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Leśniewski's *characteristica universalis*

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Abstract Leśniewski's systems deviate greatly from standard logic in some basic features. The deviant aspects are rather well known, and often cited among the reasons why Leśniewski's work enjoys little recognition. This paper is an attempt to explain why those aspects should be there at all. Leśniewski built his systems inspired by a dream close to Leibniz's *characteristica universalis*: a perfect system of deductive theories encoding our knowledge of the world, based on a perfect language. My main claim is that Leśniewski built his *characteristica universalis* following the conditions of de Jong and Betti's Classical Model of Science (2008) to an astounding degree. While showing this I give an overview of the architecture of Leśniewski's systems and of their fundamental characteristics. I suggest among others that the aesthetic constraints Leśniewski put on axioms and primitive terms have epistemological relevance.

Keywords Leśniewski's systems · Classical Model of Science · Logic · Axiomatics · *characteristica universalis*

1 Leśniewski as a follower of the Classical Model of Science

Leśniewski's work looks strange to the eyes of present-day logicians. The strangeness is conceptual as well as aesthetic—open his *Grundzüge*, and you'll see things of this sort:

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T40. $$\phi\left(\llcorner pqs \lrcorner \ulcorner\phi\left(\llcorner f \lrcorner \ulcorner\phi\left(q\phi\left(\llcorner r \lrcorner \ulcorner\phi\left(f(rr)q\right)\urcorner \llcorner r \lrcorner \ulcorner\phi\left(f(rr)\right.\right.\right.\right.\right.\right.$$

$$\left.\left.\left.\phi\left(\phi(ps)s\right)\right)\urcorner\right)\right)\urcorner \llcorner f \lrcorner \ulcorner\phi\left(q\phi\left(\llcorner r \lrcorner \ulcorner\phi\left(f(rr)q\right)\urcorner \llcorner r \lrcorner \ulcorner\phi\left(f\right.\right.\right.$$

$$\left.\left.\left.(rr)p\right)\urcorner\right)\right)\urcorner\right)\urcorner \llcorner pqs \lrcorner \ulcorner\phi\left(\phi\left(s \llcorner f \lrcorner \ulcorner\phi\left(q\phi\left(\llcorner r \lrcorner \ulcorner\phi\left(f(rr)\right.\right.\right.\right.\right.$$

$$\left.\left.q\right)\urcorner \llcorner r \lrcorner \ulcorner\phi\left(f(rr)\phi\left(\phi(ps)s\right)\right)\urcorner\right)\right)\urcorner\right)\phi\left(\llcorner f \lrcorner \ulcorner\phi\left(q\phi\left(\llcorner r \lrcorner\right.\right.\right.$$

$$\left.\left.\left.\left.\ulcorner\phi\left(f(rr)q\right)\urcorner \llcorner r \lrcorner \ulcorner\phi\left(f(rr)p\right)\urcorner\right)\right)\urcorner s\right)\right)\right)\urcorner\right) \quad \text{[T39]}$$

This is Thesis 40 of System $\mathfrak{S}5$ of Protothetic (Leśniewski 1939, p. 501). In modern notation, it reads $\forall pqs(\forall f(q \leftrightarrow (\forall r(f(rr) \leftrightarrow ((p \leftrightarrow s) \leftrightarrow s)) \leftrightarrow \forall f(q \leftrightarrow (\forall r(f(rr) \leftrightarrow q)) \leftrightarrow \forall r(f(rr) \leftrightarrow p))) \leftrightarrow \forall pqs(s \leftrightarrow (\forall f(q \leftrightarrow (\forall r(f(rr) \leftrightarrow q)) \leftrightarrow \forall r(f(rr) \leftrightarrow ((p \leftrightarrow s) \leftrightarrow s)))) \leftrightarrow (\forall f(q \leftrightarrow (\forall r(f(rr) \leftrightarrow p) \leftrightarrow \forall r(f(rr) \leftrightarrow p)) \leftrightarrow s)))))$. The main features of Leśniewski's symbolism are visible: the wheel-notation for connectives ('$\leftrightarrow$'), the parentheses of growing size, the low and high corners for the universal quantifier and its scope, and the functors ('f') written before the (propositional) arguments ('p', 'q', 'r', 's'—where '$f(rr)$' could be '$r \wedge r$', '$r \vee r \ldots$').

Formulas like the one shown above are initially inaccessible to the vast majority of formal logicians—not just those for whom a B- in Math in high-school was a reason to party. Reading Leśniewski for the first time gives a sense of frustration, discomfort, and esoteric impenetrability. If Leśniewski's deductive systems are unparalleled in their generality, elegance, precision, and intuitive foundation, they also deviate greatly from standard propositional and predicate logic in some basic features. Such deviant aspects are rather well-known in the literature on Leśniewski, and are often cited among the reasons why his work is not as popular and celebrated as it should be. Few are, however, the attempts made to explain why those aspects should be there at all. This paper is such an attempt.

As I try to make clear in the following, Leśniewski wasn't just an obsessive, maniacal perfectionist and an unbearably pedantic writer; he had a dream. And the dream was that of a perfect language in which to write a perfect system of deductive theories. The system would encode our knowledge of the world, or at least large chunks of it, and would have none of the formal shortcomings that made Frege fail, and none of the unintuitive assumptions found in Russell and Whitehead.[1] In short, Leśniewski's dream is close to Leibniz's *characteristica universalis*. The main thesis set forth in the

[1] Cf. Leśniewski (1929, pp. 5–9, Eng. trans. 1991, pp. 412–414).

following is that this element in Leśniewski's thought can be grasped in the best way with the Classical Model of Science in mind. In order to show this, I shall devote most of this paper to arguing that Leśniewski was a close follower of the Classical Model of Science as described in de Jong and Betti (2008).

The claim that Leśniewski follows closely the Classical Model of Science would seem surprising if we didn't also know that Frege was a strict adherent of this model, and that Frege exerted the single most important influence on Leśniewski's mature work.[2] Further, it is not that surprising considering that both Bolzano and Husserl, the major exponents of that Austrian tradition in which the roots of Polish analytic philosophy are found, were close followers of the same model. Nevertheless, that Leśniewski adhered to the Classical Model of Science is still important news, especially because of the extent of his adherence to this model. In fact, Leśniewski applies the Classical Model of Science with such rigour and detail that one is tempted to consider his enterprise the peak of its development, and Leśniewski himself as its last great adherent. But without undertaking similar investigations into Leśniewski's contemporaries' work, claims like these should not be considered more than strong intuitions.

