uniqueness of the product (modulo R).
3. We work modulo R (suppressing the subscript), leaving the verification of the details of the grading to the reader.
   Let $\tilde{u} = \tilde{a}\cdot\tilde{b}$ and $\tilde{v} = \tilde{b}\cdot\tilde{c}$. As $b \in a^{-1}\cdot u \cap v\cdot c^{-1}$, there is a unique $d \in a\cdot v \cap u\cdot c$ by Lemma 3.2; similarly we get $d' \in a'\cdot v \cap e^{-1}\cdot d$ and $d'' \in u\cdot c'' \cap d\cdot e$. Now let $\tilde{w} = \tilde{a}\cdot\tilde{v}$. Since $\{a,b,c,e\}$ is independent, so is $\{a,b,v,e\}$; since $b' \in a''^{-1}\cdot u \cap e^{-1}\cdot b \cap v'\cdot c^{-1}$, we can apply Corollary 3.3 in order to see that $w = d$:



   Then $w', d' \in a'\cdot v \cap e^{-1}\cdot w = a'\cdot v \cap e^{-1}\cdot d$, so $w' = d'$. Finally, $\{d,c,v,e\}$ is independent and $b \in a^{-1}\cdot u \cap v\cdot c^{-1} \cap v''\cdot c''^{-1}$, whence $w'' = d''$ again by Corollary 3.3:



   Thus $\tilde{d} = \tilde{a}\cdot(\tilde{b}\cdot\tilde{c})$; since similarly $\tilde{d} = (\tilde{a}\cdot\tilde{b})\cdot\tilde{c}$, the assertion is shown.
4. Clear.
5. Working modulo R, we have $c \in a\cdot b \cap a''\cdot b'$, $c' \in e^{-1}\cdot c \cap a'\cdot b$, and $c'' \in c\cdot e \cap a\cdot b''$. Therefore $b \in a^{-1}\cdot c \cap a'^{-1}\cdot c'$, $b' \in e^{-1}\cdot b \cap a''^{-1}\cdot c$, and $b'' \in b\cdot e \cap a^{-1}\cdot c''$. Recalling that $\tilde{a}^{-1} = (a^{-1}, a''^{-1}, a'^{-1})$, we see that $\tilde{b} = \tilde{a}^{-1}\cdot\tilde{c}$. The other implications are similar.
6. Clear (remember that e is fixed, so a'_R and a''_R are bounded over a_R).
7. The 'homomorphism' part is clear. For 'type 3', let us work modulo R. Choose quite arbitrarily $a'' \in a\cdot e$. Then $a^{-1} \in e\cdot a''^{-1} \cap b\cdot c^{-1}$, so there is a unique $b' \in e^{-1}\cdot b \cap a''^{-1}\cdot c$ by Lemma 3.2. Similarly there are unique $a' \in e^{-1}\cdot a \cap c'\cdot b^{-1}$ and $b'' \in b\cdot e \cap a^{-1}\cdot c''$. They will do.

QED

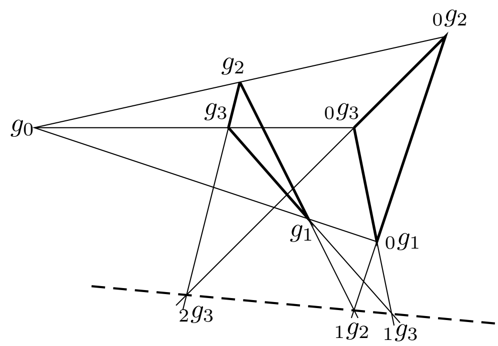
We conclude:

Theorem 3.6. *Let $S = S_0/R$ be a coreless gradedly almost hyperdefinable (over $\varnothing$) generic polygroup chunk, and $e \in S_0$. Let $\tilde{S}_0$ be as above. Then $\tilde{S} = \tilde{S}_0/R$ is a gradedly almost hyperdefinable generic group chunk over e.*

