# SIMPLE ALMOST HYPERDEFINABLE GROUPS

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Abstract. (i) We lay down the groundwork for the treatment of almost hyperdefinable groups: notions from [BTW04] are put into a natural hierarchy, and new notions, essential to the study to such groups, fit elegantly into this hierarchy.
(ii) We show that "classical" properties of definable and hyperdefinable groups in simple theories can be generalised to this context. In particular, we prove the existence of stabilisers of Lascar strong types and of the connected and locally connected components of subgroups, and that in a simple one-based theory an almost hyperdefinable group is bounded-by-abelian-by-bounded.

## Introduction

This paper is concerned with the generalisation of results from [Wag05, Wag01] to the context of $\alpha/\beta$-groups (see below for the definition), first introduced as almost hyperdefinable groups in [BTW04]. Loosely speaking, an $\alpha/\beta$-group is a group whose underlying set of elements is of the form $G = G^b/R$, where $G^b$ is a type-definable set, and $R = \bigvee_{i\in I} R_i$ is an equivalence relation which is not type-definable but is only an infinite disjunction of type-definable relations (satisfying some additional properties).

There are two aspects to our task. The first is to lay the groundwork for the model-theoretic treatment of such groups. This was partially done in [BTW04], where some basic definitions were given and the existence of well-behaved stratified local ranks was proved for almost hyperdefinable groups (and polygroups). However, in order to study a group we must consider its subgroups, and in that respect previous work leaves much to want. As we consider subgroups of $\alpha/\beta$-groups in the current paper we find ourselves forced to consider the new notion of $\beta/\beta$-subgroups, namely subgroups $H^b/R \subseteq G^b/R$ where $H^b$ is not type-definable, but again only an infinite union of type-definable sets (with some additional properties). While doing so we find ourselves working in a rather weird category, where "obvious" notions such as intersection can be somewhat surprising.

The other aspect is actually proving properties of $\alpha/\beta$-groups and their subgroups. While doing so, we shall try to skip tedious step-by-step verifications in this new context

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of proofs already given in [Wag05, Wag01], if the adaptation of the existing proofs is sufficiently trivial. We rather wish to concentrate on new ideas, such as methods to recover $\alpha$-elements and $\alpha/\beta$-groups from such $\beta/\beta$-groups, using stratified local ranks and the intermediary notion of an $\alpha^-/\beta$-group.

Finally, a word (or several) about our terminology, compared with that of [BTW04] and before. Originally, one considered definable groups, or at most type-definable groups, which lived in the real or in an imaginary sort. Then in [Wag01], one starts considering groups living in hyperimaginary sorts, namely in quotients by type-definable equivalence relations. In a hyperimaginary sort the distinction between a type-definable and a definable set is meaningless, so we call them hyperdefinable. The next level comes in [BTW04], where we have to replace the type-definable equivalence relation by which we divide by an *almost type-definable* one, namely a "nice" union of type-definable relations. The quotient is named almost hyperdefinable, in analogy with hyperdefinable. So in "almost type-definable", the "almost" qualifies the numerator (in fact there isn't necessarily a denominator), whereas in "almost hyperdefinable" it qualifies the denominator.

This is a bit of a mess (for which the author has to admit responsibility), but it works quite well, until in the current paper we encounter quotients of almost type-definable sets by almost type-definable equivalence relations. Putting these into the existing naming scheme would be complicated, since we'd have to say if the "almost" applies to the numerator, denominator, or both. In addition, the prefixes "type-" and "hyper" have somewhat lost of their original meaning, since they not longer designate generalisations of plain first-order definability (we're way past that stage), but merely serve to tell us how to interpret the adverb "almost".

We therefore decided that instead of trying to adapt by force a pretty inadequate naming scheme, we should use a new one which would be designed for the purposes of this paper; and while we're at it, why not make it more compact as well. We consider three levels of definability, $\gamma$, $\beta$, and $\alpha$, increasingly well-behaved: $\alpha$-definable is anything that was before [BTW04], namely definable, type-definable or hyperdefinable, between which we do not find the need to make any distinction (in this context!); $\gamma$-definable just means graded, that is a union of $\alpha$-definable sets with some compatibility conditions; and $\beta$-definable, which lies in between, is what we called in [BTW04] almost type-definable, namely something which is $\gamma$-definable, but is "sufficiently close" to being $\alpha$-definable. Thus almost hyperdefinable is $\alpha/\beta$, the new kind of groups we introduce is $\beta/\beta$-groups, etc.. If only the rest were just as clear and intuitive as this...

## 1. General definitions and terminology

### 1.1. Gradings.

**Convention 1.1.** In the spirit of [Ben03b], we do not distinguish between real, imaginary and hyperimaginary elements and sorts, and call them all *$\alpha$-elements* and *$\alpha$-sorts*, respectively. Similarly, classically definable, type-definable, and hyperdefinable sets are

all called *$\alpha$-sets*.
Later on we will define more general kinds of elements; however, parameters over which sets are defined are always $\alpha$-elements.

Thus, this paper fits naturally in the context of [Ben03b]. In particular, everything we say here is valid in a simple thick cat (see [Ben03c, Ben03d]). However, if the reader so wishes, she or he may assume that we work with a first order theory.

**Definition 1.2.** Let $I$ be a directed partial order.
A *$\gamma_I$-set* (over an $\alpha$-element $a$) is a family of $\alpha$-sets (defined over $a$) $X_I = \{X_i : i \in I\}$, increasing along $I$: $i \leq j \Longrightarrow X_i \subseteq X_j$. We may sometimes just write $X = \bigcup_{i\in I} X_i$, and this decomposition of $X$ into subsets is called its *grading*.
We may omit the subscript $I$ if it is clear from the context.

The idea is that the grading is an essential part of the structure on the set $X$. Classical properties are defined for $X$ with the additional requirement that they be compatible with the grading. For example:

**Definition 1.3.** Let $X_I$ and $Y_J$ be $\gamma$-sets, and $f : I \to J$ a map. Then $X_I \subseteq_f Y_J$ ($X$ is *$f$-gradedly included* in $Y$) if $X_i \subseteq Y_{f(i)}$ for all $i \in I$. If we do not care much for the particular map $f$ we may simply write $X_I \subseteq Y_J$, but it is understood that is a graded inclusion.
$X =_{f,g} Y$ ($X$ is *$f, g$-gradedly equal* to $Y$) if $X \subseteq_f Y$ and $X \supseteq_g Y$. If $I = J$ one $f : I \to I$ should suffice and we write $X =_f Y$, and of course we may omit the subscript altogether.

It is important that all relations between $\gamma$-sets (inclusion, equality, etc.) are graded in the sense defined above. On the other hand, since we never consider ungraded relations, we allow ourselves to omit the qualifier: thus *equal* always means *gradedly equal*, and so forth.

We identify equal sets, and will make sure that all the properties that we define will be invariant under equality. However, when dealing with infinitely many $\gamma$-sets, one needs to be more careful. If $\{X^j : j < \lambda\}$ and $\{Y^j : j < \lambda\}$ are two families of $\gamma_I$-sets, we say that $X^j = Y^j$ *uniformly* if exists one $f : I \to I$ such that $X^j =_f Y^j$ for all $j < \lambda$ (rather than: for all $j < \lambda$ there exists $f_j : I \to I \dots$). For example:

**Definition 1.4.** Let $X$ be a $\gamma_I$ set and $a$ an $\alpha$-element. Then $X$ is *$a$-invariant* if all $a$-conjugates of $X$ are uniformly equal.

**Lemma 1.5.** *An $a$-invariant $\gamma_I$-set $X$ is equal to a $\gamma_I$-set over $a$.*

*Proof.* Say that $X$ is defined over $b$, and write $X_i = X_i(b)$ and $q = \mathrm{tp}(b/a)$. By assumption there is $f : I \to I$ such that $X_i(b) =_f X_i(b')$ for all $b' \equiv_a b$. Let $Y_i = \{c : \exists z\, [q(z) \wedge c \in X_i(z)]\}$. Then clearly $X =_f Y$. $\mathrm{QED}_{1.5}$

Still, this allows us some liberty with the set $I$:

**Lemma 1.6.** *Let $X_I$ be a $\gamma$-set, and $J$ any directed partial order. For $(i,j) \in I \times J$, define $X'_{i,j} = X_i$. Then $X'_{I\times J}$ is a $\gamma$-set and $X = X'$.*

*Proof.* Easy. QED$_{1.6}$

Since moreover this can be done in a uniform fashion for a family of $\gamma_I$-sets, may always assume, when given several $\gamma$-sets, that they are $\gamma_I$ for the same $I$.

