Free Łukasiewicz and hoop residuation algebras

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Abstract

Hoop residuation algebras are the $\{\to, 1\}$-subreducts of hoops; they include Hilbert algebras and the $\{\to, 1\}$-reducts of MV-algebras (also known as Wajsberg algebras). The paper investigates the structure and cardinality of finitely generated algebras in varieties of k-potent hoop residuation algebras. The assumption of k-potency guarantees local finiteness of the varieties considered. It is shown that the free algebra on n generators in any of these varieties can be represented as a union of n subalgebras, each of which is a copy of the $\{\to, 1\}$-reduct of the same finite MV-algebra, i.e., of the same finite product of linearly ordered (simple) algebras. The cardinality of the product can be determined in principle, and an inclusion-exclusion type argument yields the cardinality of the free algebra. The methods are illustrated by applying them to various cases, both known (varieties generated by a finite linearly ordered Hilbert algebra) and new (residuation reducts of MV-algebras and of hoops).

1 Introduction and Preliminaries

A *partially ordered commutative residuated integral monoid* (pocrim, for short) is an ordered algebra $\boldsymbol{A} = \langle A, \cdot, \to, 1, \leq \rangle$ such that $\langle A, \cdot, 1 \rangle$ is a commutative monoid, $\leq$ is a (partial) order on A with largest element 1, and $\to$ is a *residuation* operation, i.e., for all $a, b, c \in A$

$$a \cdot c \leq b \quad \text{iff} \quad c \leq a \to b. \tag{1}$$

The partial order is equationally definable by

$$a \leq b \quad \text{iff} \quad a \to b = 1,$$

so we can drop $\leq$ from the type and treat pocrims as algebras. We refer to [6] for a study of pocrims and for further references. A *hoop* is a pocrim satisfying in addition

$$x \cdot (x \to y) \approx y \cdot (y \to x). \tag{2}$$

This identity ensures that if we define

$$x \wedge y = x \cdot (x \to y)$$

then $x \wedge y$ is the greatest lower bound of x and y with respect to the order $\leq$. The ordering of a hoop is therefore a semilattice ordering, with a term definable semilattice operation. The class of hoops can be defined as an equational class of algebras, viz. as the class of algebras $\boldsymbol{A} = \langle A, \to, \cdot, 1 \rangle$ satisfying:

(i) $\langle A, \cdot, 1 \rangle$ is a commutative monoid,

(ii) $x \to x \approx 1$,

(iii) $x \to (y \to z) \approx (x \cdot y) \to z$,

(iv) $x \cdot (x \to y) \approx y \cdot (y \to x)$.

This axiomatization is due to Bosbach [9]. Hoops were investigated also in [11], [5], [15], [7] and [8]. We will denote the variety of hoops by Ho.

Important subclasses of the variety of hoops are the variety BSL of Brouwerian semilattices, defined relative to Ho by the identity $x \cdot x \approx x$ (see [20] for a thorough study), and the variety of Wajsberg hoops (so named in [7]), defined relative to Ho by the axiom

$$(T) \quad (x \to y) \to y \approx (y \to x) \to x.$$

It is known that Brouwerian semilattices are also precisely the $\{\wedge, \to, 1\}$-subreducts of Heyting algebras (i.e., the subalgebras of the $\{\wedge, \to, 1\}$-reducts of Heyting algebras), and that Wajsberg hoops are similarly the $\{\cdot, \to, 1\}$-subreducts of Wajsberg algebras $\langle A, \cdot, \to, 0, 1 \rangle$ (introduced in [16]). Wajsberg algebras are term equivalent to MV-algebras (see [5] for a discussion), and both are extracted from the algebras used by Łukasiewicz to define his many-valued logics; see [22]. We refer to [12] for the theory and history of MV-algebras.

The $\{\to, 1\}$-subreducts of hoops will be referred to as *hoop residuation algebras*. The $\{\to, 1\}$-subreducts of pocrims are precisely the BCK-algebras; hoop residuation algebras are therefore BCK-algebras. It was conjectured by A. Wroński and proved in [15] (see [7]) that hoop residuation algebras form a *variety* that can be defined by any axiomatization of the quasivariety of BCK-algebras together with what can be viewed as an an implicational version of (1):

$$(x \to y) \to (x \to z) \approx (y \to x) \to (y \to z).$$

The following identities and quasi-identity give an axiomatization of the class of BCK-algebras:

$$x \to x \approx 1,$$

$$x \to (y \to z) \approx y \to (x \to z),$$

$$x \to 1 \approx 1,$$

$$1 \to x \approx x,$$

$$(x \to y) \to ((z \to x) \to (z \to y)) \approx 1,$$

$$(x \to y \approx 1 \ \& \ y \to x \approx 1) \implies x \approx y.$$

We will denote the variety of hoop residuation algebras by HoRA. Two important subclasses of the variety HoRA are the classes of $\{\to, 1\}$-subreducts of the varieties of Brouwerian semilattices and of the variety of Wajsberg hoops. The first of these, the class BrRA of *Brouwerian residuation algebras*, was studied by A. Diego [14], who called them *Hilbert algebras*; we will use this name for the class as well as it has become standard. He showed that the class is a variety, characterized its subdirectly irreducible algebras, and showed that it is locally finite.

The second class is that of $\{\to, 1\}$-subreducts of Wajsberg hoops (and therefore of Wajsberg algebras and MV-algebras). As the operation $\to$ was one of the two fundamental operations (together with $\neg$) present in Łukasiewicz's 'matrices' (see [22]), we will refer to these algebras as *Łukasiewicz residuation algebras*. They form a variety as well, which we will denote by $\mathsf{ŁRA}$. The lattice of varieties of these algebras was described in [21]. The subdirectly irreducible algebras in $\mathsf{ŁRA}$ are linearly ordered, and the finite ones among them are precisely the $\{\to, 1\}$-reducts $\mathbf{Ł}_n^{\to}$ of the finite linearly ordered Wajsberg algebras $\mathbf{Ł}_n$. The algebra $\mathbf{Ł}_n^{\to}$ has domain $Ł_n = \{e^0, e^1, e^2, \dots, e^{n-1}\}$, where $1 = e^0 > e^1 > e^2 > \dots > e^{n-1}$, and

$$e^i \to e^j = \begin{cases} 1 & \text{if } i \geq j, \\ e^{j-i} & \text{otherwise.} \end{cases}$$

Note that $\mathbf{Ł}_n^{\to}$ has many more subalgebras than does $\mathbf{Ł}_n$. In particular, for every $m \leq n$ the set $1 = e^0, e^1, e^2, \dots, e^{m-1}$ is a subuniverse of $\mathbf{Ł}_n^{\to}$ (although not necessarily of $\mathbf{Ł}_n$), so $\mathbf{Ł}_m^{\to}$ is a subalgebra of $\mathbf{Ł}_n$, for all m, $1 \leq m \leq n$. On the other hand, $\mathbf{Ł}_n^{\to}$ is 2-generated, for every $n < \omega$, so no variety of hoop residuation algebras that contains $\mathbf{Ł}_n^{\to}$ for infinitely many $n < \omega$ can be locally finite.

The variety of hoop residuation algebras shares some important properties with the the variety of hoops: it is congruence distributive, has the congruence extension property, and is 1-regular, i.e., a congruence of a hoop residuation algebra is determined by its 1-class. The 1-classes of the congruences of a hoop residuation algebra $\boldsymbol{A}$ are known as *filters*; they are the subsets F of A characterized by

(i) $1 \in F$,

(ii) if $a, a \to b \in F$, then $b \in F$ ('modus ponens').

In particular, every filter is upward closed. Given a filter F of $\boldsymbol{A}$, the relation

$$\Theta_F := \{(a, b) \in A^2 \,:\, a \to b \in F,\, b \to a \in F\}$$

is a congruence of $\boldsymbol{A}$ such that the class $1/\Theta_F = F$. In order to describe *filter generation*, let us write $x \to^0 y = y$, and $x \to^{k+1} y = x \to (x \to^k y)$ for $k < \omega$.

Lemma 1.1. *Let $\boldsymbol{A} \in \mathsf{HoRA}$, $F \subseteq A$, and $a \in A$. The filter $F(a)$ generated by $F \cup \{a\}$ is the set*

$$G = \{b \in A \,:\, \text{there is an } n < \omega \text{ such that } a \to^n b \in F\}.$$

Proof. Clearly $G \subseteq F(a)$. For the converse, observe that $F \subseteq G$, since for $f \in F$ we have $a \to^0 f = f$, hence $a \to f \in F$ and therefore $f \in G$. similarly $a \in G$, since $a \to a = 1 \in F$. It remains to show G satisfies (ii) of the definition of a filter. Let $b \in G$ and $b \to c \in G$. Then there are $n, m < \omega$

such that $a \to^n b \in F$, $a \to^m (b \to c) \in F$. It can be shown that in any BCK-algebra we have for all $i, j < \omega$

$$(x \to^i y) \to ((x \to^j (y \to z)) \to (x \to^{i+j} z)) \approx 1.$$

We conclude $a \to^{n+m} c \in F$ by applying modus ponens twice, and hence $c \in G$. $\square$

In particular, the filter generated by a single element—which coincides with the filter generated by $\{1\} \cup \{a\}$—is

$$\{b \in A \,:\, \text{there is an } n < \omega \text{ such that } a \to^n b = 1\}.$$

For example, in the algebras $\mathbf{Ł}_n^{\to}$ described earlier, any element $a \neq 1$ generates the largest filter $Ł_n$; hence the $\mathbf{Ł}_n^{\to}$ are simple, for all $n < \omega$.

We say that a hoop is *k-potent*, $0 < k < \omega$, if it satisfies the identity $x^k \approx x^{k+1}$; here x^k denotes the k-fold product of x with itself. Idempotent (i.e., 1-potent) hoops are just the Brouwerian semilattices mentioned above. We denote the class of k-potent hoops by $\mathsf{Ho}(k)$. This class was first studied by Büchi and Owens [11]. They showed in particular that it is locally finite for any $k < \omega$.

The present paper will be concerned with the $\{\to, 1\}$-subreducts of k-potent hoops, which we shall call *k-potent hoop residuation algebras*. It was shown in [5, Corollary 5.3] that for any variety V of k-potent hoops the class of $\{\to, 1\}$-subreducts of algebras in V is a variety. The class of all $\{\to, 1\}$-subreducts of algebras in $\mathsf{Ho}(k)$ will be denoted by $\mathsf{HoRA}(k)$. A simple equational axiomatization of the class $\mathsf{HoRA}(k)$ was given in [15]; see [7] for a discussion. For $1 \leq k < \omega$ let ε_k be the identity $x \to^k y \approx x \to^{k+1} y$.

Theorem 1.2. *The class of k-potent hoop residuation algebras is defined, relative to the variety* HoRA*, by the identity ε_k.*

In particular, the variety of idempotent hoop residuation algebras, or Hilbert algebras, can be defined relative to HoRA by ε_1; for a simpler axiomatization see [14].

Since the varieties $\mathsf{Ho}(k)$ are locally finite, so are the varieties $\mathsf{HoRA}(k)$, for $k < \omega$; conversely, it is easy to see that any locally finite variety of hoop residuation algebras must be contained in $\mathsf{HoRA}(k)$, for some $k < \omega$.

Note that $\mathbf{Ł}_n$ and $\mathbf{Ł}_n^{\to}$ are k-potent if and only if $n - 1 \leq k$. For k-potent hoop residuation algebras Lemma 1.1 simplifies to

Lemma 1.3. *Let $k < \omega$, let $\boldsymbol{A} \in \mathsf{HoRA}$ be k-potent, $F \subseteq A$, and $a \in A$. The filter generated by $F \cup \{a\}$ is the set*

$$\{b \in A : \; a \to^k b \in F\}.$$

Corollary 1.4. *Let $k < \omega$, and let $\boldsymbol{A} \in \mathsf{HoRA}$ be k-potent.*

(i) If $a \in A$, then the filter generated by a is $\{b \in A : a \to^k b = 1\}$,

(ii) If $a_1, a_2, \ldots, a_n \in A$ then the filter generated by $a_1, a_2, \ldots, a_n$ is

$$\{b \in A : a_1 \to^k (a_2 \to^k \ldots (a_n \to^k b) \ldots) = 1\}.$$

Here (ii) follows by applying the last lemma repeatedly. Note that the order in which $a_1, a_2 \ldots, a_n$ occur doesn't matter.