This paper runs as follows. After a short presentation of Leśniewski's systems (Sect. 2), I will sketch how Leśniewski's systems match every single condition of the Classical Model of Science, starting from the Domain Postulate, the Truth Postulate, and the Universality/Necessity Postulate (Sect. 3). In Sects. 4 and 5, I will show how Conditions 2 and 3 of the Model are reflected in Leśniewski informal requirements of a well-constructed axiom system. In Sect. 6, I concentrate on how axioms can be known. Finally, in the Appendix, I give a taste of Leśniewski's formal methodology and illustrate the disposition of Leśniewski's systems.

2 The architecture of Leśniewski's systems I

Who was Stanisław Leśniewski? Three things you might know about him. He was a Polish logician, he created Mereology, a formal theory of parts and wholes, and he was Alfred Tarski's teacher. All true.

Also true is that between 1919 and his death in 1939, Stanisław Leśniewski and his students, notably Tarski and Mordechaj Wajsberg, developed a nominalistic system of the foundations of mathematics split into three axiomatic, fully extensional, deductive theories: Protothetic, Ontology and Mereology. These three theories are, importantly for what follows, hierarchically ordered. The first two theories taken together form Leśniewski's logic, while Mereology is an extra-logical theory. Arithmetic can be reconstructed in Ontology (superposed to Protothetic), while Mereology (superposed to Protothetic and Ontology) provides a basis on which to develop spatiotemporal theories of topology, and Geometries such as Tarski's geometry of solids.[3] As should be

[2] On Frege, see de Jong (1996, 2008) and Korte (2008). I know of no single work dealing with Leśniewski's and Frege's axiomatics in particular.

[3] Cf. Luschei (1962, p. 28). For Geometry based on Mereology, see Tarski (1929), Sullivan (1971, 1972) and Ditchen et al. (1963).

clear at the end of these sections, the systems of Protothetic, Ontology and Mereology conform closely to the Classical Model of Science.

One general characteristic of Leśniewski's systems that is relevant to our concerns here is that the systems are *constructively nominalistic*.[4] This peculiar and important trait means that there is no system of Protothetic, Ontology, or Mereology whatsoever if it is not actually built by someone. Every system is literally constructed in time and space by some agent following explicitly and rigorously formulated rules of procedure, and the system can grow in principle endlessly. At each step or *stage* of the construction of a system the system *is* the mereological collection of its axioms and theses at that stage. You have to imagine someone at a table with sheets of paper, who, as time goes by, fills the paper with logical expressions, following strict rules governing their inscription. If pictures are taken, every now and then, of what our logician writes, they will show different stages of a system, *that* system being identical with the inscriptions (equipped with meaning) on *those* particular sheets of paper. One can loosely speak of 'the system of' Protothetic (Ontology, Mereology), but what Leśniewski left us is, in fact, a system of systems: rules to build deductive theories, and just a handful of systems of Protothetic and Mereology actually built.[5]

The language of Leśniewski's systems has a general structure given by a *theory of semantic categories* roughly corresponding to a simple theory of (linguistic) types.[6] Semantic categories are a formal correlate to the intuitive notion of part of speech in traditional grammar and have a direct ancestor in the *Bedeutungskategorien* of Husserl's *Logische Untersuchungen*.[7] Every term in Leśniewski's systems belongs to a definite semantic category (quantifiers and parentheses are not terms, while connectives are). There are two primitive semantic categories: sentence (s) and name (n). Every other category is formed from these two.

The first of Leśniewski's systems in deductive primacy, as mentioned, is Protothetic or 'science of first propositions'.[8] Protothetic presupposes no other theory for its construction. Every formal system except Protothetic presupposes at least Protothetic. Ontology presupposes Protothetic, and Mereology presupposes both Protothetic and Ontology. Presupposing another system means literally containing (a stage of) that system as proper part. For instance, a system of Ontology is obtained, roughly, by adding at a certain point new axioms, new kinds of linguistic expressions indefinable in Protothetic, and strengthening and adapting the rules.

You can see how this works with the help of the schema in the Appendix below (Fig. 1). You start building a system by inscribing an axiom of Protothetic (like for

[4] On this point see also Luschei (1962, pp. 125–128).

[5] Leśniewski did not publish any system of Ontology during his life, but only directives for systems of Ontology, see Leśniewski (1930). A system of Ontology developed by Leśniewski has been published posthumously in Srzednicki and Stachniak (1988).

[6] Or *logical* types, see Sobociński (1955/1956, p. 61). I will make extensive use of this precious paper in Sects. 4 and 5.

[7] Cf. Leśniewski (1929, p. 14, Eng. trans. 1991, p. 422).

[8] The chronological order in which the theories were developed by Leśniewski during his lifetime is reverse: Mereology (although based already intuitively on a fragment of what was to become Ontology), Ontology, Protothetic.

instance PA1 in the schema). Protothetic is a quantified propositional calculus with variable functors, that is, with variables representing functors bound by quantifiers: in PA1 '*p*', '*q*', '*r*', '*s*' are propositional variables, '*f*' is a variable representing a functor and, as you can see, quantifiers bind all of them.

Once you have inscribed an axiom, you can add a new thesis to the system (consisting at that stage only of PA1[9]) following the Protothetical *directives*—carefully and unambiguously stated rules of procedure for the effective constructions of the system formulated by Leśniewski in a strongly regimented mereological language. A series of *Terminological Explanations* supplemented by a list of abbreviations tells you how to decipher the directives. The directives do not belong to the system, but to the metalanguage (note that, unlike in Tarski, the notion of metalanguage for Leśniewski's systems is not relative, and no need is felt for a hierarchy of metalanguages). Directives also suppose Leśniewski's theory of semantic categories just mentioned (for more on what I say in this paragraph, see the Appendix).

In Protothetic, only terms of protothetical semantic categories (*s*, *s*/*s*, *s*/*ss* etc.) can appear. The double implication in PA1 is a sentence-forming functor with (two) propositional arguments (*s*/*ss*). At the stage in which the system consists only of PA1, semantic categories other than *s* (the category of '*p*', '*q*', '*r*', '*s*'), *s*/*s* ('*f*') and *s/ss* are not available, but we can introduce a new category in the system by introducing a constant expression of that category by means of a definition. Definitions can be creative, that is, some theorems might be proved with the aid of definitions that might not be proved otherwise, although no term in such theorems is directly or indirectly defined with the aid of the definitions at issue. Leśniewski formulated extremely precise rules for definitions as well.