When considering the intersection of infinitely many sets, it would seem that the right thing to do would be to define:

**Definition 1.7.** Let $(X^\alpha : \alpha < \lambda)$ be all $\gamma_I$-set. We define their intersection $\bigcap_\alpha X^\alpha$ as the $\gamma_I$-set $(\bigcap_\alpha X^\alpha)_i = \bigcap_\alpha X^\alpha_i$.

*Remark* 1.8. We do not lose generality by the uniformity requirement. Indeed, if each $X^\alpha$ is $\gamma_{I^\alpha}$ then they are all naturally $\gamma_{\prod I^\alpha}$, and we can still calculate their intersection. In this case, we obtain the intersection in the non-graded sense, which will be noted by $\bigcap^\prod_\alpha X^\alpha$. Thus, $\bigcap^\prod_\alpha X^\alpha$ in naturally a $\gamma_{\prod I^\alpha}$-set. In particular, if all the $X^\alpha$ are $\gamma_I$, then $\bigcap^\prod_\alpha X^\alpha$ is naturally a $\gamma_{I^\lambda}$-set.
If we do not have uniform gradings, this method is the best we can do. Of course, it has the disadvantage that the order type by which we grade depends on the set of subgroups which we intersect. In practice we will manage to have uniformity and keep $I$ fixed.

For sets with additional structure (groups, equivalence relations, etc.) we shall also require that the grading be compatible with the structure, and this is witnessed by some $f : I \to I$. For example:

**Definition 1.9.** A $\gamma_{I,f}$*-equivalence relation* in a sort is a $\gamma_I$-set $R$ of pairs in this sort, which is reflexive, symmetric, and $f$*-gradedly transitive*:

- $a\ R_i\ a$ for all $i \in I$ and $a$ in the sort.
- $a\ R_i\ b \implies b\ R_i\ a$ for all $i \in I$ (and $a, b$ in the sort).
- $a\ R_i\ b \wedge b\ R_i\ c \implies a\ R_{f(i)}\ c$ for every $a, b, c$ in the sort and $i \in I$.

If $R$ is a $\gamma$-equivalence relation on some sort and $X$ an $\alpha$-set in this sort, then $X^{R_i} = \{b : \exists a \in X\ b\ R_i\ a\}$. For a $\gamma_I$-set $X_I$ in this sort, we define $X^R_I = (X^{R_i}_i : i \in I)$.

**Definition 1.10.** Here $R$ is a $\gamma_{I,f}$-equivalence relation.

(i) A $\gamma_I$-set $X$ is $R$*-complete* if $X = X^R$.
This means that uniformly, every element of $X$ belongs to some $R$-class, and if $X$ intersects some $R$-class then it contains it.

(ii) A $\gamma/\gamma_{I,f}$-set is a formal quotient $X/R$ where $X$ is $R$-complete.
If $X = Y^R$ where $Y$ is $\alpha$, then $X/R = Y^R/R$ is $\alpha^-/\gamma$.
If $X$ is (equal to) an $\alpha$-set then $X/R$ is $\alpha/\gamma$.

(iii) A $\gamma$-*subset* of a $\gamma/\gamma$-set $X/R$ is a $\gamma/\gamma$-set $Y/R$ (so $Y$ is $R$-complete) where $Y \subseteq X$, and we write $Y/R \subseteq X/R$.
An $\alpha$-subset ($\alpha^-$-subset) of $X$ is a subset that is $\alpha/\gamma$ ($\alpha^-/\gamma$) as a set.

*Remark* 1.11. Note that the (graded) intersection of uniformly $R$-complete sets is $R$-complete, so we can speak of the intersection of infinitely many subsets of $X/R$, when they are uniformly such.

Intuitively, we would like to consider $X/R$ as the set $\{a^R : a \in X\}$. This is wrong, though, since this ignores all the graded information. The right way to define the structure on $X/R$ is through the category of $\gamma/\gamma$-sets.

Here we only define maps between $\alpha/\gamma$ sets. We allow ourselves this simplification, since all $\gamma/\gamma$-sets we will consider in this paper are subsets of $\alpha/\gamma$-sets, and we are only going to consider maps between them that are restrictions of maps from the surrounding $\alpha/\gamma$-sets. In fact, it does not seem at all clear what should be the "correct" definition of a map between $\gamma/\gamma$-sets beside taking the restriction of a map between ambient $\alpha/\gamma$-sets.

**Definition 1.12.** (i) Let $X/R$, $Y/R'$ be two $\alpha/\gamma$-sets, and $F \subseteq X \times Y$ an $\alpha$-set. For $a \in X$ write $F(a) = \{b' : (a, b') \in F\}$, and for $b \in Y$ write $F^{-1}(b) = \{a' : (a', b) \in F\}$. Assume that:
- $F$ is well-defined: $F(a^R) \subseteq b^{R'}$ uniformly for all $a \in X$ and $b \in F(a)$.
- $F$ is everywhere-defined: $X \subseteq F^{-1}(Y)^{R_1}$ for some $1 \in I$.

Then $F : X/R \to Y/R$ is a *(graded) map*.

(ii) Two graded maps $F, G : X/R \to Y/R'$ are equal if $F(a^R) = G(a^R)$ uniformly for all $a \in X$.

(iii) Let $F : X/R \to Y/R'$, $G : Y/R' \to Z/R''$ be maps. Define $H = \{(a, c) : \exists b\, b \in F(a) \wedge c \in G(b)\}$. Then the composition $G \circ F$ is defined as $H : X/R' \to Z/R''$.
The identity $id_{X/R} : X/R \to X/R$ is defined by the diagonal of $X$.
Consequently, if $F : X/R \to Y/R'$ and $G : Y/R' \to X/R$ are graded maps, then we say that $G = F^{-1}$ if $G \circ F = id_{X/R}$ and $F \circ G = id_{Y/R'}$.

If we were to consider $X/R$ as the set $\{a^R : a \in X\}$, then $F : X \to Y$ would be the map $a^R \mapsto F(a)^{R'}$. We remind though that this is formally wrong, and there is more structure behind.

*Remark* 1.13. (i) The condition on $F$ to be everywhere-defined is equivalent to: for some map $1 \in I$, for every $a \in X$: $F(a^{R_1}) \neq \varnothing$.

(ii) Two graded maps $F, G : X/R \to Y/R'$ are equal if and only if there is $1 \in I$ such that $F \subseteq G^{R_1 \times R'_1}$ and $G \subseteq F^{R_1 \times R'_1}$.

We defined maps between $\alpha/\gamma$-sets. A map between two $\gamma/\gamma$-sets is defined as the restriction of a map between $\alpha/\gamma$-sets containing them. More precisely:

**Definition 1.14.** Let $F : X/R \to Y/R'$ be a map between $\alpha/\gamma$-sets, and $X'/R \subseteq X/R$. Then $F(X'/R)$ is defined as $F(X')^{R'}/R'$. IF $F(X'/R) \subseteq Y'/R \subseteq Y/R'$ then $F$ induces a map $F\restriction_{X'/R}: X'/R \to Y'/R'$.

Clearly, an $\alpha$-subset is a $\alpha^-$-subset. The converse is not true, and instead we have:

**Lemma 1.15.** *Let $Y/R \subseteq X/R$ be $\gamma/\gamma$-sets (so in particular, $X$ and $Y$ are $R$-complete). Then $Y/R$ is an $\alpha^-$-subset if and only if it is an (injective) image of an $\alpha/\gamma$-set.*

*Proof.* Consider an $\alpha^-$-subset $Y/R = Z^R/R \subseteq X/R$, where $Z \subseteq X$ is an $\alpha$-set, so $Z/(R\restriction_Z)$ is an $\alpha/\gamma$-set. Let $F \subseteq Z \times Z$ be the diagonal. Then $F : Z/(R\restriction_Z) \hookrightarrow X/R$ and its image is $Y/R$.
Conversely, let $Z/R'$ be an $\alpha/\gamma$-set, and $F : Z/R' \to X/R$ a map (not necessarily injective) whose image is $Y/R$. Then $W = F(Z) \subseteq X$ is an $\alpha$-set, and one verifies easily that $Y = W^R$. $\text{QED}_{1.15}$

By $R\restriction_Z$ we mean the $\gamma$-equivalence relation $R'_I$ defined by $a \; R'_i \; b \iff (a \; R_i \; b \wedge a, b \in Z) \vee a = b$.

*Remark* 1.16. Notice that an injective image is not necessarily isomorphic to its domain. In other words, an injective and surjective map is not necessarily invertible.
Thus the difference between an $\alpha$-subset and an $\alpha^-$-subset is that between a "true" subset and an embedded set.

