

MATHEMATICAL ACCESS WORRIES AND ACCOUNTING FOR KNOWLEDGE OF LOGICAL COHERENCE

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ABSTRACT. A range of current truth-value realist philosophies of mathematics allow one to reduce the Benacerraf Problem to a problem concerning mathematicians' ability to recognize which conceptions of pure mathematical structures are coherent – in a sense which can be cashed out in terms of logical possibility. In this paper, I will clarify what it takes to solve this 'residual' access problem and then present a framework for solving it.

1. INTRODUCTION

Human beings seem to have significant mathematical knowledge. But, famously, our possession of this knowledge can seem deeply mysterious. Specifically, what could explain the match between human psychology and objective mathematical facts? Certain features of mathematics, like the apparent abstractness and causal inertness of mathematical objects, can make it seem like even modest human accuracy about mathematics could only be got by some massive lucky coincidence. Call this the access problem for realism about mathematical knowledge(broadly understood).

In this paper, I will propose an answer to the above mathematical access problem, in the following sense¹. I'll try to dispel the common impression that human possession of significant mathematical knowledge would require

¹My proposal has some affinities to a brief suggestion in REDACTED, but is much expanded and addresses issues like how knowledge of the logical coherence of conceptions of mathematical structures not statable in first order logic can be explained.

some mysterious coincidence *over and above* whatever is involved in our possession of widely accepted general purpose faculties like: first order logical deduction, observation and abduction/inference to the best explanation.²

Many popular contemporary philosophies of mathematics allow mathematical access worries to be reduced to access worries about knowledge of logical coherence. These views (views what I will call the Structuralist Consensus³) agree that mathematicians can reliably form true beliefs by making, essentially, any logically coherent pure mathematical posits they like⁴. They can then gain further mathematical knowledge by making logically valid deductions from these premises⁵.

Thus, it would suffice to dispel mathematical access worries if we could dispel analogous worries about knowledge of logical coherence — accounting for mathematicians’ ability to recognize logically coherent posits (without positing some mysterious extra coincidence)⁶. Crucially, the general purpose

²Admittedly, one might desire a more ambitious answer to mathematical access worries. However, I take it that merely answering intuitive mathematical access worries in sense above would already be a philosophically significant (c.f. companions in innocence defenses of moral realism) and, to many philosophers, counter-intuitive result.

³Different views in the structuralist consensus support the relevant claim about mathematicians’ freedom in different ways. For example, Modal Structuralists hold that mathematical claims really express modal claims like ‘It’s logically possible for there to be objects satisfying certain and logically necessary that if there were objects satisfying these axioms then...’. Plenitudinous Platonists hold that the mathematical universe is sufficiently large that all or nearly all coherent posits will express truths, as per [Balaguer, 2001] and classic set theoretic foundationalism. And neo-Carnapian Platonists hold that we have some freedom to choose how our language ‘carves up the world up into objects’, including freedom to start talking in terms of new types of objects ([Hirsch, 2011, Thomasson, 2015, Berry, 2015, Berry, 2022]).

⁴Note that pure mathematical posits are assumed to be quantifier restricted to the structure being posited. Thus, there is no danger of individually coherent posits being jointly incoherent or constraining the behavior of non-mathematical objects.

⁵Such deductions might be made via deploying standard first order logic (knowledge of which we are assuming). But in the case of non-first order logical axioms, they may also involve some more powerful inference rules for recognizing logically necessary consequences of these axioms, as discussed below. The story I’ll propose purports to account for both kinds of knowledge.

⁶Does knowledge of logical coherence require prior knowledge of abstracta (like set models or sentences)? It might if we tried to reductively analyze logical possibility using these notions. However I will instead follow Field[Field, 1984] (and to some extent Putnam in

logical abilities I'm taking for granted in this paper (ability to do first order logical deduction) don't suffice to explain this logical coherence knowledge on their own. For doing first order logical deduction can deliver knowledge that I haven't succeeded in deriving contradiction from some axioms *yet*. But this is a far cry from the knowledge we need to explain knowledge of logical coherence (i.e., $\diamond\phi$ knowledge where \diamond is the logical possibility operator, and $\diamond\phi$ ensures that no contradiction can be derived)⁷

In this paper, I'll suggest a toy model for how creatures like us (in all ways that generate intuitive access worries) could have gotten good methods of reasoning about logical coherence sufficient to explain the ability to recognize coherent pure mathematical posits and thence the kind of mathematical knowledge we seem to have.

In §2 I'll clarify how I'm thinking about access worries, and why I take them to be most naturally and directly answered by providing a kind of toy model. In §3, I'll lay out and defend a basic proposal which attempts to explain our ability to recognize coherent conceptions putative mathematical

[Putnam, 1967]) in taking the \diamond of logical possibility as primitive modal notion (that's a logical operator).

Admittedly there's now a fruitful tradition of identifying logical possibility with having a set theoretic model for various mathematical purposes (and validity with not having a counter-model). However, there are independent reasons[Gómez-Torrente, 2000, Hanson, 2006, Boolos, 1985, Etchemendy, 1990, Field, 2008] for thinking we have prior grasp on a notion of logical possibility which isn't defined in terms of set models. In a nutshell, the issue is this. It's core to our conception of this notion logical possibility that what's actual is logically possible. But if we think about logical possibility in terms of set theoretic models, then the actual world is strictly larger than the domain any set theoretic model (e.g., because it contains all the sets), so it's prima facie unclear why every sentence that truly describes the actual world must have a set theoretic model.

Also, one might feel (with Boolos) that, "one really should not lose the sense that it is somewhat peculiar that if G is a logical truth, then the statement that G is a logical truth does not count as a logical truth, but only as a set-theoretical truth" [Boolos, 1985].