Remark 3.7. Our hope is that eventually we will be able to obtain a hyperdefinable group chunk and thus a hyperdefinable group. Aiming towards that goal, the second author describes in [Tom01] a blowup procedure which, starting from the (hyperdefinable) polygroup chunk of germs $(P, *)$ obtained from a suitable partial generic multiaction π as in [BY00], yields an improved hyperdefinable partial generic multiaction π^2 and a hyperdefinable polygroup chunk $(P^2, *)$ (over a parameter e) such that $y, y' \in f(x)$ (for $f \in P^2$, $x \in \arg(\pi^2)$, $f \mathop{\smile\!\!\!\!\!\!\!\;|}_e x$) implies $\mathrm{bdd}(y) = \mathrm{bdd}(y')$ and $h, h' \in f * g$ (for $f \mathop{\smile\!\!\!\!\!\!\!\;|}_e g$ from P^2) implies $\mathrm{bdd}(h) = \mathrm{bdd}(h')$. Thus, quotienting by the core relation gives an almost hyperdefinable group chunk, but the core relation was not used in the blowup. In fact, the construction can be generalised by fixing a long sequence $(e_i)_i$ and blowing up an element a by elements $a_i \in a * e_i$ and ${}_ia \in e_i^{-1} * a$, such that every element in a product is determined in an intersection of the form $\bigcap_i a_i * {}_ib$ — such a big intersection being more likely to be hyperdefinable.

Note however that unlike for the construction described in this section, not every $f \mathop{\smile\!\!\!\!\!\!\!\;|} e$ from P can be blown up. Nevertheless, the rank situation is still good, i.e. $\mathrm{SU}(P^2) = \mathrm{SU}(P)$.

Remark 3.8. It is not too noticeable in the exposition of this section, as coreless gradedly almost hyperdefinable polygroup chunks trivially satisfy them, but certain generalised associativity properties play an important role in the construction of various blowups. It becomes apparent that they are necessary if one attempts to obtain blowups 'exactly' (in a hyperdefinable way) like in the previous remark, and not just up to the core relation. We will say that a polygroup chunk has generalised associativity for an ordinal α, if for every independent $\{g_i : i < \alpha\}$ and any choice of $\{{}_0g_i \in g_0^{-1} * g_i : 0 < i < \alpha\}$, there are $\{{}_ig_j \in g_i^{-1} * g_j : i \neq j < \alpha\}$ such that for all $\{i, j, k\}$, ${}_ig_j \in {}_ig_k * {}_kg_j$, and ${}_ig_j{}^{-1} = {}_jg_i$. The polygroup chunk of germs, as obtained in [BY00] satisfies generalised associativity for each α. This is discussed in [Tom01], where a blow-up procedure is written in a way completely parallel to the classical reconstruction of the division ring from a projective geometry by von Neumann in [vN60]. In view of this interpretation, notice that the generalised associativity for $\alpha = 4$ corresponds to the Desargues' axiom, when interpreted in the polygroup associated to a projective geometry from 2.4, as shown in the figure below.



Remark 3.9. Pasting together local blowups over parameters e_i for $i \in I$ (as in this section) of a gradedly almost hyperdefinable polygroup chunk over $\varnothing$, the second author obtains in [Tom01] a sheaf-like group chunk $\tilde{P} \xrightarrow{\pi} P$ (where π is a generic bounded covering) satisfying the universal properties for such objects. This should provide a justification for the name 'blowup', since this universal property characterizes blowing up in algebraic geometry.

4. Constructing an almost hyperdefinable group

We shall prove:

Theorem 4.1. *Let $S = \langle S_0/R, \cdot, ^{-1} \rangle$ be an I-gradedly almost hyperdefinable group chunk. Then there is an I-g.e.r. R' on S_0^2, such that $G = S_0^2/R'$ is a gradedly almost hyperdefinable group. Moreover, there is a gradedly type-definable map $\sigma : S \to G$ whose image generates G, and the couple (G, σ) is gradedly unique as such, up to a unique graded isomorphism (i.e., for every other couple (G', σ'), there is a unique isomorphism, up to graded equality of maps, rendering σ and σ' gradedly equal).*

The rest of this section will consist of the proof. It is very close to [Wag01, section 3], so we only point out the differences. A more detailed adaptation of [Wag01, section 3] to the ultraimaginary case, as well as the construction of a space from a space chunk can be found in [Tom01] and [TW01].

As R is almost type-definable, we may assume that this is witnessed by R_0.

Definition 4.2. We define R' on S_0^2. This the I-graded analogue of R from [Wag01]. We say that $(a, b)R'_i(a', b')$ if there are x, y such that:

1. $x_R \mathop{\smile\!\!\!\!\!\mid}\, aba'b'$ and $y_R \mathop{\smile\!\!\!\!\!\mid}\, aba'b'$.
2. $a \cdot xR_ia' \cdot y$ and $b \cdot xR_ib' \cdot y$ (where AR_iB means that aR_ib for some $a \in A$ and $b \in B$).

We write $[a, b]_i = (a, b)/R'_i$, and $[a, b] = (a, b)/R'$.

