# Axiomatic Investigations of the Propositional Calculus of the "Principia Mathematica"[1]

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Translation of "Axiomatische Untersuchungen des Aussagen-Kalkuls der 'Principia Mathematica,' *Mathematische Zeitschrift* 25 (1926), 305–320, by Richard Zach. From *Universal Logic: An Anthology.* New York and Basel: pp. 43–58 (2012).

## §1.

In Whitehead and Russell's "Principia Mathematica," the systematic development of the logical propositional calculus is carried out by starting with five propositional formulas as basic formulas ("primitive propositions"), from which every generally valid combination of propositions [*Aussagenverknüpfung*], i.e., one which is correct for arbitrary values of the propositional variables occurring in it, can be obtained by *substitutions* and the application of a single formal *rule of inference.*

This reduction of the theorems to the basic formulas is combined with a reduction of the logical connectives to the two operations:

of negation

$$\sim p \quad (\text{not } p)$$

and of disjunction

$$p \vee q \quad (p \text{ or } q;\ \text{"or" in the sense of the latin "vel"}),$$

from which are formed by combination:[2]

[1]The content of this article is taken to a large extent from an unpublished *Habilitationsschrift*, which was submitted by the author to the Faculty of Mathematics and Natural Sciences at Göttingen in 1918. The questions on the possibility of replacing of formulas by inference rules investigated there has been left out of the present investigation.

[2]In order to avoid a proliferation of parentheses, dots shall be used as separating symbols, just as in the Princip. Math., where the rule is that more dots separate more strongly than fewer dots. This rule, however, applies only to the separation of symbols of the *same kind.* Whenever two different symbols from among $\sim$, $\vee$, $\supset$ compete, in general the convention shall hold that for the separation of parts of the formula, $\supset$ takes precedence over $\vee$ and $\sim$, as well as $\vee$ over

the implication

$$p \supset q \quad \text{(if } p\text{, then } q\text{), as abbreviation for} \quad \sim p \vee q$$

and the conjunction[3]

$$p \,\&\, q \quad (p \text{ and } q)\text{, as abbreviation for} \quad \sim(\sim p \vee \sim q).$$

The five basic formulas are

(*Taut*) $p \vee p \supset p,$

(*Add*) $q \supset p \vee q,$

(*Perm*) $p \vee q \supset q \vee p,$

(*Assoc*) $p \vee \cdot\, q \vee r \supset q \vee \cdot\, p \vee r,$

(*Sum*) $q \supset r \cdot \supset \cdot\, p \vee q \supset p \vee r.$

When replacing the implication by the defining expression, they read:

(*Taut*) $\sim(p \vee p) \cdot \vee\, p,$

(*Add*) $\sim q \vee \cdot\, p \vee q,$

(*Perm*) $\sim(p \vee q) \cdot \vee \cdot\, q \vee p,$

(*Assoc*) $\sim(p \vee \cdot\, q \vee r) \,:\vee:\, q \vee \cdot\, p \vee r,$

(*Sum*) $\sim(\sim q \vee r) \cdot \vee : \sim(p \vee q) \cdot \vee \cdot\, p \vee r.$

The rule of inference can be formulated as follows: From two formulas[4]

$$\mathfrak{A}, \quad \mathfrak{A} \supset \mathfrak{B} \quad (\text{resp. } \sim\mathfrak{A} \vee \mathfrak{B})$$

the formula $\mathfrak{B}$ is to be obtained.

The substitution rule says that any arbitrary formula can be substituted for a propositional variable.

One may now easily convince oneself that each of the five basic formulas represents a generally valid combination of propositions when interpreted contentually. And that the application of the rules only yields formulas *of this kind*.

At the same time, the system of basic formulas is *complete* in the sense that from it *all* generally valid combinations of propositions can be obtained with using the rules. This claim can even be strengthened as follows:

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$\sim$, as long as no other separation is indictated by *parentheses*. Accordingly, the formula

$$p \supset \sim q \vee : p \vee \cdot\, q \vee r \cdot \supset \cdot\, p \supset p \vee : \sim q \vee \cdot\, q \vee r$$

has to be read in the same way as would be given without any further conventions by the following way of putting parentheses:

$$\Big(p \supset \Big((\sim q) \vee \big(p \vee (q \vee r)\big)\Big)\Big) \supset \Big(p \supset \Big(p \vee \big((\sim q) \vee (q \vee r)\big)\Big)\Big).$$

[3]In the Princip. Math. the symbol for conjunction is simply a dot.

[4]We shall use uppercase Fraktur letters as symbols used to indicate formulas of an indefinite form.