1.2. **Groups.** We recall a definition from [BTW04]:

**Definition 1.17.** Let $\mathcal{F}$ be a purely functional signature.

(i) An *$\alpha/\gamma_I$-$\mathcal{F}$-structure* is a $\alpha/\gamma_I$-set $S = S^b/R$ equipped with maps $F^S : S^{n_F} \to S$ for every $n_F$-ary function symbol $F \in \mathcal{F}$. If we have infinitely many function symbols, then we require that all $F^S$ be maps uniformly (that is, that the requirements from a map hold uniformly for all $F^S$).
(ii) If $S$ is such a structure and $S' = S'^b/R \subseteq S^b/R = S$ is a $\gamma$-subset, and in addition it is uniformly closed under the functions $F^S$ (that is to say that $F^S((S'^b)^{n_F}) \subseteq S'^b$ uniformly for all $F$), then $S'$ is a *$\gamma$-substructure*.
(iii) A $\gamma/\gamma$-structure is a $\gamma$-substructure of an $\alpha/\gamma$-structure.
(iv) If $S, S'$ are two $\gamma$-$\mathcal{F}$-structures and $G : S \to S'$ a map, then it is a *homomorphism* if $G \circ F^S = F^{S'} \circ (G, \dots, G)$ (gradedly) for every $F \in \mathcal{F}$.

*Remark* 1.18. Note that we only allow $\alpha/\gamma$-structures as ambient structures, and $\gamma/\gamma$-structures must live within such a structure. Compare this with Definition 1.12.

We can generalise Lemma 1.15 (this was also observed by Frank Wagner):

**Lemma 1.19.** *Let $S'$ be an $\gamma/\gamma$-structure, that is a substructure of an $\alpha/\gamma$ structure $S$. Then $S'$ is an $\alpha^-$-substructure if and only if it is an (injective) homomorphic image of an $\alpha/\gamma$-structure.*

*Proof.* Given Lemma 1.15, the only thing that requires proving is that if $S' = {S'^b}^R/R \subseteq S/R$ is a substructure with $S'^b$ an $\alpha$-set, then the structure can be pulled back to the $\alpha/\gamma$-set $S'' = S'^b/(R\restriction_{S'^b})$.

For sufficiently big $1 \in I$ we have $F^S(\bar{a}^{R_1}) \neq \varnothing$ for all $\bar{a} \in S^b$ and $F \in \mathcal{F}$. For a possibly bigger $2 \in I$ we have then $F^S(S'^{R_1}) \subseteq S'^{R_2}$ for every $F \in \mathcal{F}$, whereby for some yet bigger $3 \in I$: $F^S(\bar{a}^{R_1})^{R_3} \cap S'^b \neq \varnothing$ for every $\bar{a} \in S'^b$ and $F \in \mathcal{F}$, since we assumed that $S'$ was uniformly closed under the maps $F^S$.
Define $F^{S''} = \{(\bar{a}, b) : b \in F^S(\bar{a})^{R_3}\}$. Then $S''$ is an $\alpha/\gamma$-structure, and $S'' \hookrightarrow S$ is a monomorphism onto $S'$. QED$_{1.19}$

**Definition 1.20.** Let $T$ be a positive universal $\mathcal{F}$-theory: each axiom is just a universally quantified disjunction of equations of terms. Let $S = S^b/R$ be a $\alpha/\gamma$-structure. Then $S \models T$ if for some $1 \in I$, for every axiom, if we substitute elements from $S^b$ for the variables and calculate possible values for the terms, then at least one of the equations holds up to $R_1$.
Since a $\gamma/\gamma$-structure is defined as a substructure of an $\alpha/\gamma$-structure, and we only consider universal theories, we do not need to worry about satisfaction of a theory in such a structure: we will only consider substructures of models of the theory in question.

**Definition 1.21.** An $\alpha/\gamma$-group is a model in the language $\{\cdot, e, ^{-1}\}$ of the theory of groups; an $\alpha/\gamma$-homogeneous space is defined similarly, in a two-sorted language.
As said above, $\gamma/\gamma$-groups and $\gamma/\gamma$-homogeneous spaces are defined as substructures of $\alpha/\gamma$ groups and $\alpha/\gamma$-homogeneous spaces.

**Convention 1.22.** We work in an ambient $\alpha/\gamma_I$-group $G = G^b/R$. For simplicity of notation, and compatibility with [BTW04], we assume that for every $a, b, c \in G^b$:

(i) $a \cdot b \neq \varnothing$
(ii) $(a \cdot b) \cdot c \cap a \cdot (b \cdot c) \neq \varnothing$
(iii) $a \in e \cdot a \cap a \cdot e$
(iv) $e \in a \cdot a^{-1} \cap a^{-1} \cdot a$

(with the definitions given until now we only knew this up to some $R_1$, but then we can replace $\cdot$ with $\cdot^{R_1}$ and get these properties).

*Remark* 1.23. Let $H = H^b/R \subseteq G$. Then $H \leq G$ if and only if there is $f : I \to I$ such that $H_i^{b^{-1}} \cdot H_i^b \subseteq H_{f(i)}^b$ for every $i \in I$.

*Proof.* This is left as an exercise. QED$_{1.23}$

1.3. **Fullness and $\alpha^-$-subgroups.** The fullness property has a somewhat particular status: we would not expect it to hold for maps (morphisms) between structures, but we would expect it to hold for maps which are the interpretation of operations within an interpreted structure.

We recall from [BTW04]:

**Definition 1.24.** (i) A map $F : X/R \to Y/R'$ is *full* if for some $1 \in I$, $b^{R'} \subseteq F(a^R)^{R'_1}$ uniformly for all $a \in X$ and $b \in F(a)$.

(ii) Let $k < n < \omega$. A map $F : (X/R)^n \to Y/R'$ is *full in the $k$th argument* if for every $a_0, \ldots, \hat{a}_k, \ldots, a_{n-1} \in X$, the maps $F_{a_0,\ldots,\hat{a}_k,\ldots,a_{n-1}} : X/R \to Y/R'$ defined by $F_{a_0,\ldots,\hat{a}_k,\ldots,a_{n-1}}(a) = F(a_0, \ldots, a_{k-1}, a, a_{k+1}, \ldots, a_{n-1})$ are uniformly full.
$F$ is full if it is full in every argument.

(iii) A $\gamma/\gamma$-structure $S/R$ is full if $F^S$ is uniformly full for every $F \in \mathcal{F}$.

**Lemma 1.25.** *Let $X/R$ and $Y/R'$ be $\alpha/\gamma$-sets, and $F : X/R \to Y/R'$ a map. Then the following are equivalent:*

(i) *$F$ is full.*

(ii) *There is $1 \in I$ such that, if $(Z_\xi : \xi < \lambda)$ is a family of $\alpha$-subsets of $Y$, then $F^{-1}(Z_\xi^{R'})^{R_1} = F^{-1}(Z_\xi^{R_1'})^R$ uniformly.*

(iii) *There is $1 \in I$ such that if $(Z_\xi : \xi < \lambda)$ is a family of $\alpha$-subsets of $X$, then $F(Z_\xi^R)^{R_1'} = F(Z_\xi^{R_1})^{R'}$ uniformly.*

*Note that a family of $\alpha$-sets means just that: there is no uniformity requirement on the way that the $Z_\xi$ are defined.*

*Proof.* (i) $\Longrightarrow$ (ii). We may choose $1 \in I$ as in the definition of fullness, such that in addition $X \subseteq F^{-1}(Y)^{R_1}$. Let $(Z_\xi : \xi < \lambda)$ be a family of $\alpha$-subsets of $Y$, and we need to prove that $F^{-1}(Z_\xi^{R'})^{R_1} = F^{-1}(Z_\xi^{R_1'})^R$ uniformly.

By choice of $R_1$, for every $a \in X$ there is $a' \; R_1 \; a$ such that $F(a') \neq \varnothing$. If in addition $a \in F^{-1}(Z_\xi^{R_1'})^{R_i}$ then there is also $a'' \; R_i \; a$ such that $F(a'') \cap Z_\xi^{R_1'} \neq \varnothing$. Let $b \in F(a')$: then $F(a'^R) \subseteq b^{R'}$, so there is some $j$ such that $b \in Z_\xi^{R_j'}$, whereby $a \in F^{-1}(Z_\xi^{R_j'})^{R_1}$, and $j$ depends only on $i$. Thus $F^{-1}(Z_\xi^{R_1'})^R \subseteq F^{-1}(Z_\xi^{R'})^{R_1}$ uniformly, and this inclusion does not in fact depend on fullness, only on $F$'s being total.

