# Constructibility in Physics

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We pursue an approach in which space-time proves to be relational and its differential properties fulfill the strict requirements of Einstein-Weyl causality. Space-time is developed from a set theoretical foundation for a constructible mathematics. The foundation proposed is the axioms of Zermelo-Frankel (ZF) but without the power set axiom, with the axiom schema of subsets removed from the axioms of regularity and replacement and with an axiom of countable constructibility added. Four arithmetic axioms, excluding induction, are also adjoined; these formulae are contained in ZF and can be added here as axioms. All sets of finite natural numbers in this theory are finite and hence definable. The real numbers are countable, as in other constructible theories. We first show that this approach gives polynomial functions of a real variable. Eigenfunctions governing physical fields can then be effectively obtained. Furthermore, using the integral form for the field equations over a compactified space, we produce a nonlinear sigma model. The Schrödinger equation follows from a proof in the theory of the discreteness of the space-like and time-like terms of the model. This result suggests that quantum mechanics in this relational space-time framework can be considered conceptually cumulative with prior physics.

We propose the following axioms. The formulae for these axioms are given in the appendix. The

TABLE I: Axioms

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|---|---|
| Extensionality | Two sets with just the same members are equal. |
| Pairs | For every two sets, there is a set that contains just them. |
| Union | For every set of sets, there is a set with just all their members. |
| Infinity | There are infinite ordinals $\omega^*$ (i.e., sets are transitive and well-ordered by $\in$-relation). |
| Replacement | Replacing the members of a set one-for-one creates a set (i.e., bijective replacement). |
| Regularity | Every non-empty set has a minimal member (i.e. "weak" regularity). |
| Arithmetic | Four axioms for predecessor uniqueness, addition and multiplication. |
| Constructibility | The subsets of $\omega^*$ are countably constructible. |

first six axioms are the set theory of Zermelo-Frankel (ZF) without the power set axiom and with the axiom schema of subsets (a.k.a., separation) deleted from the axioms of regularity and replacement. Arithmetic is contained in ZF but must be axiomatized here. Because of the deletion of the axiom schema of subsets, a minimal $\omega^*$, usually denoted by $\omega$ and called the set of all finite natural numbers, cannot be shown to exist in this theory; instead this set theory is uniformly dependent on $\omega^*$ and then all the finite as well as infinitely many infinite natural numbers are included in $\omega^*$. These infinite numbers are one-to-one with $\omega^*$; a finite natural number is any member of $\omega^*$ that is not infinite. We can now adjoin to this sub-theory of ZF an axiom asserting that the subsets of $\omega^*$ are constructible. By constructible sets we mean sets that are generated sequentially by some process, one after the other, so that the process well-orders the sets. Gödel has shown that an axiom asserting that all sets are constructible can be consistently added to ZF [1], giving a theory usually called $\mathrm{ZFC}^+$. It is well known that no more than countably many subsets of $\omega^*$ can be shown to exist in $\mathrm{ZFC}^+$. This result will, of course, hold for the sub-theory $\mathrm{ZFC}^+$ minus the axiom schema of subsets and the power set axiom. Thus we can now add a new axiom which says that the subsets of $\omega^*$ are countably constructible. This axiom, combined with the axiom schema of bijective replacement, creates a set of constructible subsets of $\omega^*$ and deletion of the power set axiom then assures that no other subsets of $\omega^*$ exist in this theory. We shall refer to these axioms as $T$. All the sets of finite natural numbers in $T$ are finite. The general continuum hypothesis holds in this theory because all sets are countable. We now will also show that this theory is rich enough to contain some functions of a real variable. We first show $T$ has a countable real line. Recall the definition of "rational numbers" as the set of ratios, in ZF called $Q$, of any two members of the set $\omega$. In $T$, we can likewise, using the axiom of unions, establish for $\omega^*$ the set of ratios of any two of its natural numbers, finite or infinite. This will become an "enlargement" of the rational numbers and we shall call this enlargement $Q^*$. Two members of $Q^*$ are called "identical" if their ratio is 1. We now employ the symbol "$\equiv$" for "is identical to." Furthermore, an "infinitesimal" is a member of $Q^*$ "equal" to 0, i.e., letting $y$ signify the member and employing the symbol "$=$" to signify equality, $y = 0 \leftrightarrow \forall k[y < 1/k]$, where $k$ is a finite natural number. The reciprocal of an infinitesimal is "infinite". A member of $Q^*$ that is not an infinitesimal and not infinite is "finite". The constructibility axiom allows creation of the set of constructible subsets of $\omega^*$ and, in addition, provides a distance measure, giving the metric space $R^*$. The elements of $R^*$ represent the binimals forming a countable real line. In this theory $R^*$ is a subset of $Q^*$. An ***equality-preserving*** bijective map $\phi(x, u)$ between intervals $X$ and $U$ of $R^*$ in which $x \in X$ and $u \in U$ such that $\forall x_1, x_2, u_1, u_2[\phi(x_1, u_1) \wedge \phi(x_2, u_2) \rightarrow (x_1 - x_2 = 0 \leftrightarrow u_1 - u_2 = 0)]$ creates pieces which are biunique and homeomorphic. Note that $U = 0$ if and only if $X = 0$, i.e., the piece is inherently relational. We can now define "functions of a real variable in $T$". $u(x)$ is a function of a real variable in $T$ only if it is a constant or a sequence in $x$ of continuously connected biunique pieces such that the derivative of $u$ with respect to $x$ is also a function of a real variable in $T$. These functions are thus of bounded variation and locally homeomorphic with the real line $R^*$. If some derivative is a constant, they are polynomials. If no derivative is a constant, these functions do not *per se* exist in $T$ but can, however, always be represented as closely as required for physics by a sum of polynomials of sufficiently high degree obtained by an iteration of:

