AN ALGEBRAIC APPROACH TO CANONICAL FORMULAS: INTUITIONISTIC CASE

GURAM BEZHANISHVILI AND NICK BEZHANISHVILI

Abstract. We introduce partial Esakia morphisms, well partial Esakia morphisms, and strong partial Esakia morphisms between Esakia spaces and show that they provide the dual description of $(\wedge, \to)$-homomorphisms, $(\wedge, \to, 0)$-homomorphisms, and $(\wedge, \to, \vee)$-homomorphisms between Heyting algebras, thus establishing a generalization of Esakia duality. This yields an algebraic characterization of Zakharyaschev's subreductions, cofinal subreductions, dense subreductions, and the Closed Domain Condition. As a consequence, we obtain a new simplified proof (which is algebraic in nature) of Zakharyaschev's theorem that each intermediate logic can be axiomatized by canonical formulas.

1. Introduction

The study of intermediate logics (i.e. logics in between the intuitionistic propositional calculus **IPC** and the classical propositional calculus **CPC**) was initiated by Umezawa [32]. The structure of intermediate logics is rather complicated. It was shown by Jankov [19] that there are continuum many intermediate logics and that there are intermediate logics without the finite model property (fmp). By modifying Fine's construction [16] of an incomplete modal logic over **S4**, Shehtman [30] showed that there exists an incomplete intermediate logic. Later it was shown by Litak [22] that there are continuum many incomplete intermediate logics. These negative results motivated a search of the right tools for the study of intermediate logics.

Several such tools have been developed over the years. One is algebraic in nature and uses the splitting technique, which is a consequence of the powerful machinery of ultaproducts in congruence-distributive varieties developed by Jónsson [20]. The splitting technique was used successfully by Blok [6, 7, 8] for better understanding of the complicated structure of modal and intermediate logics. Another useful tool of algebraic nature is Diego's theorem [11] that the variety of implicative meet semilattices is locally finite. This result allowed McKay [23] to show that all intermediate logics axiomatizable by disjunction-free formulas have the fmp. As was shown by Zakharyaschev [40, Thm. 5.7], the class of intermediate logics axiomatizable by disjunction-free formulas coincides with the class of cofinal subframe intermediate logics.

Another useful tool is model-theoretic in nature and provides frame-based formulas introduced by Jankov [18], de Jongh [10], Fine [15, 17], and Zakharyaschev [40] for axiomatization purposes. The Jankov-de Jongh formulas provide an axiomatization of splitting logics and their joins, subframe formulas of Fine and Zakharyaschev provide an axiomatization of subframe logics, while cofinal subframe formulas of Zakharyaschev provide an axiomatization of cofinal subframe logics. For an algebraic approach to subframe and cofinal subframe intermediate logics see [2] and for a general approach to the frame-based formulas see [5].

2000 *Mathematics Subject Classification.* 03B55; 06D20; 06D50.

Key words and phrases. Intermediate logic, Heyting algebra, duality theory.

Although many intermediate logics can be axiomatized by frame-based formulas, not every intermediate logic affords such an axiomatization. To handle all intermediate logics, Zakharyaschev [36, 38] introduced canonical formulas and showed that each intermediate logic can be axiomatized by canonical formulas. This powerful result was later generalized by Zakharyaschev [39] to cover all extensions of the modal logic **K4**. Zakharyaschev's theorem has many useful consequences. To name a few, it provides a solution of the Dummett-Lemmon conjecture that the least modal companion of each complete intermediate logic is complete [38], a proof that the disjunction-free fragment of an intermediate logic with the disjunction property coincides with the disjunction-free fragment of **IPC** [37] (a result proved independently and by a different technique by Minari [25]), and an axiomatization of all subframe and cofinal subframe intermediate logics [40].

Zakharyaschev's proof is rather complicated and relies heavily on the concepts of subreduction and cofinal subreduction. It is the main goal of the present paper to give a simplified proof (which is algebraic in nature) of Zakharyaschev's theorem, and also to provide a purely algebraic explanation of Zakharyaschev's need in subreductions and cofinal subreductions. Our primary tool will be duality theory, which links the algebraic and model-theoretic techniques mentioned above with each other.