If $\mathfrak{A}$ is any formula formed from the symbols introduced, then either it is derivable from the five basic formulas according to the rules, or *any arbitrary formula* $\mathfrak{B}$ becomes derivable if $\mathfrak{A}$ is added to the basic formulas.

The justification of this claim shall be indicated briefly.[5] It results from considering the *conjunctive normal form.* A conjunctive normal form is a formula which is formed from one or more "simple disjunctions" combined using & (conjunction), i.e., such disjunctions in which every disjunct is either a variable or a variable to which a negation is applied.

For instance,

$$p \,\&\, (p \vee {\sim}q \vee r)$$

is a conjunctive normal form.

Now the familiar theorem holds, that every formula can be brought into a conjunctive normal form; i.e., we have a procedure to find, for any given formula $\mathfrak{A}$, a normal form $\mathfrak{N}$ such that

$$\mathfrak{A} \supset \mathfrak{N} \quad \text{as well as} \quad \mathfrak{N} \supset \mathfrak{A}$$

is derivable from the basic formulas. (As an aside, the determination of $\mathfrak{N}$ for $\mathfrak{A}$ is not unique.)

One now sees immediately: If $\mathfrak{A}$ is derivable, so is $\mathfrak{N}$, and conversely; furthermore, if $\mathfrak{N}$ is derivable, then so is each one of the simple disjunctions occuring in $\mathfrak{A}$ (as conjuncts).

Our claim is thus reduced to the corresponding claim for simple disjunctions. For those, however, it follows immediately; for a simple disjunction either contains two disjuncts one of which is the negation of the other (such as $p$ and ${\sim}p$); then it is derivable; or the disjunction does not contain two such disjuncts; then from it one can obtain, by substitution, a disjunction every disjunct of which is either the variable $p$ or the double negation of $p$, i.e., ${\sim}({\sim}p)$. Fom such a formula, however, the formula consisting only of the variable $p$ is derivable, and from this one can obtain any arbitrary formula by substitution.

This consideration at the same time yields a simple procedure to determine whether a given formula $\mathfrak{A}$ represents a generally valid combination of propositions, or—and this amounts to the same thing—is derivable from the basic formulas: find a conjunctive normal form $\mathfrak{N}$ of $\mathfrak{A}$. If in each simple disjunction occurring in $\mathfrak{N}$ two disjuncts occur one of which is the negation of the other, then the given formula $\mathfrak{A}$ is generally valid, otherwise it is not.

This decision procedure solves the main problem of the propositional calculus completely, and if one only wanted to characterize the generally valid logical combinations of propositions, one would be able to obtain this result more directly than by the method of Principia Mathematica.

The insight, that the listed five basic formulas suffice for the derivation of all generally valid combinations of propositions (using the rules), is nevertheless

[5] A slightly different proof can be found in the treatise by E. L. Post, "Introduction to a general theory of elementary propositions" (American Journal of Math. 43 (1921)).

significant in itself. In this light, the further question now arises whether the five basic formulas are mutually independent in the sense of formal derivability.

This is indeed *not* the case, rather, the formula *Assoc* can be derived from the other four. Therefore, the system of basic formulas can be replaced by that of the four formulas

*Taut, Add, Perm, Sum* (System 1).

Of these four formulas, none are redundant, not even if one retains *Assoc*.

These claims shall be proved below. Our investigation of dependencies is not exhausted by it, however. A number of further questions shall also be considered, to which this result naturally leads and which arise from two remarks.

The first remark is that, in the event that *Assoc* is retained, the formula *Perm* becomes provable, if instead of

$$Add: \quad q \supset p \vee q$$

the formula

$$Add^*: \quad p \supset p \vee q$$

is taken as a basic formula.

These two formulas are known to be mutually equivalent using *Perm* and *Sum*, i.e., one can, using *Perm* and *Sum*, pass from the formula *Add* to $Add^*$, and also back.

It will thus become apparent that the following four formulas suffice as a system of basic formulas:

*Taut*, $Add^*$, *Assoc, Sum* (System 2).

This fact is not so surprising, since an exchange occurs in the formula *Assoc*. One will therefore ask if the formula *Perm* is still dependent if instead of *Assoc* the formula

$$Assoc^*: \quad p \vee \cdot\, q \vee r \supset p \vee q \,\cdot\!\vee r,$$

which expresses the associative character of disjunction more purely, is chosen as a basic formula.