⁷By the completeness theorem[Gödel, 1930] first order logical axioms are coherent (intuitively satisfiable) if and only if they are syntactically consistent. So if we could perform infinitely many calculations in a finite amount of time, we could arguably recognize coherent first order logical axioms, by brute force checking syntactic coherence (going through all possible proofs). But obviously real-life mathematicians' ability to choose coherent axioms can't be explained by anything like this.

structures stated in the language of first-order logic.⁸ In §4 and §5 I'll answer some objections to this basic proposal. In §6 I'll note some reasons why many (but not all) philosophers of mathematics think our conception of mathematical structures cannot be stated in the language of first order logic. Then I'll show how the basic story told in §3 can be generalized to account for knowledge of the logical coherence of axioms in a suitably more powerful language – given plausible (but not uncontroversial) assumptions about the reliability of abduction when applied to logical possibility facts.

Overall, I aim to provide a basic story about how creatures relevantly like us could have gotten logical coherence knowledge sufficient to account for our apparent mathematical knowledge, which can be accepted by most readers – without taking a stand on vexed questions about exactly what kind of logical coherence knowledge is needed⁹.

2. BACKGROUND

Let's begin with some background about the problem to be solved: what does it take for a philosophy of mathematics to face an access problem, and what would solving such an access problem require?

⁸Examples of such conceptions are \mathbb{Q} (Robinson's arithmetic) and finite fragments of PA.

⁹Philosophers sufficiently non-skeptical and realist to face an access worry about mathematical knowledge can still disagree about how much logical knowledge is needed to account for the mathematical knowledge we actually have (because they disagree on the richness of our conceptions of mathematical structures and/or how much we know about these structures).

I won't argue for a position on this debate here. Instead, I'll try to propose a recipe for explaining knowledge of logical coherence which is broadly useful, as follows. It can be used by extreme truth value realists about mathematics (like myself) to answer access worries – provided that we happen to also be somewhat optimistic about the powers of abduction and inference to the best explanation (as sociologically tends to be the case). However, it can also be used by philosophers like Field who are more cynical about how far abduction and inference to the best explanation can take us, but also (as sociologically tends to be the case) take a more modest view of the richness of our mathematical concepts and/or the extent of our mathematical knowledge.

Following [Field, 1980, Berry, 2020], I take access worries about mathematical knowledge to involve a kind of (*ceteris paribus*) coincidence avoidance reasoning. A realist account of some domain (like mathematics) faces an access worry to the extent that accepting it seems to commit us to the existence of certain kinds of unattractive brute coincidences (regularities that cry out for explanation but have no relevant explanation), which could be avoided by adopting some relevant, less realist, alternative theory¹⁰.

Thus, access worries arise from a kind of ‘how possibly’ question — and can naturally be answered by providing a kind of toy model. They involve a ‘how possibly’ question, in the following sense. It seems unimaginable how mathematicians could possibly have acquired the accuracy they seem to have, without benefiting from some kind of striking coincidence that cries out for explanation. Yet adequate explanation seems inconceivable.

Accordingly, a natural way to answer access worries would be to dissolve this feeling of inexplicability by providing a toy model [Cassam, 2007, Nozick, 1981], i.e., a sample explanation of how mathematical knowledge could have arisen. This sample explanation doesn’t have to fit all known facts about how human mathematical knowledge actually arose. However, it does have to keep the key features of our actual situation that make adequate explanation seem inconceivable (e.g., our lack of causal contact with mathematical objects or logically possible worlds). It also cannot be buck-passing, in the sense that it explains one mysterious extra correlation the mathematical realist is committed to by appealing to another. For example one can’t solve access worries merely by explaining mathematicians’

¹⁰So, on one hand access worries can be seen as arising from a kind of informal reasoning about coincidence avoidance, which is widely accepted and has proven its fruitfulness in other areas. But, on the other hand, they only provide a *ceteris paribus* reason for favoring one theory (sometimes positing extra coincidences is, on net, the right thing to do).

acceptance of largely true theorems merely by appeal to their acceptance of largely true axioms.

In the rest of this paper, I will try to provide such a story (centering on explaining our knowledge of logical coherence facts needed to recognize acceptable mathematical posits).

3. THE BASIC PROPOSAL

To introduce the basic idea behind this proposal (and set up the toy model I'll develop), imagine creatures who speak a language much like our own¹¹ and already have the widely accepted non-mathematical faculties we are taking for granted: first order logical deduction, broadly sensory perception of non-mathematical objects and abduction/IBE.

I take it that it wouldn't be massively surprising (in the sense relevant to access worries) if such creatures acquired a kind of minimal notion of logical possibility. Specifically, I take it that we can (without question beggingly attempting to explain one coincidence by appeal to another mysterious coincidence) further imagine the protagonists of our toy model as having acquired a kind of minimal concept of logical possibility¹² which they take

¹¹For simplicity's sake, I'll suppose that they speak a fully formal language like first-order English so we can meaningfully talk about things like substitution instances.

¹²Arguably it would be surprising if they didn't develop such a notion. These creatures face a practical problem. Their language lets them form many different statements whose falsehood is guaranteed by their logical structure alone. So many plans which they can verbally represent would, ideally, be discarded as unrealizable purely on the grounds that they require something logically impossible. And there is practical benefit to recognizing this and focusing resources on plans and hypotheses, which are, at least, logically possible. Even though creatures with first-order logic will already be disposed to reject plans when they derive a contradiction from them, there are further benefits to be gained from having a positive theory (e.g., being able to infer that one scenario is logically possible only if another one is, allows one to skip searching for a contradiction in the former scenario after seeing the later scenario realized).

to satisfy the two schemas below (and the expectation that \diamond facts should follow elegant general laws)¹³.

- $\phi \rightarrow \diamond\phi$
- When $S_1 \dots S_m$ and $S'_1 \dots S'_m$ are all distinct relations with each S'_i having the same arity as S_i and no S'_i occurs in ϕ , $\diamond\phi \leftrightarrow \diamond\phi[S_1/S'_1 \dots S_m/S'_m]$

Informally speaking, the first schema embodies the idea that we are talking about a notion of possibility, saying that everything actual is logically possible. The second embodies the idea that we are talking about possibility with respect to logical form alone, so that systematically replacing one relation with another (without collision) doesn't change logical possibility facts¹⁴.