Duality theory for Heyting algebras was developed by Esakia [12] using a hybrid of topology and order. The resulting structures, called Esakia spaces, are ordered topological spaces satisfying certain conditions. Esakia showed that the category of Heyting algebras and Heyting algebra homomorphisms is dually equivalent to the category of Esakia spaces and Esakia morphisms. Since Heyting algebras provide an adequate semantics for intermediate logics, it follows that Esakia spaces also provide an adequate semantics for intermediate logics (but we have to restrict valuations of formulas into Esakia spaces to special upsets, which are topologically both open and closed). This useful link allows to transfer algebraic results to the realm of Esakia spaces and vice versa. For example, homomorphic images of Heyting algebras become closed upsets while subalgebras become special quotients of the corresponding Esakia spaces.

Esakia duality, however, is not sufficient to understand fully the algebraic export of Zakharyaschev's theorem. A generalization of Esakia duality (as well as Priestley duality) was recently developed in [3]. Using the results of [3], we develop a generalization of Esakia duality and obtain dual descriptions of $(\wedge, \to)$-homomorphisms, $(\wedge, \to, 0)$-homomorphisms, and $(\wedge, \to, \vee)$-homomorphisms between Heyting algebras. As we will see, they are characterized by means of special partial maps between Esakia spaces we call partial Esakia morphisms, well partial Esakia morphisms, and strong partial Esakia morphisms, respectively. These concepts provide sharpening of Zakharyaschev's subreductions, cofinal subreductions, and dense subreductions. Moreover, a natural generalization of strong Esakia morphisms results in the Closed Domain Condition, which sharpens Zakharyaschev's Closed Domain Condition. On the one hand, this allows us to place Zakharyaschev's results as part of a generalized Esakia duality theory; on the other hand, it opens the door for a purely algebraic proof of Zakharyaschev's theorem that each intermediate logic can be axiomatized by canonical formulas. In our opinion the proof that we offer here is simpler than Zakharyaschev's original proof.

As was pointed out by Zakharyaschev in the introduction to [41], the shape of canonical formulas does not really matter. What matters is that they provide an effective refutation tool. And indeed the shape of canonical formulas we will construct is rather different from that of Zakharyaschev's canonical formulas. Our means are algebraic, while Zakharyaschev's were model-theoretic. Nevertheless, as we will see, they do the same job in providing an axiomatization of each intermediate logic. The canonical formulas we develop in this paper generalize Jankov formulas and our proofs are close in spirit (and in fact generalize) Wronski's approach [35] to Jankov formulas. A similar approach was undertaken by Tomaszewski [31]. However, in [31] there was no attempt made to connect the algebraic approach with the model-theoretic approach of Zakharyaschev.

The paper is organized as follows. In Section 2 we recall Esakia duality for Heyting algebras. In Section 3 we introduce partial Esakia morphisms, well partial Esakia morphisms, and strong partial Esakia morphisms between Esakia spaces, and show that they give the dual description of $(\wedge, \to)$-homomorphisms, $(\wedge, \to, 0)$-homomorphisms, and $(\wedge, \to, \vee)$-homomorphisms between Esakia spaces. This yields a generalized Esakia duality. We also introduce the Closed Domain Condition and for a partial Esakia morphism f between Esakia spaces X and Y, show how announcing some anti-chains of Y as closed domains results in the corresponding $(\wedge, \to)$-homomorphism to preserve the joins of designated pairs of elements of the Heyting algebra dual to Y. In Section 4 we show that partial Esakia morphisms, well partial Esakia morphisms, and strong partial Esakia morphisms provide sharpening of Zakharyaschev's subreductions, cofinal subreductions, and dense subreductions, respectively. We also show that our Closed Domain Condition sharpens Zakharyaschev's Closed Domain Condition. Finally, in Section 5 we give an algebraic account of canonical formulas and give a simplified proof of Zakharyaschev's theorem that each intermediate logic can be axiomatized by canonical formulas. We conclude the paper by showing how to obtain Jankov formulas, subframe formulas, and cofinal subframe formulas as particular cases of canonical formulas.

2. Esakia duality

We recall that a *Heyting algebra* is a bounded (distributive) lattice $(A, \wedge, \vee, 0, 1)$ with an additional binary operation $\to: A^2 \to A$ such that for all $a, b, c \in A$ we have:

$$a \wedge c \leq b \text{ iff } c \leq a \to b.$$

It is well known (see, e.g., [28, Chapter IX] and [9, Chapter 7]) that Heyting algebras provide an adequate algebraic semantics for intermediate logics. In fact, there is a dual isomorphism between the (complete) lattice of intermediate logics and the (complete) lattice of non-trivial varieties of Heyting algebras.