At a certain point, a system of Protothetic will be expanded enough—i.e. will contain enough theses—to be able to serve as a basis for a system of Ontology. You will obtain a system of Ontology by adding an axiom like OA1 (Fig. 1, Appendix). Ontology is a calculus of names. In Leśniewski's own words, it is a 'modernized traditional logic', roughly corresponding to the predicate calculus with identity. Its propositions have the subject-predicate structure '*x* is *y*' (the 'ε' in OA1 is the copula, and *x*, *y*, *z* are nominal variables for singular, common and empty names). Once you have inscribed OA1 you can add new theses to the system following the Ontological directives, which are adapted Protothetical directives plus two new ones. In Ontology new semantic categories can appear (*n*, *s*/*n*, *s*/*nn* etc.). The copula 'ε' in OA1 is a sentence-forming functor with (two) nominal arguments, (*s*/*nn*), and new categories can be introduced in the fashion mentioned above for Protothetic.

Again, at a certain point, your system of Ontology will be expanded enough to be able to serve as a basis for a system of Mereology. Mereology is a theory of parts and wholes, of collective classes, literally composed—differently from sets—by their parts, like an armadillo is composed by the parts of his body. You will obtain a system of Mereology by adding an axiom like MA1 ('ingr' in MA1 is the functor 'is ingredient of', indefinable in Ontology).[10] Again, once you have inscribed MA1 you can add

[9] Cf. Leśniewski (1931, p. 296, n. 1, Eng. trans. 1991, p. 635, n. 15).

[10] MA1 is a ten-unit single axiom for Mereology discovered by Owen LeBlanc in 1983, see LeBlanc (1983, p. 47).

new theses to the system. In MA1 'ingr' is a name-forming functor with (one) nominal argument (*n*/*n*). In Mereology no new directives are required and no new semantic categories can be introduced. In short, the grammar of the language cannot be altered in Mereology. In fact, this aspect demarcates the logical from the non-logical: if the grammar of the language is allowed to change, it's logic, otherwise it's not.[11] Thus, Protothetics and Ontology together form logic because the grammar of their language can be changed, while Mereology in an extra-logical theory because the grammar of Mereology is given by its logical basis.

As I previously claimed, all of Leśniewski's systems obey the conditions of the Classical Model of Science as described in de Jong and Betti (2008). In the following sections, I shall set out to show in what sense and to what extent this is the case.

3 Leśniewski's systems and the Domain, Truth, and Universality/Necessity Postulates

Showing exactly in what sense Leśniewski's systems can be said to obey condition (1) of the Classical Model of Science, the Domain Postulate, is complicated. Part of the difficulty is of a general sort, and it stems, in turn, from difficulties linked to the systematization of the Postulate itself. The Domain Postulate has the appearance of a rather trivial condition, but it is that kind of deceptive triviality behind which many and intricate issues lurk, and a comprehensive treatment of the historical development of these issues has yet to be fully explicated (see also Cantù 2008). Two elements are important to Leśniewski's case: the distinction between the term-interpretation and the object-interpretation of the notion of domain, and the status of logic as a proper science obeying the Model. Here I will limit myself to sketch the way in which I think the Domain Postulate must be understood in connexion with these two elements.[12] It will be at once clear that the problem of singling out domains for Leśniewski's systems is related to the demarcation of the logical from the non-logical.

The term-interpretation of the domain of a science—according to which the domain is a collection of *terms* appearing in the truths making up a science and not of (the) objects (falling under them)—is the only one that can work for Leśniewski's systems. That this is the case can be argued on the basis of the simple fact that its rival, the object-interpretation, cannot be consistently maintained (assuming that there is no other option). As Scholz notes, both primitive and non-primitive terms of a science must belong to the same domain (Scholz 1930/1975, pp. 53, 62); since in Protothetic a constant for the false sentence can be defined, and in Ontology a constant for the empty term, and since a false sentence and an empty term do not have ontological counterparts, the domains of Protothetic and Ontology cannot be given by the objects mentioned by their terms. Therefore, the domains of Leśniewski's systems must be singled out according to the term-interpretation. The question is: how, exactly?

[11] Cf. Luschei (1962, p. 106).

[12] Or, as in de Jong and Betti (2008), the distinction between the ontological versus the semantical interpretation.

One option is this: the domain of a theory is given by the kind of constant terms that can be introduced once the primitives are given, or, in other words, the sort of truths that can be obtained from those in which the primitives occur (that is, the axioms) by following the rules of procedure. More specifically, for logical theories (Protothetics and Ontology), this can be strengthened into a formal criterion: the domains of Protothetic and Ontology can be set apart from each other and from any other non-logical theory by relying on the semantic categories that can be introduced in the language, that is, by looking at how language can be grammatically modified. Let's briefly see this. We can fix the domain of Protothetic by saying that it is a collection of truths whose constants are either propositional constants (like '0' and '1') or sentence-forming functors of propositional arguments (like the propositional negation '¬': not); we can also fix the domain of Ontology by saying that it is a collection of truths among whose constants are, in addition to Protothetical ones, also ontological constants (like '**V**': object), sentence-forming functors of ontological arguments (like '**ex**': exists), name-forming functors of ontological arguments (like the nominal negation '∼': non-). For *non-logical* theories, like Mereology, however, this won't work. In the language of such theories no new semantic categories can be introduced. It follows that our formal criterion cannot set apart non-logical theories from one another. All that can be said about Mereology from this purely formal point of view is that its domain is fixed once we say that it is a collection of truths among whose constants are, in addition to Ontological and Protothetical ones, also *other* name-forming functors of ontological arguments. Mereology would be a collection of truths in which propositions of the structure 'S is mr⟨P⟩' are allowed to appear, where 'mr' is a mereological functor indefinable in Ontology.