Now we ask, how could creatures like this gain sufficient knowledge of logical possibility to reconstruct the mathematical knowledge we seem to have? Note that no amount of mere first-order logical deduction (i.e., no first-order logical proof from empty premises) will ever let one derive even simple logical possibility facts we seem to know, like the fact that it's logically possible for there to be two distinct things $\diamond(\exists x)(\exists y)(\neg x = y)$.

I take it we can imagine creatures of the kind envisaged above getting *some* general good methods of reasoning about logical possibility via the following combination of mechanisms.

3.1. From ϕ to $\diamond\phi$. First, knowledge of non-mathematical objects (got via the faculties of sensory observation, FOL deduction and IBE we are assuming) can give one some initial data about logical possibility via the above

¹³That is, I take it we can assume this at the beginning of our story, without risk of question beggingly explaining away one apparent 'extra' coincidence by appeal to another such coincidence (which is left unexplained). See [Berry, 2020] for more details.

¹⁴Note that some natural variants on this initial conception of logical possibility would intuitively count as getting something else right (e.g., setting out to learn facts about physical, chemical, metaphysical, or psychological possibility) rather than getting logical possibility wrong.

principle that's what's actual is logically possible. For example, suppose you know that some claim ϕ is true about how the relations of friendship, nephew-hood and having been in military service together apply in just this way to the royal family of Sweden. Then you can infer that this scenario is logically possible: $\diamond\phi$. You can also infer the logical possibility of a corresponding hypothesis about which of your friends are gossiping with each other (involving relations P , Q , and R).

3.2. Abduction from regularities in what's actual to $\square\phi$. Second, patterns in these data points can suggest further facts via abduction and inference to the best explanation. These generalizations can take the form of general laws/methods of reasoning about logical possibility which let us derive additional $\diamond\phi$ claims in cases where we don't know that ϕ is actual. For example, we might learn laws/inference methods that let us derive claims of the form 'if $\diamond\phi$ then $\diamond\psi$ '.

Noticing other patterns in the behavior of non-mathematical objects (that certain states of affairs are never observed to be actual) and applying IBE can yield other kinds of logical possibility knowledge. Sometimes the best explanation for the fact that certain things never happen is that it would be logically impossible for them to happen. This provides a potential source of knowledge of $\neg\diamond\phi$ facts.

Suppose, for example, that someone thought it was logically possible for there to be 9 sundaes which differed from one another in which of three properties they had, e.g., for 9 people to choose different combinations of sundae toppings from a sundae bar containing three toppings. This person would have to explain the striking law-like regularity that, regardless of the type of items and properties in question, we never wind up observing more than 8 such items. They might postulate new physical regularities to explain

why apparently random processes of flipping three coins never generated the forbidden 9th possible outcome. However, this explanation (or some analogous one) would have to apply at every physical scale we can observe, from relationships between the tiniest particles to relationships between planets and stars (as well as to less concrete objects like poems and countries). A much more elegant explanation is that the unrealized outcome isn't logically coherent. Recognizing that the forbidden 9th outcome is forbidden in all possible domains is much more economical and a priori attractive than hypothesizing separate laws prohibiting it in each specific situation.

In this way, we can think of facts about what's actual as simultaneously a useful source of data about what's logically possible, physically possible, chemically possible, etc.

Now objector might wonder how it is possible for a single collection of data to do all these jobs? When we notice a seeming regularity, we face an in-principle choice about whether to explain it in terms of logical necessity vs. physical law, metaphysical necessity or mere *ceteris paribus* regularity. How could we ever be justified in saying that this regularity holds as a matter of (say) logical rather than merely physical necessity?

I'd reply that this is not a problem because patterns in our experience can still rationally motivate (in the sense relevant to IBE) attributing a noted regularity to logical necessity rather than physical law. For, as noted in the case above, if the right explanation for some regularity is that it holds as a matter of logical necessity, we should expect to see that all substitution instances of it (i.e., all sentences with the same logical structure) are true, whereas we'd expect the opposite if this regularity holds as a matter of merely metaphysical necessity or physical necessity.

3.3. Reflection and Generalization. Third, one could make further gains in the power and accuracy of our methods of reasoning about logical possibility by the familiar processes of deriving new consequences from whatever laws of logical possibility we currently accept, reflecting on our beliefs and recognizing when they conflict or cohere with one another.

So, to summarize, the core idea is this. We get some initial knowledge of logical possibility facts via the principle that what is actual is logically possible (just as we get some initial data about what states of affairs are chemically possible by observing what actually happens). Abduction and inference to the best explanation can then help us correct hypotheses about allowable inferences regarding logical possibility. Facts about logical possibility provide a uniform subject matter which we get initial data about from our non-mathematical faculties (via the actual to possible inference) and to which abduction and inference to the best explanation can be fruitfully applied, with the result that our knowledge of logical possibility is no more mysterious than our knowledge of physical or chemical possibility.

However, various worries can be raised about whether abduction and inference to the best explanation can give us enough logical knowledge to account for our seeming mathematical knowledge via some view in the structuralist consensus (i.e., whether it can explain our ability to recognize $\Diamond\phi$ facts, where ϕ is our conception of some mathematical structure like the natural number). I will discuss and answer a number of such worries below, proposing two important generalizations of the above story as needed to do this.

Additionally, there's a major technical problem about how to account for knowledge of the logical coherence of conceptions of pure mathematical structures that can't be stated (by finite or recursively enumerable axioms) in the language of first order logic. I will address this in §6.

4. A PRIORITY AND INNATENESS

One family of worries about the basic proposal in §3 concerns whether it can adequately allow for the possibility of (in some sense) innate or a priori mathematical knowledge.

First, one might argue that the answer to mathematical access worries proposed above can't account for our having any very *innate/hardwired* propensity to good mathematical (or logical possibility) reasoning. The basic story about logical and mathematical accuracy sketched above (involving conscious reasoning like applying abduction or inference to the best explanation) *prima facie* can't account for *innate* inclination to form true beliefs about logical possibility or mathematics¹⁵. Thus, one might object that my proposal can only solve the general mathematical access problem on the (unjustified) assumption that we won't turn out to have much of an innate push towards good logical or mathematical reasoning.¹⁶

In response to this concern, I'd like to suggest that a version of the basic story (about abduction and IBE leading us from initial datapoints to correct laws) can be realized at an evolutionary level, if our dispositions to accept good mathematical reasoning turned out to be sufficiently innate (e.g., if we were innately disposed to do something like good mathematical reasoning in a language of thought). Though evolution may not care about elegance

¹⁵See Spelke's experiments with infants in [Spelke and Kinzler, 2009] for an example of the kind of data which might suggest that certain good methods of reasoning about (something like) logical possibility or mathematics are relatively innate.