Implicative meet-semilattices (also known as Brouwerian semilattices) are obtained by dropping $\vee$ from the signature of Heyting algebras. That is, an *implicative meet-semilattice* is a bounded meet-semilattice $(A, \wedge, 0, 1)$ with an additional binary operation $\to: A^2 \to A$ such that for all $a, b, c \in A$ we have $a \wedge c \leq b$ iff $c \leq a \to b$. Clearly an implicative meet-semilattice $(A, \wedge, \to, 0, 1)$ is a Heyting algebra iff there is a binary operation $\vee : A^2 \to A$ such that $(A, \wedge, \vee)$ is a (distributive) lattice.

There are many similarities between implicative meet-semilattices and Heyting algebras. For example, in both cases homomorphisms are determined by filters. This result is well known for Heyting algebras (see, e.g., [28, Sec. I.13]); for implicative meet-semilattices it was first established by Nemitz [26] (see also Köhler [21]). As an immediate consequence, we obtain that both varieties have the congruence extension property. For the variety of implicative meet-semilattices this implies that given implicative meet-semilattices A, B, C, a 1-1 meet-semilattice homomorphism $f : A \rightarrowtail B$, and an onto meet-semilattice homomorphism $g : A \twoheadrightarrow C$, there exists an implicative meet-semilattice D, a 1-1 meet-semilattice homomorphism $h : C \rightarrowtail D$, and an onto meet-semilattice homomorphism $k : B \twoheadrightarrow D$ such that $k \circ f = h \circ g$. The same is also true for the variety of Heyting algebras.

But in some respects implicative meet-semilattices behave better than Heyting algebras. For instance, it is well known that the variety of Heyting algebras is not locally finite. In fact, as was shown by Rieger [29] and Nishimura [27], already the one-generated free Heyting algebra is infinite. On the other hand, as follows from Diego [11], the variety of implicative meet-semilattices is locally finite.

Since we will use the facts mentioned above frequently, we gather them together in the following lemma.

Lemma 2.1.

(1) *Each finitely generated implicative meet-semilattice is finite.*
(2) *There is a 1-1 correspondence between homomorphic images and filters of an implicative meet-semilattice. Consequently, there is a 1-1 correspondence between homomorphic images and filters of a Heyting algebra.*
(3) *The variety of bounded implicative meet-semilattices has the congruence extension property. Consequently, the variety of Heyting algebras has the congruence extension property.*

For a partially ordered set $(X, \leq)$ and $Y \subseteq X$, we recall that the *downset* of Y is the set

$$\downarrow Y = \{x \in X : \exists y \in Y \text{ with } x \leq y\}.$$

The *upset* of Y is defined dually and is denoted by $\uparrow Y$. If Y is a singleton set $\{y\}$, then we use $\downarrow y$ and $\uparrow y$ instead of $\downarrow\{y\}$ and $\uparrow\{y\}$, respectively. We call $U \subseteq X$ an *upset* of X if $x \in U$ and $x \leq y$ imply $y \in U$. A *downset* of X is defined dually. Let $\mathrm{Up}(X)$ and $\mathrm{Do}(X)$ denote the sets of all upsets and downsets of X, respectively. It is well known that $(\mathrm{Up}(X), \cap, \cup, \to, \emptyset, X)$ is a Heyting algebra, where for each $U, V \in \mathrm{Up}(X)$, we have:

$$U \to V = \{x \in X : \uparrow x \cap U \subseteq V\} = X - \downarrow(U - V).$$

Similarly, $(\mathrm{Do}(X), \cap, \cup, \to, \emptyset, X)$ is a Heyting algebra, but we will mainly work with the Heyting algebra of upsets of X.

Given a topological space X, we call a subset U of X *clopen* if it is both closed and open. Let $\mathrm{Cp}(X)$ denote the set of clopen subsets of X. We recall that X is *zero-dimensional* if $\mathrm{Cp}(X)$ forms a basis for the topology on X, and that X is a *Stone space* if X is compact, Hausdorff, and zero-dimensional.