But what do we have now? How informative is it to say that Mereology is the non-logical science where mereological functors appear? What makes a mereological functor mereological? For what we would have to say about *XYZ*logy, for example, is that it is the non-logical science where *xzy*logical functors appear (Leśniewski did not build any other non-logical theory than Mereology, but let us leave this aside). What this means is just that to set up a criterion for both logical and non-logical theories we need a criterion other than a purely formal one linked to logical grammar. And what this criterion would be is difficult to say. Yet this is not as bad as it seems for our purposes, because what is important here is, first, that this conclusion does not force us to abandon the term-interpretation and switch to the object-interpretation of the Domain Postulate in the case of Mereology (and in general for non-logical theories); second, that we can still claim that the domain of a science is given by certain terms even if we do not have a purely formal criterion—in the sense of grammatical, or syntactical—to single out these terms. What the above suggests, in fact, is that we need the assurance that the terms of a science *mean something definite*.[13] In the end, it is very well possible that, should we want a general description of the domain of a non-logical theory, we won't manage to go further than '*XYZ*logy is the non-logical science where *xzy*logical functors appear'.

[13] Cf. de Jong and Betti (2008, n. 33).

The whole question is, therefore, brought back to the meaning of the primitive terms, and it suggests that, in general, a theory can be said to obey the Domain Postulate only insofar as its primitive terms are *meaningful*, so that its theses *say something* even if they contain empty or non-denotative (of you prefer, non-representing) terms. Note that by this I do not mean theories in which the primitive terms are *capable of being interpreted* (or reinterpreted) or *are successively interpreted* by the usual means of some set-theoretical mappings once the syntax of the theory is given. Theories for which the notion of interpretation in this fashion makes sense do not, I take it, obey the Domain Postulate (although de Jong and Betti (2008) do not say so).

I am well aware that by this nothing is said on either the general issue of how the meaningfulness of primitive terms of a deductive system is granted or on the more specific issue of how this works for Leśniewski's systems. The matter is of paramount importance, but also extremely difficult, and a satisfactory treatment would by far exceed the scope of this paper. I can only attract attention to this point as a crucial one in the development of axiomatics, and to the fact that, insofar as the Domain Postulate grants meaningfulness to the terms of a science, it connects immediately to two others: condition (4), the Truth Postulate, and condition (5), the Universality (and Necessity) Postulate. Leśniewski's systems obey both—and to this I shall now turn.

As to condition (4), every thesis of Leśniewski's system is true. Axioms are just special cases of true theses of the system of which they are axioms. Though exceptionally few passages support this, these few are clear enough:

> *Truth1*. The psychological 'source' of my axioms is my '*intuitions*', which means simply that I *believe* in the truth of my axioms (Leśniewski 1916, p. 6, Eng. trans. 1991, p. 130).

> *Truth2*. Since I have no predilection for various 'mathematical games' that consist in writing according to this or that conventional rule various more or less picturesque formulas which need not be meaningful, or even—as some 'mathematical gamers' might prefer—which should necessarily be meaningless,—I would not have taken the trouble to systematize and to check often and scrupulously the directives of my system had I not ascribed to its theses a certain specific and completely determined sense in virtue of which the axioms of the system, and the method of definition and of inference encoded in the directives of this system, have for me an irresistibly intuitive validity (Leśniewski 1929, p. 78, Eng. trans. 1991, p. 487, amended here).[14]

So, axioms are true, and every other sentence that can be proved in the theory from the axioms by following the derivation rules at our disposal, or every other thesis that can be introduced as a definition, is also true. As to the 'certain specific and completely determined sense', it's worth noting that Leśniewski found it perfectly normal to write treating formal theses as meaningful expressions of ordinary language, for instance:

[14] See also the criticism of von Neumann (ibid., p. 80, Eng. trans. 1991, p. 489). Nothing in Leśniewski's writings points towards a meaning of 'validity' that would not count as truth here.

I established […] that
(b) $[g, p] : g(p, p) . g(\neg(p), p) . \equiv . [q] . g(q, p)$
and
(c) $[g, p] : : \neg(p) . \equiv : p . [q] . q \therefore \supset : g(\neg(p), p) . \equiv . g(p \equiv . [q] . q, p)$.
Considering that
(d) $[p] \therefore \neg(p) . \equiv : p \equiv . [q] .q$ (Leśniewski 1929, p. 32, Eng. trans. 1991, p. 440),

(it goes on a little more like this).

Leśniewski's axioms are not axiom-schemata, nor postulates in the sense of Hilbert's if-then axiomatics.[15] Least of all are they propositional functions like in the so-called American Postulate Theorists. The theses of Leśniewski's systems contain no free variables—exactly like the theses just mentioned—and the same holds for axioms, of course. PA1, for instance, *asserts that* for *all* sentences p and q, p and q if and only if … and so on.

The above connects immediately to the Universality Postulate. There can be no doubt that Leśniewski's systems match this postulate, for the theories have the highest possible degree of formal universality in two senses. The first sense is linked, among others, to what we saw in the last paragraph and concerns some technical aspects of the systems, specifically, to what in logic is called the *expressive power* of a theory. Protothetic is a most general theory of propositional logic because it is—in principle—of unboundedly high order. The propositional logic one learns usually in logic courses is first-order, namely a logic in which the propositional functors (the propositional connectives) take only propositions as arguments and there is no propositional quantification, while in second-order propositional logic functors can take first-order functors as arguments and quantifiers bind propositional variables. Protothetic is of even higher order because functors can bind other functors of whatever order and there are variable functors, that is, functors can be quantified over. Since Ontology contains Protothetic as a proper part, and Mereology, Ontology, and quantification in Ontology functions in a similar way, all Leśniewski's systems are collections of propositions of utmost generality. The second sense in which Leśniewski's systems match the Universality Postulate regards their freedom from existential assumptions: a deductive theory, in particular a logical one, should not come with presuppositions concerning the sort of objects that are in the world or their number.

This, in turn, connects to the Necessity Postulate. Necessity of the axioms and necessity of the theorems pose different problems. I will discuss the necessity of the axioms in the next section in connection with the Postulate of Grounded Knowledge. There is not much to be said about the necessity of theorems.[16] One way to look at the Necessity Postulate is to link it to aprioricity.[17] Leśniewski is almost silent on the topic, but in an early paper he adheres to a position similar to Frege's insofar as he

[15] This is the case notwithstanding the quote I mark as *Axioms* below.

[16] Definitions raise problems of their own, and in view of their importance, they need separate treatment. They won't receive it here, but it does not seem particularly difficult to fit them in what follows.

