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The Gödel Paradox and Wittgenstein's Reasons

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An interpretation of Wittgenstein's much criticized remarks on Gödel's First Incompleteness Theorem is provided in the light of paraconsistent arithmetic: in taking Gödel's proof as a paradoxical derivation, Wittgenstein was drawing the consequences of his deliberate rejection of the standard distinction between theory and metatheory. The reasoning behind the proof of the truth of the Gödel sentence is then performed within the formal system itself, which turns out to be inconsistent. It is shown that the features of paraconsistent arithmetics match with some intuitions underlying Wittgenstein's philosophy of mathematics, such as its strict finitism and the insistence on the decidability of any mathematical question.

1. The Implausible Wittgenstein

Wittgenstein's comments on Gödel's First Incompleteness Theorem (henceforth: G1) in the *Remarks on the Foundations of Mathematics* were dismissed by early commentators such as Kreisel, Dummett, and Bernays as an unfortunate episode in the career of a great philosopher. Critics were particularly struck by the fact that Wittgenstein seems to take the undecidable formula of the formal system to which G1 applies as a *paradoxical* sentence, not too different from the usual Liar—and Gödel's proof, therefore, as the deduction of an inconsistency:

> 11. Let us suppose I prove the unprovability (in Russell's system) of P [where P is the Gödel sentence of the system]; then by this proof I have proved P. Now if this proof were one in Russell's system—I should in this case have proved at once that it belonged and did not belong to Russell's system.—That is what comes of making up such sentences. But there is a contradiction here!—Well, then there is a contradiction here. Does it do any harm here? [Wittgenstein, 1956, p. 51e]

Zermelo, Perelman, and probably Russell made similar mistakes in the interpretation of G1.[1] It is usually maintained that the error rests on a

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[1] See [Perelman, 1936], [Dawson, 1984] on Russell and on Zermelo's letter to Gödel on this issue.

confusion between a theory and its metatheory, or between syntax and semantics, which makes it impossible to distinguish the truth predicate, inexpressible (by Tarski's theorem) within the theory to which G1 applies, from the provability predicate, which, on the contrary, is (weakly) expressible. Alan Ross Anderson, for instance, explicitly charges Wittgenstein with such a confusion [1958, p. 486]. Until a few years ago, the discussion on Wittgenstein's remarks seemed to be summed up by the verdict of Gödel himself, who privately stated that Wittgenstein 'advance[d] a completely trivial and uninteresting misinterpretation'[2] of G1.

However, in recent years some interpreters have attempted to extract interesting philosophical theses from the comments of the *Bemerkungen*. Floyd and Putnam [2000] have claimed that Wittgenstein's intuitions anticipate some metamathematical results on non-standard models of arithmetic. And the debate is rapidly evolving, with authoritative commentators taking a stance on Wittgenstein's real thoughts.[3]

In this paper I will provide an interpretation of Wittgenstein's position on Gödel's First Theorem, based on one single, simple argument naturally connecting Wittgenstein's stance to a *paraconsistent* (philosophy of) mathematics. The paraconsistent argument will also vindicate another idea which has harmed Wittgenstein's reputation: the view that we should not dramatize the possibility that a calculus turns out to be inconsistent—such a Wittgensteinian tolerance towards inconsistency being precisely of the kind that paraconsistent logics, as is well known, are designed to formalize.

2. 'There Is No Metamathematics'

Let us begin with a review of Wittgenstein's claims on the subject of G1. At the core of the 'single argument' is the idea that, in maintaining an interpretation of Gödel's proof that made of it a paradoxical derivation, Wittgenstein was just drawing the consequences of his bold denial of the standard distinction between theory and metatheory (therefore, between formalized arithmetic and metamathematics). After Gödel and Tarski, logicians know that they should distinguish carefully between theory and metatheory and between syntax and semantics. Unlike Zermelo and Perelman, however, Wittgenstein *knowingly* denied several aspects of such distinctions. He never had second thoughts, for instance, on his rejection of Hilbert's metamathematics:

> If [the word 'calculus'] is used in a calculus, that doesn't make the calculus into a metacalculus; in such a case the word is just a chessman like all the others.

[2] Quoted in [Dawson, 1984], p. 89.
[3] See [Hintikka, 1999], [Rodych, 1999; 2003], [Floyd, 2001].