We will write

$$(\bigotimes{}^k \{a_1, a_2, \ldots, a_n\}) \to b$$

for

$$a_1 \to^k (a_2 \to^k \ldots (a_n \to^k b) \ldots).$$

These remarks imply that the class of k-potent hoop residuation algebras has *equationally definable principal congruences* (EDPC, for short). Indeed, define

$$p(x, y, z) = (\bigotimes{}^k \{x \to y, y \to x\}) \to z.$$

In [5] this term is called a *ternary deductive term* for $\mathsf{HoRA}(k)$. It is not difficult to verify that for any k-potent residuation hoop $\boldsymbol{A}$ and $a, b, c, d \in A$ we have

$$c \equiv d \mod \Theta(a, b) \quad \text{iff} \quad p(a, b, c) = p(a, b, d).$$

A result from [15] (described also in [7]) that is especially important to the present paper concerns the structure of the subdirectly irreducible algebras in HoRA. Given two algebras $\boldsymbol{A} = \langle A, \to^{\boldsymbol{A}}, 1\rangle$, $\boldsymbol{B} = \langle B, \to^{\boldsymbol{B}}, 1\rangle \in \mathsf{HoRA}$ let $\boldsymbol{A} \oplus \boldsymbol{B}$ be the algebra obtained from $\boldsymbol{A}$ by replacing the element 1 by a copy of $\boldsymbol{B}$. More precisely, assuming $A \cap B = \{1\}$, $\boldsymbol{A} \oplus \boldsymbol{B}$ is the algebra $\langle A \cup B, \to, 1\rangle$ where

$$x \to y = \begin{cases} x \to^{\boldsymbol{A}} y & \text{if } x, y \in A, \\ x \to^{\boldsymbol{B}} y & \text{if } x, y \in B, \\ y & \text{if } x \in B,\, y \in A, \\ 1 & \text{if } x \in A,\, x \neq 1,\, y \in B. \end{cases}$$

This is a hoop residuation algebra, and it is k-potent if and only if both $\boldsymbol{A}$ and $\boldsymbol{B}$ are. We recall from [7]:

Theorem 1.5. *An algebra $\boldsymbol{A} \in \mathsf{HoRA}$ is subdirectly irreducible if and only if $\boldsymbol{A} \cong \boldsymbol{B} \oplus \boldsymbol{C}$ for some $\boldsymbol{B} \in \mathsf{HoRA}$ and $\boldsymbol{C} \in \mathsf{ŁRA}$, with $\boldsymbol{C}$ subdirectly irreducible.*

We conclude:

Corollary 1.6. *A k-potent hoop residuation algebra $\boldsymbol{A}$ is subdirectly irreducible if and only if $\boldsymbol{A} \cong \boldsymbol{B} \oplus \boldsymbol{Ł}_n^{\rightarrow}$ for some k-potent hoop residuation algebra $\boldsymbol{B}$ and $1 \leq n-1 \leq k$.*

In the representation of a k-potent hoop residuation algebra $\boldsymbol{A}$ given by the corollary, $\boldsymbol{B}$ can be taken to be a subalgebra of $\boldsymbol{A}$, called the *support* $\sigma(\boldsymbol{A})$ of $\boldsymbol{A}$, with universe $\sigma(A)$, and $\boldsymbol{Ł}_n^{\rightarrow}$ is isomorphic to a subalgebra $\mu(\boldsymbol{A})$ of $\boldsymbol{A}$, called the *monolith of* $\boldsymbol{A}$ with universe $\mu(A)$; we will from now on write $\boldsymbol{A} \cong \sigma(\boldsymbol{A}) \oplus \mu(\boldsymbol{A})$. The domain $\mu(A)$ of $\mu(\boldsymbol{A})$ is the smallest non-trivial filter of $\boldsymbol{A}$, and hence we have:

Lemma 1.7. *Let $\boldsymbol{A}$ be a subdirectly irreducible k-potent hoop, say $\boldsymbol{A} = \sigma(\boldsymbol{A}) \oplus \mu(\boldsymbol{A})$, and let $h : \boldsymbol{A} \to \boldsymbol{D}$ be a homomorphism. The following are equivalent:*

(i) h is 1-1,

(ii) $h(x) \neq 1$ for some $x \in \mu(A)$.

The algebra $\sigma(\boldsymbol{A})$ is not only a subalgebra of $\boldsymbol{A}$, but also a homomorphic image of $\boldsymbol{A}$: the map $h : \boldsymbol{A} \to \sigma(\boldsymbol{A})$ given by $h(a) = a$ for $a \in \sigma(A)$, $h(a) = 1^{\sigma(\boldsymbol{A})}$ for $a \in \mu(A)$ is a homomorphism. The following technical lemma will be used in the sequel:

Lemma 1.8. *Let $\boldsymbol{A}$ be a k-potent subdirectly irreducible hoop residuation algebra generated by a set $X \subseteq A$. Then*

(i) $\sigma(\boldsymbol{A})$ is generated by $X \cap (\sigma(A) - \{1\})$,

(ii) $\mu(\boldsymbol{A})$ is generated by $X \cap (\mu(A) - \{1\})$.

Proof. The results follow from the fact that $\sigma(\boldsymbol{A})$ and $\mu(\boldsymbol{A})$ are subalgebras of $\boldsymbol{A}$, and, furthermore, that if $a \in \mu(A) - \{1\}$, $b \in \sigma(A) - \{1\}$, then $a \to b = b$ and $b \to a = 1$. Statement (i), alternatively, follows using the homomorphism $h : \boldsymbol{A} \to \sigma(\boldsymbol{A})$ given above: we see that $\sigma(\boldsymbol{A})$ is generated by $h(X) - \{1\} = X \cap (\sigma(A) - \{1\})$, as claimed. □

For idempotent hoop residuation algebras, i.e., the Brouwerian residuation algebras or Hilbert algebras, Corollary 1.6 specializes to the well-known result by Diego [14] that the subdirectly irreducible Hilbert algebras are precisely the algebras of the form $\boldsymbol{B} \oplus \mathbf{Ł}_2^{\to}$, for any Hilbert algebra $\boldsymbol{B}$.

It is implicit in Corollary 1.6, and easy to verify, that for $k < \omega$ the class $\mathsf{HoRA}(k)$ is closed under the operation $\_ \oplus \mathbf{Ł}_k^{\to}$ that assigns to a hoop residuation algebra $\boldsymbol{B}$ the algebra $\boldsymbol{B} \oplus \mathbf{Ł}_k^{\to}$. Conversely, for a given $k < \omega$, any variety closed under the operation $\_ \oplus \mathbf{Ł}_k^{\to}$ must contain $\mathbf{Ł}_k^{\to}$, and it can be shown that $\mathsf{HoRA}(k)$ is the smallest variety of hoop residuation algebras closed under the operation. The proof is similar to that of a similar result for k-potent hoops; see [7, Theorem 3.4].

The variety of hoops is known to be congruence permutable; in contrast, non-trivial varieties of hoop residuation algebras are not. Indeed, the smallest non-trivial variety of hoop residuation algebras is the variety BoRA of $\{\to, 1\}$-subreducts of Boolean algebras (also known as the variety of *implication algebras* or *Tarski* algebras), which fails to be congruence permutable.

2 Free hoop residuation algebras

Free implicative Boolean algebras (in our terminoloy, free Boolean residuation algebras) were studied by Abbott [1] and Monteiro [23]. Diego [14] determined the free Hilbert algebra (or Brouwerian residuation algebra) on two generators, and he gave a bound on the size of the free Hilbert algebra on three generators. Urquhart [25] showed that the finitely generated free Hilbert algebra on n-generators is a union of n copies of the $\{\to, 1\}$-reduct of a Boolean algebra, and, improving on Diego's result, he deduced that the cardinality of the free Hilbert algebra on three generators is $\leq 3 * 2^{23}$. Hendriks [18], using techniques developed by De Bruijn [10], determined that the cardinality of the free Hilbert algebra on three generators is in fact $3*2^{23}-22$; see Theorem 3.7 and the comments that follow it. Guzmán and Lynch [17] gave a description of the finitely generated free algebras and their cardinalities in varieties generated by what we have called (see [3]) finite *pure Hilbert algebras*; these include varieties of Hilbert algebras generated by finite chains.

The results of this section can be viewed as a generalization of Urquhart's work referred to above. Given a variety V of hoop residuation algebras, let V_{S} denote the class of simple algebras in V. If V is a non-trivial variety of Hilbert algebras, V_{S} consists of the $\{\rightarrow, 1\}$-reduct $\mathbf{Ł}_2^{\rightarrow}$ of the 2-element Boolean algebra, and there exists a power of $\mathbf{Ł}_2^{\rightarrow}$, i.e., an algebra in $\mathbb{P}(\mathsf{V}_{\mathrm{S}})$, such that the finitely generated free algebra on n generators in V can be written as a union of n subalgebras, each a copy of that same algebra in $\mathbb{P}(\mathsf{V}_{\mathrm{S}})$. If V is a variety of k-potent hoop residuation algebras, then V_{S} consists of reducts $\mathbf{Ł}_n^{\rightarrow}$ of the n-element Wajsberg algebra $\mathbf{Ł}_n$, for $n \leq m$, where m is some fixed number $\leq k+1$. We show in this section that for each $n < \omega$ there exists an algebra in $\overline{\mathbf{T}}_{\leq 1}(n) \in \mathbb{P}(\mathsf{V}_{\mathrm{S}})$ such that the finitely generated free algebra on n generators in V is a union of n subalgebras, each a copy of that same algebra $\overline{\mathbf{T}}_{\leq 1}(n) \in \mathbb{P}(\mathsf{V}_{\mathrm{S}})$. In the process we will learn how to determine $\overline{\mathbf{T}}_{\leq 1}(n)$, and how to compute the cardinality of the free algebra.

For X a set of variables, $\mathcal{L}$ a set of operation symbols, let $\mathrm{T}_{\mathcal{L}}(X)$ denote the set of all terms built in the usual recursive way from the variables in X using the operation symbols in $\mathcal{L}$. If X consists of the n variables $x_1, x_2, \ldots, x_n$, $1 \leq n \leq \omega$, we will also write $\mathrm{T}_{\mathcal{L}}(n)$ for $\mathrm{T}_{\mathcal{L}}(X)$. For $s \in \mathrm{T}_{\mathcal{L}}(X)$, let $\mathrm{Va}(s)$ denote the set of variables occurring in s.

Throughout this section V will be a locally finite variety of hoop residuation algebras; i.e., $\mathsf{V} \subseteq \mathsf{HoRA}(k)$, for some $k < \omega$. Set $\mathcal{L} = \{\rightarrow, 1\}$. The elements of the free algebra in V on n free generators can be represented as $\bar{s}$, where $s \in \mathrm{T}_{\mathcal{L}}(n)$, and for $s, t \in \mathrm{T}_{\mathcal{L}}(n)$ we have $\bar{s} = \bar{t}$ if and only if $\mathsf{V} \models s \approx t$. For $s \in T_{\mathcal{L}}(n)$ the element $\bar{s}$ thus depends on V, and we should really write $\bar{s}^{\mathsf{V}}$ or something of that sort; but since we will always be concerned with one variety of hoop residuation algebras V, we will not make the dependence explicit in the notation. For $1 \leq n < \omega$ let $\mathbf{F}_{\mathsf{V}}(n)$ denote the free algebra in V generated by the n free generators $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$; its universe is $\mathrm{F}_{\mathsf{V}}(n) = \{\bar{s} \, : \, s \in T_{\mathcal{L}}(n)\}$. Since we have $\mathsf{V} \models (x \rightarrow x) \approx 1$, for every term $s \in \mathrm{T}_{\mathcal{L}}(n)$ there is a term $s' \in \mathrm{T}_{\mathcal{L}}(n)$ in which the symbol 1 does not occur such that $\mathsf{V} \models s \approx s'$: just replace every occurrence of the symbol 1 by $(x_1 \rightarrow x_1)$. We may therefore replace $\mathcal{L}$ by $\mathcal{L}' = \{\rightarrow\}$, and will write $\mathrm{T}(X)$ for $\mathrm{T}_{\mathcal{L}'}(X)$ and $\mathrm{T}(n)$ for $\mathrm{T}_{\mathcal{L}'}(n)$.