It can be shown that this is not the case; i.e., in system 2 the formula *Assoc* cannot be replaced by $Assoc^*$. It can, however, be replaced by *Add* and $Assoc^*$ together, so that the five formulas

*Taut, Add*, $Add^*$, $Assoc^*$, *Sum* (System 3)

suffice as a system of basic formulas. (That the formulas *Assoc* and $Assoc^*$ do not have the same inferential power when *Perm* is removed does of course not contradict the known fact that these two formulas are equivalent in the presence of *Perm* and *Sum*.)

The second remark is that in system 1, the formula *Add* can be replaced by the specialized formula

$$\textit{Simp}: \quad q \supset \cdot\, p \supset q$$

resp.

$$\sim q \vee \cdot \sim p \vee q,$$

so that the four formulas

$$\textit{Taut, Simp, Perm, Sum} \qquad \text{(System 4)}$$

suffice as basic formulas.

By an analogous specialization as that of *Add* to *Simp*, the formula

$$\textit{Comm}: \quad p \supset \cdot\, q \supset r : \supset : q \supset \cdot\, p \supset r$$

resp.

$$\sim(\sim p \vee \cdot \sim q \vee r) \vee : \sim q \vee \cdot \sim p \vee r$$

results from *Assoc*, and the formula

$$\textit{Syll}: \quad q \supset r \cdot \supset : p \supset q \cdot \supset \cdot\, p \supset r$$

resp.

$$\sim(\sim q \vee r) \vee : \sim(\sim p \vee q) \vee \cdot \sim p \vee r$$

from *Sum*. Now the question arises whether *Assoc* might not also be replaced by *Comm*, or *Sum* by *Syll*. This question is decided in the negative. Neither *Assoc* in system 2, nor *Assoc*$^*$ in system 3 can be replaced by *Comm*, and *Sum* cannot be replaced by *Syll* in any of the systems 1, 2, 3, or 4. Furthermore, it is shown that the forumla *Add* cannot be replaced by *Simp* in system 3.

Finally the relation of the formula

$$\textit{Id}: \quad p \supset p$$

resp.

$$\sim p \vee p$$

to the formulas of systems 1, 2, 3, 4 shall be considered. *Id* is distinguished by being the simplest generally valid combination of propositions. One would therefore hope to use it as a basic formula. However, it cannot replace any of the formulas in any of the systems 1, 2, 3, 4.

The simplest derivation of *Id* is that from

$$\textit{Taut}, \textit{Add}, \textit{Syll}$$

where one first obtains from *Add* by substitution the formula

$$Id^*: \quad p \supset p \vee p$$

resp.

$$\sim p \vee \cdot\, p \vee p.$$

Without applying *Sum*, one can derive *Id*

from *Add* and *Assoc*,
even already from *Simp* and *Comm*;
moreover from *Taut*, *Add**, *Assoc**,
and also from *Add**, *Perm*, *Assoc**.

Carrying out these derivations, which are found easily, shall be left to the reader. In every one of these derivations either *Add* or *Simp* or *Add** is used. We will show that without applying one of these formulas, the formula *Id* is *no longer* provable from the remaining formulas in systems 1, 2, 3, 4.

Of course the formula *Id* is not a sufficient replacement for either of the formulas *Add*, *Add**. It will be shown that by adding *Id* to the formulas so that the four formulas

*Taut*, *Perm*, *Assoc*, *Sum*

not even the formula *Id** is provable, and moreover, that even if *Id** is taken in addition to *Id*, the formulas *Add* and *Add** as well as *Simp* remain unprovable.

The claims put forward shall now be established, specifically, the claimed *dependencies* on the basis of which the formula systems 1, 2, 3, 4 are recognized to be sufficient systems of basic formulas will be proved in §2; in §3, proofs for the *independence* claims will be given, and it shall be shown that *none of the systems* 1*, 2,* 3*,* 4 *contains a redundant formula.*

For the sake of convenience, let us collect the systems here again:

System 1: *Taut*, *Add*, *Perm*, *Sum*,
,, 2: *Taut*, *Add**, *Assoc*, *Sum*,
,, 3: *Taut*, *Add*, *Add**, *Assoc**, *Sum*,
,, 4: *Taut*, *Simp*, *Perm*, *Sum*.

## §2.

By way of explanation of the derivations below, we start with a few remarks.