Lemma 4.3. *R' is a gradedly almost type-definable equivalence relation.*

Proof. First, each R'_i is type-definable by 2.24. It is clearly symmetric and reflexive. For graded transitivity, just adapt the proof from [Wag01]. We are left with almost type-definability.
So consider a pair (a, b), and fix $\bar{a}' = \{a'_j : j < \kappa\}$ and $\bar{b}' = \{b'_j : j < \kappa\}$ be such that $\bigcup_j a'_j/R_0 = a/R$ and $\bigcup_j b'_j/R_0 = b/R$.
Assume now that $(a, b)R'_i(c, d)$, as witnessed by x, y. We may assume that $xy \mathop{\smile\!\!\!\!\!\mid}_{abcd} \bar{a}'\bar{b}'$, whereby $x_R \mathop{\smile\!\!\!\!\!\mid}\, \bar{a}'\bar{b}'cd$ and $y_R \mathop{\smile\!\!\!\!\!\mid}\, \bar{a}'\bar{b}'cd$. Now, there is $z \in c \cdot y$ such that $z \in_i a \cdot x$; if $z' \in_0 z$ with $z' \mathop{\smile\!\!\!\!\!\mid}\, x_R$ and $a'' \in z' \cdot x^{-1}$, we obtain $z \in_1 a'' \cdot x$ for some $1 \in I$. Therefore $a'' \in a_R$ and $a'' \cdot xR_1c \cdot y$. Then there is $j < \kappa$ such that $a'' \in_0 a'_j$, and then $a'_j \cdot xR_2c \cdot y$ for some $2 \in I$. Similarly, we find $j' < \kappa$ such that $b'_{j'} \cdot xR_2c \cdot y$. This shows that $(a'_j, b'_{j'})R'_2(c, d)$. More generally, we saw that $[a, b] = \bigcup_{j,j'}[a'_j, b'_{j'}]_2$, and R'_2 witnesses that R' is almost type-definable. QED

From here, we follow [Wag01] almost word-by-word, allowing at each claim for refinement in the same style as above. In other words, every claim will turn out to be: For any $i \in I$ there is $j \in I$, such that if the parameters are given up to R_i or R'_i then the conclusion holds up to R'_j. Independence is of course the independence of a_R and $[a, b]$ as given above (it makes sense for $[a, b]$ as well, as R' is almost type-definable), not that of a and a, b. We leave the verification of the details to the reader. As for the uniqueness, we diverge somewhat from [Wag01], and give a stronger result.

Theorem 4.4. *With the notations and assumptions of 4.1, assume that $P = P_0/R''$ is a coreless polygroup, and $\tau : S_0/R \to P_0/R''$ is a gradedly type-definable generic homomorphism (i.e. $\tau(a^{-1}) = \tau(a)^{-1}$, and $\tau(a \cdot b) \in \tau(a) \cdot \tau(b)$, gradedly, for any independent $a, b \in S_0$), such that every element in the image of τ is generic. Then there is a unique gradedly type-definable homomorphism $\hat{\tau} : G \to P$ with $\tau = \hat{\tau} \circ \sigma$ (i.e. any $\hat{\tau}'$ with the same properties is gradedly equal to $\hat{\tau}$).*
Moreover, for every $g \in G$ (we omit the subscript R'), if we write it as $a \cdot b$ where these are generics each of which independent of g, then $\hat{\tau}(g) = \tau(a) \cdot \tau(b) \cap \mathrm{dcl}(g) = \tau(a) \cdot \tau(b) \cap \mathrm{bdd}(g)$.

Remark 4.5. If P is a group, or more generally if the image of τ is closed under products of independent elements, then the assumption that every element in the image of τ be generic is not necessary: By an analogue of [Wag01, Lemma 1.13], in this case the image is precisely the set of generic elements of the generated sub-polygroup.

Proof. We proceed in several steps:

1. For $(a, b) \in S_0^2$ and m that will be fixed later define:
$$X(a, b) = \{(c, d) \in S_0^2 : cd \mathop{\smile\!\!\!\!\!\!\!\;|}\, ab \text{ and } c \mathop{\smile\!\!\!\!\!\!\!\;|}\, d\}$$
$$Y(a, b, c, d) = (\tau(a \cdot c) \cdot \tau((b \cdot c)^{-1}))_{R''_m} \cap (\tau(a \cdot d) \cdot \tau((b \cdot d)^{-1}))_{R''_m}$$
$$\hat{\tau}(a, b) = \bigcup_{(c,d) \in X(a,b)} Y(a, b, c, d)$$
So $X(a, b)$, $Y(a, b, c, d)$ and $\hat{\tau}(a, b)$ are type-definable. As τ is generically homomorphic, there is m such that for $(c, d) \in X(a, b)$
$$\tau((b \cdot c)^{-1}) \in_m \tau(c^{-1} \cdot d) \cdot \tau((b \cdot d)^{-1})$$
$$\tau(a \cdot d) \in_m \tau(a \cdot c) \cdot \tau(c^{-1} \cdot d)$$
thus by associativity in P we can fix m sufficiently big such that $Y(a, b, c, d) \neq \varnothing$, whence $\hat{\tau}(a, b) \neq \varnothing$. Assume that $(c, d), (c', d') \in X(a, b)$; we wish to show that any $f \in Y(a, b, c, d)$ and $f' \in Y(a, b, c', d')$ are R''-equivalent. We may assume that $c = c'$ and $d \mathop{\smile\!\!\!\!\!\!\!\;|}_{abc} d'$ (if not, choose $c'' \mathop{\smile\!\!\!\!\!\!\!\;|}\, abcdc'd'$, and pass through steps $(c, d) \to (c'', d) \to (c'', d') \to (c', d')$). We obtain, modulo some bounded degree of equivalence:
$$\begin{aligned} c \mathop{\smile\!\!\!\!\!\!\!\;|}\, abdd' &\Longrightarrow (a \cdot c)_R \mathop{\smile\!\!\!\!\!\!\!\;|}\, abdd' \\ &\Longrightarrow (a \cdot c)_R \mathop{\smile\!\!\!\!\!\!\!\;|}\, (a \cdot d)_R (b \cdot d)_R (a \cdot d')_R (b \cdot d')_R \\ &\Longrightarrow \tau(a \cdot c)_{R''} \mathop{\smile\!\!\!\!\!\!\!\;|}\, f_{R''} f'_{R''} \end{aligned}$$
(meaning that at each step there are independent representatives, and we can bound the degree of R or R'' needed to show that these are indeed representatives). As in addition $f, f' \in (\tau(a \cdot c) \cdot \tau((b \cdot c)^{-1}))_{R''_m}$ and $\tau(a \cdot c)_{R''}$ is generic, we see that $f \in_{m'} f'$ for some m' that can be calculated from R, R'' and τ. Thus $\hat{\tau} : S_0^2 \to P$ is a gradedly definable map.
2. We wish to show that $\hat{\tau}$ induces a gradedly definable map $G \to P$. Assume that $(a, b) R'_i (a', b')$, as witnessed by some x, y. Let $x'y' \mathop{\smile\!\!\!\!\!\!\!\;|}_{aba'b'} xy$ realise the same type. Then $(x, x'), (y, y') \in X(a, b) \cap X(a', b')$, and $Y(a, b, x, x') R''_j Y(a', b', y, y')$ for some j that depends only on i. Thus $\hat{\tau}(a, b) R''_j \hat{\tau}(a', b')$.

3. We show compatibility with multiplication. For some R_3' we defined $(a,b)\cdot(c,d)$ as the set of (f,h) such that $(f,g)R_3'(a,b)$ and $(g,h)R_3'(c,d)$, and this is contained in some $[f,h]_4$. It is therefore sufficient to show the compatibility for the case $(a,c)\in(a,b)\cdot(b,c)$. However, taking $x \mathop{\smile\!\!\!\!\!\!\mid}\, abc$ and $y \mathop{\smile\!\!\!\!\!\!\mid}\, abcx$, we get $(x,y)\in X(a,b)\cap X(a,c)\cap X(b,c)$, and the compatibility follows from the definition. Similarly for the inverse, as $(a,b)^{-1}=(b,a)$.
4. $\hat\tau$ extends τ: In the definition of Y we can replace m with any $m'\geq m$ without changing anything essential (some grading bounds will change). An element $a\in S_0$ is represented in S_0 by $\sigma(a)=(a\cdot b,b)$ for $b \mathop{\smile\!\!\!\!\!\!\mid}\, a$, which is well defined up to a bounded degree of equivalence. Clearly, for sufficiently large m' we get $\tau(a)\in_{m'} Y(a,b,c,d)$ for every $(c,d)\in X(a\cdot b,b)$. Therefore $\tau=\hat\tau\circ\sigma$ gradedly.
5. Uniqueness of $\hat\tau$: The compatibility with the multiplication implies that if $\hat\tau'$ is another candidate, then there is m' such that for every $f\in\hat\tau'(a,b)$ we necessarily have $f\in_{m'} Y(a,b,c,d)$ for every $c,d\in X(a,b)$, whence the graded equality.