[17] See de Jong and Betti (2008, Sect. 2).

links aprioricity with justification (of non-fundamentals) from (non-factual) general laws.[18] If aprioricity is interpreted as freedom from empirical assumptions and as following from most general laws, then Leśniewski's systems are necessary in the sense of a priori thanks to their ontological neutrality—and this applies to both logical and non-logical theories.[19]

4 The fundamental terms and the Composition Postulate

According to (one reading of) condition (2) of the Classical Model of Science, a science has a number of fundamental terms and all other terms are defined from them. Traditionally, philosophers would prescribe or concretely strive to reduce the number of fundamental terms as much as they could. Not only do all of Leśniewski's deductive theories share this concern, but they do it to an extreme extent. For one thing, each theory can be based on one single primitive, that is, Leśniewski had a most rigorous take on condition (2a). But that in which Leśniewski remains unsurpassed is his rules for definitions—that is, a most rigorous formulation of condition (2b). And, as we shall see, there is more.

Concerning (2a), Leśniewski required that primitive terms be *mutually independent*. For example, in the case in which in an axiom system—i.e. the axioms of a theory—several primitive terms occur, none of them could be defined by means of the others (Sobociński 1955/1956, p. 57). Furthermore, Leśniewski found it strongly desirable that formal systems be based on *a single primitive*. Leśniewski also required that primitive terms should be as *simple* as possible, that is, they should admit the smallest number of arguments of the lowest possible semantical category (Ibid. 58–59).

The independence of primitives is a standard requirement. But while the single primitive remained a dream for many philosophers, Leśniewski actually managed to base each of his theories—including logic—on one primitive. The result is extremely elegant. Consider a system of Protothetic based on PA1 (Fig. 1, Appendix). The single primitive term of this system, which is indicated by the curved arrow above it, is the double implication. This is the best primitive term for Protothetic one can possibly hope for, for several reasons, all matching the ideal captured in the Model, especially condition (2). One reason is simply that Leśniewskian definitions are formulated with the aid of the double implication. This depends also on Leśniewski's general conception of definitions, according to which definitions are not metalinguistic abbreviations, but (true) *theses* of the system. If definitions belong to the system, they must be

[18] Frege (1884, p. 4). The 'general laws' Frege has in mind, one can safely assume, are the fundamental propositions of the Classical Model.

[19] On a more historical note, Leśniewski's general orientation as to the Model dates from his very first contributions, but it must be excluded that Leśniewski was already conversant with Frege's work in 1911. The main difference with the later phase of the formal systems in this respect is that Leśniewski's 'proper sciences' were shaped by Leśniewski's growing concern for extreme degrees of precision, and the status of logic as a real science, and as the one on which the other sciences should be based. On this, Frege had—one can safely say—a strong influence. But Leśniewski's early source for the Classical Model in general was, I think, Husserl. Cf. Husserl (1900/1901, Sect. 63).

formulated with the aid of its primitive terms: Leśniewski found it unacceptable to introduce a special term like '$=_{df}$' to the sole aim of formulating definitions.[20] Another reason is due to the form of Leśniewskian axioms: as you see in PA1, OA1 and MA1 (Fig. 1, Appendix), axioms are themselves equivalences, so in a system of Protothetic based on PA1, the single primitive term is also the only constant functor appearing in the axiom system.

As mentioned above, Leśniewski's systems also obey condition (2b), the 'Composition Postulate', and this relates interestingly both to the Domain Postulate and to the Postulate of Grounded Knowledge. In each Leśniewskian theory, every non-primitive term is definable with the aid of its primitive terms (and rules) alone. For instance, in Protothetic, (propositional) negation and conjunction can be defined as follows:

D1 $\forall p(\neg p \leftrightarrow (p \leftrightarrow \forall q(q))$[21]
D2 $\forall pq\,(p \wedge q \leftrightarrow \forall f(p \leftrightarrow (fp \leftrightarrow fq)))$.

And once you have conjunction, that is, you have added D2, you can, for instance, define implication:

D3 $\forall pq((p \rightarrow q) \leftrightarrow (p \leftrightarrow p \wedge q))$.[22]

In Ontology, identity, nominal negation ('$\sim$'), a constant for the empty name ('$\mathbf{\Lambda}$') and nominal conjunction ('$\oplus$') can be defined as follows on the basis of 'ε':

D4 $\forall xy(x = y \leftrightarrow x\,\varepsilon\,y \wedge y\,\varepsilon\,x)$
D5 $\forall xy(x\,\varepsilon \sim\! y \leftrightarrow x\,\varepsilon\,y \wedge \neg(x\,\varepsilon\,y))$
D6 $\forall x(x\,\varepsilon\,\mathbf{\Lambda} \leftrightarrow x\,\varepsilon\,x \wedge \neg(x\,\varepsilon\,x))$
D7 $\forall x(\forall yz(x\,\varepsilon\,y{\oplus}z \leftrightarrow x\,\varepsilon\,x \wedge x\,\varepsilon\,y \wedge x\,\varepsilon\,z))$

In Mereology, 'proper part' (ppt), for instance, can be defined on the basis of 'ingr' as follows:

D8 $\forall xy(x\,\varepsilon\,\mathrm{ppt}(y) \leftrightarrow x\,\varepsilon\,\mathrm{ingr}(y) \wedge \neg(x = y))$

Leśniewski was concerned with the Composition Postulate to such a degree that he turned, as it were, Port-Royal Logic's Rule 1 into a technical requirement for non-fundamental terms.[23] For Leśniewski, explaining the meaning of a term comes down to adding to the system a definition of the term at issue. Since the meaning of a term is given by its definition, at any stage of development of a system at which its definition is not yet added, the term in question is meaningless and cannot be (yet) used. Given the peculiar hierarchical disposition of Leśniewski's theories, this means that one cannot just build a system of Ontology without letting an axiom (system)

[20] Cf. Leśniewski (1929, p. 11, Eng. trans. 1991, p. 418).

[21] '$\forall p(p)$' is the expression that Leśniewski uses to define the constant for the false proposition, '0'.

[22] Note that you do need the quantifier to be able to do this, but the quantifier is not a term in Leśniewski's systems. The first double implication in D1, D2 and D3 is responsible for the equivalential form of the definitions.

[23] See de Jong and Betti (2008, Sect. 1).

of Ontology be preceded by its Protothetical basis, that is, by the axiom (system) of Protothetic and a finite number of individual theses according to the directives subjoined to the axiom (system). Any term occurring in an axiom of Ontology that is not a variable, not inside quantifiers, and not a term peculiar to Ontology, must be a protothetical constant already introduced by a thesis of Protothetic: if this is not the case, your axiom of Ontology will, strictly speaking, be meaningless. For instance, if you want to add D7 above to your system of Mereology, you must have D4 in your ontological basis (which, in turn, must include D2). This connects directly to the second part of condition (7) in the Model, the Postulate of Grounded Knowledge about terms, according to which non-fundamental terms are adequately known through their definitions.