If a formula is obtained by literal repetition or by substitution from a basic formula or a formula already derived, then the reference label of said formula is given on the *left.* A *new reference label* for a formula obtained is given on its *right.*

The application of the inference rule follows the schema

$$\begin{array}{c}\mathfrak{A}\\ \mathfrak{A}\supset\mathfrak{B}\\ \hline \mathfrak{B}\end{array}$$

Furthermore, we will use—in order to give an *abbreviated description* of proofs—the schema

$$\begin{array}{c}\mathfrak{A}\supset\mathfrak{B}\\ \mathfrak{B}\supset\mathfrak{C}\\ \hline \mathfrak{A}\supset\mathfrak{C}\end{array}$$

This is explained as follows: From the basic formula *Sum* one obtains, as mentioned already, the formula

$$\textit{Syll}: \quad q\supset r\cdot\supset : p\supset q\cdot\supset\cdot p\supset r$$

by substitution. By using this formula twice in conjunction with the inference rule, one can derive the formula

$$\mathfrak{A}\supset\mathfrak{C}$$

from the formulas

$$\mathfrak{A}\supset\mathfrak{B} \quad \text{and} \quad \mathfrak{B}\supset\mathfrak{C}.$$

Therefore, we may proceed, wherever *it is permitted to use the basic formula Sum*, as if we had an inference rule according to which the formula

$$\mathfrak{A}\supset\mathfrak{C}$$

can be obtained from

$$\mathfrak{A}\supset\mathfrak{B}, \qquad \mathfrak{B}\supset\mathfrak{C}.$$

And this we shall do, in order to avoid unnecessary complexities.

Reference labels introduced in a proof for derived formulas are given by numerals in brackets; these labels need to be fixed only *within a proof.*

Furthermore let us remark that two formulas which result from one another by applying the abbreviation

$$p\supset q \quad \text{for} \quad {\sim}p\vee q$$

shall count as equivalent in our considerations. Passing from one of the ways of writing the formula to the other is indicated by "resp."

We will now proceed to the proofs of the claimed dependencies. The theorems will be numbered in such a way that *the number of each theorem coincides with that of the system of formulas* which is established as complete by it.[6]

1. Derivation of *Assoc* from *Taut*, *Add*, *Perm*, *Sum*.

$$\begin{array}{ll}(\textit{Add}) & r\supset p\vee r\\ (\textit{Sum}) & \underline{r\supset p\vee r\cdot\supset\cdot q\vee r\supset q\vee\cdot p\vee r}\\ & q\vee r\supset q\vee\cdot p\vee r\end{array}$$

[6]Note that negation does *not occur explicitly anywhere* in the derivations 1, 2, 3. The dependencies 1, 2, 3 therefore still obtain if implication is considered as a *primitive connective* instead of as a combination of negation and disjunction.

(*Sum*) $\dfrac{q \lor r \supset q \lor \cdot p \lor r \cdot \supset \cdot p \lor \cdot q \lor r \supset p \lor : q \lor \cdot p \lor r}{p \lor \cdot q \lor r \supset p \lor : q \lor \cdot p \lor r}$

(*Perm*) $\dfrac{p \lor : q \lor \cdot p \lor r \supset q \lor \cdot p \lor r : \lor p}{p \lor \cdot q \lor r \supset q \lor \cdot p \lor r : \lor p}$ (1)

(*Add*) $p \supset r \lor p$

(*Perm*) $\dfrac{r \lor p \supset p \lor r}{p \supset p \lor r}$

(*Add*) $\dfrac{p \lor r \supset q \lor \cdot p \lor r}{p \supset q \lor \cdot p \lor r}$

(*Sum*) $\dfrac{p \supset q \lor \cdot p \lor r \cdot \supset \cdot q \lor \cdot p \lor r : \lor p \supset q \lor \cdot p \lor r : \lor : q \lor \cdot p \lor r}{q \lor \cdot p \lor r : \lor p \supset q \lor \cdot p \lor r : \lor : q \lor \cdot p \lor r}$

(*Taut*) $\dfrac{q \lor \cdot p \lor r : \lor : q \lor \cdot p \lor r \supset q \lor \cdot p \lor r}{q \lor \cdot p \lor r : \lor p \supset q \lor \cdot p \lor r}$ (2)

(1) $p \lor \cdot q \lor r \supset q \lor \cdot p \lor r : \lor p$

(2) $\dfrac{q \lor \cdot p \lor r : \lor p \supset q \lor \cdot p \lor r}{p \lor \cdot q \lor r \supset q \lor \cdot p \lor r}$

2. Derivation of *Perm* from *Taut*, *Add**, *Assoc*, *Sum*.

(*Taut*) $p \lor p \supset p$

(*Sum*) $\dfrac{p \lor p \supset p \cdot \supset \cdot q \lor \cdot p \lor p \supset q \lor p}{q \lor \cdot p \lor p \supset q \lor p}$ (1)

(*Add**) $q \supset q \lor p$

(*Sum*) $\dfrac{q \supset q \lor p \cdot \supset \cdot p \lor q \supset p \lor \cdot q \lor p}{p \lor q \supset p \lor \cdot q \lor p}$

