Constructibility and Quantum Mechanics

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We pursue an approach in which space-time proves to be relational and its differential properties fulfill the strict requirements of Einstein-Weyl causality. Space-time emerges here from a set theoretical foundation for a constructible mathematics. In this theory the Schrödinger equation can be obtained by adjoining a physical postulate of action symmetry in generalized wave phenomena. This result now allows quantum mechanics to be considered conceptually cumulative with prior physics. A set theoretical foundation is proposed in which the axioms are the theory of Zermelo-Frankel (ZF) but without the power set axiom, with the axiom schema of subsets removed from the axioms of regularity and replacement and with an axiom of countable constructibility added. Four arithmetic axioms are also adjoined; these formulae are contained in ZF but must be added here as axioms. All sets of finite natural numbers in this theory are finite and hence definable. The real numbers are countable. We first show that this constructible approach gives polynomial functions of a real variable, which are of bounded variation and locally homeomorphic. Eigenfunctions governing physical fields can then be effectively obtained. By using an integral of the Lagrange density of a field over a compactified space, we produce a nonlinear sigma model. The Schrödinger equation follows directly from the postulate and a sui generis proof in this theory of the discreteness of the space-like and time-like terms of the model.

We propose the following axioms. The formulae for these axioms are given in the appendix.

TABLE I: Axioms

Extensionality	Two sets with just the same members are equal.
Pairs	For every two sets, there is a set that contains just them.
Union	For every set of sets, there is a set with just all their members.
Infinity	There are infinite ordinals ω^*.
Replacement	Replacing the members of a set one-for-one creates a set (i.e., bijective replacement).
Regularity	Every non-empty set has a minimal member (i.e. "weak" regularity).
Arithmetic	Four axioms for predecessor uniqueness, addition and multiplication.
Constructibility	The subsets of ω^* are countably constructible.

The first six axioms are the set theory of Zermelo-Frankel (ZF) without the power set axiom and with the axiom schema of subsets (a.k.a., separation) deleted from the axioms of regularity and replacement. Arithmetic is contained in ZF but must be axiomatized here. Because of the deletion of the axiom schema of subsets, a minimal ω^*, usually denoted by ω and called the set of all finite natural numbers, cannot be shown to exist in this theory; instead this set theory is uniformly dependent on ω^* and then all the finite as well as infinitely many infinite natural numbers are included in ω^*. These infinite numbers are one-to-one with ω^*; a finite natural number is any member of ω^* that is not infinite.

We can now adjoin to this sub-theory of ZF an axiom asserting that the subsets of ω^* are constructible. By constructible sets we mean sets that are generated sequentially by some process, one after the other, so that the process well-orders the sets. Gödel has shown that an axiom asserting that all sets are constructible can be consistently added to ZF [1], giving a theory usually called ZFC$^+$. It is well known that no more than countably many subsets of ω^* can be shown to exist in ZFC$^+$. This result will, of course, hold for the sub-theory ZFC$^+$ minus the axiom schema of subsets and the power set axiom. Thus we can now add a new axiom which says that the subsets of ω^* are countably constructible. This axiom, combined with the axiom schema of bijective replacement, creates a set of constructible subsets of ω^* and deletion of the power set axiom then assures that no other subsets of ω^* exist in this theory.

We shall refer to these eleven axioms as T. All the sets of finite natural numbers in T are finite. The general continuum hypothesis holds in this theory because all sets are countable. We now will also show that this theory is rich enough to contain functions of a real variable effectively governing physical fields.

We first show T has a countable real line. Recall the definition of "rational numbers" as the set of ratios, in ZF called Q, of any two members of the set ω. In T, we can likewise, using the axiom of unions, establish for ω^* the set of ratios of any two of its natural numbers, finite or infinite. This will become an "enlargement" of the rational numbers and we shall call this enlargement Q^*. Two members of Q^* are called "identical" if their ratio is 1. We now employ the symbol "$\equiv$" for "is identical to." Furthermore, an "infinitesimal" is a member of Q^* "equal" to 0, i.e., letting y signify the member and employing the symbol "=" to signify equality, $y = 0 \leftrightarrow \forall k[y < 1/k]$, where k is a finite natural number. The reciprocal of an infinitesimal is "infinite". A member of Q^* that is not an infinitesimal and not infinite is "finite". The constructibility axiom well-orders the set of constructible subsets of ω^* and gives a metric space. These subsets then represent the binimals forming a countable real line R^*. In this theory R^* is a subset of Q^*.