Leśniewski took great pains to formulate rules for definitions. I say more on this in the Appendix below.

5 The fundamental propositions and the Proof Postulate

As is the case with condition (2), every Leśniewskian system follows (3) in a very strict fashion. Here, again, if tradition required axioms to be reduced in number, Leśniewski joined in, made the requirement far more challenging and succeeded on all fronts. One of Leśniewski's greatest concerns was to construct systems based on a single axiom. Not only did he take condition (3) so seriously to actually construct single-axiom systems of Protothetic, Ontology, and Mereology (in keeping with 3a), but the exceptional thing is that he stated his rules of procedures, the directives, with an astoundingly high degree of formal precision (in keeping with 3b).[24] And, yet again, there is more.

As Sobociński tells us, Leśniewski formulated requirements for well-constructed axioms systems to apply both to any axiomatic theory whatsoever and to his own systems, although he never worked this out in a systematic manner (Sobociński 1955/1956, p. 55). The requirements described by Sobociński regard the aesthetics, not the correctness of the axiomatic systems. As we shall see in Sect. 6, this is important for the aims of this paper.

First of all, axioms had to be *mutually independent*, that is, it had to be impossible to derive axioms from one another following the rules of procedure of the theory. Moreover, the axiom system had to be *consistent*. The consistency of the axiom system was kept separate from that of a theory. Considering a theory whose consistency is presupposed, the requirement was equivalent to demanding that first, all the theses singled out as axioms belong to 'the field of the theory', second, that the rules of procedure be adjusted to the primitive terms occurring in the axioms. The first part of this requirement, as we can see, is related to the Domain Postulate. The axiom system should also be *adequate*. Sobociński safely put this as the requirement that all true *and desirable* theses belonging to the theory be derivable

[24] This is also what puts off most readers, see the Appendix below.

from the theses accepted as axioms.[25] The requirements of mutual independence, consistency, and adequacy (completeness) were fundamental and also fairly standard lore by then.

Leśniewski also formulated some special requirements for axioms of any theory whatsoever. First, the *shorter* an axiom system, the better; this meant that the axioms system should contain the lowest possible amount of symbols—brackets, quantifiers, and variables in the quantifiers excluded. Second, of two axiom systems based on the same primitive term(s) with the same length, the one with the *lower number of different variables* was considered better.[26] Third, an axiom should be *organic*,[27] that is, no segment of the axiom should be a thesis of the system or become a thesis once variables get bound. Organicity turned out to have bearings on the length of the axiom system, and for this reason this requirement was considered extremely significant.[28] But fourth, and most importantly, a well-constructed axiom system should have, possibly, just *one* axiom of a certain kind—a *proper* axiom.[29] A proper axiom is a thesis placed at the head of a system as a starting point, and from which it cannot be deduced that something exists. For instance axioms of infinity, which state that there exists an infinite number of objects, are not axioms proper. Leśniewski thought that such axioms should be avoided, since, as already mentioned, deductive theories should be ontologically neutral.

In addition to these, Leśniewski formulated requirements applying to his systems in particular. Among the latter, we find the requirement according to which an axiom system is *canonical*: (a) it contains a single axiom that is based on a single primitive term; (b) it has the form of an equivalence whose left-hand side is an expression of the form '$f(abc)$' where 'f' is a sentence-forming functor or an expression of the form '$A\varepsilon f(abc)$' where 'f' is a name-forming functor, 'A', 'a', 'b', 'c' being variables—that is, the left-hand side contains only the simplest possible sentence that includes the primitive term; and (c) the quantifier at the head of the axiom (the main quantifier) 'should bind the variables appearing in the expression on the left-hand side of the equivalence and no others'.[30] Canonical axiom systems grant the "shortest and most intuitive axioms for all theories that can be based on Leśniewski's system of logic" (Sobociński 1955/1956, p. 62), and it's worth noting

[25] Cf. Sobociński (1955/1956, p. 57). Adequacy is, of course, completeness minus Gödel. Sobociński's care here is obviously in place, but this should not make us think that the relationship between Leśniewski's systems and Gödel's incompleteness results is a closed chapter. Quite the contrary, a discussion of this point from the philosophical point of view would deserve a separate study, including a discussion of Leśniewski's appalled reaction to Gödel's proof—as the rumour has it. A technical investigation on these matters is to be found in Canty (1967).

[26] Cf. Sobociński (1955/1956, pp. 62a, b), and cf. Leśniewski (1938, p. 23, Eng. trans. 1991, p. 671).

[27] 'Organisch nämlich in bezug auf ein System X heisst eine Aussage $y \epsilon X$, deren kein (sinnvoller) Teil Element von X ist (die Bezeichnung 'organisch' rührt von Hrn. S. Leśniewski her, die Definition der 'organischen Aussage' von Hrn. M. Wajsberg)', Łukasiewicz and Tarski (1930, p. 37). Note that 'ϵ' in this quotation is not the copula, but set inclusion.

[28] See for more requirements, among which *ontological uniformity* and *internal independence of single axioms*, Sobociński (1955/1956, pp. 62–63), LeBlanc (1983, pp. 101, 109–110).

[29] Cf. Sobociński (1955/1956, p. 60).

[30] Cf. LeBlanc (1983, p. 101).

that the form of a Leśniewskian canonical axiom resembles that of a Leśniewskian definition.

6 The Postulate of (Grounded) Knowledge

On the traditional analysis of knowledge, knowledge that p is justified true belief of p, and p can be justified either immediately or mediately. The Proof Postulate (3b) takes care of mediate justification and together with the Truth Postulate (4) yields that in a science *truths follow from truths*. However, for Leśniewski truth has to follow from truth on the basis of *intuitively* valid rules of inference—that is, on one reading, epistemically acceptable.

But then the truths from which one starts, the axioms, must be known. And how does *this* work? Again, Leśniewski told us very little about these matters. In 1916, before the architecture of his systems was complete, he wrote:

> *Axioms.* Now, I treat my system expressly as a hypothetico-deductive system, from which it follows that, actually, I assert only that from the propositions I call 'axioms' follow propositions which I call 'theorems'. [...] My axioms don't have a logical 'source', which means simply that these axioms do not possess a proof in my system, like no other axiom whatsoever is, in the nature of things, proved in the system for which it is an axiom (Leśniewski 1916, p. 6, Eng. trans. 1991, p. 130, amended here).