> Logic isn't metamathematics either; that is, work within the logical calculus can't bring to light essential truths about mathematics. *Cf.* here the 'decision problem' and similar topics in modern mathematical logic. [. . .]
>
> (Hilbert sets up rules of a particular calculus as rules of metamathematics.) [Wittgenstein, 1974, pp. 296–297]

That is to say: Hilbert's metamathematics is nothing but mathematics. It is not a metacalculus, because there are no metacalculi: it is *just one more* calculus. It would take too much space here to discuss Wittgenstein's reasons for rejecting Hilbert's conception. (Roughly, they are closely connected to a rejection of the Platonic idea that mathematical sentences describe a domain of independently existing objects.) But Wittgenstein's claim that Gödel's proof is actually the derivation of a paradox follows from such a rejection. Let us see why.

Take the usual Peano Arithmetic (PA), obtained by simply adding to ordinary first-order predicate logic with identity the first-order version of the Peano axioms (including the induction principle in schematic form). Let γ be the Gödel sentence of the system. G1 has it that if PA is consistent, then $\nvdash_{PA} \gamma$, and if PA is ω-consistent, then $\nvdash_{PA} \neg\gamma$. As such, this is a syntactic result. But the proof of G1 usually goes hand in hand with the following short 'semantic' story, which Wittgenstein would have labelled as the *prose* (as opposed to the real mathematical proof): since γ 'claims' (*via* arithmetization) to be not provable, and we have just proved that it is not provable, then γ just is what it claims to be; hence, it is *true*.[4]

However, this intuitive reasoning cannot be mirrored within the theory. The truth predicate for PA, were it expressible within PA, would under the usual conditions lead to the Liar paradox; whereas the provability predicate is expressible. Gödel himself pointed out the analogies between his undecidable sentence and such paradoxes as Richard's or the Liar. But it seems clear that, whereas the Liar, 'This sentence is false', produces an antinomy, with the Gödel sentence, metamathematically read as 'This sentence is not provable', no contradiction is forthcoming. Thus G1 appears to establish a fundamental divergence between provability and truth—or so the usual story goes.

That the truth of the Gödel sentence is established by means of a detour through the metatheory was clearly stated by Gödel in the opening paragraph of his [1931], where he declared that 'the proposition that is undecidable *in the system PM* still was decided by metamathematical considerations' (p. 599). It was probably this claim that initially perplexed Wittgenstein, for in the *Philosophical Remarks* he had observed:

[4] To be sure, under the hypothesis of the consistency of PA.

> 3. There cannot be a hierarchy of proofs! [...] There can't in any fundamental sense be such a thing as meta-mathematics. Everything must be of one type (or, what comes to the same thing, not of a type). [...]
>
> Thus, it isn't enough to say that p is provable, what we must say is: provable according to a particular system.
>
> Further, the proposition doesn't assert that p is provable in the system S, but in *its own* system, the system of p.
>
> That p belongs to the system S cannot be asserted, but must show itself.
>
> You can't say p belongs to the system S; you can't ask which system p belongs to; you can't search for the system of p. Understanding p means understanding its system. *If p appears to go over from one system into another, then p has, in reality, changed its sense.* [Wittgenstein, 1964, p. 180] (emphasis added)

Within this framework, it cannot happen that the very same sentence (say, γ), turns out to be expressible, but undecidable, in a formal system (say, PA) and demonstrably true (under the aforementioned consistency hypothesis) in a different theory (the metatheory). If, as Wittgenstein maintained as one of the basic tenets of his philosophy of mathematics, proofs establish the very meaning of the proved sentences, then it is not possible for the same sentence (that is, for a sentence with the same meaning) to be undecidable in a formal system, but decided in a different system (the metasystem). And if the meaning of a mathematical sentence is determined by the rules that govern its use in the calculus, and in particular by its own demonstration, then a demonstrative incompleteness *in the theory* would become *eo ipso* an incompleteness *of meaning*:

> The edifice of rules must be complete, if we are to work with a concept at all—*we cannot make any discoveries in syntax.*—For, only the group of rules defines the sense of our signs, and any alteration (*e.g.* supplementation) of the rules means an alteration of the sense. [...]
>
> Mathematics cannot be incomplete; any more than a sense can be incomplete. [*Ibid*, pp. 182, 188]

Also the separation between provability and truth, allegedly established as a consequence of G1, has to go:

> 7. 'But may there not be true propositions which are written in this symbolism, but are not provable in Russell's system?'—'True propositions', hence propositions which are true in *another* system, *i.e.* can