For terms $s \in \mathrm{T}(X)$ we define the *right-most variable* $\mathrm{rv}(s)$ *of* s by induction on the complexity of s as follows: for $x \in X$ a variable, $\mathrm{rv}(x) = x$, and if $s = (t \rightarrow t')$, $t, t' \in \mathrm{T}(X)$ then $\mathrm{rv}(s) = \mathrm{rv}(t')$. For $1 \leq i \leq n$, let

$\mathrm{T}_i(n) = \{s \in \mathrm{T}(n) : \mathrm{rv}(s) = x_i\}$, and let $\overline{\mathrm{T}}_i(n)$ be the set of elements $\{\bar{s} \in \mathrm{F}_{\mathsf{V}}(n) : s \in \mathrm{T}_i(n)\}$. Of course, the sets $\overline{\mathrm{T}}_i(n)$, $1 \leq i \leq n$, are not pairwise disjoint; for example, since $\mathsf{V} \models x \to x \approx y \to y$ we have

$$\overline{x_1 \to x_1} = \overline{x_2 \to x_2} = \ldots = \overline{x_n \to x_n},$$

and hence $\overline{x_1 \to x_1} \in \bigcap\{\overline{\mathrm{T}}_i(n) : 1 \leq i \leq n\}$. We write $\overline{\mathrm{T}}_{\leq \ell}(n) = \bigcap_{1 \leq i \leq \ell} \overline{\mathrm{T}}_i(n)$.

It is easy to see that the sets $\overline{\mathrm{T}}_i(n)$ are actually subuniverses of $\mathbf{F}_{\mathsf{V}}(n)$, and hence so are the sets $\overline{\mathrm{T}}_{\leq \ell}(n)$. We will denote the corresponding subalgebras by $\overline{\mathbf{T}}_i(n)$ and $\overline{\mathbf{T}}_{\leq \ell}(n)$ respectively. We will see that the algebras $\overline{\mathbf{T}}_{\leq \ell}(n)$ are direct products of (finite) simple Łukasiewicz residuation algebras, and we will provide a precise description of these products. The algebra $\mathbf{F}_{\mathsf{V}}(n)$ is a union of subalgebras $\overline{\mathbf{T}}_i(n)$, $1 \leq i \leq n$. The next theorem gives the cardinality of $\mathbf{F}_{\mathsf{V}}(n)$ in terms of the cardinalities of the sets $\overline{\mathrm{T}}_{\leq \ell}(n)$, $1 \leq \ell \leq n$.

Theorem 2.1. *Suppose* $\mathsf{V} \subseteq \mathsf{HoRA}(k)$*, for some* k*,* $1 \leq k < \omega$*. For* $1 \leq n < \omega$ *we have*

$$|\mathbf{F}_{\mathsf{V}}(n)| = \sum_{\ell=1}^{n} (-1)^{\ell-1} \binom{n}{\ell} |\overline{\mathrm{T}}_{\leq \ell}(n)|. \tag{3}$$

Proof. Firstly note that V is locally finite, and hence $\mathrm{F}_{\mathsf{V}}(n)$ is a finite set. If $z \in \mathrm{F}_{\mathsf{V}}(n)$, then $z = \bar{s}$ for some $s \in \mathrm{T}(n)$; more precisely, $s \in \mathrm{T}_i(n)$, $1 \leq i \leq n$, where $\mathrm{rv}(s) = x_i$. Thus

$$\mathrm{F}_{\mathsf{V}}(n) = \bigcup_{i \in \{1,\ldots,n\}} \overline{\mathrm{T}}_i(n).$$

Applying the inclusion-exclusion principle, we see that

$$|\mathbf{F}_{\mathsf{V}}(n)| = \sum \{(-1)^{\ell-1} |\overline{\mathrm{T}}_{i_1} \cap \overline{\mathrm{T}}_{i_2} \cap \ldots \cap \overline{\mathrm{T}}_{i_\ell}| : 1 \leq i_1 < i_2 < \ldots < i_\ell \leq n\}.$$

Given an $s \in \mathrm{T}(n)$, let

$$J_s = \{j : 1 \leq j \leq n, \text{there is a } t \in \mathrm{T}(n) \text{ such that } \bar{s} = \bar{t} \text{ and } \mathrm{rv}(t) = x_j\}.$$

For $1 \leq i_1 < i_2 < \ldots < i_\ell \leq n$ we have

$$\overline{\mathrm{T}}_{i_1} \cap \overline{\mathrm{T}}_{i_2} \cap \ldots \cap \overline{\mathrm{T}}_{i_\ell} = \{\bar{s} : J_s \supseteq \{i_1, i_2, \ldots, i_\ell\}\}.$$

The automorphism of $\mathbf{F}_{\mathsf{V}}(n)$ that sends $\bar{x}_j$ to $\bar{x}_{i_j}$, $j = 1, 2, \ldots, \ell$, maps the set $\overline{\mathrm{T}}_1 \cap \overline{\mathrm{T}}_2 \cap \ldots \cap \overline{\mathrm{T}}_\ell$ onto the set $\overline{\mathrm{T}}_{i_1} \cap \overline{\mathrm{T}}_{i_2} \cap \ldots \cap \overline{\mathrm{T}}_{i_\ell}$; hence

$$|\overline{\mathrm{T}}_{i_1} \cap \overline{\mathrm{T}}_{i_2} \cap \ldots \cap \overline{\mathrm{T}}_{i_\ell}| = |\overline{\mathrm{T}}_1 \cap \overline{\mathrm{T}}_2 \cap \ldots \cap \overline{\mathrm{T}}_\ell| = |\overline{\mathrm{T}}_{\leq \ell}|.$$

For each ℓ, $1 \leq \ell \leq n$, we have $\binom{n}{\ell}$ sets of the form $\overline{\mathrm{T}}_{i_1} \cap \overline{\mathrm{T}}_{i_2} \cap \ldots \cap \overline{\mathrm{T}}_{i_\ell}$, each of size $|\overline{\mathrm{T}}_{\leq \ell}|$, yielding the desired formula (3). $\square$

We will now give a representation of the sets $\overline{\mathrm{T}}_{\leq \ell}(n)$ that will enable us to determine their size in the case V is locally finite, i.e., $\mathsf{V} \subseteq \mathsf{HoRA}(k)$, for some $k < \omega$. We need some preliminary lemmas.

By a *valuation* v into an algebra $\boldsymbol{A}$ we mean a map $v : \{x_1, \ldots, x_n\} \to A$, or also its natural extension $v : \mathbf{T}(n) \to \boldsymbol{A}$ from the algebra of terms $\mathbf{T}(n)$ into $\boldsymbol{A}$, with the property that $\boldsymbol{A}$ is generated by the set $\{v(x_1), \ldots, v(x_n)\}$.

Lemma 2.2. *Let $\boldsymbol{A} = \boldsymbol{B} \oplus \boldsymbol{C} \in \mathsf{V}$, and let $v : \{x_1, \ldots, x_n\} \to A$ be a valuation into $\boldsymbol{A}$.*

(i) If $s \in \mathrm{T}_i(n)$ then $v(s) \geq v(x_i)$.

(ii) If $s \in \mathrm{T}_i(n)$, and $v(x_i) \in B$, then $v(s) \in B$ as well.

Proof. (i). Observe that in any hoop we have $x \to y \geq y$. The claim now follows by induction on the complexity of s. Indeed, suppose $\mathrm{rv}(s) = x_i$. Then $s = x_i$ or $s = t \to t'$, with $\mathrm{rv}(t') = x_i$. In the first case $v(s) = v(x_i)$, in the second, assuming $v(t') \geq v(x_i)$ by induction hypothesis, we have $v(s) \geq v(t') \geq v(x_i)$ as well.

(ii). Note that if $a \in C$, $b \in B$, $b < 1$, then in $\boldsymbol{A}$ we have $a \to b = b$. Let $s \in \mathrm{T}_i(n)$, $\mathrm{rv}(s) = x_i$, and suppose $v(x_i) \in B$. If $v(x_i) = 1$, then by (i) $v(s) \geq v(x_i) = 1$, so $v(s) = 1 \in B$. Next assume $s = t \to t'$, with $\mathrm{rv}(t') = x_i$. By induction hypothesis we have $v(t') \in B$. If $v(t) \in B$ as well, then $v(s) = v(t) \to v(t') \in B$. So assume $v(t) \notin B$, i.e., $v(t) \in C - \{1\}$. As before, if $v(t') = 1$, then $v(s) = v(t) \to v(t') = v(t) \to 1 = 1$. If $v(t') < 1$, then $v(s) = v(t) \to v(t') = v(t')$. $\square$

Lemma 2.3. *Let $s \in \mathrm{T}_i(n)$. If $\mathsf{V} \not\models s \approx 1$, then there is an n-generated subdirectly irreducible algebra $\boldsymbol{A} = \sigma(\boldsymbol{A}) \oplus \mu(\boldsymbol{A}) \in \mathsf{V}$ and a valuation $v : \{x_1, \ldots, x_n\} \to \boldsymbol{A}$ such that $v(x_i) \in \mu(\boldsymbol{A})$, $v(s) \neq 1^{\boldsymbol{A}}$.*

Proof. Since $\mathsf{V} \not\models s \approx 1$, there is an n-generated subdirectly irreducible algebra $\boldsymbol{A} = \sigma(\boldsymbol{A}) \oplus \mu(\boldsymbol{A}) \in \mathsf{V}$ such that $\boldsymbol{A} \not\models s \approx 1$, and $\boldsymbol{A}$ can be chosen to be minimal in the sense that no proper homomorphic image $\boldsymbol{A}'$ of $\boldsymbol{A}$ satisfies $\boldsymbol{A}' \not\models s \approx 1$. Let v be a valuation into $\boldsymbol{A}$ such that $v(s) \neq 1^{\boldsymbol{A}}$. Suppose $v(s) \in \sigma(\boldsymbol{A})$. Let $h : \boldsymbol{A} \to \sigma(\boldsymbol{A})$ be the homomorphism discussed just before Lemma 1.8; it is given by $h(a) = a$ if $a \in \sigma(\boldsymbol{A})$, $h(a) = 1^{\sigma(\boldsymbol{A})}$ if $a \in \mu(\boldsymbol{A})$.