An ***equality-preserving*** bijective map $\phi(x,u)$ between intervals X and U of R^* in which $x \in X$ and $u \in U$ such that $\forall x_1, x_2, u_1, u_2[\phi(x_1,u_1) \wedge \phi(x_2,u_2) \rightarrow (x_1 - x_2 = 0 \leftrightarrow u_1 - u_2 = 0)]$ creates pieces which are biunique and homeomorphic. Note that $U = 0$ if and only if $X = 0$, i.e., the piece is inherently relational.

We can now define "functions of a real variable in T". $u(x)$ is a function of a real variable in T only if it is a constant or a sequence in x of continuously connected biunique pieces such that the derivative of u with respect to x is also a function of a real variable in T. These functions are thus of bounded variation and locally homeomorphic. If some derivative is a constant, they are polynomials. If no derivative is a constant, these functions do not *per se* exist in T but can, however, always be represented as closely as required for physics by a sum of polynomials of sufficiently high degree obtained by an iteration of:

$$\int_a^b \left[p \left(\frac{du}{dx} \right)^2 - qu^2 \right] dx \equiv \lambda \int_a^b ru^2 dx \tag{1}$$

where λ is minimized subject to:

$$\int_a^b ru^2 dx \equiv \text{const} \tag{2}$$

where:

$$a \neq b, \quad u \left(\frac{du}{dx} \right) \equiv 0 \tag{3}$$

at a and b; p, q, and r are functions of the real variable x.

Letting n denote the n^{th} iteration, $\forall k \exists n[\lambda_{n-1} - \lambda_n < 1/k]$ where k is a finite natural number. So, a polynomial such that, say, $1/k < 10^{-50}$ should be sufficient for physics as it is effectively a Sturm-Liouville "eigenfunction". These can be decomposed, since they are polynomials, into biunique "irreducible eigenfunction pieces" obeying the boundary conditions.

As a bridge to physics, let x_1 be space and x_2 be time. We now postulate the following integral relation for a one-dimensional string $\Psi = u_1(x_1)u_2(x_2)$:

$$\int \left[\left(\frac{\partial \Psi}{\partial x_1} \right)^2 - \left(\frac{\partial \Psi}{\partial x_2} \right)^2 \right] dx_1 dx_2 \equiv 0 \tag{4}$$

The eigenvalues λ_{1m} are determined by the spatial boundary conditions. For each eigenstate m, we can use this identity to iterate the eigenfunctions u_{1m} and u_{2m} constrained by the indicial expression $\lambda_{1m} \equiv \lambda_{2m}$.

A more general string in finitely many space-like and time-like dimensions can likewise be produced. Let $u_{\ell mi}(x_i)$ and $u_{\ell mj}(x_j)$ be eigenfunctions with non-negative eigenvalues $\lambda_{\ell mi}$ and $\lambda_{\ell mj}$ respectively.

We define a "field" as a sum of eigenstates:

$$\underline{\Psi}_m = \sum_\ell \Psi_{\ell m} \underline{i_\ell}, \Psi_{\ell m} = \mathrm{C} \prod_i u_{\ell mi} \prod_j u_{\ell mj} \tag{5}$$

with the postulate: ***for every eigenstate*** m ***in a compactified space the integral of the Lagrange density is identically null***.

The physics behind this postulate will become clear. Let ds represent $\prod_i r_i dx_i$ and $d\tau$ represent $\prod_j r_j dx_j$. Then for all m,

$$\begin{aligned} &\int \sum_{\ell i} \frac{1}{r_i} \left[P_{\ell mi} \left(\frac{\partial \Psi_{\ell m}}{\partial x_i} \right)^2 - Q_{\ell mi} \Psi_{\ell m}^2 \right] ds d\tau \\ &- \int \sum_{\ell j} \frac{1}{r_j} \left[P_{\ell mj} \left(\frac{\partial \Psi_{\ell m}}{\partial x_j} \right)^2 - Q_{\ell mj} \Psi_{\ell m}^2 \right] ds d\tau \equiv 0 \end{aligned} \tag{6}$$

In this integral expression the P, Q, and R can be functions of any of the x_i and x_j, thus of any $\Psi_{\ell m}$ as well.

This is a ***nonlinear sigma model***. As seen in the case of a one-dimensional string, these Ψ_m can in principle be obtained by iterations constrained by an indicial expression, $\sum_{\ell i} \lambda_{\ell mi} \equiv \sum_{\ell j} \lambda_{\ell mj}$ for all m.