This passage can be interpreted as saying two things. First, axioms are unproved truths; second, that's all. It's important to point out about the latter that since this passage includes the one I indicated above as *Truth1*, it's out of the question that we should interpret its initial part as speaking of axioms as assumptions. Leśniewski is not denying here that axioms are truths, he is denying that axiomatic structures—in the sense of an *ordo essendi*—reflect metaphysical primacy.[31]

Concerning the first thing, it follows that axioms can only be immediately justified. And, since, as we know, they are true, they can only be known to be true immediately *or by themselves*[32]—that is, they have to be self-evident. One way to fulfil the Necessity Postulate for axioms, actually, is to demand this. Did Leśniewski think of axioms as self-evident? In no case does he say so. But is there really no evidence of this?

The overall rhetoric of the passage above, in which Leśniewski signals that he does not want to go into the matter at all ('I assert only') means, at any rate, that he is well aware of the problem and that he has no solution for it—at least in 1916. But there is more. Before I add anything to this, however, note that what follows is meant at most as a first approach to the difficult problem of the status of the axioms in theories, like Leśniewski, that both follow the Classical Model of Science and admit of several different axiomatizations.

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[31] See on the distinction between *ordo essendi* and *ordo cognoscendi* see de Jong and Betti (2008, Sect. 1).

[32] This is 'the Evidence Postulate' of de Jong and Betti (2008, Sect. 2).

Consider the notion of self-evidence highlighted by Scholz in connection with Aristotle:

> These indemonstrable sentences are distinguished by the fact that one need only understand the meaning of the terms they contain in order to know that the assertions they make are true. In short, they are distinguished by the fact that they are evident *ex terminis* (Scholz 1930/1975, p. 56).

This is the notion of self-evidence we are after. And if this is the notion we can rely on, I suggest we see Leśniewski's informal, aesthetic requirements upon axioms and primitive terms I mention in Sects. 4 and 5 as obeying the Knowledge Postulate in this sense. Two circumstances support this. First of all, the aesthetic requirements prescribe that the primitive term be a single concept of the lowest grammatical degree placed in the initial segment of the shortest possible single axiom proper with the form of an equivalence with the lowest possible amount of variables. Secondly, and consequently, such single canonical proper axiom is, to speak loosely, a *self-definition* of the primitive term.

This is as close as it gets to a solution of the problem of knowing the axioms. For what's the idea beyond all this, if not the attempt to reduce a *certain kind* of theoretical effort to a minimum? Note that this is not to say that it is *easy* to understand things like PA1. It is, in fact, very difficult, and the opposite would be surprising. In a well-constructed axiom system, the single canonical axiom proper should present us with an extremely compact, though neatly packaged, content that we then are supposed to unfold completely mechanically following perfect rules. But *understanding* an axiom like PA1 is by no means a mechanical chore. The traditional take on this (see for an excellent example, Leibniz >1684), has it that different epistemic tasks are associated with understanding the axioms and with inferring; in inferring one would not do anything other than *calculare* or perform mental operations on mere symbols—following "an Ariadne's thread, a rough means of leading the mind".[33] Intuition or imagination should not come into play in this inferential step, but only in grasping the concepts involved in the axioms (and the definitions). Saying that a thesis is self-evident is not saying that it is trivial or obvious to everyone, it is saying that everything you need to understand it, and see that it is true, is available in the thesis itself. In this specific sense, Leśniewski's systems are conceived to offer to the feeble human mind a theoretical grip that is as adequate and full as possible, both in what Leibniz would call (*lingua*) *characteristica*, the encoding in adequate symbols, and in what he would call *calculus ratiocinator*, the rules of inference to apply to the encoding so conceived. Leśniewski's 'authentic symbolism' in Sect. 1 above is linked to this.

In this connexion it becomes easier to see that profound epistemological concerns inform both the informal requirements and the precision with which Leśniewski's formal language and methodology are constructed and laid out. In the Appendix, I give an idea of the degree of this precision by showing in some detail how terminological explanations, directives (metalanguage, fixed), and the system (language, growing) are related.

[33] Leibniz (1677, p. 570, Eng. trans. 1987, p. 164, n. 2).

7 Conclusion

The upshot of all this is that Leśniewski's formal enterprise, an enterprise close to that of a *characteristica universalis* in Leibniz's sense, was marked by strict adherence to the Classical Model of Science. One of the most important points worth considering in this perspective regards the extreme care in which Leśniewski formulated the directives for his systems, a care that goes far beyond Frege's, if only because Leśniewski's systems are paradox-free. The way in which this is accomplished has strong epistemological motivations, like, among others, excluding error and offering the most adequate theoretical support to the mind. This may sound paradoxical in light of the frustration one feels when confronted with Leśniewski's directives, his terrifying list of carefully specified terminological explanations of the terms occurring in the directives, the daunting task of learning the vocabulary necessary to understand the terms occurring in the explanations, a list of abbreviations, and the examples. However, the end of all this is clear: encoding large bodies of knowledge of the world, in ontologically neutral, paradox-free, axiomatic deductive systems of the highest generality, based on a completely unambiguous language equipped with completely unambiguous truth-preserving rules to pass from axioms, resting for their truth upon nothing else than the meanings of the terms used in them, to theorems.

Appendix: the architecture of Leśniewski's systems II

Here below you find the directives for system $\mathfrak{S}5$ of Protothetic. In keeping with the constructively nominalistic character of the system, they do not apply to the system *tout court* but make explicit reference to system stages:

> If a thesis A is the last thesis already belonging to the system, an expression B may be added to the system as a new thesis only if at least one of the five following conditions is fulfilled:
> (1) B ε defp (A)
> (2) $[\exists C]$. C ε thp (A). B ε cnsqrprtqntf (C)
> (3) $[\exists C, D]$. C ε thp (A). D ε thp (A). B ε cnsqeqvl (C, D)
> (4) $[\exists C]$. C ε thp (A). B ε cnsqsbstp (A, C)
> (5) B ε extnsnlp (A) (Leśniewski 1929, p. 76, Eng. trans. 1991, p. 485).