Conversely, assume that $a \in F^{-1}(Z_\xi^{R_i'})$, so there exists $b \in F(a) \cap Z_\xi^{R_i'}$. We know that $F(a^R)^{R_1'} \supseteq b^{R'}$ uniformly for all such $a, b$, so there is $j$ which depends only on $i$ such that $Z_\xi \cap F(a^{R_j})^{R_1'} \neq \varnothing$, whereby $a \in F^{-1}(Z_\xi^{R_1'})^{R_j}$. We obtain $F^{-1}(Z_\xi^{R'}) \subseteq F^{-1}(Z_\xi^{R_1'})^R$ uniformly.

(ii) $\Longrightarrow$ (iii). Let $(Z_\xi : \xi < \lambda)$ be a family of $\alpha$-subsets of $X$. We may assume that $1 \in I$ is as in the antecedent, and in addition satisfying $X \subseteq F^{-1}(Y)^{R_1}$. The uniform inclusion $F(Z_\xi^R)^{R_1'} \subseteq F(Z_\xi^{R_1})^{R'}$ follows from the definition of a well-defined map. For the other, let $b \in F(Z_\xi^{R_1})^{R_i'}$, so $F^{-1}(b^{R_i'})^{R_1} \cap Z_\xi \neq \varnothing$. On the other hand, by assumption there is $j$ which depends only on $i$ such that $F^{-1}(b^{R_i'})^{R_1} \subseteq F^{-1}(b^{R_1'})^{R_j}$, whereby $b \in F(Z_\xi^{R_j})^{R_1'}$, as required.

(iii) $\Longrightarrow$ (i). Taking $Z_a = \{a\}$, $F(a^{R_1})^{R'} \subseteq F(a^R)^{R_1'}$ uniformly for all $a$. $\text{QED}_{1.25}$

In groups the situation is rather simple, due to the existence of inverses. The following is proved in [BTW04]:

**Fact 1.26.** (i) *Every $\alpha/\gamma$-group is full.*

(ii) *if $\langle G, X\rangle$ is a homogeneous space, with $\cdot_X : G \times X \to X$ the group action, then it is full if and only if $\cdot_X$ is full in the first argument.*

*Remark* 1.27. With Convention 1.22, we obtain a stronger version of fullness for $G$, namely that $a^R \cdot b = a \cdot b^R = c^R$ uniformly for all $a, b \in G^b$ and $c \in a \cdot b$.

The issue of fullness of homogeneous spaces is essential for the usefulness of stratified ranks (in a simple theory, see [BTW04]).

Assume that $\langle G, X\rangle = \langle G^b/R, X^b/R'\rangle$ is an $\alpha/\gamma$-homogeneous space. For $x \in X^b$, let $r_x : G \to X$ be defined by $r_x(g) = g \cdot x$. We define $G^b_x = r_x^{-1}(x^{R'})^R = r_x^{-1}(x^{R'})^{R_1}$ (for some $1 \in I$, which we know must exist), and $G_x = G^b_x/R$ is defined as the stabiliser of $x$.

Conversely, if $H \leq G$ we can define $R^{H,l}_i = \{(g, g') : g = g' \vee g \in g' \cdot H^b_i\}$. By Convention 1.22, this is an equivalence relation; the quotient $G/H$ is defined as $G^b/R^{H,l}$, and $\langle G, G/H\rangle$ has a natural structure of an $\alpha/\gamma$-homogeneous space. Finally, it can be verified by the reader that in the latter case $G_{gH} = gHg^{-1}$, and in the former $G/G_x$ is isomorphic to the orbit of $x$ (all gradedly, of course).

This is directly connected with the fullness of homogeneous spaces:

**Proposition 1.28.** *Assume that $G$ is $\alpha/\gamma$. A subgroup $H \leq G$ is $\alpha^-$ if and only if it is the stabiliser of an element in a full homogeneous space over $G$.*

*Proof.* Assume that $H = {H^b}^R/R \leq G = G^b/R$ where $H^b$ is $\alpha$. Take $R^{H,l}_i = \{(g, g') : g = g' \vee g \in (g' \cdot H^b)^{R_i}\}$. This is not precisely what we defined above, but in this particular case it is gradedly equal to it, and therefore just as good.
Let $f : I \to I$ witness the fullness of $G$, namely $(g \cdot h)^R \subseteq_f g^R \cdot h$ for all $g, h \in G^b$. Then:

$$\begin{aligned}
(g^{R_{f^2(i)}} \cdot g')^{R^{H,l}_1} &= ((g^{R_{f^2(i)}} \cdot g') \cdot H^b)^{R_1} \\
&\supseteq (g \cdot g')^{R_{f(i)}} \cdot H^b \\
&\supseteq ((g \cdot g') \cdot H^b)^{R_i} = (g \cdot g')^{R^{H,l}_i}
\end{aligned}$$

Thus $f^2$ and any $1 \in I$ witness the fullness of $\langle G, G/H\rangle$, and clearly $H = G_{eH}$.
The converse is a special case of Lemma 1.25. $\text{QED}_{1.28}$

1.4. $\beta$ **notions.** The notion of a $\beta$-definable object lies somewhere between $\alpha$-definable and $\gamma$-definable objects. A $\beta_I$-object is a $\gamma_I$-object $X_I$ such that there is $1 \in I$ for which $X_1$ is very similar to the whole of $X$, the precise definition of which depending on the nature of the object in question. In order to make 1 explicit, we may speak of a $\beta_{I,1}$-object.

We give the basic definitions, as well as a few example and properties, without assuming that the theory is simple. It should be noted, however, that the notion of $\beta$-object is closely related to simplicity, and is most useful in that context.

As usual, we start we equivalence relations:

**Definition 1.29.** A $\beta_{I,f,1}$-equivalence relation $R_I$ is a $\gamma_{I,f}$-equivalence relation on some sort $X$ such that there exists a bound $\nu$ on the size of an $R_1$-anti-clique in an $R$-class (an $R_1$-anti-clique is a sequence $(a_\xi : \xi < \alpha)$ such that $\neg(a_\xi \ R_1 \ a_\zeta)$ for every $\xi < \zeta < \alpha$).

In fact, we encounter relations which we would like to call $\beta$ as well. The right definition would seem to be:

**Definition 1.30.** (i) Let $R$ be a $\gamma_I$-relation on a sort $X$, that is a $\gamma_I$-set in the sort $X \times X$. Then $R^*$ is the equivalence relation generated by $R$, which is naturally $\gamma_{I\times\omega}$.
(ii) A $\beta_{I,1}$-relation $R_I$ is a $\gamma_I$-relation such that there is a bound on the size of an $R_1$-anti-clique in an $R^*$-class.

*Remark* 1.31. (i) A $\gamma$-equivalence relation is $\beta_1$ as an equivalence relation if and only if it is $\beta_1$ as a relation. If $R$ is a $\beta$-relation, then in particular $R^*$ is a $\beta$-equivalence relation.
(ii) If $R$ is a $\beta_{I,f,1}$-equivalence relation then every $R$-class can be covered by $\nu$ sets of the form $a^{R_1}$. Conversely, if $R$ is a $\gamma_{I,f}$-equivalence relation having this property, then it is $\beta_{I,f,f(1)}$.

If $X/R$ is an $\alpha/\gamma$-set and $R$ is a $\beta$-equivalence relation then we say that $X/R$ is an $\alpha/\beta$-set, and similarly in other cases ($\alpha/\beta$-group, etc.).

*Example* 1.32. One example of a $\beta$-equivalence relation is the core equivalence defined in [BTW04]. In fact, it is defined as the transitive closure of a $\beta$-relation.

If this example originated from a $\gamma$-equivalence relation with bounded classes, then on the other extremity we have:

**Lemma 1.33.** *Let $c$ be an $\alpha$-element and $X$ a sort. Define an equivalence relation on $X$ by saying that $a \ R_1 \ b$ if they lie on some $c$-indiscernible sequence, and let $R_n$ be the $n$-iterate of $R_1$. Then $R_1$ is a $\beta$-relation, so the transitive closure $R = R^* = \bigvee R_i$ is a $\beta_{\omega,1}$-equivalence relation over $c$. It coincides with equality of Lascar strong type: it has boundedly many classes and it is finest as such.*
*Moreover, $R$ is the finest bounded $\beta$-equivalence relation over $c$ in the following sense: if $R'$ is any bounded $\beta_{I,f,1}$-equivalence relation over $c$, and $a \ R_n \ b$, then $a \ R'_{f^{\lfloor \lg_2 n \rfloor}(1)} \ b$.*

*Proof.* That $R$ is the finest bounded $c$-invariant equivalence relation is a classical result. It is clearly $\gamma_\omega$. Notice that in the classical proof that $R$ is bounded, one in fact proves that there is a bound on the size of an $R_1$-anti-clique in the entire sort, so *a fortiori* in every class, and both $R_1$ and $R$ are $\beta_{\omega,1}$.
For the moreover part: clearly, $R_1 \vdash R'_1$ since there is a bound on the size on an $R'_1$-anti-clique within an $R'$-class, and there are boundedly many $R'$-classes. We obtain $R_{2^n} \vdash R'_{f^n(1)}$ by easy induction. $\text{QED}_{1.33}$

This result is of interest even in a simple theory where equality of Lascar strong types is an $\alpha$-equivalence relation. See Lemma 2.2 below.