$$\int_a^b \left[ p \left( \frac{du}{dx} \right)^2 - qu^2 \right] dx \equiv \lambda \int_a^b ru^2 dx \qquad (1)$$

where $\lambda$ is minimized subject to:

$$\int_a^b ru^2 dx \equiv \text{const} \qquad (2)$$

where:

$$a \neq b, \quad u \left( \frac{du}{dx} \right) \equiv 0 \qquad (3)$$

at $a$ and $b$; $p$, $q$, and $r$ are functions of the real variable $x$. Letting $n$ denote the $n^{th}$ iteration, $\forall k \exists n[\lambda_{n-1} - \lambda_n < 1/k]$ where $k$ is a finite natural number. So, a polynomial such that, say, $1/k < 10^{-50}$ should be sufficient for physics as it is effectively a Sturm-Liouville "eigenfunction". These can be decomposed, since they are polynomials, into biunique "irreducible eigenfunction pieces" obeying the boundary conditions. As a bridge to physics, let $x_1$ be space and $x_2$ be time. We now postulate the following integral expression for a one-dimensional string $\Psi = u_1(x_1)u_2(x_2)$:

$$\int \left[ \left( \frac{\partial \Psi}{\partial x_1} \right)^2 - \left( \frac{\partial \Psi}{\partial x_2} \right)^2 \right] dx_1 dx_2 \equiv 0 \qquad (4)$$

The eigenvalues $\lambda_{1m}$ are determined by the spatial boundary conditions. For each eigenstate $m$, we can use this integral expression constrained by the indicial relation $\lambda_{1m} \equiv \lambda_{2m}$ to iterate the eigenfunctions $u_{1m}$ and $u_{2m}$. A more general string in finitely many space-like and time-like dimensions can likewise be produced. Let $u_{\ell mi}(x_i)$ and $u_{\ell mj}(x_j)$ be eigenfunctions with non-negative eigenvalues $\lambda_{\ell mi}$ and $\lambda_{\ell mj}$ respectively. We define a "field" as a sum of eigenstates:

$$\underline{\Psi}_m = \sum_\ell \Psi_{\ell m} \underline{i_\ell}, \Psi_{\ell m} = \mathrm{C} \prod_i u_{\ell mi} \prod_j u_{\ell mj} \qquad (5)$$

with the postulate: ***for every eigenstate*** $m$ ***the integral form for the field equations in a compactified space-time is identical to 0***. Let $ds$ represent $\prod_i r_i dx_i$ and $d\tau$ represent $\prod_j r_j dx_j$. Then for all $m$,

$$\int \sum_{\ell i} \frac{1}{r_i} \left[ P_{\ell mi} \left( \frac{\partial \Psi_{\ell m}}{\partial x_i} \right)^2 - Q_{\ell mi} \Psi_{\ell m}^2 \right] ds d\tau \qquad (6)$$
$$- \int \sum_{\ell j} \frac{1}{r_j} \left[ P_{\ell mj} \left( \frac{\partial \Psi_{\ell m}}{\partial x_j} \right)^2 - Q_{\ell mj} \Psi_{\ell m}^2 \right] ds d\tau \equiv 0$$