Then $h \circ v$ is a valuation into $\sigma(\boldsymbol{A})$, and $h \circ v(s) = v(s) \neq 1^{\boldsymbol{A}} = 1^{\sigma(\boldsymbol{A})}$, contradicting our assumption that $\boldsymbol{A}$ was minimal such that $\boldsymbol{A} \not\models s \approx 1$. Thus $v(s)$ does not belong to $\sigma(\boldsymbol{A})$, and by (ii) of the previous lemma it follows that $v(x_i) \notin \sigma(\boldsymbol{A})$ either; hence $v(x_i) \in \mu(\boldsymbol{A})$, and $v(x_i) \neq 1^{\boldsymbol{A}}$. $\square$

Corollary 2.4. *Let $\bar{s} \in \overline{\mathrm{T}}_{\leq \ell}(n)$, for some ℓ, $1 \leq \ell \leq n$. If $\bar{s} \neq 1$, then there is an n-generated subdirectly irreducible algebra $\boldsymbol{A} = \sigma(\boldsymbol{A}) \oplus \mu(\boldsymbol{A}) \in \mathsf{V}$ and a valuation $v : \{x_1, \ldots, x_n\} \to \boldsymbol{A}$ such that $v(x_i) \in \mu(\boldsymbol{A})$, $1 \leq i \leq \ell$, $v(s) \geq v(x_i)$, for all i, $1 \leq i \leq \ell$, and $v(s) \neq 1^{\boldsymbol{A}}$.*

Proof. In particular $\bar{s} \in \overline{\mathrm{T}}_1(n)$, so we may assume $s \in \mathrm{T}_1(n)$. By the previous lemma there is an n-generated subdirectly irreducible algebra $\boldsymbol{A} = \sigma(\boldsymbol{A}) \oplus \mu(\boldsymbol{A}) \in \mathsf{V}$ and a valuation $v : \{x_1, \ldots, x_n\} \to A$ such that $v(x_1) \in \mu(\boldsymbol{A})$, $v(s) \in \mu(\boldsymbol{A})$, and $v(s) \neq 1^{\boldsymbol{A}}$. Let $1 \leq i \leq \ell$. Then $\bar{s} \in \overline{\mathrm{T}}_i(n)$ as well, so there is a term $s' \in \mathrm{T}_i(n)$ such that $\bar{s} = \bar{s}'$. Since $\boldsymbol{A} \in \mathsf{V}$, it follows that $v(s') = v(s)$. Thus $v(s') \in \mu(\boldsymbol{A})$, and since $v(s') \neq 1^{\boldsymbol{A}}$, $v(x_i) \in \mu(\boldsymbol{A})$ as well by Lemma 2.2 (ii), and $v(s) = v(s') \geq v(x_i)$. $\square$

Fix ℓ, $1 \leq \ell \leq n$. Given two valuations $v : \{x_1, \ldots, x_n\} \to \boldsymbol{A}$ and $v' : \{x_1, \ldots, x_n\} \to \boldsymbol{A}'$, with $\boldsymbol{A}$ and $\boldsymbol{A}'$ (n-generated) subdirectly irreducible algebras in V, we say that $v' \leq v$ if there is a homomorphism $h : \boldsymbol{A} \to \boldsymbol{A}'$ such that $v' = h \circ v$. The relation $\leq$ is a quasi-order on the set of all valuations $v : \{x_1, \ldots, x_n\} \to \boldsymbol{A}$ into n-generated subdirectly irreducible algebras $\boldsymbol{A} \in \mathsf{V}$. We say that two such valuations v, v' are *equivalent* if $v \leq v'$ and $v' \leq v$. Let $\mathcal{V}$ be a set containing precisely one valuation from each equivalence class; the quasi-order $\leq$ restricted to $\mathcal{V}$ is then a partial order.

We now define for $1 \leq \ell \leq n$ the set $\mathcal{V}_\ell \subseteq \mathcal{V}$ to be the set of valuations $v \in \mathcal{V}$, say $v : \{x_1, \ldots, x_n\} \to \boldsymbol{A} = \sigma(\boldsymbol{A}) \oplus \mu(\boldsymbol{A})$, with the property that $v(x_i) \in \mu(\boldsymbol{A})$, $v(x_i) \neq 1^{\boldsymbol{A}}$, $1 \leq i \leq \ell$. Note that any two different valuations $v, v' \in \mathcal{V}_\ell$ are incomparable w.r.t. $\leq$. For suppose $v' \leq v$; say $h : \boldsymbol{A} \to \boldsymbol{A}'$ is a homomorphism such that $v' = h \circ v$. Then $1 \neq v'(x_1) = h(v(x_1))$, and since $v(x_1) \in \mu(\boldsymbol{A})$, it follows from Lemma 1.7 that h is 1-1. Since $v'(\{x_1, \ldots, x_n\})$ generates $\boldsymbol{A}'$, h is onto as well, and hence an isomorphism. But then $v = h^{-1} \circ v'$, so $v \leq v'$ and therefore $v = v'$.

Since V is a locally finite variety, and hence contains, up to isomorphism, only a finite number of n-generated subdirectly irreducible algebras, the sets $\mathcal{V}$ and $\mathcal{V}_\ell$ are finite. We write $\mathcal{V}_\ell = \{v_1, \ldots, v_m\}$, with $v_j : \{x_1, \ldots, x_n\} \to \boldsymbol{A}_j = \sigma(\boldsymbol{A}_j) \oplus \mu(\boldsymbol{A}_j)$, $1 \leq j \leq m$, and $v_j \neq v_{j'}$ when $j \neq j'$. The valuations

$v_1, \dots, v_m$ are thus, in particular, pairwise incomparable with respect to $\leq$. The number m of valuations in $\mathcal{V}_\ell$ depends of course on ℓ.

Let $d_j = \max\{v_j(x_i),\, 1 \leq i \leq \ell\}$, and let $\boldsymbol{D}_j$ be the subalgebra of $\mu(\boldsymbol{A}_j)$ with domain the set $\{c \in \mu(\boldsymbol{A}_j) \,:\, c \geq d_j\}$. We know that $\mu(\boldsymbol{A}_j)$ is isomorphic to $\mathbf{Ł}_r^{\rightarrow}$, for some $1 \leq r \leq k$; $\boldsymbol{D}_j$ is then isomorphic to $\mathbf{Ł}_s^{\rightarrow}$, for some s, $1 \leq s \leq r$. Let $\boldsymbol{D} = \prod_{j=1}^m \boldsymbol{D}_j$.

Theorem 2.5. *The map* $\varphi : \overline{\mathbf{T}}_{\leq\ell}(n) \to \boldsymbol{D}$ *given by*

$$\bar{s} \mapsto (v_j(s))_{j=1}^m \in \prod_{j=1}^m \boldsymbol{D}_j$$

is an embedding.

Proof. For $1 \leq j \leq m$ the valuation v_j can be thought of as a homomorphism from the algebra of terms $\mathbf{T}(n)$ to a certain algebra $\boldsymbol{A}_j \in \mathsf{V}$, and the map $\varphi : \mathbf{T}(n) \to \boldsymbol{A}$ given by

$$s \mapsto (v_j(s))_{j=1}^m \in \boldsymbol{A} = \prod_{j=1}^m \boldsymbol{A}_j$$

is thus a homomorphism from $\mathbf{T}(n)$ to the algebra $\boldsymbol{A} \in \mathsf{V}$. The kernel of that map contains the congruence $\{(s,t) \,:\, s,t \in \mathrm{T}(n),\, \mathsf{V} \models s \approx t\}$, and the map $\overline{\varphi} : \mathbf{F}_{\mathsf{V}}(n) \to \boldsymbol{A}$ defined by

$$\bar{s} \mapsto (v_j(s))_{j=1}^m \in \boldsymbol{A} = \prod_{j=1}^m \boldsymbol{A}_j$$

is therefore a well-defined homomorphism as well. Recall that $\overline{\mathbf{T}}_{\leq\ell}(n)$ is a subalgebra of $\mathbf{F}_{\mathsf{V}}(n)$. We now verify that the image of $\overline{\mathbf{T}}_{\leq\ell}(n)$ under $\overline{\varphi}$ is contained in $\prod_{j=1}^m D_j \subseteq \prod_{j=1}^m A_j$. Let $\bar{s} \in \overline{\mathrm{T}}_{\leq\ell}(n)$. Since $\bar{s} \in \overline{\mathrm{T}}_i(n)$, for $1 \leq i \leq \ell$, it follows from Lemma 2.2 (i) that $v_j(s) \geq v_j(x_i)$, for $1 \leq i \leq \ell$; thus $v_j(s) \geq d_j$, and hence $v_j(s) \in D_j$, for all j, $1 \leq j \leq m$. We can therefore write $\overline{\varphi} : \overline{\mathbf{T}}_{\leq\ell}(n) \to \prod_{j=1}^m \boldsymbol{D}_j$.

It remains to show that $\overline{\varphi}$ is 1-1. Since we are dealing with hoop residuation algebras, to see this, it suffices to show that for $\bar{s} \in \overline{\mathrm{T}}_{\leq\ell}(n)$, if $\bar{s} \neq 1$, then $\overline{\varphi}(\bar{s}) \neq 1$. So assume $\bar{s} \in \overline{\mathrm{T}}_{\leq\ell}(n)$, $\bar{s} \neq 1$. It follows from Corollary 2.4 that there is a valuation $v : \{x_1, \dots, x_n\} \to \boldsymbol{B}$, for some subdirectly irreducible algebra $\boldsymbol{B} = \sigma(\boldsymbol{B}) \oplus \mu(\boldsymbol{B}) \in \mathsf{V}$, such that $v(x_i) \in \mu(\boldsymbol{B})$, $1 \leq i \leq \ell$, and $v(s) \neq 1$. Then $v(x_i) \neq 1$, for $1 \leq i \leq \ell$, so v is equivalent to a valuation $v_j \in \mathcal{V}_\ell$. More explicitly, there is an isomorphism $h : \boldsymbol{B} \to \boldsymbol{A}_j$ such that $v_j = h \circ v$. Hence $v_j(s) = h(v(s)) \neq 1$, and therefore $\overline{\varphi}(\bar{s}) \neq 1$. □

In order to show that the map $\overline{\varphi}$ is onto, and hence an isomorphism, we need to review some properties of Łukasiewicz residuation algebras and hoop residuation algebras.

If a Łukasiewicz residuation algebra has a smallest element, then it is the reduct (rather than a subreduct) of a Wajsberg hoop $\langle A, \cdot, \to, 1\rangle$, and therefore of a Wajsberg algebra or, equivalently, of an MV-algebra. Indeed, if $\boldsymbol{A} = \langle A, \to, 1\rangle$ is a Łukasiewicz residuation algebra with smallest element $d \in A$, then the following operations are definable in $\langle A, \to, d, 1\rangle$:

$$\neg a := a \to d, \tag{4}$$
$$a \cdot b := \neg(a \to \neg b), \tag{5}$$
$$a \vee b := (a \to b) \to b = (b \to a) \to a, \tag{6}$$
$$a \wedge b := \neg(\neg a \vee \neg b). \tag{7}$$

With these definitions $\boldsymbol{A} = \langle A, \to, \cdot, 1\rangle$ becomes a Wajsberg hoop, and hence $\boldsymbol{A} = \langle A, \to, \cdot, d, 1\rangle$ is a Wajsberg algebra. Here $\vee$ and $\wedge$ are the least upper bound and greatest lower bound operations, as usual, with respect to the partial order given by $a \leq b$ if and only if $a \to b = 1$.

Let $j(x, y)$ denote the term $(((x \to y) \to y) \to x) \to x$. It follows from (7) that in any Łukasiewicz residuation algebra $\boldsymbol{A}$ we have $j^{\boldsymbol{A}}(a, b) = j^{\boldsymbol{A}}(b, a)$ (in fact, $j^{\boldsymbol{A}}(a, b) = \text{l.u.b.}\{a, b\}$), for any $a, b \in A$, so $\mathsf{ŁRA}$ satisfies the identity

$$(J) \quad j(x, y) \approx j(y, x). \tag{8}$$

This identity was introduced and studied by Cornish [13] in his work on BCK algebras. It was pointed out in [7] that the identity (J) also holds in all hoop residuation algebras. Therefore $\overline{j(x_1, x_2)} \in \overline{\mathrm{T}}_{\leq 2}(n)$. We define recursively for $n \geq 1$ the terms $j_n(x_1, \ldots, x_n)$ by $j_1(x_1) = x_1$, $j_2(x_1, x_2) = j(x_1, x_2)$, and

$$j_{n+1}(x_1, \ldots, x_n, x_{n+1}) = j(j_n(x_1, \ldots, x_n), x_{n+1}).$$

Lemma 2.6. *(i)* $\bar{j}_\ell \in \overline{\mathrm{T}}_{\leq \ell}(n)$, *for* $1 \leq \ell \leq n$.

(ii) For $\boldsymbol{A} \in \mathsf{ŁRA}$, $n \geq 1$, *and* $a_1, \ldots, a_n \in A$, *we have*

$$j_n(a_1, \ldots, a_n) = l.u.b.\{a_1, \ldots, a_n\}.$$

For the remainder of this section we fix $k < \omega$, and consider only algebras in $\mathsf{HoRA}(k)$. We define for a finitely generated (and hence finite) subdirectly irreducible hoop residuation algebra $\boldsymbol{A} \in \mathsf{HoRA}(k)$ a *characteristic term* χ. This is a generalization of a term first introduced by Jankov [19] for finite subdirectly irreducible Heyting algebras.