¹⁶A third way of realizing the explanatory strategy proposed above involves something like meme selection on mathematical textbooks and/or practices. We can imagine the relevant process of using IBE to generate and correct beliefs occurring either within an individual's lifetime or over 100s years of intellectual history/via meme selection on social norms for reasoning about logical possibility. Perhaps each creature learns how to reason about logical possibility and mathematics from the society around them, and individuals only very rarely suggest revisions to these methods – but theories which elegantly predict and explain regularities in what's actual are much more likely to spread once suggested. Considering the development of probability theory textbooks (with older theories leading to countries dutch booking themselves and thus consistently losing money)[Hacking, 1995] may provide a real life model for such a process.

and theoretical economy in quite the sense that we do, mental resources are expensive and those methods of reasoning that could be encoded in the simplest manner and handle the most general situations would be favored¹⁷

Second, one might worry that accepting the kind of story about knowledge of logical possibility (and thence mathematics) developed above commits one to a controversial empiricism about mathematical knowledge. As our mathematical knowledge is generally assumed to be a priori, this presents a prima facie problem (though some, like Quine and Mill [Mill, 2002, Quine, 1961], are happy to bite the bullet).

However, I don't think any such commitment to empiricism is incurred. For note that experience playing an important causal role in explaining how we got accurate methods of reasoning about logical possibility and thence mathematical reasoning (whether via conscious reasoning or evolutionary selection) doesn't prevent the knowledge gained by using these faculties from qualifying as a priori. Sometimes (in a kind of 'epistemic Stockholm syndrome') conscious experience and inference to the best explanation leads us to accept some method of reasoning, and then we decide that we should have reasoned that way all along (so facts discovered using these methods are a priori knowable).

The online supplement to a New York Times article [Tierney, 2008] on the Monty Haul problem provides a cute demonstration of this psychological fact. It used a computer simulation using a random number generator to change readers' opinions about how one ought to analyze probabilities in that case (and hence whether it would be beneficial to change doors). So contingent experiences with a computer simulation seemingly changed

¹⁷One might question whether something analogous to abduction and inference to the best explanation can apply at the level of evolutionary selection. But I suspect that most readers are already committed to a fair amount of optimism on this front; we don't tend to think there is any access problem about the fact that human infants have seemingly correct inclinations to fear heights or avoid poisonous foods.

readers' minds about which methods of reasoning about probability are appropriate for use *a priori* (whether or not you've done experiments with a simulated Monty Haul problem).

5. DOES IBE GO FAR ENOUGH?

Now let me turn to a series of (progressively more radical) worries about how far abduction and inference to the best explanation can take us.

5.1. Scientific Induction Unreliable in Mathematics? Most radically, someone might reject the story above because they hold that abduction and inference to the best explanation are completely unreliable with regard to mathematics (and hence plausibly also logical possibility)¹⁸. If this were correct, it would certainly raise a problem for the answer to access worries about logical possibility sketched above. Someone pressing this worry will doubt that the mechanisms above could even yield laws that correctly predict what's logically possible for finite collections of objects¹⁹.

In response to this, I would note that there's strong independent reason to reject insinuations that generalization from cases is completely unreliable in mathematics. Mathematicians frequently use hunches developed from past experience, judgments of general plausibility or theoretical attractiveness and the results of computational searches²⁰ to guide their research. For example, the widespread expectation that Fermat's last theorem was true *before* any proof was found was (partially) motivated by consistent failure to find a counterexample. If we want to make sense of the apparent success of this aspect of mathematical practice, we can't suppose that abduction and

¹⁸See [Frege, 1980] pg. 16 for a version of this objection.

¹⁹Here I have in mind claims of the form $\diamond\Phi$, where Φ logically entails the Fregean translation into purely logical vocabulary of 'there are at most n things'.

²⁰Of course, mathematicians don't do this naively. If they already know that any counterexample must be large, they won't change their judgments because no small counterexamples were found.

inference to the best explanation are *completely unreliable* when applied to the mathematical realm.

Also note that the idea that something like the abduction/inference to the best explanation found in the sciences can also reliably be applied to mathematics, is a controversial but modestly popular position in the literature on the search for new axioms in set theory. Gödel famously suggested that we can reliably add new axioms by choosing principles which unify and explain the mathematical beliefs which we already have²¹. If this is true, then it seems plausible that the creatures in our just-so story could reliably extend an initial collection of good methods of reasoning about logical possibility in the same way.

5.2. A Gap Between the Finite and the Infinite? Next, there's a worry that the story suggested above cannot explain the *degree* of mathematical knowledge we take ourselves to have (specifically) because there's a big gap between the laws of logical possibility which apply to the finite and the infinite.

One might allow that the above mechanisms can explain human accuracy about logical possibility facts involving finite collections, but argue as follows. All the 'inputs' to the abductive story above (i.e., knowledge of what's actually true via sensory perception and inference to the best explanation) involve finite structures. So abduction from mere knowledge that certain finite structures are logically coherent couldn't plausibly lead us to (correct) laws about what scenarios involving infinitely many objects are logically possible. Many elegant generalizations that hold for finite collections fail for infinite structures, e.g., consider Hilbert's hotel. Thus, (unless we can allow

²¹In [Gödel, 1947] Gödel writes, "There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole field, and yielding such powerful methods for solving problems... that, no matter whether or not they are intrinsically necessary, they would have to be accepted at least in the same sense as any well-established physical theory." See [Koellner, 2010] for more on this.

for some inputs concerning infinite structures) the above proposal can't account for our knowledge of logical possibility of axioms describing even the smallest mathematical structures involving infinitely many objects, like the natural number structure.²²

I propose that we can answer this concern by adding an additional quasi-Quinean twist to the basic story outlined in §3. Recall that Quine famously suggested that we can learn mathematical objects *exist* via the fact that our best scientific theories quantify over them [Quine, 1961]. This claim is extremely controversial.