In this integral expression the $P$, $Q$, and $R$ can be functions of any of the $x_i$ and $x_j$, thus of any $\Psi_{\ell m}$ as well. This is a ***nonlinear sigma model***. As seen in the case of a one-dimensional string, these $\Psi_m$ can in principle be obtained by iterations constrained by an indicial relation, $\sum_{\ell i} \lambda_{\ell mi} \equiv \sum_{\ell j} \lambda_{\ell mj}$ for all $m$. We see that the postulate asserts a fundamental identity of the magnitudes of the two components of the integral. A sui generis proof in $T$ that these components have only discrete values will now be shown. Let expressions (7) and (8) both be represented by $\alpha$, since they are identical:

$$\sum_{\ell mi} \int \frac{1}{r_i} \left[ P_{\ell mi} \left( \frac{\partial \Psi_{\ell m}}{\partial x_i} \right)^2 - Q_{\ell mi} \Psi_{\ell m}^2 \right] ds d\tau \qquad (7)$$

$$\sum_{\ell mj} \int \frac{1}{r_j} \left[ P_{\ell mj} \left( \frac{\partial \Psi_{\ell m}}{\partial x_j} \right)^2 - Q_{\ell mj} \Psi_{\ell m}^2 \right] ds d\tau \qquad (8)$$

I. $\alpha(\Psi)$ is positive and closed to addition and to the absolute value of subtraction, so it is either continuous or discrete.

II. $\alpha(\Psi) \equiv 0 \leftrightarrow \Psi \equiv 0$; otherwise, as $\Psi$ is a function of real variables, the range $\Psi \neq 0$ and $\alpha(\Psi) \neq 0$, thus $\alpha(\Psi)$ is not continuous.

III. Therefore $\alpha$ is discrete; $\alpha \equiv n\kappa$, where $n$ is any finite natural number and $\kappa$ is some finite positive unit. $\kappa$ must, of course, be determined empirically.

With this result and without any additional physical postulates, we can now obtain the Schrödinger equation from the nonlinear sigma model in one time-like dimension and finitely many space-like dimensions. Let $\ell = 1, 2$, $r_t = P_{1mt} = P_{2mt} = 1$, $Q_{1mt} = Q_{2mt} = 0$, $\tau = \omega_m t$ and we normalize $\Psi$ as follows:

$$\Psi_m = \sqrt{(C/2\pi)} \prod_i u_{im}(x_i)[u_{1m}(\tau) + i \cdot u_{2m}(\tau)] \qquad (9)$$

where $i = \sqrt{-1}$ with

$$\int \sum_m \prod_i u_{im}^2 ds(u_{1m}^2 + u_{2m}^2) \equiv 1 \tag{10}$$

then:

$$\frac{du_{1m}}{d\tau} = -u_{2m} \quad \text{and} \quad \frac{du_{2m}}{d\tau} = u_{1m} \tag{11}$$

or

$$\frac{du_{1m}}{d\tau} = u_{2m} \quad \text{and} \quad \frac{du_{2m}}{d\tau} = -u_{1m} \tag{12}$$

For the minimal non-vanishing field, $\alpha$ has its least finite value $\kappa$. Thus,

$$(C/2\pi) \sum_m \oint \int \left[ \left( \frac{du_{1m}}{d\tau} \right)^2 + \left( \frac{du_{2m}}{d\tau} \right)^2 \right] \prod_i u_{im}^2(x_i) ds d\tau \equiv C \equiv \kappa \tag{13}$$

Substituting the Planck constant $h$ for $\kappa$, this can now be put into the familiar Lagrangian form for the time term in the Schrödinger equation,

$$\frac{h}{2i} \sum_m \oint \int \left[ \Psi_m^* \left( \frac{\partial \Psi_m}{\partial t} \right) - \left( \frac{\partial \Psi_m^*}{\partial t} \right) \Psi_m \right] ds dt \tag{14}$$

Since the Schrödinger equation is well confirmed by experiment, this can be considered an empirical determination of $\kappa$. Also note that $\alpha \equiv n\kappa$ is actually the Heisenberg Uncertainty Principle. Returning to the statement that, for $\Psi \neq 0$, $\alpha(\Psi) \neq 0$, we recognize that we have assumed that space-time exists, i.e., that the upper and lower limits of at least one of the integrals over the space-time dimensions are not equal. Otherwise, if space-time does not exist, then, for any $\Psi$, $\alpha(\Psi) = 0$; accordingly, the theory provides us with a necessary and sufficient condition: space-time exists if and only if there exists a $\Psi$ such that $\alpha(\Psi) \neq 0$. Space-time is therefore relational. Moreover, in this theory space-time is countable and fields are locally homeomorphic with the real line $R^*$. This is sufficient for its differential structure to fulfill the strict requirements of Einstein-Weyl causality [3]. We have also shown that:

- The Schrödinger equation is obtained in a constructible theory without reference to the statistical interpretation of the wave function, which, it can be argued, may be inferred from the equation itself and a requirement that quantum mechanics will reduce to its classical limit. [4].
- This suggests that quantum mechanics in this relational space-time framework could be considered conceptually cumulative with prior physics. If so, it would resolve a long-standing controversy.

In addition, though we do not have the opportunity to discuss these points, we note that:

- There are inherently no singularities in the physical fields obtained in this theory.
- The solution to the QED divergence problem posed by Dyson [5] is provided, since the actual convergence or divergence of the essential perturbation series is undecidable in this theory.
- Wigner's metaphysical question regarding the apparent unreasonable effectiveness of mathematics in physics [6] is directly answered, since the foundations of mathematics and physics are now linked.