Clearly, if $(R^\alpha : \alpha < \lambda)$ are uniformly $\gamma$-equivalence relations, meaning that they are all $\gamma_{I,f}$-equivalence relations for some $I$ and $f : I \to I$, then $\bigcap R^\alpha$ is also a $\gamma_{I,f}$-equivalence relations. We can also prove that the $\beta$-property is preserved, and moreover that in this case the graded intersection is not so far from the non-graded one:

**Lemma 1.34.** *Let $(R^\alpha : \alpha < \lambda)$ be $\beta_{I,1}$-relations. Then $\bigcap_\alpha R^\alpha$ is a $\beta_{I,1}$-relation, and in fact there is a bound on a $(\bigcap R_1^\alpha)$-anti-clique in a $(\bigcap^\Pi R^\alpha)^*$-class (which is stronger than a bound on an anti-clique in a $(\bigcap R^\alpha)^*$-class).*
*Moreover, if all the $R^\alpha$ are $\beta_{I,f,1}$-equivalence relations, then so is $\bigcap_\alpha R^\alpha$, and every $\bigcap^\Pi R^\alpha$-class contains boundedly many $\bigcap R^\alpha$-classes.*

*Proof.* Let $\nu$ be a bound on the size of $R_1^\alpha$-anti-cliques in an $R^{\alpha*}$-class, for all $\alpha < \lambda$. We may assume that $\nu \geq \lambda$.
Consider an $\bigcap_\alpha R_1$-anti-clique $\{a_\xi : \xi < \mu = (2^\nu)^+\}$. Paint each pair $\{\xi, \zeta\} \in [\mu]^2$ (say that $\xi < \zeta$) with the minimal $\alpha < \lambda$ such that $\neg(a_\xi \; R_1^\alpha \; a_\zeta)$. We have at most $\nu$ colours, so by the Erdős-Rado Theorem there is a homogeneous subset of cardinality $\nu^+$. This would be an $R_1^\beta$-anti-clique where $\beta$ is the colour of this homogeneous set, which therefore cannot be contained, by assumption, in an $R^{\beta*}$-class, and *a fortiori* it cannot be contained in a $(\bigcap R^\alpha)^*$- or a $(\bigcap^\Pi R^\alpha)^*$-class.
The moreover part follows. $\text{QED}_{1.34}$

*Remark* 1.35. As for arbitrary sets, if we do not have uniformity, and every $R^\alpha$ is a $\beta_{I^\sigma,1^\alpha}$-relation then they are all $\beta_{\prod I^\alpha, \prod 1^\alpha}$-relations, and if every one is a $\gamma_{I^\alpha,f^\alpha}$-equivalence relation then they are all $\gamma_{\prod I^\alpha, \prod f^\alpha}$-equivalence relations.

We pass to subsets of groups. We keep the same conventions as before.

**Definition 1.36.**
(i) A *$\beta_2$-subset* of $G$ is a subset $X/R \subseteq G$ such that there exist a cardinal $\nu$ and $(g_\alpha, g'_\alpha \in G^b : \alpha < \nu)$ such that $X \subseteq \bigcup_\alpha g_\alpha \cdot X_2 \cdot g'_\alpha$.
We say loosely that $X$ can be covered by boundedly many two-sided translates of $X_2$.
(ii) A *left $\beta_2$-subgroup* of $G$ is a subgroup $H \leq G$ such that $R^{H,l}$ is a $\beta_2$-equivalence relation.

**Lemma 1.37.** *Every left $\beta_2$-subgroup of $G$ is a $\beta_2$-subset.*

*Proof.* Let $H \leq G$ be a left $\beta_2$-subgroup. Since there is a bound on the size of an $R_2^{H,l}$-anti-clique in $H^b$, we can cover $H^b$ by boundedly many sets of the form $g \cdot H_2^b \subseteq g \cdot H_2^b \cdot e$. $\text{QED}_{1.37}$

**Lemma 1.38.** *If $G$ is $\alpha/\beta_1$, then every $\alpha^-$-subset is a $\beta$-subset. An $\alpha^-$-subgroup is left $\beta$.*

*Proof.* Let $e^R = \bigcup e_\alpha^{R_1}$ be the identity of $G$. Then there is $2 \in I$ such that if $X \subseteq G^b$ is $\alpha$, then $X^R = \bigcup e_\alpha \cdot X^{R_2} = \bigcup X^{R_2} \cdot e_\alpha$.
If $X^R/R = H$ is a subgroup, a similar argument show that $R^{H,l}$ is $\beta_2$: for every $g \in G^b$ we have $g^{R^{H,l}} = g \cdot X^R = g^R \cdot X = \bigcup g_\alpha^{R_1} \cdot X = \bigcup g_\alpha \cdot X^{R_2}$, for well chosen $2 \in I$ and $g_\alpha \in g^R$. $\mathrm{QED}_{1.38}$

This is fortunate, since in an $\alpha/\beta$-group, we ordinarily only wish to consider $\beta$-subsets.

We get an analogue for Lemma 1.34:

**Lemma 1.39.** (i) *Let $(H^\alpha : \alpha < \lambda)$ be $\gamma_{I,f}$-subgroups of $G$. Then $\bigcap H^\alpha$ is a $\gamma_{I,f}$-subgroup*
(ii) *If the $H^\alpha$ are all left $\beta$, then $\bigcap H^\alpha$ is left $\beta$ and has bounded index in $\bigcap^\Pi H^\alpha$.*

*Proof.* (i) Clear.
(ii) Apply Lemma 1.34 to $R^{H^\alpha,l}$. Although $R^{\bigcap H^\alpha,l}$ is not defined in the same way as $\bigcap R^{H^\alpha,l}$, one verifies easily that they are (gradedly) equal. $\mathrm{QED}_{1.39}$

Therefore, the graded intersection of $\alpha^-$-subgroups is at least left $\beta$, but we do not know any reason why it should be $\alpha^-$.

## 2. The simple case

We move on to study $\alpha/\beta$-groups in simple theories.

**Convention 2.1.** We assume that $T$ is simple. For general facts about simple theories we refer the reader to [Wag00].

We keep Convention 1.22, and add the assumption that $G$ is $\alpha/\beta_{I,1}$.

We will use local stratified $D$-ranks as defined (for polygroups, and *a fortiori* for groups) in [BTW04]. $D_G(-,\varphi,\psi)$ denotes local $D$-ranks stratified by $G$ on both sides.

**Lemma 2.2.** *If $R$ is a bounded $\beta_{I,f,2}$-equivalence relation over $c$ and $a \equiv_c^{\mathrm{Ls}} b$ then $a\ R_{f(2)}\ b$.*

*Proof.* Just apply Lemma 1.33, recalling that in a simple theory two iterates suffices in order to generate the equality of Lascar strong type. $\mathrm{QED}_{2.2}$

**Definition 2.3.** $d_{\varphi,\psi,n}(x)$ is the partial type that says that $D_G(x,\varphi,\psi) \geq n$ (for example, the partial type that says that there exists a tree witnessing $D_G(-,\varphi,\psi) \geq n$, the elements on whose leaves being an indiscernible $n$-dimensional array containing $x$).