For the moreover part, assume that $f\in\tau(a)\cdot\tau(b)\cap\mathrm{bdd}(g)$. Let a_0,g_0 be representatives of a and g, respectively, such that $a_0 \mathop{\smile\!\!\!\!\!\!\mid}\, g_0$, and let a_0^i be a Morley sequence for a_0 over g_0. Let $a^i=a_0^i/R'$ and $b^i={a^i}^{-1}\cdot g$, so in particular $a=a^0$ and $b=b^0$. As $f\in\tau(a^0)\cdot\tau(b^0)\cap\mathrm{bdd}(g)$, we get $f\in\tau(a^i)\cdot\tau(a^i)$ for every i. It suffices to consider any two different values of i to see that the construction of $\hat\tau$ yields: $f=\hat\tau(g)$. QED

Remark 4.6. A homomorphism $\tau: S\to P$ of polygroups is (generically) of *type 3* if for (independent generic) $x,y\in P$ and $c\in S$ with $\tau(c)=z\in x*y$ there are $a,b\in S$ with $c\in a*b$ and $\tau(a)=x$ and $\tau(b)=y$. It is not to difficult to see that in the previous theorem, if τ is generically onto of type 3, the induced map $\bar\tau$ is onto of type 3.

Proof. Let us show that $\hat\tau$ is of type 3, working modulo R. Consider some $x,y\in P$ and $\hat\tau([c_0,c_1])=z\in x*y$. We may assume $c_0 \mathop{\smile\!\!\!\!\!\!\mid}\, xy$ and $c_1 \mathop{\smile\!\!\!\!\!\!\mid}\, xy$ by translation; choose $d \mathop{\smile\!\!\!\!\!\!\mid}\, c_0c_1xy$ and put $c_0'=c_0*d$, $c_1'=c_1*d$, $\tau(c_0)=z_0$, $\tau(c_1)=z_1$, $\tau(c_0')=z_0'$ and $\tau(c_1')=z_1'$. Then $z=f*z_0^{-1}\cap f'*{z_0'}^{-1}$ by definition of $\hat\tau$. Since τ is a homomorphism, $\tau(d)=: u\in z_0^{-1}*z_0'\cap z_1^{-1}*z_1'$. Since $z\in z_1'*{z_0'}^{-1}\cap x*y$, by transposition we can find $v'\in x^{-1}*z_1'\cap y*z_0'$. By associativity $x^{-1}*(z_1'*x^{-1})=(x^{-1}*z_1')*u^{-1}$ implies $x^{-1}*z_1\approx v'*u^{-1}$, and similarly $y*z_0\approx v'*u^{-1}$. Since $u \mathop{\smile\!\!\!\!\!\!\mid}\, z_0z_1xy$, any two elements from the two intersections are core-equivalent, and we can choose $v\in x^{-1}*z_1\cap y*z_0\cap v'*u^{-1}$. By the fact that u, v and v' are generic and τ is generically onto of type 3, we find e and e' such that $e'=e*d$, $\tau(e)=v$ and $\tau(e')=v'$. Now, it is easily checked that $[c_0,e]=[c_0',e']$, $[e,c_1]=[e',c_1']$, and they witness that $x=\hat\tau([c_0,e])$, $y=\hat\tau([e,c_1])$ and $[c_0,c_1]=[c_0,e]\circ[e,c_1]$. QED

Remark 4.7. In [BY01a] the first author has shown a strong structure theorem for coreless polygroups in simple theories: $P\cong G/\!\!/H$ for some $H<G$. This improved earlier results of the second and third author, who used stabilizers to show that they are 'poly-isogenous' to groups. In fact the universal property from 4.4 can be used to derive an intermediate result: the generic part S of a gradedly almost hyperdefinable polygroup P in a simple theory can be blown up with respect to some $e\in S$, yielding $\pi:\tilde S_e\to S_e$ which is (generically) onto of type 3 with bounded

fibres, and lifting to a homomorphism $\hat{\pi} : G_e \to P$, where G is a group, again onto of type 3 with bounded fibres. The classical theory (see e.g. [Com84]) yields $P \cong G/\!\!/\hat{\pi}$, where $G/\!\!/\hat{\pi} = \{\hat{\pi}^{-1}(\hat{\pi}(a)) : a \in G\}$ and $a_{\hat{\pi}} * b_{\hat{\pi}} := \{c_{\hat{\pi}} : c \in a_{\hat{\pi}} \cdot b_{\hat{\pi}}\}$.

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