> rightly be asserted in another game. [. . .] [A] proposition which cannot be proved in Russell's system is 'true' or 'false' in a different sense from a proposition of *Principia Mathematica.* [Wittgenstein, 1956, p. 50e]

In the end, '"True in Russell's system" means, as was said, proved in Russell's system; and "false in Russell's system" means the opposite has been proved in Russell's system' [*Ibid*, p. 51e]. That at the core of Wittgenstein's rejection of the Platonic 'prose' associated with Gödel's proof of G1 is his identification of truth with provability has been argued in detail by Victor Rodych and S.G. Shanker in various essays.[5]

So much for Wittgenstein's views. Now for the 'single argument' that embeds them in a paraconsistent setting.

3. The Single Argument

My strategy exploits an idea proposed by Richard Routley and Graham Priest in various influential essays,[6] which allows us to interpret Gödel's proof precisely as a *paradoxical* derivation. The core thought is to see what happens when one applies G1 to the theory that captures *our intuitive, or naïve, notion of proof*. By 'naïve notion of proof' Routley and Priest mean the one underlying ordinary mathematical activity: 'proof, as understood by mathematicians (not logicians), is that process of deductive argumentation by which we establish certain mathematical claims to be true' [Priest, 1987, p. 40]. Since Hilbert, formal logicians treat proofs as purely syntactic objects. However, proving something, for a working mathematician, amounts to establishing that some sentence is *true*. When we want to settle the question whether some mathematical sentence is true or false, we try to deduce it, or its negation, from other mathematical sentences which are already known to be true. The process cannot go backwards *ad infinitum*. We should therefore eventually reach mathematical sentences which are known to be true without having to be proved—*e.g.*, because they are 'self-evident'. However, this is not important (nor is it important to establish which are the primal truths; concerning arithmetic, they may be, for instance, principles such as those of Peano, that is, claims according to which every number has a successor, *etc*.).

Naïve proofs are phrased in informal, standard mathematical English. But 'it is accepted by mathematicians that informal mathematics could be formalized if there were ever a point to doing so, and the belief seems quite legitimate' [*Ibid*, p. 41]. The primal truths may be written down in the

[5] See [Rodych, 1999; 2003], [Shanker, 1988].

[6] [Routley, 1979], [Priest, 1979; 1984; 1987]. To be sure, Priest may disagree with the picture of Wittgenstein's attitude towards Gödel proposed here (see his [2004]).

formal language and taken, say, as axioms; and proofs may be expressed as formal arguments. After the translation, the naïve theory would certainly be sufficiently strong, *i.e.*, capable of representing all (primitive) recursive functions.

Is the naïve notion of proof effectively decidable (*i.e.* recursive, given Church's Thesis)? This is much less straightforward, and it is likely that the crux of the argument lies here.[7] To assume that the proof relation of naïve arithmetic is decidable challenges the standard perspective, taken as established precisely by Gödel's results. I will come back to this point, though, after exposing the paraconsistent argument, which goes as follows.

Let T be the formalisation of our naïve intuitive mathematical theory. If T is sufficiently strong and its notion of proof is decidable, then G1 applies. So *if* T is consistent, then there is a sentence ϕ which is not a theorem of T, but which can be established as true via a naïve proof, and therefore *is* a theorem of T. (Of course, anything that is naïvely-intuitively provable is provable within the naïve-intuitive theory.) So 'assuming its consistency, it would, therefore, seem to be both complete and incomplete in the relevant sense' [Priest, 1984, p. 165]. Now we have no way to avoid a paradox: *either* we accept *this* one, *i.e.*, $\vdash_T \phi$ and $\nvdash_T \phi$;[8] *or* we have to admit that our naïve mathematical theory, with its naïve notion of proof, is inconsistent.

Routley and Priest claim that, under a few plausible assumptions, the Gödel sentence ϕ for T can be proved within T itself, together with its negation; so one of the inconsistencies hosted by T is to the effect that $\vdash_T \phi$ and $\vdash_T \neg\phi$. The philosophical point is that 'This sentence is not provable' now has its 'provable' understood as meaning 'demonstrably true', and, precisely as Wittgenstein conjectured, Gödel's proof becomes the derivation of a real paradox:[9]

> In informal terms, the paradox is this. Consider the sentence 'This sentence is not provably true'. Suppose the sentence is false. Then it is provably true, and hence true. By *reductio* it is true. Moreover, we have just proved this. Hence it is provably true. And since it is

[7] As an anonymous referee has appropriately pointed out to me.