A sui generis proof in T that the integrals in the nonlinear sigma model have only discrete values will now be shown. Let expressions (7) and (8) both be represented by α, since they are identical:

$$\sum_{\ell mi} \int \frac{1}{r_i} \left[P_{\ell mi} \left(\frac{\partial \Psi_{\ell m}}{\partial x_i} \right)^2 - Q_{\ell mi} \Psi_{\ell m}^2 \right] ds d\tau \tag{7}$$

$$\sum_{\ell mj} \int \frac{1}{r_j} \left[P_{\ell mj} \left(\frac{\partial \Psi_{\ell m}}{\partial x_j} \right)^2 - Q_{\ell mj} \Psi_{\ell m}^2 \right] ds d\tau \tag{8}$$

I. α is positive and must be closed to addition and to the absolute value of subtraction. In T, α is an infinitesmal unit times an infinite natural number or a finite unit times a finite natural number.

II. If space-time exists but the field is not present, $\alpha \equiv 0$; otherwise, if the field is present, then in T α must be finite, $\alpha \neq 0$; thus $\alpha = 0 \leftrightarrow \alpha \equiv 0$.

III. $\therefore \alpha \equiv n\kappa$, where n is a natural number and κ is some finite positive constant such that $\alpha = 0 \leftrightarrow n \equiv 0$.

If we give κ the dimensions of action, then the postulate (6) expresses a fundamental symmetry of the potential and kinetic components of action.

With this result and without any additional physical postulates, we can now obtain quantum mechanics from this nonlinear sigma model in one time-like dimension and finitely many space-like dimensions.

Let $\ell = 1, 2$, $r_t = P_{1mt} = P_{2mt} = 1$, $Q_{1mt} = Q_{2mt} = 0$, $\tau = \omega_m t$ and we normalize Ψ as follows:

$$\Psi_m = \sqrt{(C/2\pi)} \prod_i u_{im}(x_i)[u_{1m}(\tau) + i \cdot u_{2m}(\tau)] \quad (9)$$

where $i = \sqrt{-1}$ with

$$\int \sum_m \prod_i u_{im}^2 ds(u_{1m}^2 + u_{2m}^2) \equiv 1 \quad (10)$$

then:

$$\frac{du_{1m}}{d\tau} = -u_{2m} \quad \text{and} \quad \frac{du_{2m}}{d\tau} = u_{1m} \quad (11)$$

or

$$\frac{du_{1m}}{d\tau} = u_{2m} \quad \text{and} \quad \frac{du_{2m}}{d\tau} = -u_{1m} \quad (12)$$

For the minimal non-vanishing field, the integral α over each irreducible eigenfunction piece of $u_{1m}(\tau)$ and $u_{2m}(\tau)$ is κ; over a cycle it is 4κ. Thus,

$$(C/2\pi) \sum_m \oint \int \left[\left(\frac{du_{1m}}{d\tau} \right)^2 + \left(\frac{du_{2m}}{d\tau} \right)^2 \right] \prod_i u_{im}^2(x_i) ds d\tau \equiv C \equiv 4\kappa \quad (13)$$

Substituting the Planck h for 4κ and ωt for τ, the integral α can now be put into the conventional Lagrangian form for the Schrödinger equation,

$$\frac{1}{2i} \sum_m \oint \int \left[\Psi_m^* \left(\frac{\partial \Psi_m}{\partial t} \right) - \left(\frac{\partial \Psi_m^*}{\partial t} \right) \Psi_m \right] ds dt \quad (14)$$

Therefore, the Schrödinger equation is a special case of a nonlinear sigma model in one time dimension and finitely-many spatial dimensions.

Finally, if we return to the above proof of the discreteness of α, we discover that the proof assumed absolute space-time. The theory admits another possibility: $\alpha = 0$ if and only if space-time is likewise infinitesimal. Thus, finite space-time exists if and only if there are finite fields; space-time is relational. In addition, in this theory the real line is countable and the quantum fields are locally homeomorphic with the real line, thus the differential properties of space-time fulfill the strict requirements of Einstein-Weyl causality [3], suggesting a possible foundation for quantum gravity.

In the process of this discussion, we have shown that:

- Quantum mechanics is obtained without requiring introduction of the statistical interpretation of the wave function, thereby resolving a long-standing controversy [4].

- Quantum mechanics is instead derived in a constructible theory using an action symmetry postulate.

In addition, though we do not have the opportunity to discuss these points, we note that:

- There are inherently no singularities in the physical fields obtained in this theory.

- The solution to the QED divergence problem posed by Dyson [5] is provided, since the actual convergence or divergence of the essential perturbation series is undecidable in this theory.

- Wigner's metaphysical question regarding the apparent unreasonable effectiveness of mathematics in physics [6] is directly answered, since the foundations of mathematics and physics are now linked.