(*Assoc*) $\dfrac{p \lor \cdot q \lor p \supset q \lor \cdot p \lor p}{p \lor q \supset q \lor \cdot p \lor p}$

(1) $\dfrac{q \lor \cdot p \lor p \lor p \supset q \lor p}{p \lor q \supset q \lor p}$

3. Derivation of *Assoc* from *Taut*, *Add*, *Assoc**, *Sum*.

(*Add*) $r \supset p \lor r$

(*Sum*) $\dfrac{r \supset p \lor r \cdot \supset \cdot q \lor r \supset q \lor \cdot p \lor r}{q \lor r \supset q \lor \cdot p \lor r}$

(*Add*) $\dfrac{q \lor \cdot p \lor r \supset r \lor : q \lor \cdot p \lor r}{q \lor r \supset r \lor : q \lor \cdot p \lor r}$

(*Sum*) $\dfrac{q \lor r \supset r \lor : q \lor \cdot p \lor r \cdot \supset \cdot p \lor \cdot q \lor r \supset p \lor \therefore r \lor : q \lor \cdot p \lor r}{p \lor \cdot q \lor r \supset p \lor \therefore r \lor : q \lor \cdot p \lor r}$

(*Assoc**) $\dfrac{p \lor \therefore r \lor : q \lor \cdot p \lor r \supset p \lor r \cdot \lor : q \lor \cdot p \lor r}{p \lor \cdot q \lor r \supset p \lor r \cdot \lor : q \lor \cdot p \lor r}$

(*Add*) $\underline{p \lor r \cdot \lor : q \lor \cdot p \lor r \supset q \lor \therefore p \lor r \cdot \lor : q \lor \cdot p \lor r}$

$$p \lor \cdot q \lor r \supset q \lor \therefore p \lor r \cdot \lor : q \lor \cdot p \lor r$$

(*Assoc**) $$\frac{q \lor \therefore p \lor r \cdot \lor : q \lor \cdot p \lor r \supset q \lor \cdot p \lor r : \lor : q \lor \cdot p \lor r}{p \lor \cdot q \lor r \supset q \lor \cdot p \lor r : \lor : q \lor \cdot p \lor r}$$

(*Taut*) $$\frac{q \lor \cdot p \lor r : \lor : q \lor \cdot p \lor r \supset q \lor \cdot p \lor r}{p \lor \cdot q \lor r \supset q \lor \cdot p \lor r}$$

4. Derivation of *Add* from *Taut*, *Simp*, *Perm*, *Sum*.

(*Taut*) $$\sim p \lor \sim p \supset \sim p$$

(*Sum*) $$\frac{\sim p \lor \sim p \supset \sim p \cdot \supset \cdot \sim(\sim p) \lor \cdot \sim p \lor \sim p \lor \sim p \supset \sim(\sim p) \lor \sim p}{\sim(\sim p) \lor \cdot \sim p \lor \sim p \supset \sim(\sim p) \lor \sim p} \tag{1}$$

(*Simp*) $$\sim p \supset \cdot p \supset \sim p$$

resp.

$$\sim(\sim p) \lor \cdot \sim p \lor \sim p$$

(1) $$\frac{\sim(\sim p) \lor \cdot \sim p \lor \sim p \supset \sim(\sim p) \lor \sim p}{\sim(\sim p) \lor \sim p}$$

(*Perm*) $$\frac{\sim(\sim p) \lor \sim p \supset \sim p \lor \sim(\sim p)}{\sim p \lor \sim(\sim p)}$$

resp.

$$p \supset \sim(\sim p)$$

(*Sum*) $$\frac{p \supset \sim(\sim p) \cdot \supset \cdot q \lor p \supset q \lor \sim(\sim p)}{q \lor p \supset q \lor \sim(\sim p)}$$

(*Perm*) $$\frac{q \lor \sim(\sim p) \supset \sim(\sim p) \lor q}{q \lor p \supset \sim(\sim p) \lor q}$$

resp.

$$q \lor p \supset \cdot \sim p \supset q \tag{2}$$

(*Taut*) $$p \lor p \supset p$$

resp.

$$\sim(p \lor p) \lor p$$

(*Perm*) $$\frac{\sim(p \lor p) \lor p \supset p \lor \sim(p \lor p)}{p \lor \sim(p \lor p)}$$

(2) $$\frac{p \lor \sim(p \lor p) \supset \cdot \sim\big(\sim(p \lor p)\big) \supset p}{\sim\big(\sim(p \lor p)\big) \supset p}$$