[8] This is quite close to Wittgenstein's remark, quoted at the beginning: 'let us suppose I prove the unprovability (in Russell's system) of P; then by this proof I have proved P. Now if this proof were one in Russell's system—I should in this case have proved at once that it belonged and did not belong to Russell's system.'

[9] Therefore, Anderson's comment on Wittgenstein, according to which 'the conclusion to draw would not be that P at once 'belonged and did not belong' to Russell's system, but rather that Russell's system was inconsistent' [1958, p. 488] is really of little importance: either horn of the dilemma makes us end up in a contradiction; and both contradictions (*i.e.*, a system proving both its Gödel sentence and its negation, and a system both proving and not proving something) are expected in a thoroughly paraconsistent framework, as is shown in [Priest, 1987], pp. 239–243.

> true, it is not provably true. Contradiction. This paradox is not the only one forthcoming in the theory. For, as the theory can prove its own soundness, it must be capable of giving its own semantics. In particular, [every instance of] the T-scheme for the language of the theory is provable in the theory. Hence [. . .] the semantic paradoxes will all be provable in the theory. Gödel's 'paradox' is just a special case of this. [Priest, 1987, pp. 46–47][10]

The 'semantic prose' on G1 attacked by Wittgenstein has it that the truth of the Gödel sentence is established in the metatheory. However, T, formalizing as it does our naïve notion of proof, should absorb the metatheory within the theory. After all, as Wittgenstein might have added, mathematicians use ordinary English, and ordinary English may well be (and, according to many philosophers of language, actually is) semantically closed. As Routley [1979, p. 326] stressed, 'everyday arithmetic as presented within a natural language like English appears, unlike say first-order Peano arithmetic, appropriately closed'. And 'is provable in arithmetic' and 'is arithmetically true' are 'English, and in a good sense arithmetical, predicates.' Mirroring this fact, T is semantically closed in the Tarskian sense, and inconsistent. The reasoning behind the proof of the truth of the Gödel sentence is now performed *within* the formal system itself. There is no metasystem in which one establishes that (if the object system is consistent, then) the Gödel sentence is true: there are no metasystems. One cannot 'get out' of the system and solve, in its metasystem, problems that were meaningfully expressible but undecidable within it.

Back now to the key assumption that the naïve notion of proof is effectively decidable (thus, given Church's Thesis, recursive). The first thing to notice is that this may well have been Wittgenstein's assumption too. Wittgenstein believed that the naïve (*i.e.*, the working mathematician's) notion of proof had to be decidable, for lack of decidability meant to him simply lack of mathematical *meaning*: Wittgenstein believed that everything had to be decidable in mathematics, so the argument coheres with Wittgenstein's position on this point, too. But Routley and Priest also have positive arguments for the view. That the naïve notion of proof is decidable means that we can in principle effectively recognize a naïve proof when we see one. Now, Priest [1987, p. 41] stresses, 'it is part of the very notion of proof that a proof should be effectively recognizable as such'—for the point of a naïve proof is that it is a way of settling the issue whether a given mathematical claim is true or not. As Alonzo Church claims,

> Consider the situation which arises if the notion of proof is non-effective. There is then no certain means by which, when a sequence of formulas has been put forward as a proof, the auditor may

[10] See also [Priest, 1984, p. 172].

> determine whether it is in fact a proof. Therefore he may fairly demand a proof, in any given case, that the sequence of formulas put forward is a proof; and until the supplementary proof is provided, he may refuse to be convinced that the alleged theorem is proved. This supplementary proof ought to be regarded, it seems, as part of the whole proof of the theorem . . . [1956, p. 5]

Besides, by acknowledging that the naïve proof relation is decidable we can explain how we *learn* arithmetic—that is, via an effective procedure:

> We appear to obtain our grasp of arithmetic by learning a set of basic and effective procedures for counting, adding, *etc.*; in other words, by knowledge encoded in a decidable set of axioms. If this is right, then arithmetic truth would seem to be just what is determined by these procedures. It must therefore be axiomatic. If it is not, the situation is very puzzling. The only real alternative seems to be Platonism, together with the possession of some kind of sixth sense, 'mathematical intuition'. [Priest, 1994, p. 343]