Let $v : \{x_1, \dots, x_n\} \to \boldsymbol{A}$ be a valuation, for some finite subdirectly irreducible hoop residuation algebra $\boldsymbol{A} \in \mathsf{HoRA}(k)$, let $1 \le \ell \le n$, and suppose $v(x_i) \in \mu(\boldsymbol{A})$, $1 \le i \le \ell$. For $a \in A$ let $t_a(x_1, \dots, x_n)$ be a term such that $v(t_a) = a$. Define a set of terms

$$T = \{t_{b \to c} \to (t_b \to t_c),\ (t_b \to t_c) \to t_{b \to c} : b, c \in A\},$$

and let $\chi_{\boldsymbol{A},v,\ell}(x_1, \dots, x_n) = ((\bigotimes^k T) \to j_\ell)(x_1, \dots, x_n)$.

Proposition 2.7. *Let $\boldsymbol{A}$, v, ℓ and $\chi = \chi_{\boldsymbol{A},v,\ell}$ be as above. Let $\boldsymbol{B}$ be a (finite) subdirectly irreducible hoop residuation algebra in $\mathsf{HoRA}(k)$, and let $v' : \{x_1, \dots, x_n\} \to \boldsymbol{B}$ be a map such that $v'(x_i) \in \mu(\boldsymbol{B})$ for $1 \le i \le \ell$. Then*

(i) $v'(\chi) = max\{v'(x_1), \dots, v'(x_\ell)\}$ *if* $v' = h \circ v$ *for some homomorphism* $h : \boldsymbol{A} \to \boldsymbol{B}$,

(ii) $v'(\chi) = 1^{\boldsymbol{B}}$ *if there is no homomorphism* $h : \boldsymbol{A} \to \boldsymbol{B}$ *such that* $v' = h \circ v$,

(iii) $\chi \in \overline{\mathrm{T}}_{\le \ell}(n)$.

Proof. Firstly, suppose $v' = h \circ v$ for some homomorphism $h : \boldsymbol{A} \to \boldsymbol{B}$. Since $v(t_a) = a$, for $a \in A$, we have $v(t) = 1^{\boldsymbol{A}}$, for all $t \in T$, and hence $v'(t) = h \circ v(t) = 1^{\boldsymbol{B}}$, for all $t \in T$. Therefore $v'(\chi) = v'((\bigotimes^k T) \to j_\ell) = v'(j_\ell) = \text{l.u.b.}\{v'(x_1), \dots, v'(x_\ell)\} = \max\{v'(x_1), \dots, v'(x_\ell)\}$, since $\mu(\boldsymbol{B})$ is linearly ordered.

For the second claim, if there is no homomorphism $h : \boldsymbol{A} \to \boldsymbol{B}$ such that $v' = h \circ v$, then in particular the map $h : \boldsymbol{A} \to \boldsymbol{B}$ given by

$$a \mapsto t_a^{\boldsymbol{B}}(v'(x_1), \dots, v'(x_n))$$

fails to be a homomorphism. Hence for some $a, b, c \in \boldsymbol{A}$ such that $a = b \to^{\boldsymbol{A}} c$ we have $v'(t_a) \neq v'(t_b) \to^{\boldsymbol{B}} v'(t_c)$. Then for at least one term $t \in T$ we have $v'(t) \neq 1^{\boldsymbol{B}}$. Hence the filter of $\boldsymbol{B}$ generated by the set $v'(T) \subseteq B$ is different from $\{1^{\boldsymbol{B}}\}$, and must contain the monolith $\mu(\boldsymbol{B})$ of $\boldsymbol{B}$, and in particular the element $v'(j_\ell) = \max\{v'(x_1), \dots, v'(x_\ell)\}$. By Corollary 1.4 we have $v'(\chi) = 1^{\boldsymbol{B}}$ in this case.

The third claim follows immediately from Lemma 2.6 (i). Indeed, since $\chi = ((\bigotimes^k T) \to j_\ell)$, and $\bar{j}_\ell \in \overline{\mathrm{T}}_{\le \ell}(n)$, we have $\bar{\chi} \in \overline{\mathrm{T}}_{\le \ell}(n)$. ◻

Jankov used the characteristic formula to define what are now known as ‘splitting varieties’ of Heyting algebras. Although we won’t use splitting varieties in this paper, it is interesting to see how the last proposition yields a version of Jankov’s result for $\mathsf{HoRA}(k)$:

Proposition 2.8. *Let $\boldsymbol{A}$ and v be as above, with $v(x_1) \neq 1^{\boldsymbol{A}}$, and let $\ell = 1$. For $\boldsymbol{B} \in \mathsf{HoRA}(k)$ the following are equivalent:*

(i) $\boldsymbol{B} \models \chi_{\boldsymbol{A},v,1} \approx 1$,

(ii) $\boldsymbol{A} \notin V(\boldsymbol{B})$.

Proof. Choosing for h the identity homomorphism in Proposition 2.7 (i) we see $v(\chi_{\boldsymbol{A},v,1}) = v(x_1) < 1^{\boldsymbol{A}}$, so $\boldsymbol{A} \not\models \chi_{\boldsymbol{A},v,1} \approx 1$, and (i) implies (ii). Conversely, if $\boldsymbol{B} \not\models \chi_{\boldsymbol{A},v,1} \approx 1$, then by Lemma 2.3 there is a subdirectly irreducible algebra $\boldsymbol{B}' \in V(\boldsymbol{B})$ and a valuation $v' : \{x_1, \ldots, x_n\} \to \boldsymbol{B}'$ such that $v'(x_1) \in \mu(\boldsymbol{B}')$ and $v'(\chi_{\boldsymbol{A},v,1}) \neq 1^{\boldsymbol{B}'}$. Applying Proposition 2.7 we see there is a homomorphism $h : \boldsymbol{A} \to \boldsymbol{B}'$ such that $v' = h \circ v$ and $v'(\chi_{\boldsymbol{A},v,1}) = v'(x_1) \neq 1^{\boldsymbol{B}'}$. So $h(v(x_1)) \neq 1^{\boldsymbol{B}'}$, while $v(x_1) \in \mu(\boldsymbol{A})$, hence h is 1-1. Thus $\boldsymbol{A} \in S(\boldsymbol{B}') \subseteq V(\boldsymbol{B})$, as desired. ☐

The last proposition shows that every finite subdirectly irreducible algebra in $\boldsymbol{A} \in \mathsf{HoRA}(k)$ is a ‘splitting algebra’ in $\mathsf{HoRA}(k)$. Since $\mathsf{HoRA}(k)$ has EDPC (see the remarks following Corollary 1.4), this follows also from the general fact that in varieties with EDPC every finite subdirectly irreducible algebra is ‘splitting’ (see [4]).

We are now ready to prove:

Theorem 2.9. *The map $\varphi : \overline{\mathbf{T}}_{\leq \ell}(n) \to \boldsymbol{D}$ given by*

$$\bar{s} \mapsto (v_j(s))_{j=1}^{m} \in \prod_{j=1}^{m} \boldsymbol{D}_j$$

is an isomorphism.

Proof. We already showed that φ is an embedding; it remains to show it is onto.

Recall that $d_j = \max\{v_j(x_i), 1 \leq i \leq \ell\} \in \boldsymbol{A}_j$, for $1 \leq j \leq m$, and that $D_j = \{c \in \mu(A_j) : d_j \leq c\}$. Observe that since $v_j(x_1), \ldots, v_j(x_\ell) \in \mu(\boldsymbol{A}_j)$, and $\mu(\boldsymbol{A}_j)$ is a (linearly ordered) Łukasiewicz residuation algebra, $\pi_j \circ \varphi(\bar{j}_\ell) =$

$v_j(\bar{j}_\ell) = \max\{v_j(x_1), \ldots, v_j(x_\ell)\} = d_j$. Therefore $d = (d_j)_{j=1}^m = \varphi(\bar{j}_\ell)$, the smallest element of $\prod_{j=1}^m \boldsymbol{D}_j$. The subalgebra $\varphi(\overline{\mathbf{T}}_{\leq\ell}(n))$ of $\prod_{j=1}^m \boldsymbol{D}_j$ is thus the reduct of a Łukasiewicz algebra, and in particular closed under $\wedge$ as defined in (7).

To show φ is onto, we will show that for $1 \leq j \leq m$ and $c \in D_j$, there is a term $t_{j,c} \in \overline{\mathrm{T}}_{\leq\ell}(n)$ with the property $\pi_j \circ \varphi(\bar{t}_{j,c}) = c$, while $\pi_{j'} \circ \varphi(\bar{t}_{j,c}) = 1^{\boldsymbol{D}_{j'}}$, for all $j' \neq j$, $1 \leq j' \leq m$. Since $\prod_{j=1}^m \boldsymbol{D}_j$ is generated as a meet semilattice by the elements $\varphi(\bar{t}_{j,c})$, this will show that $\varphi(\overline{\mathrm{T}}_{\leq\ell}(n)) = \prod_{j=1}^m D_j$.

So let $v_j : \{x_1, \ldots, x_n\} \to \boldsymbol{A}_j$, as before, and let $c \geq d_j$. By assumption, the algebra $\boldsymbol{A}_j$ is generated by the set $\{v_j(x_i) : 1 \leq i \leq n\}$, so there is a term $s \in \mathrm{T}(n)$ such that $v_j(s) = c$. Recall the definition of the characteristic formula $\chi = \chi_{\boldsymbol{A}_j, v_j, \ell}$ given above. Now define

$$t_{j,c}(x_1, \ldots, x_n) = (s \to \chi) \to \chi.$$

Firstly, since $v_j = \mathrm{id} \circ v_j$, we have $v_j(\chi) = \max\{v_j(x_1), \ldots, v_j(x_\ell)\} = d_j$, by Proposition 2.7 (i). Hence

$$v_j(t_{j,c}) = (v_j(s) \to^{\boldsymbol{A}_j} d_j) \to^{\boldsymbol{A}_j} d_j = (c \to^{\boldsymbol{D}_j} d_j) \to^{\boldsymbol{D}_j} d_j) = \neg^{\boldsymbol{D}_j}\neg^{\boldsymbol{D}_j} c = c,$$

as desired. Secondly, for $j' \neq j$, $1 \leq j' \leq m$, there is no homomorphism $h : \boldsymbol{A}_j \to \boldsymbol{A}_{j'}$ such that $v_{j'} = h \circ v_j$, since the valuations in $\mathcal{V}_\ell$ are pairwise incomparable with respect to the partial order $\leq$. Hence by Proposition 2.7 (ii) we have $v_{j'}(\chi) = 1^{\boldsymbol{D}_{j'}}$, and therefore $v_{j'}(t_{j,c}) = 1^{\boldsymbol{D}_{j'}}$ as well. Finally, since $\bar{\chi} \in \overline{\mathrm{T}}_{\leq\ell}(n)$, we have also $\bar{t}_{j,c} \in \overline{\mathrm{T}}_{\leq\ell}(n)$. This completes the proof that φ is onto. □

3 Examples

We now use our results to compute the cardinalities of finitely generated free algebras in some specific varieties of hoop residuation algebras.

Throughout this section we write $\mathbf{F}_{\mathsf{V}}(n)$ for the free V algebra on n free generators. The set $X = \{x_1, \ldots, x_n\}$ is the free generating set for the free algebra $\mathbf{F}_{\mathsf{V}}(X)$. For $1 \leq \ell \leq n$ we let $X_{\leq\ell} = \{x_1, \ldots, x_\ell\}$ and $X_{>\ell} = \{x_{\ell+1}, \ldots, x_n\}$. If $v : X \to \boldsymbol{A}$ is a map, then $v_{\leq\ell}$ and $v_{>\ell}$ denote v restricted to $X_{\leq\ell}$ and v restricted to $X_{>\ell}$, respectively.