(*Sum*) $$\frac{\sim\big(\sim(p \lor p)\big) \supset p \cdot \supset \cdot q \lor \sim\big(\sim(p \lor p)\big) \supset q \lor p}{q \lor \sim\big(\sim(p \lor p)\big) \supset q \lor p} \tag{3}$$

(*Perm*) $$\sim\big(\sim(p \lor p)\big) \lor q \supset q \lor \sim\big(\sim(p \lor p)\big)$$

(3) $$\frac{q \lor \sim\big(\sim(p \lor p)\big) \supset q \lor p}{\sim\big(\sim(p \lor p)\big) \lor q \supset q \lor p}$$

(*Sum*) $$\frac{\sim\big(\sim(p \lor p)\big) \lor q \supset q \lor p \cdot \supset \cdot \sim q \lor \cdot \sim\big(\sim(p \lor p)\big) \lor q \supset \sim q \lor \cdot q \lor p}{\sim q \lor \cdot \sim\big(\sim(p \lor p)\big) \lor q \supset \sim q \lor \cdot q \lor p} \tag{4}$$

(*Simp*) $q \supset \cdot \sim(p \vee p) \supset q$

resp.

$$\frac{\begin{array}{c}\sim q \vee \cdot \sim\big(\sim(p \vee p)\big) \vee q \\ \sim q \vee \cdot \sim\big(\sim(p \vee p)\big) \vee q \supset \sim q \vee \cdot q \vee p\end{array}}{\sim q \vee \cdot q \vee p} \quad (4)$$

resp.

$$\frac{\begin{array}{c}q \supset q \vee p \\ q \vee p \supset p \vee q\end{array}}{q \supset p \vee q} \quad (Perm)$$

## §3.

The task is now to prove the theorems claimed in §1 about *independence*—to which also the claims about non-replaceability belong. That a formula $\mathfrak{A}$ is not dispensable within a system of formulas, resp., that it cannot be replaced by another formula $\mathfrak{B}$, is established if we show that some generally valid formula $\mathfrak{C}$ cannot be derived from the formula system which remains after deleting the formula $\mathfrak{A}$, resp., which results by replacing the formula $\mathfrak{A}$ by the formula $\mathfrak{B}$.

Taking the established *dependencies* into account, one sees that the claimed independence claims are justified, provided the following independencies are shown:

| | | | |
|---|---|---|---|
| I. | *Taut* | cannot be derived from | *Add*, *Perm*, *Assoc*, *Sum*; |
| II. | *Sum* | ,, ,, ,, ,, | *Taut*, *Add*, *Add**, *Perm*, *Assoc*, *Assoc**, *Syll*; |
| III. | *Simp* | ,, ,, ,, ,, | *Taut*, *Perm*, *Assoc*, *Sum*, *Id*, *Id**; |
| IV. | *Id** | ,, ,, ,, ,, | *Taut*, *Perm*, *Assoc*, *Sum*, *Id*; |
| V. | *Id* | ,, ,, ,, ,, | *Taut*, *Perm*, *Assoc*, *Sum*; |
| VI. | *Add* | ,, ,, ,, ,, | *Taut*, *Add**, *Simp*, *Assoc*, *Sum*; |
| VII. | *Add** | ,, ,, ,, ,, | *Taut*, *Add*, *Assoc**, *Sum*; |
| VIII. | *Assoc** | ,, ,, ,, ,, | *Taut*, *Add*, *Add**, *Comm*, *Sum*; |
| IX. | *Assoc** | ,, ,, ,, ,, | *Taut*, *Add*, *Assoc*, *Sum*. |

The proofs of these theorems will be given according to the usual method used for such investigations, viz., the *method of exhibition:* in each case, a finite group (in the extended sense of the word) is given, i.e., a finite totality of elements, for which the "disjunction" is defined as a two-place operation, and "negation" as a one-place operation, by giving the course-of-values (the operation in general need not be associative, uniquely invertible, nor commutative).[7]

Furthermore, a subtotality of "designated values" is singled out from the group.

A formula is then called a "correct formula" with respect to the group under consideration, if it always yields a designated value for any substitution of elements of the group for the propositional variables.

[7]The elements will be designated by lowercase Greek letters.

From this it is first of all clear that every propositional formula which results from a correct formula by substitution is also correct. Furthermore, the groups will be set up in such a way that one obtains applying the inference rule to two correct formulas yields another correct formula. For this it is sufficient that the following condition $B$ is satisfied:

The disjunction $\sim p \vee q$, which is formed from the negation of a *designated value* $p$ and an arbitrary element $q$, has a designated value only if $q$ is a designated value. If this condition is satisfied, it follows that every formula which is derivable from correct formulas is also corect.