This passage is at first rather disconcerting: the language used in it is a mixture of Russell–Peano language and freakish looking expressions like 'cnsqrprtqntf'. If one does her homework though, things become perfectly clear. The meaning of all the alien-like expressions is fixed in a list of Terminological Expanations (TE): 'defp (A)' is fixed by TE XLIV, 'thp (A)' by TE XXXII, 'cnsqrprtqntf (C)' 'cnsqeqvl (C, D)' 'cnsqsbstp (A, C)' and 'extnsnlp (A)' respectively by TE XLV, XLVI XLVIII and IL. TE XLIV explains the first directive: it says that B is a protothetical definition immediately after thesis A (B ε defp (A)) if, and only if, 18 conditions are fulfilled (for ontological definitions, the conditions are 21).[34]

[34] Cf. Leśniewski (1930, p. 123, LVI°, Eng. trans. 1991, p. 619). See also Luschei (1962, pp. 272–276, 7.3).

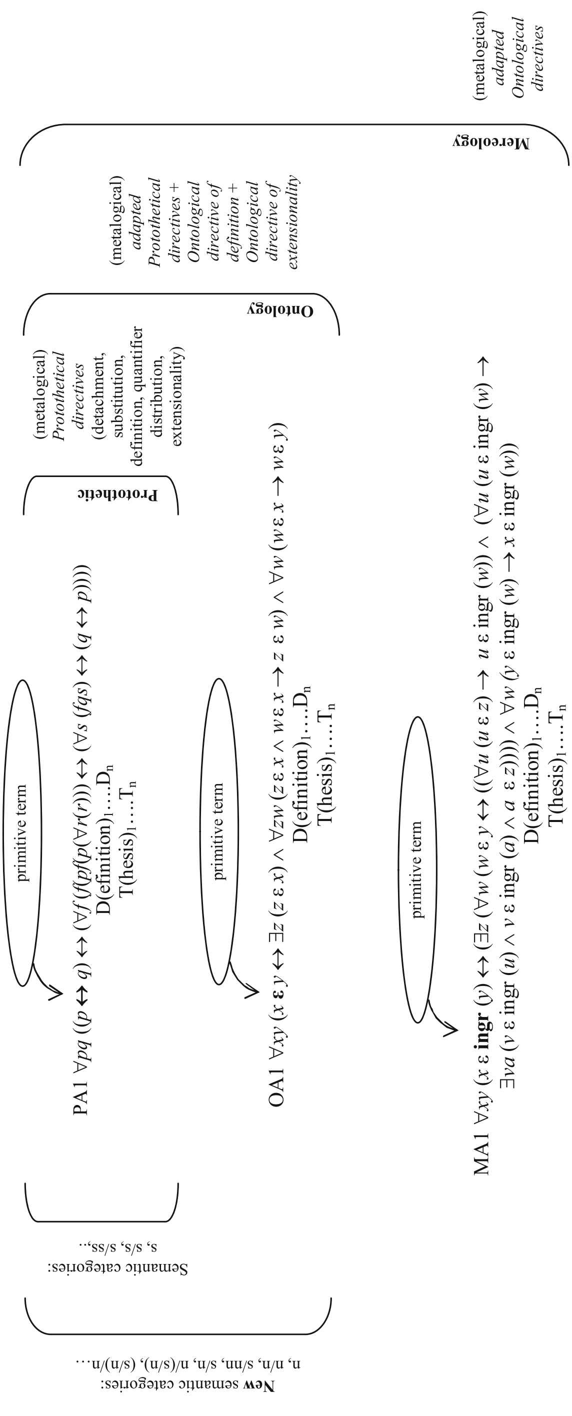

Fig. 1 The disposition of Leśniewski's systems

The directives for 𝔖5 appear at the end of the *Grundzüge*, after the terminological explanation and a list of abbreviations employed in them (and some clarifications to it), containing among others the following abbreviations (on the left, corresponding meaning on the right)

vrb	Word
A1	Axiom A1
thp	Thesis of this system of Protothetic
1ingr (A)	First word belonging to A

Leśniewski adds some clarifications to this list, sometimes by means of examples, for instance:

> *Ad* ‘vrb’. Expressions—‘man’, ‘word’, ‘*p*’ […] ‘(‘—are examples of words. Expressions—the man’, ‘(*p*)’ […] $f\lfloor$) word’—are examples of objects that are combinations of words but are not words. […] Single letters of a word consisting of at least two letters are not words. Expressions consisting of at least two words are not words (Leśniewski 1929, p. 61, Eng. trans. 1991, pp. 469–470).

After the list of abbreviations and the clarifications, Leśniewski introduces the TE. The TE form an abbreviated dictionary with built-in ostensive elements, sometimes resting ultimately on examples. The first two of them are:

TE I. [A] : A ε vrb1. =. A ε cnf (1ingr (A1))
TE II. [A] : A ε vrb2. =. A ε cnf (5ingr (A1))

The list of abbreviations, together with one additional instruction, tells us how to read these strings. For instance, TE I is to be read like this: ‘of an object *A* I say that it is a word1 if and only if *A* is equiform to the first word of A1’. From the abbreviations we know that ‘A1’ in TE I means ‘Axiom I’.[35] ‘Axiom I’ functions like a demonstrative, since it gets its reference by ostension: A1 had been displayed by Leśniewski by actually inscribing and naming the axiom some pages in advance in his paper—and it will be displayed by me now:

A1 $\forall pqr(((p \leftrightarrow r) \leftrightarrow (q \leftrightarrow p)) \leftrightarrow (r \leftrightarrow q))$.[36]

So, in TE I, the name ‘A1’ functions like a singular name of a material object you can ostensively refer to. The list of TE is actually hardly understandable if you do not grasp this ostensive element, that is, the connection between the expression ‘A1’ and a particular inscription. One could of course have chosen another axiom, say Wajsberg’s 1923–1926 single axiom,[37] and adapted the TE to this axiom (for instance, TE II could

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[35] Vrb1 we can call, following LeBlanc (1991, p. 14) ‘quantifier indicator’ and is equiform to this symbol: $\lfloor$. The four quantifier indicators are: $\lfloor , \rfloor$ $\lceil , \rceil$, see Thesis is 40 above.

[36] Leśniewski (1929, p. 33, Eng. trans. 1991, p. 441). A1 is one of the three axioms forming the axiom system of system 𝔖1 of Protothetic.

[37] Cf. Leśniewski (1929, p. 59, Eng. trans. 1991, pp. 467–468).

contain ‘8ingr (A1)’ in place of ‘5ingr (A1)’). However, axioms are part of the system, while directives and everything that serves to their comprehension are not.

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