This point, too, meets some Wittgensteinian concerns on teaching and learning mathematical calculi as a public—social—phenomenon. Perhaps the most amazing fact about mathematics as a discipline is the unanimity (generally speaking) of mathematicians on what counts as a proof. As Wittgenstein remarked, the whole 'language game' of mathematical proofs would be rendered impossible by lack of consensus among mathematicians. If the notion of arithmetic proof were not effectively recognizable, then the process whereby mathematics is learnt, and the general agreement of working mathematicians on what counts as a mathematical proof, would turn out to be a mystery (of course, this is but a particular case of a famous, more general argument to the effect that *language* can only be learnt recursively, and so the grammar of a learnable language must be generated by a decidable set of rules).[11] On the contrary, as Routley [1979, p. 327] claims, if the truths of mathematics are effective or effectively enumerable we can understand 'how one generation of mathematicians learns what counts as true from the previous generation, namely they learn certain basic mathematical truths and how to prove others by making deductions'.

Of course, one can speak against the decidability of the naïve notion of proof on the basis of Gödel's results themselves. But one may argue that, in the context, this would beg the question against paraconsistentists—and against Wittgenstein, too. *Both* Wittgenstein and the paraconsistentists, on one side, *and* the followers of the standard view on the other, agree on the following thesis: the decidability of the notion of proof and its

[11] On which see, famously, [Davidson, 1984, Ch. 1].

consistency are incompatible. But to infer from this that the naïve notion of proof is not decidable invokes the indispensability of consistency, which is exactly what Wittgenstein and the paraconsistent argument call into question. Contrary to what Bernays [1959, p. 523] claimed, the discussion of G1 in the *Bemerkungen* does *not* 'suffer from the defect that Gödel's quite explicit premise of the consistency of the considered formal system is ignored.' Bernays's charge just begs the question against Wittgenstein, for, as Victor Rodych[12] has forcefully argued, the consistency of the relevant system is precisely what is called into question by Wittgenstein's reasoning: '"Perhaps", Wittgenstein might say, "all calculi that admit such sentence-constructions [as the Gödel sentences] are syntactically inconsistent"' [Rodych, 1999, p. 190].

4. Inconsistent Arithmetic

This last point brings into play the other aspect of Wittgenstein's philosophy of mathematics which the paraconsistent interpretation can recapture, namely, the attitude towards contradictions.

After taking Gödel's proof as a paradox, Wittgenstein asks rhetorically: 'But there is a contradiction here!—Well, then there is a contradiction here. Does it do any harm here?' [Wittgenstein, 1956, p. 51e]. For he believed that a calculus within which one can derive a contradiction does not thereby become useless:

> Can we say: 'Contradiction is harmless if it can be sealed off'? But what prevents us from sealing it off? [...]
>
> Let us imagine having been taught Frege's calculus, contradiction and all. But the contradiction is not presented as a disease. It is, rather, an accepted part of the calculus, and we calculate with it. [...]
>
> For might we not possibly have wanted to produce a contradiction? Have said—with pride in a mathematical discovery: 'Look, this is how we produce a contradiction'? [Wittgenstein, 1956, pp. 104e–106e]

Because of these insights, Wittgenstein has been considered a precursor of paraconsistent logics. He anticipated the intuition that, by rejecting *ex falso quodlibet* (*i.e.*, the classically and intuitionistically valid logical law according to which a contradiction entails everything), we may admit that an inconsistent calculus does not thereby become trivial.[13] Now, if

[12] See [Rodych, 2002, pp. 384–385].
[13] On this point, see [Marconi, 1984].

we adopt a paraconsistent logic, the theory T capturing our naïve-intuitive notion of proof is not just an argumentative trick anymore. It is possible to provide a respectable logical framework for Wittgenstein's idea according to which Gödel's proof is paradoxical, and nevertheless the derivation of such paradoxes does not render the relevant system(s) useless. Inconsistent arithmetics, *i.e.*, non-classical arithmetics based on a paraconsistent logic, are nowadays a reality, both from the proof-theoretic and from the model-theoretic points of view. By not fulfilling the consistency requirement, they avoid G1, as well as Church's undecidability result: they are, that is, demonstrably *complete* and *decidable* (and so axiomatizable).[14] Therefore, such theories fulfil precisely Wittgenstein's request according to which there should not be mathematical problems that can be meaningfully formulated within a system that adequately represents actual mathematical practice, but which the rules of the system cannot decide.