In order to prove that a formula $\mathfrak{F}$ is independent of certain other formulas $\mathfrak{A}, \ldots, \mathfrak{K}$, one only has to find a group which satisifes condition $B$ and for which $\mathfrak{A}, \ldots, \mathfrak{K}$ are correct formulas while $\mathfrak{F}$ is not a correct formula.

For each of the independence theorems I through IX we will now give a corresponding group. They are specified as follows: First the elements are enumerated within parentheses, and the designated values between braces. Then negation and disjunction are are defined.[8] (If a defining equation contains a propositional variable, this means that the equation shall hold for *every* value of the variable.) Subsequently the formulas that have to be verified as correct are listed.

This verification is left to the reader; for the more complex formulas it can be abbreviated significantly by suitable case distinctions.

Finally, a specific substitution of values is given for the formula to be shown independent, under which the formula yields a *non-designated value*, which shows that it is not a correct formula.

In order to facilitate the verification, the following should be remarked: If the formula *Id* is a correct formula for a group, then

if disjunction is commutative, *Perm* is also a correct formula;

if disjunction is associative, *Assoc** is also a correct formula;

if disjunction is commutative as well as associative, *Assoc* is also a correct formula;

if $p \vee p = p$ (for every value of $p$), then *Taut* is a correct formula;

For *each of the groups, condition B is satisfied.* This has to be established for each group separately, but we will not mention it every time.

**Group I.**

$$(\alpha, \beta, \gamma); \qquad \{\alpha\}.$$
$$\sim\alpha = \beta, \quad \sim\beta = \alpha, \quad \sim\gamma = \gamma,$$
$$\alpha \vee p = p \vee \alpha = \alpha,$$
$$\beta \vee p = p \vee \beta = p,$$
$$\gamma \vee \gamma = \alpha$$

The disjunction so defined can be represented arithmetically as multiplication of the three congruence classes 0, 1, 2 mod 4.

[8] Here the equality sign is used in the sense of definitional equality.

*Id*, *Add*, *Perm*, *Assoc*, *Sum* are correct formulas; by contrast, *Taut* is not a correct formula, since

$$\sim(\gamma \vee \gamma) \vee \gamma = \sim\alpha \vee \gamma = \beta \vee \gamma = \gamma$$

**Group II.**

$$(\alpha, \beta, \gamma, \delta); \qquad \{\alpha\}.$$
$$\sim\alpha = \beta, \quad \sim\beta = \alpha, \quad \sim\gamma = \delta, \quad \sim\delta = \alpha$$
$$\begin{aligned} &\alpha \vee p = p \vee \alpha = \alpha, \\ &\beta \vee p = p \vee \beta = p, \\ &p \vee p = p, \\ &\gamma \vee \delta = \delta \vee \gamma = \alpha \end{aligned}$$

The disjunction can be represented arithmetically as multiplication of the congruence classes 0, 1, 3, 4 mod 6.

*Id*, *Taut*, *Add*, *Add**, *Perm*, *Assoc*, *Assoc**, *Syll* are correct formulas; but not *Sum*, since

$$\begin{aligned} &\sim(\sim\delta \vee \beta) \vee \cdot \sim(\gamma \vee \delta) \vee (\gamma \vee \beta) \\ &= \sim(\alpha \vee \beta) \vee \cdot \sim\alpha \vee \gamma = \sim\alpha \vee \cdot \beta \vee \gamma \\ &= \beta \vee \gamma = \gamma. \end{aligned}$$

**Group III.**

$$(\alpha, \beta, \gamma, \delta); \qquad \{\alpha, \gamma\}.$$
$$\sim\alpha = \beta, \quad \sim\beta = \alpha, \quad \sim\gamma = \delta, \quad \sim\delta = \gamma$$
$$\begin{aligned} &\alpha \vee p = p \vee \alpha = \alpha, \\ &\beta \vee p = p \vee \beta = \beta \text{ for } p \neq \alpha, \\ &p \vee p = p, \\ &\gamma \vee \delta = \delta \vee \gamma = \gamma \end{aligned}$$

*Id*, *Id**, *Taut*, *Perm*, *Assoc*, *Sum* are correct formulas; but not *Simp* (and consequently neither is *Add*), since

$$\sim\gamma \vee \cdot \sim\alpha \vee \gamma = \delta \vee \cdot \beta \vee \gamma = \delta \vee \beta = \beta.$$