We introduce some notation for the inclusion-exclusion arguments that are used in this section. Let S be a set and $Z_1, \ldots, Z_t$ subsets of S. For

$1 \leq j \leq t$ let S_j denote the sum of the cardinalities of all sets of the form $Z_{i_1} \cap Z_{i_2} \cap \cdots \cap Z_{i_t}$ for $1 \leq i_1 < i_2 < \cdots < i_j \leq t$. By $N(S; Z_1, \ldots, Z_t)$ we denote the number of elements in S that are not in any of $Z_1, \ldots, Z_t$. The standard inclusion-exclusion formula is

$$N(S; Z_1, \ldots, Z_t) = |S| - S_1 + S_2 + \cdots + (-1)^j S_j + \cdots + (-1)^t S_t. \qquad (9)$$

Example 3.1. Let V be the variety of 2-potent hoop residuation algebras. We describe $\mathbf{F}_{\mathsf{V}}(1)$ and $\mathbf{F}_{\mathsf{V}}(2)$ using our methods.

The only 1-generated subdirectly irreducible algebra in V is $\mathbf{\L}_2^{\to}$ and there is only one valuation in $\mathcal{V}_1$. So the free hoop residuation algebra on one free generator in V is $\mathbf{\L}_2^{\to}$. In fact, $\mathbf{F}_{\mathsf{V}}(1)$ is $\mathbf{\L}_2^{\to}$ even if $\mathsf{V} = \mathsf{HoRA}$.

To determine the cardinality of $\mathbf{F}_{\mathsf{V}}(2)$ we first note that there are three 2-potent subdirectly irreducible hoop residuation algebras that are at most 2-generated: $\mathbf{\L}_2^{\to}$, $\mathbf{\L}_3^{\to}$ and $\mathbf{C}_3$. Recall that the domain of $\mathbf{C}_3$ is $\{1, e^1, b\}$. There are five maps v from $X = \{x_1, x_2\}$ to $\boldsymbol{A}$ where $\boldsymbol{A}$ is one of $\mathbf{\L}_2^{\to}$, $\mathbf{\L}_3^{\to}$ and $\mathbf{C}_3$ and v has the properties that $v(X)$ generates $\boldsymbol{A}$, $v(x_1)$ is in the monolith $\mu(\boldsymbol{A})$, and $v(x_1) \neq 1$. These five valuations are schematically drawn in Figure 1. They are pairwise inequivalent and thereby constitute $\mathcal{V}_1$.

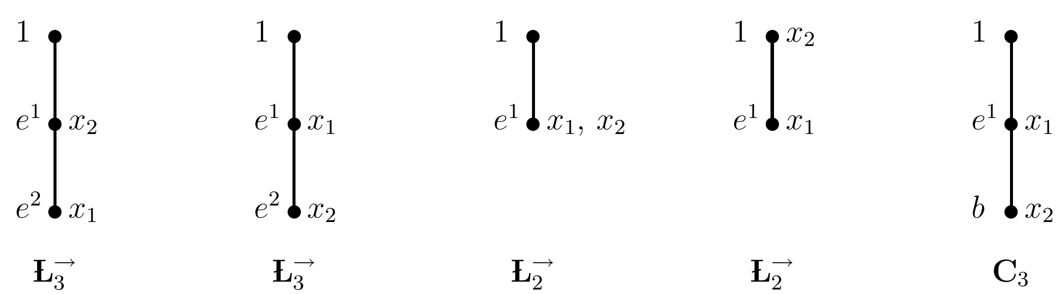

Figure 1

We apply Theorem 2.9 to determine the subalgebras $\overline{\mathbf{T}}_{\leq \ell}(2)$. An examination of the valuations shows that the first contributes a factor of $\mathbf{\L}_3^{\to}$ to $\overline{\mathbf{T}}_{\leq 1}(2)$. The second valuation, since $\max\{v(x_1), v(x_2)\} = e^1$, contributes a factor of $[e^1, 1] \simeq \mathbf{\L}_2^{\to}$ to $\overline{\mathbf{T}}_{\leq 1}(2)$. The remaining three valuations each contribute a factor of $\mathbf{\L}_2^{\to}$. Thus, $\overline{\mathbf{T}}_{\leq 1}(2) = \mathbf{\L}_3^{\to} \times (\mathbf{\L}_2^{\to})^4$. To determine $\overline{\mathbf{T}}_{\leq 2}(2)$ we note that $\mathcal{V}_2$ consists only of the first three valuations in Figure 1 since the fourth has $v(x_2) = 1$ and the fifth has $v(x_2) \notin \mu(\mathbf{C}_3)$. Since each of the first three valuations has $\max\{v(x_1), v(x_2)\} = e^1$, each contributes a factor of $\mathbf{Ł}_2^{\to}$ to $\overline{\mathbf{T}}_{\leq 2}(2)$. Therefore $\overline{\mathbf{T}}_{\leq 2}(2) = (\mathbf{Ł}_2^{\to})^3$. By Theorem 2.1 we have that the free hoop residuation algebra on 2 free generators has cardinality

$$2\,|\overline{\mathrm{T}}_{\leq 1}(2)| - |\overline{\mathrm{T}}_{\leq 2}(2)| = 96 - 8 = 88.$$

We next provide a more complex illustrative example.

Example 3.2. Let V be the variety generated by the algebra $\mathbf{Ł}_2^{\to} \oplus \mathbf{Ł}_3^{\to}$. Applying Theorem 2.9 we determine $\overline{\mathbf{T}}_{\leq \ell}(3)$ for $1 \leq \ell \leq 3$ and use the results to compute $|\mathbf{F}_{\mathsf{V}}(3)|$. The only subdirectly irreducible algebras in V are $\mathbf{Ł}_2^{\to} \oplus \mathbf{Ł}_3^{\to}$, $\mathbf{Ł}_3^{\to}$, $\mathbf{Ł}_2^{\to}$ and the 3-element Hilbert algebra $\mathbf{C}_3$. For every ℓ, a valuation $v \in \mathcal{V}_\ell$ is a function $v : X \to \boldsymbol{A}$ for $\boldsymbol{A}$ a subdirectly irreducible algebra in V with the property that $v(X)$ generates $\boldsymbol{A}$ and $v(X_{\leq \ell}) \subseteq \mu(\boldsymbol{A}) - \{1\}$. Let m be such that $e^m = \max(v(X_{\leq \ell}))$. Every $v \in \mathcal{V}_\ell$ is associated with a subdirectly irreducible algebra $\boldsymbol{A}$ and an integer m. The valuation v contributes a factor algebra $[e^m, 1]$ to the product algebra $\overline{\mathbf{T}}_{\leq \ell}(3)$. The number of valuations $v \in \mathcal{V}_\ell$ with given $\boldsymbol{A}$ and m can be determined by inspection using diagrams as in Example 3.1. The results of the analysis are summarized in the following table.

SI algebra	m	$\ell = 1$	$\ell = 2$	$\ell = 3$
$\mathbf{Ł}_2^{\to}$	1	4	2	1
$\mathbf{C}_3$	1	5	1	0
$\mathbf{Ł}_3^{\to}$	1	5	7	6
$\mathbf{Ł}_3^{\to}$	2	5	1	0
$\mathbf{Ł}_2^{\to} \oplus \mathbf{Ł}_3^{\to}$	1	2	2	0
$\mathbf{Ł}_2^{\to} \oplus \mathbf{Ł}_3^{\to}$	2	2	0	0

Thus,

$$\begin{aligned}
\overline{\mathbf{T}}_{\leq 1}(3) &= (\mathbf{Ł}_2^{\to})^{4+5+5+2} \times (\mathbf{Ł}_3^{\to})^{5+2} = (\mathbf{Ł}_2^{\to})^{16} \times (\mathbf{Ł}_3^{\to})^{7}, \\
\overline{\mathbf{T}}_{\leq 2}(3) &= (\mathbf{Ł}_2^{\to})^{2+1+7+2} \times (\mathbf{Ł}_3^{\to})^{1} = (\mathbf{Ł}_2^{\to})^{12} \times (\mathbf{Ł}_3^{\to})^{1}, \\
\overline{\mathbf{T}}_{\leq 3}(3) &= (\mathbf{Ł}_2^{\to})^{1+6} = (\mathbf{Ł}_2^{\to})^{7}.
\end{aligned}$$

From Theorem 2.1 we see that the cardinality of the free algebra on 3 free generators in the variety generated by $\mathbf{Ł}_2^{\to} \oplus \mathbf{Ł}_3^{\to}$ is $3 * 2^{16}3^7 - 3 * 2^{12}3^1 + 1 * 2^7$.

3.1 Hilbert algebras

We next consider some examples of subvarieties V of $\mathsf{HoRA}(1)$, i.e., of the variety BrRA of Brouwerian residuation algebras, commonly known as the variety of Hilbert algebras. We obtain a formula for the size of the finitely generated free algebras in V. For $\mathsf{V} \subseteq \mathsf{HoRA}(1)$ and $1 \leq \ell \leq n$, the set $\mathcal{V}_\ell$ consists all valuations $v : \{x_1, \ldots, x_n\} \to \boldsymbol{A} = \boldsymbol{B} \oplus \mathbf{Ł}_2^{\to}$, where $\boldsymbol{A} \in \mathsf{V}$, $Ł_2^{\to} = \{1, e\}$, for which $v(x_i) = e$ for $1 \leq i \leq \ell$. For each $k \geq 2$ let $\mathbf{C}_k$ denote the Hilbert algebra whose domain is the k-element chain $1 > e > b_1 > b_2 > \cdots > b_{k-2}$; note $\mathbf{C}_2 = \mathbf{Ł}_2^{\to}$, and in general $\mu(\mathbf{C}_k) = \mathbf{Ł}_2^{\to}$, with domain $\{1, e\}$. Neither $\mathbf{C}_k$ nor $\sigma(\mathbf{C}_k)$ have any nontrivial automorphisms.

Example 3.3. Let V be the variety generated by the 2-element Hilbert algebra $\mathbf{C}_2$, which of course coincides with the 2-element Łukasiewicz hoop residuation algebra $\mathbf{Ł}_2^{\to}$; V is the variety of Boolean residuation algebras. The only subdirectly irreducible algebra in V is $\mathbf{C}_2$. For every $\ell \geq 1$ a valuation $v : X \to \mathbf{C}_2$ is in $\mathcal{V}_\ell$ if and only if $v(x_i) = e$ for $1 \leq i \leq \ell$. Therefore $|\mathcal{V}_\ell| = 2^{n-\ell}$ and $\overline{\mathbf{T}}_{\leq \ell}(n)$ is isomorphic to $(\mathbf{Ł}_2^{\to})^{2^{n-\ell}}$. By Theorem 2.1 we have

$$|\mathbf{F}_{\mathsf{V}}(n)| = \sum_{\ell=1}^{n} (-1)^{\ell-1} \binom{n}{\ell} 2^{2^{n-\ell}}.$$

The cardinalities of $\mathrm{F}_{\mathsf{V}}(n)$ for $n = 1, 2, 3$ and 4 are 2, 6, 38 and 942, respectively. This formula for the cardinality of the free Boolean residuation algebra appears in Monteiro's work; see [23].

Example 3.4. Next, let V denote the variety generated by $\mathbf{C}_3$. The only subdirectly irreducibles in this variety are $\mathbf{C}_2$ and $\mathbf{C}_3$. As in the previous example there are $2^{n-\ell}$ valuations in $\mathcal{V}_\ell$ that map into $\mathbf{C}_2$. The valuations in $\mathcal{V}_\ell$ that map into $\mathbf{C}_3$ are those v for which $v(x_i) = e$ for $1 \leq i \leq \ell$ and $v(x_i)$ has value b_1 for at least one $i > \ell$. There are $3^{n-\ell} - 2^{n-\ell}$ such valuations into $\mathbf{C}_3$. Thus, the cardinality of $\mathcal{V}_\ell$ is $3^{n-\ell}$. So $\overline{\mathbf{T}}_{\leq \ell}(n)$ is isomorphic to $(\mathbf{Ł}_2^{\to})^{3^{n-\ell}}$ and by Theorem 2.1

$$|\mathbf{F}_{\mathsf{V}}(n)| = \sum_{\ell=1}^{n} (-1)^{\ell-1} \binom{n}{\ell} 2^{3^{n-\ell}}.$$

The values for $n = 1, 2, 3$ and 4 are 2, 14, 1514 and 536,867,870 respectively.