However, note that it would suffice for the story about knowledge of logical possibility above if scientific use of claims about infinity gave us knowledge of logical possibility claims directly (rather than knowledge of truths about non-mathematical objects which can be used to infer logical possibility claims). And many people, even philosophers who reject this Quinean idea, tend to allow that *either* something like long use of some axioms without deriving a contradiction or scientifically explanatory use of these axioms is a (*ceteris paribus*) reliable guide to the *logical coherence* of these axioms.

Thus, we can plausibly appeal to the scientific usefulness of theories requiring the existence of infinitely many objects as a reliable source of input regarding the logical possibility of certain scenarios involving infinitely many different objects. For example, we might say the scientific-explanatory usefulness (or long harmless use of) of reasoning with a space of 'possible words'

²²Arguably faculties continuous with the sensory observation and inference to the best explanation we are taking for granted can deliver knowledge of the truth (and hence logical possibility) of a few claims implying the existence of infinitely many non-mathematical objects. Consider the following first order logical claim, 'For every spatial region in the path of Zeno's arrow there is a shorter one'. It's not clear whether or not infinitely divisible regions of space turn out to be part of fundamental physics. However, one might argue such spatial regions are part of the manifest image and can be known to literally exist much as holes, shadows, heatwaves, marriages and contracts can be known to literally exist.

(abstract objects witnessing possible ways concatenating some letters from the alphabet A-Z) that's taken to satisfy certain closure conditions, gives us some reason to believe in the logical coherence — if not the truth — of axioms describing this space. And the same goes for explicitly mathematical structures used in scientific theories, like the natural numbers and the reals. If initial data points involving the logical coherence of *few central* infinite mathematical (or abstract) structures can be secured in this way, the worry above will be answered. Applying inference to the best explanation/abduction concerning logical possibility (as in the basic just-so story told above) can then account for our knowledge of the logical coherence of other (less scientifically useful or frequently studied) conceptions of infinite mathematical structures²³.

Note that the above proposal suggests that long and/or scientifically explanatory use can be a *good* guide to logical coherence (so that we can add an extra source of initial data points regarding $\diamond\phi$ facts) as follows, not that it's an infallible guide. I allow that it's sometimes useful and rational to temporarily adopt (in some sense of the word) logically incoherent scientific theories²⁴ But I would argue that such cases are rare²⁵. Also what's useful (and what people actually do) isn't accepting these theories without caveat²⁶, so we might accommodate this point by saying that long scientific explanatory use without caveat is a good guide to truth.

²³Note that I don't presume (or need to presume) that concrete reality forces any single unique such structure on us. As Penelope Maddy emphasizes in [Maddy, 2011], science and philosophy of science may under-determine what logico-mathematical structure to ascribe to a physical system.

²⁴For example, consider the scientific and explanatory use of the Dirac delta function in physics.

²⁵See [Vickers, 2013] for extended discussion of a number of cases studies supporting this point.

²⁶We might also refine the idea of scientific usefulness above to something like following claim: useful scientific theories tend to be consistent or have their inconsistency flagged and paired with a, formal or informal, way of using these theories while avoiding first order logical explosion associated with them (e.g., like the idea that quantum mechanics is to be used to describe the very small and relativity to describe the very large).

One might fear that adding the above new quasi-Quinean element to the core proposal made above re-introduces known problems for Quinean empiricism. However, the fact that our current story only endorses an indirect relationship between scientific and mathematical beliefs (mediated by acceptance of general laws of logical possibility) and doesn't require scientific usefulness for mathematical existence lets us continue to avoid many important problems for Quine's account. For example, we have no trouble accommodating recreational mathematics [Hahn and Schilpp, 1986] or mathematicians' tendency to learn about scientifically useful mathematical objects *before* any scientific usefulness is discovered [Friedman, 2001].

5.3. Extent of Knowledge. Finally, one might worry that the above story can't account for our apparent knowledge of facts about logical coherence (and necessity) involving large infinite collections, such as are plausibly needed to account for knowledge of set theory. REDACTED discusses a version of this worry (posed as a response to a simpler predecessor of the view defended here) in some detail, "A critic might advance the following analogy: saying [knowledge of logical possibility facts involving finite and countably many objects yields general principles that can give knowledge of logical possibility facts involving larger structures as needed to reconstruct set theory] is like saying that inference to the best explanation plus observations of birds in New Mexico explains our possession of true beliefs about birds in Canada as well. Presumably, in the ornithological case, we need to go gather more data in order to get many true beliefs about birds in Canada. But, in the mathematical case, we can't gather more data. Thus, our apparent possession of substantial true beliefs about what is logically coherent for larger infinite collections remains mysterious."

There I respond to this worry by arguing that abduction and inference to the best explanation plausibly do give us *some* knowledge of birds in distant locals, we just expect this knowledge to be relatively sparse and less confident than knowledge of birds near by. So the above argument only suggests that our beliefs about logical possibility should get fewer and less confident as we consider larger and larger infinite structures. But this is just what happens with our beliefs about logical coherence and large collections: as one moves from finite collections to countably infinite collections (like the natural numbers) to uncountable collections (like the sets) our beliefs do get more sparse and less confident. For example, the continuum hypothesis²⁷ (CH) is a fairly simple statement involving sets of (relatively) small infinite size, yet it is known that both the truth and the falsity of CH are compatible with ZFC.

Now however, I would like to make an additional defensive point. As discussed in §2, the amount of logical knowledge we need to account for will depend on the style of realism and optimism about mathematical knowledge one embraces. Personally, I'm inclined to think that logical possibility facts are sufficiently uniform for the process of reflective equilibrium outlined above to account (via the structuralist consensus) for our knowledge of the coherence all the theorems of mainstream ZFC set theory) and (as we'll see below) our ability to recognize the logical coherence of non first order logical axioms categorically describing the natural numbers. But philosophers more skeptical of the extent of our mathematical knowledge (and who take more modest logical knowledge to be needed to account for it) are even better situated. For, less ambitious and/or extensive knowledge of logical possibility

²⁷The continuum hypothesis states that there are no sets whose cardinality is intermediate between the cardinality of the real numbers and that of the natural numbers.

facts needs to be explained, in order to account for the degree and kind of mathematical knowledge *they* take us to have.