**Lemma 2.4.** *Let $X \subseteq G$ be a $\beta_2$-set. Let $p = \bigwedge_{k<\gamma} d_{\varphi_k,\psi_k,n_k}$ be a conjunction of some partial types of this form. Then for any given pair $\varphi,\psi$, $D_G(X_i \wedge p,\varphi,\psi)$ is constant for all $i \geq 2$.*

*Proof.* It suffices to prove that $D_G(X_i \wedge p, \varphi, \psi) = D_G(X_2 \wedge p, \varphi, \psi)$ for $i \geq 2$, for every fixed pair $\varphi, \psi$. Let $c \vDash X_2 \wedge p$ be such that $D_G(c, \varphi, \psi) = D_G(X_2 \wedge p, \varphi, \psi)$. We know that $X_i \subseteq \bigcup a_\alpha \cdot X_2 \cdot b_\alpha$, and we may assume that $c \downarrow \bar{a}\bar{b}$. Then $c \in a_\alpha \cdot d \cdot b_\alpha$ for some $\alpha$ and $d \in X_2$. Therefore:

$$\begin{aligned} D_G(d, \varphi_k, \psi_k) &\geq D_G(d/a_\alpha b_\alpha, \varphi_k, \psi_k) = D_G(c/a_\alpha b_\alpha, \varphi_k, \psi_k) \\ &= D_G(c, \varphi_k, \psi_k) \geq n_k \end{aligned}$$

Thus $d \vDash p$, whereby:

$$\begin{aligned} D_G(X_2 \wedge p, \varphi, \psi) &\geq D_G(d, \varphi, \psi) \geq D_G(c, \varphi, \psi) \\ &= D_G(X_i \wedge p, \varphi, \psi) \end{aligned}$$

QED$_{2.4}$

We obtain:

**Proposition 2.5.** *For a subgroup $H \leq G$, the following conditions are equivalent:*

(i) *$H$ is a left $\beta_2$-subgroup of $G$ (for some $2 \in I$).*
(ii) *$H$ is a $\beta_2$-subset of $G$ (for some $2 \in I$).*
(iii) *For some $2 \in I$, for every pair $\varphi, \psi$, $D_G(X_i, \varphi, \psi)$ is constant for every $i \geq 2$.*

*Moreover, we can keep the same 2 from top to bottom.*

*Proof.* (i) $\Longrightarrow$ (ii). Is already known.

(ii) $\Longrightarrow$ (iii). By Lemma 2.4.

(iii) $\Longrightarrow$ (i). Take 2 as in the assumption. As the local ranks of $X_i$ do not depend on $i$ for $i \geq 2$, there is a bound on the size of a sequence $(a_\alpha) \subseteq H^b$ such that $(a_\alpha \cdot H_2^b)^{R_1}$ are all disjoint. This gives a bound on an $R_3^{H,l}$-anti-clique in $H^b$, and therefore in any class $g \cdot H^b$. In fact, assuming that $e \in H_2^b$ (which holds in any case from some point on) we can have $2 = 3$. QED$_{2.5}$

*Remark* 2.6. By passing to inverses, one sees that $H \leq G$ is a $\beta$-subset if and only if it is a right $\beta$-subgroup, the definition being the obvious one.

**Corollary 2.7.** *If $H \leq G$ is $\beta$, then $G/H$ and $G/\!\!/H$ are $\alpha/\beta$-homogeneous space and $\alpha/\beta$-polygroup, respectively.*

We prove the graded analogue of [Wag01, Lemma 3.12]:

**Lemma 2.8.** *Let $X/R \subseteq G$ be a $\beta_2$-set. Assume also that there is $f : I \to I$ such that, whenever $x, y \in X_i$ and $x \downarrow y$, then $x^{-1} \cdot y \subseteq X_{f(i)}$. Write $H_i^b = X_i \cdot X_i$. Then $H = H^b/R$ is a $\beta$-subgroup of $G$.*
*Moreover, if $X$ is $\alpha^-$, then so is $H$.*

*Proof.* We begin with the following observation: We assumed that $x, y \in X_i$ and $x \downarrow y$ imply that $x^{-1} \cdot y \subseteq X_{f(i)}$. If we only assume $x_R \downarrow y_R$ we can find $x' \in x^{R_2}$, $y' \in y^{R_2}$ such that $x' \downarrow y'$. Since $X$ is $R$-complete we find $f' : I \to I$ such that $x_R \downarrow y_R$ and $x, y \in X_i$

imply that $x^{-1} \cdot y \subseteq X_{f'(i)}$, and we may assume that this is already true for $f$.
As in the proof of [Wag01, Lemma 3.12], we may assume that each $X_i$ is closed for inverses.
Fix an enumeration of all pairs $(\varphi_k, \psi_k : k < \lambda)$, and find a maximal tuple $\bar{n}$ (in lexicographical order) such that $X_2 \wedge \bigwedge d_{\varphi_k,\psi_k,n_k}$ is consistent. By Lemma 2.4, $\bar{n}$ is also maximal such that $X_i \wedge \bigwedge d_{\varphi_k,\psi_k,n_k}$ is consistent, for every $i \geq 2$. We write $p = \bigwedge d_{\varphi_k,\psi_k,n_k}$.
We now proceed as in the proof of [Wag01, Lemma 3.12]. Assume that $a \vDash X \wedge p$, $b \in X$, $a_R \mathop{\smile\!\!\!\!\!\!\mid}\, b$ and $c \in a \cdot b$. Then $c \in X$, and we get for every $k$: $D_G(a, \varphi_k, \psi_k) = D_G(a/b, \varphi_k, \psi_k) = D_G(c/b, \varphi_k, \psi_k) \leq D_G(c, \varphi_k, \psi_k)$, and by the maximality of $\bar{n}$ we obtain in particular $c_R \mathop{\smile\!\!\!\!\!\!\mid}\, b$ and $c \vDash p$.
Assume now that $b \in X \cdot X \cdot X$. We may find $a \vDash X_2 \wedge p$ such that $a \mathop{\smile\!\!\!\!\!\!\mid}\, b$, and moreover find $b_0, b_1, b_2 \in X$ such that $b \in b_0 \cdot b_1 \cdot b_2$ and $a \mathop{\smile\!\!\!\!\!\!\mid}\, b_0 b_1 b_2$. Then we have $c_0 \in a \cdot b_0^{-1}$, $c_1 \in a \cdot b_1$ and $c_2 \in c_1 \cdot b_2$ such that $b \in c_0^{-1} \cdot c_1$. By the previous argument $c_2 \vDash X \wedge p$ and $c_{2R} \mathop{\smile\!\!\!\!\!\!\mid}\, b_2$. Since clearly $c_{2R} \mathop{\smile\!\!\!\!\!\!\mid}_{b_2} b_0 b_1$ we get $c_{2R} \mathop{\smile\!\!\!\!\!\!\mid}\, b_1$ whereby $c_1 \in X$ as well. Similarly $c_0 \in X$ so $b \in X$. Since we took $a$ to be in $X_2$, we obtain $g : I \to I$ which depends on 2 and $f$ such that $X \cdot X \cdot X \subseteq_g X \cdot X$. Since $X$ is assumed to be closed for inverses we get $H \cdot H^{-1} = H$ gradedly.
It follows that $H = H^b/R$ is a subgroup. It is $\beta$, since it follows from the calculations above that $D_G(H_i, \varphi_k, \psi_k) = D_G(X_i, \varphi_k, \psi_k) = n_k$ for all $i \geq 2$ and $k < \lambda$.
The moreover part is clear from the construction. $\text{QED}_{2.8}$

**Corollary 2.9.** *If $H \leq G$ is a $\beta_2$-subgroup then generic elements exist for $H$ and are precisely those whose stratified ranks are equal to those of $H_i^b$ for some (any) $i \geq 2$.*

*Example* 2.10. The subgroup $H$ from [Ben03a] is a bounded $\beta$-subgroup; if we extend the definitions to polygroups, then the core of an $\alpha/\beta_I$ polygroup is $\beta_{\omega\times I}$. (To be more precise: by their constructions, these examples are $\gamma$; then $\beta$ follows from boundedness.)

*Example* 2.11. Let $p \in S(c)$ be a Lascar strong type, $p(x) \vdash x \in G^b$. Define

$$S_i(p) = \{a \in G^b : \exists b, b' \, [b, b' \vDash p \wedge b_R \mathop{\smile\!\!\!\!\!\!\mid}_c a_R \wedge a \in (b' \cdot b^{-1})^{R_i}]\}$$

Then for every $i$ there is $i'$ such that $S_i(p) \subseteq S_1(p)^{R_i} \subseteq S_{i'}(p)$, whereby $S_I(p)$ is $\alpha^-$.
Assume that $b, b'$ witness that $a \in S_i(p)$. As $b_R \mathop{\smile\!\!\!\!\!\!\mid}_c a_R$ there are $a_0 \in a^{R_1}$ and $b_0 \in b^{R_1}$ such that $b_0 \mathop{\smile\!\!\!\!\!\!\mid}_c a_0$, and then there is $b_1 \in b_0^{R_1}$ such that $b_1 \vDash p$ and $b_1 \mathop{\smile\!\!\!\!\!\!\mid}_c a_0$. We also have:

$$\begin{aligned} D_G(b'/c, \varphi, \psi) = D_G(p, \varphi, \psi) &= D_G(b/c, \varphi, \psi) \\ &= D_G(b/a_0 c, \varphi, \psi) = D_G(b'/a_0 c, \varphi, \psi) \end{aligned}$$

Whereby $b'_R \mathop{\smile\!\!\!\!\!\!\mid}_c a_0$, so there is $b'_1 \in b'^{R_1^2}$ such that $b'_1 \mathop{\smile\!\!\!\!\!\!\mid}_c a_0$ and $b'_1 \vDash p$.
Note that then there is some fixed $2 \in I$ such that there is always $(b' \cdot b^{-1})^{R_i} \subseteq ((b'_1 \cdot$

$b_1^{-1})^{R_i})^{R_1}$, so a definition like:

$$S'_i(p) = \{a \in G^b : \exists b, b', a_0 \, [b, b' \vDash p \wedge a_0 \; R_1 \; a \wedge b \underset{c}{\perp} a_0 \wedge b' \underset{c}{\perp} a_0 \wedge a \in (b' \cdot b^{-1})^{R_i}]\}$$

Would have given something gradedly equal. Moreover, this shows that $S_I(p) = S_I(p)^{-1}$ gradedly.