Besides, the perspective of inconsistent arithmetics is typically involved in a form of strict *finitism*. The underlying intuition would be that there is a finite (albeit hardly imaginable and unknown to us) number of things in the world. Although we cannot specify the number, we know that it must be 'a number larger than the number of combinations of fundamental particles in the cosmos, larger than any number that could be sensibly specified in a lifetime'.[15] This also agrees with a persistent tendency in Wittgenstein's philosophy of mathematics, emphasized by Dummett, Kreisel, Kielkopf, and others.[16] As Rodych has remarked, the intermediate Wittgenstein is

[14] I cannot spell out the details due to lack of space, but for a quick review of paraconsistent formal arithmetics one can see [Bremer, 2005, Ch. 13]. More details in [Priest, 1987, Ch. 17].

[15] [Priest, 1994, p. 338]. A paraconsistent finite model of arithmetic with a largest (and inconsistent) number n can be obtained by applying to the standard model of arithmetic $\mathbb{N}$ an appropriate filter that reduces its cardinality (see [Meyer and Mortensen, 1984] for the technical details). The domain of the finite model is $\{m : m \leq n\}$. Roughly, 'up to n' things work like in ordinary arithmetic, but n has any atomic or negated property iff some number $\geq n$ has it, so n is an inconsistent number. Now, let N be the theory of $\mathbb{N}$, that is, the set of arithmetic sentences true in the standard model; and let N_n be the set of sentences true in the paraconsistent finite model with the largest number n. We may take as the underlying logic of N_n some mainstream paraconsistent logic, such as LP (Priest's Logic of Paradox), or FDE (Belnap and Dunn's First Degree Entailment). As Priest [1994; 1987, pp. 234–237] has shown, such a theory as N_n has the following pleasing properties: it is, of course, *inconsistent*, including as theorems both its own Gödel sentence and its negation; but it is provably *non-trivial* (that is, it is provable that, despite admitting inconsistencies, the theory does not prove everything)—and its non-triviality proof can be formalized within it. It fully contains N, that is, it includes all the sentences true in the standard model. It is syntactically *complete* (that is, for every formula α, it includes either α or $\neg\alpha$), and *decidable*, therefore axiomatizable. The inconsistent arithmetic thus avoids G1; it also avoids Gödel's Second Theorem, in the sense that its non-triviality can be established within the theory.

[16] See [Dummett, 1959, pp. 504–505], [Bernays, 1959, p. 519]. See also [Kielkopf, 1970].

dominated by 'his finitism and his [. . .] view of mathematical meaningfulness as algorithmic decidability', according to which only 'finite logical sums and products (containing only decidable arithmetic predicates) *are* meaningful because they are *algorithmically decidable*'. But this tendency remains also in the later phase:

> as in the middle period, the later Wittgenstein seems to maintain that an expression is a meaningful proposition only *within* a given calculus, and *iff* we knowingly have in hand an applicable and effective DP [decision procedure] by means of which we can decide it. [Rodych, 1999, pp. 174–176].

The theoretical features of paraconsistent arithmetics, therefore, match with various Wittgensteinian views in the philosophy of mathematics. The cost of accepting such theories is clear: we have to revise some well-established acquisitions of classical mathematical logic. On the other hand, such an audacious rethinking in a paraconsistent framework may nowadays vindicate some of Wittgenstein's 'outrageous claims', too swiftly dismissed by commentators who took the logic of Russell and Frege as the One True Logic.

References

Anderson, A.R. [1958]: 'Mathematics and the "Language Game"', in [Benacerraf and Putnam, 1964], pp. 481–490.

Benacerraf, P., and H. Putnam, eds [1964]: *Philosophy of Mathematics. Selected Readings*. Englewood Cliffs, N.J.: Prentice Hall.

Bernays, P. [1959]: 'Comments on Ludwig Wittgenstein's *Remarks on the Foundations of Mathematics*', in [Benacerraf and Putnam, 1964], pp. 510–536.

Bremer, M. [2005]: *An Introduction to Paraconsistent Logics*. Frankfurt: Peter Lang.

Church, A. [1956]: *Introduction to Mathematical Logic*. Princeton: Princeton University Press.

Davidson, D. [1984]: *Inquiries into Truth and Interpretation*. Oxford: Oxford University Press.