6. MORE AMBITIOUS MATHEMATICAL AXIOMS

Now let's turn to the second major problem I hope to solve in this paper: that of accounting for mathematicians' ability to recognize good *non-first order logical* conceptions of mathematical structures.

As noted above, many popular philosophies of mathematics agree that we could solve mathematical access problems if we could account for mathematicians' ability to recognize the logical coherence of axioms stating our conceptions of mathematical structures (and then truth preservingly derive suitable kinds of logical consequences from these axioms). However (for reasons to be explained below) many philosophers think our conceptions of some paradigmatic mathematical structures can't be fully stated in the language of first order logic. This creates a problem for the story above. For, at first glance, the basic explanatory mechanism proposed in §3 can only explain knowledge of the logical possibility or impossibility of claims in the language of first order logic (e.g., $\diamond\phi$ claims where ϕ is a sentence in the language of first order logic). It appears that this mechanism can't account for our ability to recognize logical coherence of 'rich' (non-first order logical) conceptions of mathematical structures.

Philosophers who accept only a very weak form of mathematical realism — on which only sentences derivable or refutable from some first-order conception of a mathematical structure have a determinate truth-value — may be satisfied with a story about the ability to recognize the logical coherence of these conceptions of mathematical structures.

However, as noted above, many philosophers and mathematicians think we must have some conception of certain pure mathematical structures that goes beyond anything stateable in the language of first order logic.

For one thing, many take us to have a conception of the natural numbers which uniquely pins down their structure, and thereby ensures the truth or falsehood of all sentences in the language of number theory. And it's a theorem that no such conception of mathematical structures can be formulated using the language of first-order logic alone (by a single sentence or even an infinite collection of recursively enumerable axioms).

Indeed, the problem gets worse. For Gödel's theorem actually shows that each FOL theory of the kind mentioned above fails to determine an answer to some $\text{Con}(T)$ sentence. These are sentences that only use mathematical vocabulary, but intuitively say that no number codes a proof of '0=1' from premises in a certain algorithmically described first-order logical theory T . Thus we're disposed to accept (and treat as ad a conceptually central truth, constraining acceptable interpretations of 'number') a biconditional of the following form:

- $\text{Con}(T)$ iff 0=1 isn't provable from the axioms of T .

Thus, if you accept that there are determinate facts about provability, the attractiveness of the biconditional above creates pressure to accept that there are also determinate truth values for all $\text{Con}(T)$ sentences. Yet it's a theorem that no consistent finite (or recursively axiomatizable) collection of first-order logical sentences (extending Peano Arithmetic) candidate for our conception of the natural numbers can decide all such Con sentences.

For these and other reasons, many philosophers accept that our conception of paradigmatic mathematical structures like the natural numbers cannot be expressed in the language of first order logic alone. It follows that to explain the degree of mathematical knowledge these philosophers take us

to have (via the structuralist consensus proposal assumed in this paper), we must explain mathematicians' ability to recognize the coherence of mathematical axioms stated using some more powerful logical vocabulary than that of first order logic (e.g. axioms using second order quantification).

Now if we could (somehow) presume some *initial* knowledge of some basic statements Φ involving second-order quantification, then maybe we could use the story about actual-to-possible inference and generalization above to explain knowledge of claims about the logical coherence of axioms expressing our conceptions of pure mathematical structures (e.g., knowledge that $\Diamond PA_2$, where PA_2 denotes the second-order Peano Axioms).

But there's a problem. For now we want to explain our knowledge of the coherence of PA_2 . But we can't assume that the protagonists of our toy model will have any knowledge of non-mathematical facts involving second order logical quantification, which could then be used as initial data points for our process of abduction and inference to the best explanation. While the widely accepted general purpose faculties (of sensory observation, inference to the best explanation and FOL) we're taking for granted can clearly give us knowledge that certain *first-order* states of affairs are actual (and hence logically possible, by the inference from actual to possible mentioned above), it is less clear how we could get knowledge of any claims involving second order quantification.

Insofar as we can't see or touch or taste etc. the objects of second order quantification (as opposed to the concrete objects which can figure in first-order reasoning), these objects can seem to raise all the same access worries as mathematical objects, and our knowledge of these objects cannot be presumed. For example, many would say knowledge of facts like $(\exists X)(\forall x)(X(x) \text{ iff } x \text{ is a brown egg})$ requires the existence of a second-order object. And knowledge of (abstract causally inert) second-order objects can

seem mysterious in exactly the same way knowledge of sets would be. It's not like we can just "see" second-order objects. We don't see sets of eggs floating over an egg carton²⁸. Thus (at least some philosophers who accept the structuralist consensus will feel²⁹), we can't just assume input knowledge of second-order logic without risking begging the question³⁰. But if we don't presume knowledge of some second-order facts as a starting point, then the abductive story above cannot be used to account for knowledge that various second-order states of affairs are logically possible.

I propose to address this problem by appealing to a notion of **conditional** logical possibility, which has been independently motivated in the literature on Potentialist set theory as a response to the Burali-Forti paradox³¹.

The notion of conditional logical possibility naturally extends the notion of logical possibility simpliciter, and we will see that it has the following pair of useful features. On the one hand, it offers all the expressive power of second-order logic. But on the other hand, facts about conditional logical possibility are sufficiently similar to (and, one might, say continuous with) facts about logical possibility simpliciter that we can generalize the story about knowledge of logical possibility above to account for some knowledge

²⁸Or at least, the suggestion that we do in [Maddy, 1992] has proved deeply controversial.

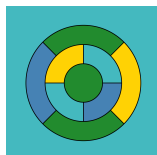
²⁹I have in mind nominalists in the structuralist consensus, like Hellman in [Hellman, 1996] and thereafter, who reject second order logic as objectionably ontologically committal.