Assume now that $a \perp_c a'$ are in $S_i(p) \subseteq S'_{i'}(p) \subseteq S_{i'}(p)$. We can find witnesses $a_0, b, b'$ such that: $a_0 \; R_1 \; a$, $b, b' \vDash p$, $b \perp_c a_0$, $b' \perp_c a_0$ and $a \in (b' \cdot b^{-1})^{R_{i'}}$, and similarly $a'_0, c, c'$ witnessing $a' \in S'_{i'}(p)$, such that in addition $a_0 \perp_c a'_0$. By the independence theorem we may assume that $b = c'$, whereby $a \cdot a' \subseteq S_{i''}(p)$ for some $i''$ which only depends on $i$. Since $S_I(p)$ is $a^-$ it is in particular $\beta$, and we may apply Lemma 2.8 to get an $\alpha^-$-subgroup $Stab(p) = S(p)^2 \leq G$. In fact, it is gradedly equal to $(S_1(p)^2)^R$.

**Proposition 2.12.** *Let $H \leq G$ be a $\beta$-subgroup, say over $A$.*

(i) *Let $g \in H$, and $p = \mathrm{lstp}(g/A)$. Then $Stab(p) \leq H$, and it is of bounded index if and only if $g$ is generic in $H$.*

(ii) *Let $(K^\alpha : \alpha < \lambda)$ be uniformly $\beta$-subgroups of $H$ defined over $\mathrm{bdd}(A)$, and $[H : K^\alpha] < \infty$ for all $\alpha$. Assume also that for some $i$ we have $g^\alpha \in H_i^b$ generic for $H$ over $A$, for all $\alpha < \lambda$. Then $Stab(\mathrm{lstp}(g^\alpha/A)) \subseteq K^\alpha$ uniformly.*

(iii) *For every $i \in I$, $Stab(\mathrm{lstp}(g/A))$ is uniformly unique for all generic $g \in H_i^b$, meaning that all choices of such $g$ give uniformly equal subgroups of $H$, and this unique subgroup is over $A$. It is also a gradedly normal subgroup of $H$, in the sense that that all its group-theoretic conjugates in $H$ are uniformly equal.*

*Proof.* (i) Clearly $Stab(p) \subseteq H$, and it is a subgroup.
If $g$ is generic, then $Stab(p)$ contains many independent generics of $H$, whereby it is of bounded index. Conversely, if $Stab(p)$ is of bounded index then it contains generics, whereby $S(p)$ contains ones. But then there are $g \vDash p$, $h \in S(p)$ such that $h$ is generic over $g$ and $h \cdot g \vDash p$, whereby $p$ is generic.

(ii) Let $p = \mathrm{lstp}(g/A)$ be generic for $H$, and $K \leq H$ a $\beta_{I,f,2}$-subgroup of bounded index in $H$, defined over $\mathrm{bdd}(A)$. The cosets of $K$ induce a bounded $\mathrm{bdd}(A)$-$\beta$-equivalence relation on the realisations of $p$. By Lemma 2.2 there is $3 \in I$ which depends only on $f$ and 2 such that $p(x) \vDash x \in g \cdot K_3$. Therefore there are $4, 5 \in I$ that depend only on $f$ and 3, such that $S_1(p) \subseteq K_4$ and $Stab(p)_1 \leq K_5$. We get $Stab(p) \leq K$ gradedly.
The uniformity now follows.

(iii) Uniform uniqueness follows from the two previous items.
This unique subgroup is also uniformly gradedly equal to any of its model theoretic conjugates over $A$ (as these are uniformly $\alpha^-$ and of bounded index), so it is over $A$.
By considering group-theoretic conjugates the same argument yields that it is normal.

QED$_{2.12}$

**Definition 2.13.** The *connected component (over $A$)* of a $\beta$-subgroup $H$ over $A$ is defined as $Stab(\mathrm{lstp}(g/A))$ for any generic element $g$ of $H$, and is noted $H^0_A$. It is $\alpha^-$.

**Lemma 2.14.** *Let $\mathfrak{H}$ be a type-definable family of uniformly $\beta$-subgroups of $G$, having the same stratified local ranks. Then commensurativity is type-definable for members of $\mathfrak{H}$.*
*In other words, let $p$ be a partial type, and $H(x)$ be $\gamma_I$-sets depending on a parameter, such that $\{H(a) : a \vDash p\}$ are uniformly $\beta$-subgroup their $D_G(-,\varphi,\psi)$-ranks are equal. Then there is a type-definable equivalence relation $E(x,y)$ on $p$ such that $\vDash E(a,b)$ if and only if $H(a)$ and $H(b)$ are commensurate.*

*Proof.* Compare the local ranks of the intersection with the common local ranks. Uniformity assures us that there is 2 such that it suffices to check the rank of $H_2(x) \cap H_2(y)$. $\text{QED}_{2.14}$

*Remark* 2.15. Given a $\beta$-subgroup $H$, the family of all model-theoretic and group-theoretic conjugates of $H$ satisfies the assumptions.

**Definition 2.16.** A $\beta$-subgroup $H \leq G$ is *locally connected* if whenever $H'$ is a group-theoretic or model-theoretic conjugate of $H$ commensurate with $H$, then $H = H'$ gradedly, and uniformly for all such $H'$.
The *locally connected component* of a $\beta$-subgroup $H$ is a minimal locally connected $\beta$-subgroup commensurate with $H$.

**Fact 2.17.** *Every $\beta$-subgroup $H$ has a unique locally connected component, noted $H^c$.*

*Proof.* The construction of locally connected components generalises from [Wag01] using Lemma 2.8, Lemma 1.39 and graded intersections. $\text{QED}_{2.17}$

Locally connected components serve us as canonical representatives of commensurativity classes (note however that the addition of constants to the language may change the notion of a locally connected subgroup, and therefore the subgroup $H^c$). As the connected component of a locally connected subgroup is clearly locally connected, we see $H^c$ is $\alpha^-$.

**Corollary 2.18.** *Let $H \leq G$ be a locally connected $\beta$-subgroup. Then $H$ has an canonical $\alpha$-parameter. If $g \in G$, then $H \cdot g$ has a canonical $\beta$-parameter.*

*Proof.* Say that $H$ is defined over $a$, and write $H = H(a)$. Let $B = \{b : b \equiv a\}$, and $\mathfrak{H} = \{H(b) : b \in B\}$. By Lemma 2.14 there is a type-definable equivalence relation $E$ such that $E(b,b') \iff [H(b) : H(b) \cap H(b')] < \infty$, but the latter is equivalent to $H(b) = H(b')$. Therefore $u = a_E$ is an $\alpha$-canonical parameter for $H$.
Divide $G^b \times B$ by the relation $F_I$ where $F_i(g,b,g',b') \iff E(b,b') \wedge [g' \cdot g^{-1} \cap H_i(b) \cap H_i(b') \neq \varnothing]$. This is a $\beta_I$-equivalence relation. Fix $g \in G^b$ and consider $(g,a)_F$. Let $\varphi$ be an automorphism, $g', a' = \varphi(g,a)$, and assume that $F_i(g,a,g',a')$. Then $\varphi$ fixes $u$, and therefore it fixes $H$ gradedly (set-wise). Then $g' \cdot g^{-1} \cap H_i \neq \varnothing$ implies that $Hg = \varphi(Hg)$

gradedly, where the gradedness depends only on $i$ (the smaller $i$ is, the closer are the gradings on $Hg$ and $\varphi(Hg)$). Conversely, if $Hg = \varphi(Hg)$ gradedly, then there is $i$ such that $F_i(g, a, g', a')$, and the closer the gradings are on $Hg$ and $\varphi(Hg)$, the smaller we can take $i$. Therefore $(g, a)_F$ is a $\beta_I$-canonical parameter for the $\beta/\beta$-coset $Hg$. QED$_{2.18}$

## 3. One-based groups

We conclude by showing that classical properties of hyperdefinable groups in one-based simple theories hold for $\alpha/\beta$-groups.