Example 3.5. Let V denote the variety of Hilbert algebras generated by all finite chains $\mathbf{C}_k$, for $2 \leq k < \omega$. From Jónsson's Theorem we know that the $\mathbf{C}_k$ are the only finite subdirectly irreducible algebras in V. If $v \in \mathcal{V}_\ell$, say, $v : X \to \mathbf{C}_k$, with $v(x_i) = e$ for $1 \leq i \leq \ell$, then the domain of $\mathbf{C}_k$ is $\{1, e\} \cup v(X_{>\ell})$. $v : X \to \mathbf{C}_k$ with these properties is a valuation in $\mathcal{V}_\ell$. For $1 \leq \ell \leq n$ and $2 \leq m \leq n + 2 - \ell$ let $N(\ell, m, n)$ denote the number of functions $f : X_{>\ell} \to C_m$ for which $C_m - \{1, e\}$

$$\mathcal{V}_\ell = N(\ell, 2, n) + N(\ell, 3, n) + \cdots + N(\ell, n - \ell + 2, n).$$

For example, we have already observed that $N(\ell, 2, n) = 2^{n-\ell}$ and $N(\ell, 3, n) = 3^{n-\ell} - 2^{n-\ell}$. An inclusion-exclusion argument yields

$$N(\ell, m, n) = \sum_{j=0}^{m-2} (-1)^j \binom{m-2}{j} (m - j)^{n-\ell}.$$

Thus,

$$|\mathcal{V}_\ell| = \sum_{m=2}^{n-\ell+2} N(\ell, m, n) = \sum_{m=2}^{n-\ell+2} \left(\sum_{q=0}^{n-\ell-m+2} (-1)^q \binom{m+q-2}{q} \right) m^{n-\ell}.$$

By Theorem 2.1 we have

$$|\mathbf{F}_{\mathsf{V}}(n)| = \sum_{\ell=1}^{n} (-1)^{\ell-1} \binom{n}{\ell} 2^{\sum_{m=2}^{n-\ell+2} N(\ell,m,n)}.$$

The cardinalities of $\mathrm{F}_{\mathsf{V}}(n)$ for $n = 1, 2$ and 3 are 2, 14 and 6122, respectively.

F. Guzmán and C. Lynch [17] consider free algebras in the variety V generated by a single finite *pure* Hilbert algebra, i.e., a Hilbert algebra in which $\to$ satisfies

$$a \to b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise.} \end{cases}$$

They use inclusion-exclusion to argue that

$$|\mathbf{F}_{\mathsf{V}}(n)| = \sum_{\ell=1}^{n} (-1)^{\ell-1} \binom{n}{\ell} 2^{|\mathcal{V}_\ell|}.$$

They then give a formula for the cardinality of the free Hilbert algebra in a variety of Hilbert algebras generated by a finite chain.

We conclude this subsection by providing a recursive method for computing an upper bound on the size of $\mathcal{V}_\ell$ for any variety V of Hilbert algebras and all $1 \leq \ell \leq n$. In the event that V is the variety of all Hilbert algebras the method gives the exact value for $|\mathcal{V}_\ell|$ and the analysis can be used, in principle, to determine the cardinality of the free Hilbert algebra on n free generators.

If $\boldsymbol{A}$ is any subdirectly irreducible Hilbert algebra, then the universe of the monolith $\mu(\boldsymbol{A})$ is $\{1, e\}$ and the universe of $\sigma(\boldsymbol{A})$ is $A - \{e\}$. We observed in the comments following Corollary 1.6 that $\sigma(\boldsymbol{A})$ is a subalgebra of $\boldsymbol{A}$.

Let V be any nontrivial variety of Hilbert algebras. For $1 \leq \ell \leq n$ consider any valuation $v : X \to \boldsymbol{A}$ in $\mathcal{V}_\ell$. Then $\boldsymbol{A}$ is a subdirectly irreducible algebra in V and $v(X)$ generates $\boldsymbol{A}$. As usual, the monolith of $\boldsymbol{A}$ is $\{1, e\}$. Let $E_v = v^{-1}(\{e\}) \cap X_{>\ell}$. By Lemma 1.8 the set $v(X_{>\ell} - E_v)$ generates the subalgebra $\sigma(\boldsymbol{A})$ of $\boldsymbol{A}$. Hence $\sigma(\boldsymbol{A})$ is a homomorphic image of $\mathbf{F}_{\mathsf{V}}(X_{>\ell} - E_v)$ by a homomorphism onto $\sigma(\boldsymbol{A})$ that agrees with v on $X_{>\ell} - E_v$. Let θ_v denote the kernel of this homomorphism. We may assume that $\boldsymbol{A}$ is $\mathbf{F}_{\mathsf{V}}(X_{>\ell} - E_v)/\theta_v) \oplus \mathbf{Ł}_2^{\rightarrow}$. We claim the map $v \mapsto (E_v, \theta_v)$ is 1-1 on $\mathcal{V}_\ell$. For if $v, w \in \mathcal{V}_\ell$ are such that $(E_v, \theta_v) = (E_w, \theta_w)$, where $v : X \to \boldsymbol{A}$ and $w : X \to \boldsymbol{B}$, then $\boldsymbol{A}$ and $\boldsymbol{B}$ are isomorphic algebras via an isomorphism $h : \boldsymbol{A} \to \boldsymbol{B}$ for which $h(e) = e$ and $h(x_i/\theta_v) = x_i/\theta_w$ for all $x_i \in X_{>\ell} - E_v$. We have $w = hv$, so $w \leq v$. As $v, w \in \mathcal{V}_\ell$, and the valuations in $\mathcal{V}_\ell$ are pairwise incomparable, we conclude $v = w$. Thus we obtain the following upper bound.

Theorem 3.6. *In any variety* V *of Hilbert algebras*

$$|\mathcal{V}_\ell| \leq \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} |\mathrm{Con}(\mathbf{F}_{\mathsf{V}}(n - \ell - j))|.$$

Suppose that a variety V of Hilbert algebras has the property that for all $\boldsymbol{B} \in \mathsf{V}$ the algebra $\boldsymbol{B} \oplus \mathbf{Ł}_2^{\rightarrow}$ is also in V. We show that the upper bound in Theorem 3.6 is achieved for V. It follows from Corollary 1.6 that the variety $\mathsf{HoRA}(1)$ of all Hilbert algebras has this property, and we observed in comments following Lemma 1.8 that $\mathsf{HoRA}(1)$ is the only variety of Hilbert algebras with the property. For V, if E is any nonvoid subset of $X_{>\ell}$ and θ is any congruence relation on $\mathbf{F}_{\mathsf{V}}(X_{>\ell} - E)$, then the valuation

$$v : X \to (\mathbf{F}_{\mathsf{V}}(X - E)/\theta) \oplus \mathbf{Ł}_2$$

given by $v(x_i) = e$ for $x_i \in X_{\leq\ell} \cup E$ by $v^{-1}(\{e\}) = E$ and $v(x_i) = x_i/\theta$ for $x_i \in X_{>\ell} - E$ is such that $(E, \theta) = (E_v, \theta_v)$. We conclude

Theorem 3.7. *Let* V *be the variety* $\mathsf{HoRA}(1)$ *of all Hilbert algebras. Then*

$$|\mathcal{V}_\ell| = \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} |\mathrm{Con}(\mathbf{F}_{\mathsf{V}}(n-\ell-j))|. \tag{10}$$

It is known and is easily checked that the free Hilbert algebras on 0, 1 and 2 free generators have 1, 2, and 14 elements respectively. The corresponding congruence lattices have 1, 2 and 18 elements. We use (10) with $n = 3$ to determine the size of $\mathcal{V}_\ell$ for $1 \le \ell \le 3$. We have $|\mathcal{V}_1| = 1*18+2*2+1*1 = 23$, $|\mathcal{V}_2| = 3$, and $|\mathcal{V}_3| = 1$. So by Theorem 2.1 we see that the free Hilbert algebra on three free generators has cardinality

$$3 * 2^{23} - 3 * 2^3 + 2^1 = 25,165,802.$$

This is in agreement with the value found by A. Hendriks in [18, p. 92]. G. Renardel de Lavalette (see [18, p. 92]) has calculated that the cardinality of the free Hilbert algebra on 4 free generators is $2^{623,662,965,552,393} - 50,331,618$.

3.2 Łukasiewicz Residuation Algebras

We next consider the free algebras in the variety of Łukasiewicz residuation algebras, $\mathsf{ŁRA}$. Recall that the domain of $\mathbf{Ł}_k^{\rightarrow}$, denoted Ł_k, consists of the k elements $1 = e^0 > e^1 > e^2 > \ldots > e^{k-1}$.

For $1 \le \ell \le n$ and $1 \le m \le k-1$ let $V(k, n, \ell, m)$ denote the number of valuations $v : X \to \mathbf{Ł}_k^{\rightarrow}$ for which $v(X)$ generates all of $\mathbf{Ł}_k^{\rightarrow}$ and $e^m = \max(v(X_{\le \ell}))$.

We fix $r \ge 2$ and let V be the variety generated by $\mathbf{Ł}_r^{\rightarrow}$. The only subdirectly irreducible algebras in V are the $\mathbf{Ł}_k^{\rightarrow}$ for $2 \le k \le r$. For this variety

$$|\mathcal{V}_\ell| = \sum_{k=2}^{r} \sum_{m=1}^{k-1} V(k, n, \ell, m).$$

By Theorem 2.9 we can write

$$\overline{\mathbf{T}}_{\le \ell}(n) \cong \prod_{k=2}^{r} (\mathbf{Ł}_k^{\rightarrow})^{q_k}.$$

Here the exponents q_k are given by

$$q_k = \sum_{i=k}^{r} V(i, n, \ell, k-1).$$

Hence, by finding the values of $V(k,n,\ell,m)$ we can determine the structure of the $\overline{\mathbf{T}}_{\leq\ell}(n)$ and the size of $\mathbf{F}_{\mathsf{V}}(n)$.

We require some additional notation. Let $I_m = \{e^m, e^{m+1}, \ldots, e^{k-1}\}$ for $1 \leq m \leq k-1$ and let $I_k = \emptyset$.

The maximal proper subuniverses of $\mathbf{Ł}_k^{\rightarrow}$ are $\{1, e^1, \ldots, e^{k-2}\}$ and the sets $\{1, e^d, e^{2d}, \ldots, e^{k-1}\}$, where d is any prime number dividing $k-1$. Let $M_1, \ldots, M_u$ be the maximal proper subuniverses of $\mathbf{Ł}_k^{\rightarrow}$.

If $A, B \subseteq Ł_k$, then let A^ℓ denote all $v : X \to Ł_k$ for which $v(X_{\leq\ell}) \subseteq A$ and let $A^\ell B^{n-\ell}$ denote all $v : X \to Ł_k$ with $v(X_{\leq\ell}) \subseteq A$ and $v(X_{>\ell}) \subseteq B$.

A map $v : X \to Ł_k$ is in $V(k,n,\ell,m)$ if and only if $e^m \in v(X_{\leq\ell}) \subseteq I_m$ and $v(X)$ generates all of $\mathbf{Ł}_k^{\rightarrow}$. These conditions on v are equivalent to v being in $I_m^\ell Ł_k^{n-\ell} - I_{m+1}^\ell Ł_k^{n-\ell}$ and $v(X)$ is not a subset of any maximal proper subuniverse of $\mathbf{Ł}_k^{\rightarrow}$.

If for $1 \leq i \leq u$ we let

$$Z_i = ((I_m \cap M_i)^\ell - (I_{m+1} \cap M_i)^\ell) M_i^{n-\ell},$$

then an inclusion-exclusion formula using the notation of (9) gives

$$V(k,n,\ell,m) = N(I_m^\ell Ł_k^{n-\ell} - I_{m+1}^\ell Ł_k^{n-\ell}; Z_1, \ldots, Z_u). \tag{11}$$

For example, if $2 \leq m < k-1$ with m and $k-1$ relatively prime, then the only maximal proper subuniverse that contains e^m is $\{1, e^1, \ldots, e^{k-2}\}$. Thus (11) yields

$$\begin{aligned} V(k,n,\ell,m) = (k-m)^\ell k^{n-\ell} - (k-m-1)^\ell k^{n-\ell} \\ - (k-m-1)^\ell (k-1)^{n-\ell} + (k-m-2)^\ell (k-1)^{n-\ell}. \end{aligned}$$

Similarly, if $m = k-1$ and $k-1$ is prime, then the only maximal proper subuniverse containing $e^m = e^{k-1}$ is $\{1, e^{k-1}\}$ and the formula in (11) simplifies to

$$V(k,n,\ell,k-1) = k^{n-\ell} - 2^{n-\ell}.$$

The following table gives the values of $V(k,n,\ell,m)$ for $2 \leq k \leq 5$ and $1 \leq m \leq 4$.