Appendix: ZF - Subsets - Power Set + Constructibility + Arithmetic

Extensionality. Two sets with just the same members are equal. $\forall x \forall y \, [\forall z \, [z \in x \leftrightarrow z \in y] \rightarrow x = y]$

Pairs. For every two sets, there is a set that contains just them. $\forall x \forall y \exists z [\forall w w \in z \leftrightarrow w = x \vee w = y]$

Union. For every set of sets, there is a set with just all their members. $\forall x \exists y \forall z [z \in y \leftrightarrow \exists u [z \in u \wedge u \in x]]$

Infinity. There are infinite ordinals ω^* (i.e., sets are transitive and well-ordered by $\in$-relation). $\exists \omega^* [O \in \omega^* \wedge \forall x [x \in \omega^* \rightarrow x \cup \{x\} \in \omega^*]]$

Replacement. Replacing members of a set one-for-one creates a set (i.e., "bijective" replacement). Let $\phi(x, y)$ a formula in which x and y are free,
$\forall z \forall x \in z \forall y [\phi(x, y) \wedge \forall u \in z \forall v [\phi(u, v) \rightarrow u = x \leftrightarrow y = v]] \rightarrow \exists r \forall t [t \in r \leftrightarrow \exists s \in z \phi(s, t)]$

Regularity. Every non-empty set has a minimal member (i.e. "weak" regularity). $\forall x [\exists y y \in x \rightarrow \exists y [y \in x \wedge \forall z \neg [z \in x \wedge z \in y]]]$

Constructibility. The subsets of ω^* are countably constructible. $\forall \omega^* \exists S [(\omega^*, O) \in S \wedge \forall y \forall z [(y, z) \in S \rightarrow [[\exists m_y [m_y \in y \wedge \forall v \neg [v \in y \wedge v \in m_y]] \leftrightarrow [\exists t_y \forall u [u \in t_y \leftrightarrow u \in y \wedge u \neq m_y]] \wedge (t_y \cup m_y, z \cup \{z\}) \in S]]]]$

The four formulae (a) to (d) below can be derived in ZF but must be adjoined to T as axioms.

Let $x' = xU\{x\}$

(a) $\forall x \in \omega^* (x \neq O \leftrightarrow \exists y \in \omega^* (y' = x))$

(b) $\forall x \forall y (x' = y' \rightarrow x = y)$

Let x and y be members of ω^* and $[x, y]$ and $[[x, y], z]$ represent ordered pairs.

(c) $\exists A \forall x \in \omega^* \forall y \in \omega^* E!z[[x, O], x] \in A \wedge [[x, y], z] \in A \wedge [[x, y'], z'] \in A$; addition: $x + y = z$

(d) $\exists M \forall x \in \omega^* \forall y \in \omega^* E!z[[x, O], O] \in M \wedge [[x, y], z] \in M \wedge [[x, y'], z + x] \in M$; multiplication: $x \cdot y = z$

Define $[a, b]_r$ such that $[a_1, b]_r + [a_2, b]_r \equiv [a_1 + a_2, b]_r$ and $[a_1, b_1]_r \equiv [a_2, b_2]_r \leftrightarrow a_1 \cdot b_2 \equiv a_2 \cdot b_1$. The extended set of rationals Q^* is the set of such pairs for all a and b in ω^*.

Theorems

Using the axioms (a), (b) and axioms of regularity, unions and bijective replacement one can show that ω^* is a set which contains the predecessor of every member of itself, except for the null set.

From axiom (c) and the axiom of regularity, there is, for any given $x \in \omega^*$, a one-to-one relation between all $y \in \omega^*$ and z. It can then be shown by the axiom of bijective replacement that the sets $t(\omega^*, x)$ exist:

$$\forall \omega^* \forall x \in \omega^* (\forall y_1 \forall y_2 \forall z_1 \forall z_2 ([[x, y_1], z_1] \in A \wedge [[x, y_2], z_2] \in A \rightarrow ((z_1 = z_2) \leftrightarrow (y_1 = y_2))) \rightarrow \exists t(\omega^*, x) \forall z (z \in t(\omega^*, x) \leftrightarrow \exists y \in \omega^* ([[x, y], z] \in A)))$$

Using the axiom of regularity, one can show that $\forall \omega^* \forall x \in \omega^* \forall u \in \omega^* (u \in x \veebar u \in \{x\} \veebar u \in t(\omega^*, x'))$. This is a "trichotomy".

These theorems allow the members of ω^* to be considered natural numbers. Therefore, the axioms of arithmetic are directly applicable.

From the axiom of constructibility, the set Z^* such that $[O, Z^*] \in S$ maps to the real line R^*. Since Z^* is an ω^* by the axiom of infinity, the arithmetic axioms and theorems are directly applicable to R^* as well. The metric between members y_1 and y_2 of the real line $(0, 1)$ is given by $[|z_1 - z_2|, Z^*]_r$ where $[y, z] \in S$.

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