³⁰That is, a story which did this would intuitively fail to solve access worries leaving a mystery about how this knowledge of second order objects could have arisen.

³¹Doing this lets one simplify existing formulations and eliminate unnecessary and potentially problematic de re modal claims (claims about what's possible for a specific object) in favor of claims about what's possible given the structure of how some relations apply and thereby avoids modal shyness worries of a kind noted by Linnebo in [Linnebo, 2018].

of the logical possibility of axioms stated using the conditional logical possibility operator³²³³

6.1. Conditional Logical Possibility. To quickly motivate and introduce the notion of conditional logical possibility, suppose we have a map like this:



I might say, ‘It’s logically impossible, given the facts about how ‘is adjacent to’ and ‘is a country’ apply on the map above, that each country is either yellow, green or blue and no two adjacent countries are the same color.’ Because if we consider all the possibilities consistent with these relations applying as they actually do, each involves two adjacent map regions having the same color.

As noted above, the notion of conditional possibility generalizes the notion of logical possibility simpliciter. When evaluating claims about traditional logical possibility operator \diamond , we ignore all limits on the size of the universe. We consider only the most general combinatorial constraints on how any relations could apply to any objects (c.f. Frege [Frege, 1980]). And we ignore subject matter specific metaphysical constraints so, e.g.,

³²The story about the acquisition of correct laws and good general methods of reasoning about logical possibility proposed below will also account for the ability to reliably derive various further (logically necessary) consequences from such axioms, and thereby gain further mathematical knowledge in something like the way mathematicians seem to gain such knowledge.

³³Extant work like [Berry, 2018, Berry, 2022] argues that reformulating Hellman’s potentialist set theory in terms of a notion of ‘conditional logical possibility’ operator \diamond allows for some conceptual simplification, and perhaps has certain other philosophical advantages. That work also shows how using this notion lets us eliminate appeals to second order logic (or plural quantification) in our characterization of other mathematical structures. An alternative approach to the problem at hand (following Hellman [Hellman, 1996]), would be to employ plural quantification. Perhaps something similar to my proposal could be articulated using plural quantification. However, in this paper I will work with the conditional possibility operator because doing so is (at least) expositively helpful.

$\diamond\exists x(Raven(x) \wedge Vegetable(x))$ comes out true. When evaluating conditional logical possibility $\diamond_{R_1 \dots R_n}$ we do almost exactly the same, but we hold fixed (structurally speaking) how certain specific relations $R_1 \dots R_n$ ³⁴ See the appendix A for further clarification of what holding structural facts fixed means by comparison with claims about set theoretic models.

Using the conditional logical possibility operator, we can formalize the non-three colorability claim above as follows:

$\neg\diamond_{adjacent, country}$ Each country is either yellow, green or blue and no two adjacent countries are the same color.

We can also categorically describe the intended structure of the natural numbers using the conditional logical possibility operator. Recall that we can categorically describe the natural numbers via the second-order Peano Axioms (a combination of all the first order Peano Axioms except for instances of the induction schema with the following second order statement of induction.).

³⁴One can further explain and motivate the notion of conditional logical possibility by relating it to Stuart Shapiro’s notion of systems and structures [Shapiro, 1997]. Shapiro’s ‘systems’ involve some objects and a choice of relations $R_1 \dots R_n$. For example, “An extended family is a system of people with blood and marital relationships [and] a chess configuration is a system of pieces under spatial and ‘possible move’ relationships” [Shapiro, 1997]. And a structure is ‘the abstract form’ of a system, which we get by “highlighting the interrelationships among the objects and ignoring any features of them that do not affect how they relate to other objects in the system.” [Shapiro, 1997]. So, for example, the natural-number structure is equally well exemplified by “the strings on a finite alphabet in lexical order, an infinite sequence of strokes... and so on.” [Shapiro, 1997] Note that adding or subtracting objects to the world outside of a given system will make no difference to which structure that system instantiates.

Although I propose the logical possibility operator as a conceptual primitive, we can (roughly) explain it in Shapiro’s terms as follows. It is logical possible, given the $R_1 \dots R_n$ facts, that ϕ (i.e., $\diamond_{R_1 \dots R_n}$) iff some logically possible scenario makes ϕ true while holding fixed what structure the system formed by the objects related by $R_1 \dots R_n$ (considered under the relations $R_1 \dots R_n$) instantiates.

$$\mathbf{Induct}_2(\forall X) [(X(0) \wedge (\forall n) (X(n) \rightarrow X(n+1))) \rightarrow (\forall n)(X(n))]$$

We can reformulate this claim using conditional logical possibility as follows³⁵.

- **Induct_◇**: ‘ $\Box_{\mathbb{N},S}$ If 0 is happy and the successor of every happy number is happy then every number is happy.’

In other words: it is logically necessary, given how \mathbb{N} and S apply, then if 0 is happy and the successor of every happy number is happy then every number is happy.’

Thus, we can write a sentence PA_\diamond , (purely in terms of first order logic plus the conditional logical possibility operator) which categorically describes the natural numbers.³⁶ And [Berry, 2018] argues that we can similarly rewrite other second-order conceptions of pure mathematical structures.

Thus, plausibly (given the structuralist consensus), it suffices to answer mathematical access worries to account for mathematicians’ ability to recognize that categorical descriptions of mathematical structures like PA_\diamond are logically coherent. That is, we need to account for knowledge of facts like $\diamond PA_\diamond$.

Above I argued that we can attractively explain knowledge of $\diamond\phi$ facts in cases where ϕ is first order, by appealing to initial observations about which other first-order sentences are actually true, together with knowledge that

³⁵I write ‘0’ below for readability, but recall that one can contextually define away all uses of 0 in a familiar Russellian fashion in terms of only relational vocabulary

³⁶Just use the fact above to replace the second-order induction axiom in second order Peano Arithmetic with a version stated in terms of conditional logical possibility. Recall that the Second Order Peano Axioms are the familiar first order Peano Axioms for number theory, with the induction schema replaced by a single induction axiom using second order quantification.

what's actual is logically possible and inference to the best explanation. I now argue we can use the same basic mechanisms to account for knowledge of $\diamond\phi$ facts like $\diamond\text{PA}_\diamond$, where the state of affairs ϕ being recognized as logically possible or impossible is described using the conditional logical possibility operator $\diamond\dots$.