	$m=1$	$m=2$	$m=3$	$m=4$
$\mathbf{Ł}_2^{\rightarrow}$	$2^{n-\ell}$			
$\mathbf{Ł}_3^{\rightarrow}$	$(2^\ell-1)3^{n-\ell} - 2^{n-\ell}$	$3^{n-\ell} - 2^{n-\ell}$		
$\mathbf{Ł}_4^{\rightarrow}$	$(3^\ell - 2^\ell)4^{n-\ell} - (2^\ell-1)3^{n-\ell}$	$(2^\ell-1)4^{n-\ell} - 3^{n-\ell}$	$4^{n-\ell} - 2^{n-\ell}$	
$\mathbf{Ł}_5^{\rightarrow}$	$(4^\ell - 3^\ell)5^{n-\ell} - (3^\ell - 2^\ell)4^{n-\ell}$	$(3^\ell - 2^\ell)5^{n-\ell}$ $-(2^\ell-1)(4^{n-\ell}+3^{n-\ell}) + 2^{n-\ell}$	$(2^\ell-1)5^{n-\ell} - 4^{n-\ell}$	$5^{n-\ell} - 3^{n-\ell}$

From this table it is possible to determine for each of the varieties V generated by $\mathbf{Ł}_k^{\rightarrow}$ for $2 \leq k \leq 5$, the set $\mathcal{V}_\ell$ and the algebra $\overline{\mathbf{T}}_{\leq \ell}(n)$, and by means of Theorem 2.1, the cardinality of $\mathbf{F}_{\mathsf{V}}(n)$.

The variety generated by $\mathbf{Ł}_2^{\rightarrow}$ is the same as the variety of Boolean residuation algebras discussed in Example 3.3.

For the variety generated by $\mathbf{Ł}_3^{\rightarrow}$ we have

$$\overline{\mathbf{T}}_{\leq \ell}(n) = (\mathbf{Ł}_2^{\rightarrow})^{(2^\ell-1)3^{n-\ell}}(\mathbf{Ł}_3^{\rightarrow})^{3^{n-\ell}-2^{n-\ell}}$$

and so the free algebra on n free generators in this variety has cardinality

$$\sum_{\ell=1}^{n}(-1)^{\ell-1}\binom{n}{\ell}2^{(2^\ell-1)3^{n-\ell}}3^{3^{n-\ell}-2^{n-\ell}}.$$

The values for $n = 1, 2$ and 3 are 2, 40 and 368,768 respectively.

For the variety generated by $\mathbf{Ł}_4^{\rightarrow}$ we see that

$$|\overline{\mathbf{T}}_{\leq \ell}(n)| = 2^{(3^\ell-2^\ell)4^{n-\ell}}3^{(2^\ell-1)4^{n-\ell}-2^{n-\ell}}4^{4^{n-\ell}-2^{n-\ell}}.$$

From this an expression for the size of the free algebra can be obtained. The free algebra on two free generators in this variety has 4320 elements. A similar analysis for the variety generated by $\mathbf{Ł}_5^{\rightarrow}$ can be carried out. In this case the free algebra on two free generators has cardinality 940,032.

3.3 Hoop residuation algebras of bounded potency

We present a recursive method for describing the free algebras in the variety $\mathsf{HoRA}(r)$ for $r \geq 2$. We let V denote $\mathsf{HoRA}(r)$ and we choose and fix $1 \leq \ell \leq k \leq r$. We wish to determine the number $V(k, n, \ell, m)$ of those $v \in \mathcal{V}_\ell$ for which $v(X)$ generates a subdirectly irreducible algebra $\boldsymbol{A}$ with $\mu(\boldsymbol{A}) = \mathbf{Ł}_k^{\rightarrow}$ and having $v(X_{\leq \ell}) \subseteq \{e^1, \dots, e^{k-1}\}$ with $e^m = \max(v(X_{\leq \ell}))$. Let v be such a valuation. For this v we form the $(k+1)$-tuple $(v_{\leq \ell}, E_1, \dots, E_{k-1}, \theta_v)$ where

$v_{\leq \ell} \in \{e^m, \dots, e^{k-1}\}^{X_{\leq \ell}}$,

$e^m \in v_{\leq \ell}(X_{\leq \ell})$,

$E_i = v^{-1}(\{e^i\}) \cap X_{>\ell}$ for $1 \leq i \leq k-1$, and

$\theta_v \in \mathrm{Con}(\mathbf{F}_{\mathsf{V}}(X_{>\ell} - (E_1 \cup \cdots \cup E_{k-1})))$ is the kernel of the homomorphic extension of v restricted to $X_{>\ell} - (E_1 \cup \cdots \cup E_{k-1})$.

Using Lemma 1.8 we see that $\mathbf{F}_{\mathsf{V}}(X_{>\ell} - (E_1 \cup \cdots \cup E_{k-1}))/\theta_v$ is isomorphic to $\sigma(\boldsymbol{A})$. As argued previously for Hilbert algebras, the map $v \mapsto (v_{\leq\ell}, E_1, \ldots, E_{k-1}, \theta_v)$ is 1-1 on $\mathcal{V}_\ell$.

Let S denote all $(k+1)$-tuples of the form $(s, Y_1, \ldots, Y_{k-1}, \theta)$ for which

$$s \in \{e^m, \ldots, e^{k-1}\}^{X_{\leq\ell}} - \{e^{m+1}, \ldots, e^{k-1}\}^{X_{\leq\ell}},$$

$$Y_i \subseteq X_{>\ell} \text{ for } 1 \leq i \leq k-1,$$

$$Y_i \cap Y_j = \emptyset \text{ for } i \neq j, \text{ and}$$

$$\theta_v \in \mathrm{Con}(\mathbf{F}_{\mathsf{V}}(X_{>\ell} - (Y_1 \cup \cdots \cup Y_{k-1}))).$$

We know from Corollary 1.6 that if $\boldsymbol{B} \in \mathsf{V}$, then $\boldsymbol{B} \oplus \mathbf{Ł}_k^{\rightarrow} \in \mathsf{V}$. Hence, every $(k+1)$-tuple $(s, Y_1, \ldots, Y_{k-1}, \theta)$ in S is of the form $(v_{\leq\ell}, E_1, \ldots, E_{k-1}, \theta_v)$ for some valuation

$$v : X \to (\mathbf{F}_{\mathsf{V}}(X_{>\ell} - (Y_1 \cup \cdots \cup Y_{k-1}))/\theta) \oplus \mathbf{Ł}_k^{\rightarrow}.$$

Using Lemma 1.8 (ii) we see that v will be in $\mathcal{V}_\ell$ provided the subuniverse generated by $v(X_{\leq\ell} \cup Y_1 \cup \cdots \cup Y_{k-1})$ contains all of $\{e^1, \ldots, e^{k-1}\}$. This condition is equivalent to $v(X_{\leq\ell} \cup Y_1 \cup \cdots \cup Y_{k-1})$ not being contained in any proper subuniverse of $\mathbf{Ł}_k^{\rightarrow}$. Let $M_1, \ldots, M_u$ denote all the maximal proper subuniverses of $\mathbf{Ł}_k^{\rightarrow}$. For $1 \leq i \leq u$ let $W_i \subseteq S$ consist of all $(s, Y_1, \ldots, Y_{k-1}, \theta)$ for which $s \in (M_i)^{X_{\leq\ell}}$ and if Y_j is nonvoid, then $e^j \in M_i$ for $1 \leq j \leq k-1$. Then $V(k, n, \ell, m)$ is given by the inclusion-exclusion formula (9), that is, $V(k, n, \ell, m) = N(S; W_1, \ldots, W_u)$. From these values we can determine $\overline{\mathbf{T}}_{\leq\ell}(n)$ and $|\mathbf{F}_{\mathsf{V}}(n)|$.

We present the details of this computation for $\mathsf{HoRA}(3)$. In the following formulas for $V(k, n, \ell, m)$ we let q_i denote $|E_i|$ for $i = 1, 2$ while p denotes $n - \ell - q_1 - q_2$. The maximal proper subuniverses of $\mathbf{Ł}_3^{\rightarrow}$ are $\{1, e^1\}$ and $\{1, e^2\}$. Thus, for $k = 3$ and $m = 1$ there are $2^\ell - 1$ choices for $v_{\leq\ell}$ in S. The only maximal subuniverse of $\mathbf{Ł}_3^{\rightarrow}$ that contains the range of such a $v_{\leq\ell}$ is $\{1, e^1\}$ and there is only one choice for $v_{\leq\ell}$ here. So for this k and m the inclusion-exclusion formula gives

$$V(3,n,\ell,1) = (2^\ell - 1)\left(\sum_{\substack{0\leq p,q_1,q_2\\ p+q_1+q_2=n-\ell}} \binom{n-\ell}{p\ q_1\ q_2} |\mathrm{Con}(\mathbf{F}_{\mathsf{V}}(p))|\right)$$
$$- \sum_{q_1=0}^{n-\ell} \binom{n-\ell}{q_1} |\mathrm{Con}(\mathbf{F}_{\mathsf{V}}(n-\ell-q_1))|.$$

For the other values of k and m we have

$$V(3,n,\ell,2) = \left(\sum_{\substack{0\leq p,q_1,q_2\\ p+q_1+q_2=n-\ell}} \binom{n-\ell}{p\ q_1\ q_2} |\mathrm{Con}(\mathbf{F}_{\mathsf{V}}(p))|\right)$$
$$- \sum_{q_2=0}^{n-\ell} \binom{n-\ell}{q_2} |\mathrm{Con}(\mathbf{F}_{\mathsf{V}}(n-\ell-q_2))|$$

and

$$V(2,n,\ell,1) = \sum_{q_1=0}^{n-\ell} \binom{n-\ell}{q_1} |\mathrm{Con}(\mathbf{F}_{\mathsf{V}}(n-\ell-q_1))|.$$

We use these formulas to compute the cardinality of the free algebra on 3 free generators in $\mathsf{HoRA}(3)$. The free algebras on 0 and 1 free generators have 1 and 2 elements respectively, as do their congruence lattices. In the variety $\mathsf{HoRA}(3)$ there are precisely three subdirectly irreducible algebras that are at most 2-generated: $\mathbf{Ł}_3^{\rightarrow}$, $\mathbf{Ł}_2^{\rightarrow}$ and the Hilbert algebra $\mathbf{C}_3$. By considering valuations into these algebras we see that $\mathbf{F}_{\mathsf{HoRA}(3)}(2)$ has cardinality 88; its congruence lattice has 72 elements. We evaluate $V(3,3,\ell,1)$, $V(3,3,\ell,2)$ and $V(2,3,\ell,1)$ for $\ell = 1,2,3$ using the formulas above to determine $\overline{\mathbf{T}}_{\leq\ell}(3)$ for $\ell = 1,2$ and 3. We obtain

$$\overline{\mathbf{T}}_{\leq 1}(3) = (\mathbf{Ł}_2^{\rightarrow})^{84} \times (\mathbf{Ł}_3^{\rightarrow})^{7},$$
$$\overline{\mathbf{T}}_{\leq 2}(3) = (\mathbf{Ł}_2^{\rightarrow})^{12} \times (\mathbf{Ł}_3^{\rightarrow})^{1},$$
$$\overline{\mathbf{T}}_{\leq 3}(3) = (\mathbf{Ł}_2^{\rightarrow})^{7}.$$

An application of Theorem 2.1 gives $|\mathbf{F}_{\mathsf{HoRA}(3)}(3)| = 3 * 2^{84}3^7 - 3 * 2^{12}3^1 + 2^7$.

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