## Appendix: ZF - Subsets - Power Set + Constructibility + Arithmetic

*Extensionality.* Two sets with just the same members are equal. $\forall x \forall y \, (\forall z \, (z \in x \leftrightarrow z \in y) \rightarrow x = y)$ *Pairs.* For every two sets, there is a set that contains just them. $\forall x \forall y \exists z (\forall w w \in z \leftrightarrow w = x \vee w = y)$ *Union.* For every set of sets, there is a set with just all their members. $\forall x \exists y \forall z (z \in y \leftrightarrow \exists u (z \in u \wedge u \in x))$ *Infinity.* There are infinite ordinals $\omega^*$ (i.e., sets are transitive and well-ordered by $\in$-relation). $\exists \omega^* (O \in \omega^* \wedge \forall x (x \in \omega^* \rightarrow x \cup \{x\} \in \omega^*))$ *Replacement.* Replacing members of a set one-for-one creates a set (i.e., "bijective" replacement). Let $\phi(x, y)$ a formula in which $x$ and $y$ are free,
$\forall z \forall x \in z \forall y (\phi(x, y) \wedge \forall u \in z \forall v (\phi(u, v) \rightarrow u = x \leftrightarrow y = v)) \rightarrow \exists r \forall t (t \in r \leftrightarrow \exists s \in z \phi(s, t))$ *Regularity.* Every non-empty set has a minimal member (i.e. "weak" regularity). $\forall x (\exists y y \in x \rightarrow \exists y (y \in x \wedge \forall z \neg (z \in x \wedge z \in y)))$ *Constructibility.* The subsets of $\omega^*$ are countably constructible. $\forall \omega^* \exists S ([\omega^*, O] \in S \wedge \forall y \forall z ([y, z] \in S \wedge \exists m_y (m_y \in y \wedge \forall v \neg (v \in y \wedge v \in m_y)) \leftrightarrow \exists t_y \forall u (u \in t_y \leftrightarrow u \in y \wedge u \neq m_y) \wedge [t_y \cup m_y, z \cup \{z\}] \in S))$. The four formulae (a) to (d) below are contained in ZF but must be adjoined to T as axioms. Let $x' = xU\{x\}$

(a) $\forall x \in \omega^* (x \neq O \leftrightarrow \exists y \in \omega^* (y' = x))$

(b) $\forall x \forall y (x' = y' \rightarrow x = y)$

Let $x$ and $y$ be members of $\omega^*$ and $[x, y]$ and $[[x, y], z]$ represent ordered pairs.

(c) $\exists A \forall x \in \omega^* \forall y \in \omega^* E! z \in \omega^* ([[O, O], O] \in A \wedge [[x, y], z] \in A \rightarrow [[x, y'], z'] \in A \wedge [[x', y], z'] \in A)$; addition: $x + y = z$

(d) $\exists M \forall x \in \omega^* \forall y \in \omega^* E! z \in \omega^* ([[O, O], O] \in M \wedge [[x, y], z] \in M \rightarrow [[x, y'], z + x] \in M \wedge [[x', y], z + y] \in M)$; multiplication: $x \cdot y = z$

Define $[a,b]_r$ such that $[a_1,b]_r + [a_2,b]_r \equiv [a_1 + a_2,b]_r$ and $[a_1,b_1]_r \equiv [a_2,b_2]_r \leftrightarrow a_1 \cdot b_2 \equiv a_2 \cdot b_1$. The extended set of rationals $Q^*$ is the set of such pairs for all $a$ and $b$ in $\omega^*$.

*Theorems*

Using the axioms (a), (b) and axioms of regularity, unions and bijective replacement one can show that $\omega^*$ is a set which contains the predecessor of every member of itself, except for the null set. From axiom (c), there is, for $x$ fixed, a one-to-one relation between all $y \in \omega^*$ and some $z$. A set $g_x(z)$ can thus be created by the axiom of bijective replacement. Then $z \notin g_x(z) \veebar (z = x \veebar (z \in g_x(z) \wedge z \neq x))$. This is a "trichotomy". These theorems allow the members of $\omega^*$ to be considered natural numbers. Therefore, the axioms of arithmetic are directly applicable. From the axiom of constructibility, the set $Z^*$ such that $[O, Z^*] \in S$ maps to the real line $R^*$. Since $Z^* \in \omega^*$ by the axiom of infinity, the arithmetic axioms and theorems are directly applicable to $R^*$ as well. The metric between members $y_1$ and $y_2$ of the real line $(0,1)$ is given by $[|z_1 - z_2|, Z^*]_r$ where $[y,z] \in S$.

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