**Group IV.**

This group differs from group III only in that

$$\gamma \vee \gamma = \beta$$

is specified:

$$(\alpha, \beta, \gamma, \delta); \qquad \{\alpha, \gamma\}.$$
$$\sim\alpha = \beta, \quad \sim\beta = \alpha, \quad \sim\gamma = \delta, \quad \sim\delta = \gamma$$
$$\alpha \vee p = p \vee \alpha = \alpha,$$
$$\beta \vee p = p \vee \beta = \beta \text{ for } p \neq \alpha,$$
$$\gamma \vee \gamma = \beta,$$
$$\gamma \vee \delta = \delta \vee \gamma = \gamma$$
$$\delta \vee \delta = \delta.$$

*Id*, *Taut*, *Perm*, *Assoc*, *Sum* are correct formulas; but not *Id**, since

$$\sim\gamma \vee \cdot\, \gamma \vee \gamma = \delta \vee \beta = \beta.$$

**Group V.**

$$(\alpha, \beta, \gamma); \qquad \{\alpha\}.$$
$$\sim\alpha = \beta, \quad \sim\beta = \alpha, \quad \sim\gamma = \gamma$$
$$\alpha \vee p = p \vee \alpha = \alpha,$$
$$\beta \vee \beta = \beta \vee \gamma = \gamma \vee \beta = \gamma \vee \gamma = \beta.$$

*Taut*, *Perm*, *Assoc*, *Sum* are correct formulas; but not *Id*, since

$$\sim\gamma \vee \gamma = \gamma \vee \gamma = \beta$$

**Group VI.**

$$(\alpha, \beta, \gamma); \qquad \{\alpha\}.$$
$$\sim\alpha = \beta, \quad \sim\beta = \alpha, \quad \sim\gamma = \gamma$$
$$\alpha \vee p = \alpha, \quad \beta \vee p = p, \quad \gamma \vee p = \gamma.$$

*Id*, *Taut*, *Add**, *Simp*, *Assoc**, *Comm*, *Sum* are correct formulas; but not *Add* (and consequently neither is *Assoc*), since

$$\sim\alpha \vee \cdot\, \gamma \vee \alpha = \beta \vee \gamma = \gamma.$$

**Group VII.**

$$(\alpha, \beta, \gamma); \qquad \{\alpha, \gamma\}.$$
$$\sim\alpha = \beta, \quad \sim\beta = \alpha, \quad \sim\gamma = \beta$$
$$\alpha \vee p = \alpha,$$
$$\beta \vee p = \gamma \vee p = p.$$

*Id*, *Taut*, *Add*, *Assoc**, *Sum* are correct formulas; but not *Add**, since

$$\sim\gamma \vee \cdot\, \gamma \vee \beta = \beta \vee \beta = \beta.$$

**Group VIII.**

$$(\alpha, \beta, \gamma, \delta); \qquad \{\alpha\}.$$

$$\sim\alpha = \beta, \quad \sim\beta = \alpha, \quad \sim\gamma = \alpha, \quad \sim\delta = \gamma$$

$$\alpha \vee p = p \vee \alpha = \alpha,$$

$$\beta \vee p = p \vee \beta = p,$$

$$p \vee p = p,$$

$$\gamma \vee \delta = \alpha, \quad \delta \vee \gamma = \delta.$$

*Id*, *Taut*, *Add*, *Add*$^*$, *Comm*, *Sum* are correct formulas; but not *Perm* (and consequently neither are *Assoc* and *Assoc*$^*$), since

$$\sim(\gamma \vee \delta) \vee \cdot\, \delta \vee \gamma = \sim\alpha \vee \delta = \beta \vee \delta = \delta.$$

**Group IX.**

$$(\alpha, \beta, \gamma, \delta); \qquad \{\alpha, \gamma\}.$$

$$\sim\alpha = \beta, \quad \sim\beta = \alpha, \quad \sim\gamma = \delta, \quad \sim\delta = \gamma$$

$$\alpha \vee p = p \vee \alpha = \alpha,$$

$$\beta \vee p = \delta \vee p = p,$$

$$\gamma \vee \beta = \beta,$$

$$\gamma \vee \gamma = \gamma \vee \delta = \gamma.$$

*Id*, *Taut*, *Add*, *Assoc*, *Sum* are correct formulas; but not *Assoc*$^*$ (and consequently neither are *Add*$^*$ and *Perm*), since

$$\begin{aligned}&\sim(\gamma \vee \cdot\, \beta \vee \delta) \vee : \gamma \vee \beta \cdot \vee\ \delta \\ &= \sim(\gamma \vee \delta) \vee \cdot\, \beta \vee \delta \\ &= \sim\gamma \vee \delta = \delta \vee \delta = \delta.\end{aligned}$$

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