**Convention 3.1.** We keep the assumption that $T$ is simple, and assume furthermore it is one-based.

**Proposition 3.2.** *Assume that $H \leq G$ is locally connected and $\beta_2$. Let $u$ be the canonical parameter for $H$. Then $u \in \mathrm{bdd}(\varnothing)$.*

*Proof.* Let $h \in H_2^b$ be (a representative of) a generic element of $H$. Let $g \in G^b$ be a generic element of $G$ over $h, u$. Let $f_R = (h \cdot g)_R$, so $f_R \mathop{\smile\!\!\!\!\!|}\, h, u$ and $f$ is generic for $G$ over $h, u$. Since $R$ is $\beta_1$ we may choose $f$ such that $f \in (H_2^b \cdot g)^{R_1} \subseteq H_3^b \cdot g$ and $f \mathop{\smile\!\!\!\!\!|}\, h, u$ (for some $3 \in I$). On the other hand, $h \mathop{\smile\!\!\!\!\!|}_u g$, so $f$ is generic for $H \cdot g$ over $g, u$.
Let $(f_\xi : \xi < \omega)$ be a Morley sequence in $\mathrm{tp}(f/g, u)$, and let $\varphi$ be an automorphism fixing $(f_\xi)$. Write $u' = \varphi(u)$, $H' = \varphi(H)$, $g' = \varphi(g)$. Then $(f_\xi) \subseteq H_3^b \cdot g \cap {H'}_3^b \cdot g' \subseteq H_4^b \cdot f_0 \cap {H'}_4^b \cdot f_0 \subseteq (H_5^b \cap {H'}_5^b) \cdot f_0$ for some 4 and 5 in $I$. As $f_\xi$ is a Morley sequence in $\mathrm{tp}(f/g, u)$, we may assume that $f_1 \mathop{\smile\!\!\!\!\!|}_{g,u} u', f_0$. We can take $s \in f_1 \cdot f_0^{-1} \cap H_5^b \cap {H'}_5^b$, and we get for every $i \geq 2$:

$$
\begin{aligned}
D_G(\mathrm{tp}(s/u, u'), \varphi, \psi) &\geq D_G(\mathrm{tp}(s/g, f_0, u, u'), \varphi, \psi) = D_G(\mathrm{tp}(f_1/g, f_0, u, u'), \varphi, \psi) \\
&= D_G(\mathrm{tp}(f_1/g, u), \varphi, \psi) = D_G(\mathrm{tp}(f/g, u), \varphi, \psi) = D_G(\mathrm{tp}(h/g, u), \varphi, \psi) \\
&= D_G(H_i^b, \varphi, \psi) = D_G({H'}_i^b, \varphi, \psi),
\end{aligned}
$$

But then $s$ is generic for both $H$ and $H'$, so $H$ and $H'$ are commensurate and $u' = u$.
This shows that $u \in \mathrm{dcl}((f_\xi))$, whereby $(f_\xi) \mathop{\smile\!\!\!\!\!|}_{\mathrm{Cb}(f/g,u)} g, u \Longrightarrow u \in \mathrm{bdd}(\mathrm{Cb}(f/g, u)) \subseteq \mathrm{bdd}(f)$, by one-basedness. Finally: $u \mathop{\smile\!\!\!\!\!|}\, f \Longrightarrow u \mathop{\smile\!\!\!\!\!|}\, u \Longrightarrow u \in \mathrm{bdd}(\varnothing)$. QED$_{3.2}$

**Corollary 3.3.** *$G$ is ($\beta_{I\times\omega}$-bounded)-by-($\alpha$-abelian)-by-bounded. That is to say that there is an $\alpha$-subgroup $\tilde{Z} \leq G$ of bounded index such that $\tilde{Z}'$ is bounded and $\beta_{\omega\times I}$.*
*Moreover, there exists a $\beta_I$-subgroup $N \lhd G$, of bounded index such that $N'$ is central in $N$.*

*Proof.* We follow [Wag05, Theorem 6.3].
For $g \in G^b$, write:

$$\begin{aligned} C_{G_I^b}(g) &= (\{h \in G^b : h \in (g \cdot h \cdot g^{-1})^{R_i}\} : i \in I) \\ C_G(g) &= C_{G_I^b}(g)/R \leq G \\ \Delta_g^b &= \{(h, h') : h \in G^b, h' \in g \cdot h \cdot g^{-1}\} \subseteq G^{b2} \\ \Delta_g &= {\Delta_g^b}^{R \times R}/R \times R \leq G^2 \end{aligned}$$

The class $\{\Delta_g : g \in G^b\}$ is uniformly $\alpha^-$, so it is uniformly $\beta_I$, and contains boundedly many commensurativity classes. Then the class $\{\Delta_g \cap \Delta_e : g \in G^b\}$ is uniformly $\beta_I$; as the projection of $\Delta_g \cap \Delta_e$ is gradedly equal to $C_G(g)$, and uniformly so, the class $\{C_G(g) : g \in G^b\}$ is uniformly $\beta_I$. Therefore the set:

$$\begin{aligned} \tilde{Z}^b &= \{g \in G^b : \Delta_g \text{ is commensurate to } \Delta_e\} \\ &= \{g : [G : C_G(g)] < \infty\} \end{aligned}$$

is an $R$-complete $\alpha$-set, closed for product and inverse, whereby $\tilde{Z} = \tilde{Z}^b/R \lhd G$ is a normal $\alpha$-subgroup (that is, $\alpha/\beta$-subgroup).
Continue as in [Wag05, Theorem 6.3] to conclude that $\tilde{Z}$ has bounded index in $G$ and boundedly many commutators. The set of commutators is a $\alpha^-$-set which generates $\tilde{Z}'$ in $\omega$ steps, whereby the latter is $\gamma_{\omega \times I}$, and in fact $\beta_{\omega \times I}$ as it is bounded.
Finally, as $\tilde{Z}'$ is bounded and $R$ is $\beta$, we can find $(g_\xi : \xi < \lambda) \subseteq G^b$ such that $\tilde{Z}' = (\bigcup_\xi g_\xi^{R_1})/R$. By the same argument as above, $C_{\tilde{Z}}(g_\xi)$ are uniformly $\beta_I$. Then the graded intersection $\bigcap_\xi C_{\tilde{Z}}(g_\xi)$ is $\beta_I$, and in fact uniformly so for all possible choices of $(g_\xi)$. If $(h_\zeta)$ is another such choice, then it differs from $(g_\xi)$ at most by $R_1$, so there is a (uniform) bound on the differences in the gradings of $\bigcap_\xi C_{\tilde{Z}}(g_\xi)$ and $\bigcap_\zeta C_{\tilde{Z}}(h_\zeta)$. This shows that $\bigcap_\xi C_{\tilde{Z}}(g_\xi)$ is gradedly and uniformly unique, and we may define $C_{\tilde{Z}}(\tilde{Z}') = \bigcap_\xi C_{\tilde{Z}}(g_\xi)$ without ambiguity (although it may be somewhat smaller than the group-theoretic $C_{\tilde{Z}}(\tilde{Z}')$, as we require each element to centralise $\tilde{Z}'$ uniformly). Then $N = C_{\tilde{Z}}(\tilde{Z}') \lhd G$ is a normal $\beta_I$-subgroup of bounded index, and $N'$ is central in $N$. QED$_{3.3}$

**Corollary 3.4.** *If $T$ is one-based then $G$ is naturally isogenous to the abelian $\alpha/\beta$ group $\tilde{Z}(G)/\tilde{Z}(G)'$.*

*Remark* 3.5. It is important that $\tilde{Z}(G)$ is $\alpha/\beta$ (and not $\beta/\beta$, as are most of the subgroups of $G$ we consider): it can serve as an ambient group without modification, and $G/\tilde{Z}(G)$ is a hyperdefinable group in the classical sense (this is true also if $T$ is not one-based, and this is in fact the more interesting case, as then $G/\tilde{Z}(G)$ is not necessarily bounded). Moreover, even if $G$ were $\alpha$ (in opposition to $\alpha/\beta$), then the best abelian group we know that is isogenous to it is $\alpha/\beta$. This is an overlooked pre-[BTW04] example of a $\alpha/\beta_\omega$ group.

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