Dawson, W. [1984]: 'The reception of Gödel's Incompleteness Theorems', *Philosophy of Science Association 2*. Reprinted in [Shanker, 1988], pp. 74–95.

Dummett, M. [1959]: 'Wittgenstein's philosophy of mathematics', in [Benacerraf and Putnam, 1964], pp. 491–509.

Floyd, J. [2001]: 'Prose *versus* proof: Wittgenstein on Gödel, Tarski, and truth', *Philosophia Mathematica* (3) **9**, 280–307.

Floyd, J., and H. Putnam [2000]: 'A note on Wittgenstein's "Notorious Paragraph" about the Gödel Theorem', *Journal of Philosophy* **97**, 624–632.

Gödel, K. [1931]: 'Über formal unentscheidbare Sätze der *Principia Mathematica* und verwandter Systeme I', *Monatshefte für Mathematik und Physik* **38**, 173–198. Translated by van Heijenoort as 'On formally undecidable propositions of *Principia Mathematica* and related systems I', in [van Heijenoort, 1967], pp. 596–616.

VAN HEIJENOORT, J., ed. [1967]: *From Frege to Gödel: A Source Book in Mathematical Logic*. Cambridge, Mass.: Harvard University Press.

HINTIKKA, J. [1999]: 'Ludwig Wittgenstein: Half truths and one-and-a-half truths', in J. Hintikka, *Selected Papers*. Vol. I, Dordrecht: Kluwer.

KIELKOPF, C. [1970]: *Strict Finitism: An Examination of Ludwig Wittgenstein's* Remarks on the Foundations of Mathematics. The Hague: Mouton.

KÖLBEL, M., and B. WEISS, eds, [2004]: *Wittgenstein's Lasting Significance*. London and New York: Routledge.

KREISEL, G. [1958]: 'Review of Wittgenstein's *Remarks on the Foundations of Mathematics*', *British Journal for the Philosophy of Science* **9**, 135–158.

MARCONI, D. [1984]: 'Wittgenstein on contradiction and the philosophy of paraconsistent logics', *History of Philosophy Quarterly* **1**, 333–352.

MEYER, R.K., and C. MORTENSEN [1984]: 'Inconsistent models for relevant arithmetic', *Journal of Symbolic Logic* **49**, 917–929.

PERELMAN, C. [1936]: 'L'antinomie de M. Gödel', *Académie Royale de Belgique. Bulletin de la Classe des Sciences* (5) **22**, 730–733.

PRIEST, G. [1979]: 'The logic of paradox', *Journal of Philosophical Logic* **8**, 219–241.

——— [1984]: 'Logic of paradox revisited', *Journal of Philosophical Logic* **13**, 153–179.

——— [1987]: *In Contradiction: A Study of the Transconsistent*. Dordrecht: Martinus Nijhoff. 2nd expanded ed., Oxford: Oxford University Press, 2006.

——— [1994]: 'Is arithmetic consistent?', *Mind* **103**, 337–349.

——— [1995]: *Beyond the Limits of Thought*. Cambridge: Cambridge University Press.

——— [2004]: 'Wittgenstein's remarks on Gödel's Theorem', in [Kölbel and Weiss, 2004], pp. 206–225.

RODYCH, V. [1999]: 'Wittgenstein's inversion of Gödel's Theorem', *Erkenntnis* **51**, 173–206.

——— [2002]: 'Wittgenstein on Gödel: The newly published remarks', *Erkenntnis* **56**, 379–397.

——— [2003]: 'Misunderstanding Gödel: New arguments about Wittgenstein and new remarks by Wittgenstein', *Dialectica* **57**, 279–313.

ROUTLEY, R. [1979]: 'Dialectical logic, semantics and metamathematics', *Erkenntnis* **14**, 301–331.

SHANKER, S.G., ed. [1988]: *Gödel's Theorem in Focus*. London: Croom Helm.

WITTGENSTEIN, L. [1953]: *Philosophische Untersuchungen/Philosophical Investigations*. Oxford: Basil Blackwell.

——— [1956]: *Bemerkungen über die Grundlagen der Mathematik/Remarks on the Foundations of Mathematics.* Oxford: Basil Blackwell.

——— [1964]: *Philosophische Bemerkungen/Philosophical Remarks*. Oxford: Basil Blackwell.

——— [1974]: *Philosophische Grammatik/Philosophical Grammar*. Oxford: Basil Blackwell.