First, note that if we could establish initial knowledge of a suitably large class of conditional logical possibility claims as either true or false, we could leverage the basic story about knowledge via inference from actual to possible, IBE, abduction etc. proposed above to explain knowledge of general good methods of reasoning about such conditional logical possibility claims.

We can no longer rely on observation of concrete scenarios to gain this initial knowledge. However, I propose that we can explain our knowledge of the truth-values of a large class of subscripted \diamond claims by applying a version of the story about generalization above!

First, note that what's actual is automatically conditionally logically possible fixing the facts about how any list of relations $R_1 \dots R_n$ apply. So we have some initial knowledge of $\diamond_{R_1 \dots R_n} \phi$ facts and $\diamond_{R_1 \dots R_n}$ facts.

Second, inference to the best explanation can seemingly give us knowledge of $\neg\diamond_{R_1 \dots R_n}$ facts. For example, the best explanation for the fact that no one ever three colors some map might be that the map isn't three colorable (i.e., it would be logically impossible to do so, given the facts about which map regions are adjacent to one another³⁷). Thus, we can get some initial knowledge of $\neg\diamond_{R_1 \dots R_n}$ facts (and thence, by inference from actual to logically possible) the corresponding $\diamond\neg\diamond_{R_1 \dots R_n}$ facts.

In this way, I propose, we can (in principle) gain knowledge of a bunch of $\diamond\phi$ statements where ϕ uses the conditional logical possibility operator.

³⁷Note that a prediction which follows from this explanation (and *not* from alternative theories like that three coloring is merely physically impossible) is that we shouldn't expect the map to be three textured or three scented either.

So, finally, we can notice patterns in these conditional logical possibility facts. We can use abduction and inference to the best explanation to get general laws of what's logically possible or necessary involving conditional logical possibility claims, which imply the logical possibility of states of affairs (described in terms of conditional possibility) that aren't actual³⁸. In this way, creatures like us (in all ways that generate mathematical access worries) could have gotten correct general methods of reasoning about logical possibility with sufficient power to yield knowledge of some logical possibility claims like $\Diamond PA_{\Diamond}$.

7. CONCLUSION

Many philosophies of mathematics allow us to reduce access worries about mathematics to access worries concerning our knowledge of logical possibility, by saying that any logically coherent axioms pure mathematicians chose would express truths (for one reason or another). In this paper, I have tried to solve the 'residual access problem' of how to account for relevant knowledge of logical possibility.

To do this, I've developed and defended a toy model for how creatures like us (in all ways that drive access worries) could have gotten armchair reasoning methods able to deliver this knowledge of logical possibility. On the basic picture being proposed, sensory and scientific knowledge leads (via the fact that what's actual is possible) to initial knowledge of logical possibility. Applying abduction and inference to the best explanation from this data can then yield good general laws of reasoning about *logical possibility* which allows us to recognize logically coherent mathematical axioms. In this way, logical possibility need be no more deeply mysterious than knowledge of physical or chemical possibility.

³⁸See [Berry, 2022] for an example of some candidate general laws of logical possibility and a proof that they have sufficient power to reconstruct set theory.

However, I've suggested that we can certain worries about this basic idea by making two small additions to the basic picture above: the quasi-Quinian move in §5.2 and appeal to a notion of conditional logical possibility which is independently motivated by the literature on potentialist set theory.

APPENDIX A. SET THEORETIC MIMICRY

I will now describe how to use the familiar formal background of set theory to *mimic* intended truth conditions for statements in a language containing the logical possibility operator \diamond alongside usual first order logical vocabulary (where distinct relation symbols R_1 and R_2 always express distinct relations) as follows.

A formula ψ is true relative to a model \mathcal{M} ($\mathcal{M} \models \psi$) and an assignment ρ which takes the free variables in ψ to elements in the domain of \mathcal{M} ³⁹ just if:

- $\psi = R_n^k(x_1 \dots x_k)$ and $\mathcal{M} \models R_n^k(\rho(x_1), \dots, \rho(x_k))$.
- $\psi = x = y$ and $\rho(x) = \rho(y)$.
- $\psi = \neg\phi$ and ϕ is not true relative to \mathcal{M}, ρ .
- $\psi = \phi \wedge \psi$ and both ϕ and ψ are true relative to \mathcal{M}, ρ .
- $\psi = \phi \vee \psi$ and either ϕ or ψ are true relative to \mathcal{M}, ρ .
- $\psi = \exists x\phi(x)$ and there is an assignment ρ' which extends ρ by assigning a value to an additional variable v not in ϕ and $\phi[x/v]$ is true relative to \mathcal{M}, ρ' ⁴⁰.
- $\psi = \diamond_{R_1 \dots R_n} \phi$ and there is another model \mathcal{M}' which assigns the same tuples to the extensions of $R_1 \dots R_n$ as \mathcal{M} and $\mathcal{M}' \models \phi$.⁴¹

³⁹Here 'an assignment' means a partial function ρ from the collection of variables in the language of logical possibility to objects in \mathcal{M} , such that the domain of ρ is finite and includes (at least) all free variables in ψ

⁴⁰As usual (?) $\phi[x/v]$ substitutes v for x everywhere where x occurs free in ϕ

⁴¹As usual, I am taking \square to abbreviate $\neg\diamond\neg$

Note that this means that \perp is not true relative to any model \mathcal{M} and assignment ρ .

If we ignore the possibility of sentences which demand something coherent but fail to have set models because their truth would require the existence of too many objects, we could then characterize logical possibility as follows:

Set Theoretic Approximation: A sentence in the language of logical possibility is true (on some interpretation of the quantifier and atomic relation symbols of the language of logical possibility) iff it is true relative to a set theoretic model whose domain and extensions for atomic relations captures what objects there are and how these atomic relations actually apply (according to this interpretation) and the empty assignment function ρ .

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