Definition 2.2. [12] We call a pair $(X, \leq)$ an *Esakia space* if:

(1) X is a Stone space.
(2) $\leq$ is a partial order on X.
(3) $\uparrow x$ is closed for each $x \in X$.
(4) $U \in \mathrm{Cp}(X)$ implies $\downarrow U \in \mathrm{Cp}(X)$.

By Esakia duality, each Heyting algebra A gives rise to the Esakia space $A_* = (X, \leq)$, where X is the set of prime filters of X, $\leq$ is set-theoretic inclusion, and the topology on X is given by the following basis $\{\varphi(a) - \varphi(b) : a, b \in A\}$, where

$$\varphi(a) = \{x \in X : a \in x\}$$

is the Stone map. Conversely, each Esakia space $(X, \leq)$ gives rise to the Heyting algebra $X^* = (\mathrm{CpUp}(X), \cap, \cup, \to, \emptyset, X)$, where $\mathrm{CpUp}(X)$ is the set of clopen upsets of X, and the Heyting algebra operations on $\mathrm{CpUp}(X)$ are the restrictions of the Heyting algebra operations on $\mathrm{Up}(X)$; that is, X^* is a Heyting subalgebra of $(\mathrm{Up}(X), \cap, \cup, \to, \emptyset, X)$.

Let $(X, \leq)$ and $(Y, \leq)$ be partially ordered sets and $f : X \to Y$ a map. We recall that f is *order-preserving* if $x \leq z$ implies $f(x) \leq f(z)$ for each $x, z \in X$, and that f is a *bounded morphism* (or a *p-morphism*) if in addition for each $x \in X$ and $y \in Y$, from $f(x) \leq y$ it follows that there exists $z \in X$ such that $x \leq z$ and $f(z) = y$. If $(X, \leq)$ and $(Y, \leq)$ are Esakia spaces, then we call a map $f : X \to Y$ an *Esakia morphism* if it is a continuous bounded morphism.

Given Heyting algebras A and B and a Heyting algebra homomorphism $f : A \to B$, we define $f_* : B_* \to A_*$ by $f_*(y) = f^{-1}(y)$ for each prime filter y of B. Then f_* is an Esakia morphism. Moreover, if $f : A \to B$ and $g : B \to C$ are Heyting algebra homomorphisms, then $f_* \circ g_* = (g \circ f)_*$. Conversely, if $(X, \leq)$ and $(Y, \leq)$ are Esakia spaces and $f : X \to Y$ is an Esakia morphism, then $f^* : Y^* \to X^*$ is a Heyting algebra homomorphism, where $f^*(U) = f^{-1}(U)$ for each clopen upset U of Y. Moreover, if $f : X \to Y$ and $g : Y \to Z$ are Esakia morphisms, then $f^* \circ g^* = (g \circ f)^*$. Put together, these observations provide Esakia duality between the category **Heyt** of Heyting algebras and Heyting algebra homomorphisms and the category **Esa** of Esakia spaces and Esakia morphisms:

Theorem 2.3. [12] *The categories* **Heyt** *and* **Esa** *are dually equivalent.*

In fact, given a Heyting algebra A, the Stone map $\varphi : A \to {A_*}^*$ establishes the desired isomorphism of Heyting algebras, and given an Esakia space X, the map $\varepsilon : X \to {X^*}_*$, given by

$$\varepsilon(x) = \{U \in \mathrm{CpUp}(X) : x \in U\},$$

establishes the desired order-homeomorphism of Esakia spaces.

Esakia duality is an extremely useful tool in giving dual descriptions of algebraic concepts important for the study of Heyting algebras. For instance, it is well known (see, e.g., [12, 13]) that filters of a Heyting algebra A dually correspond to closed upsets of A_*, while ideals of A dually correspond to open upsets of A_*; also, subdirectly irreducible Heyting algebras dually correspond to those rooted Esakia spaces in which the root is an isolated point. Here we recall that a Heyting algebra A is *subdirectly irreducible* if $A - \{1\}$ has the largest element s, called the *second largest element* of A; and that an Esakia space X is *rooted* if there exists $x \in X$, called the *root* of X, such that $X = {\uparrow}x$.

There are several other well-known results about Esakia duality that we will use frequently. They can be found, e.g., in [12, 13, 14]. We gather them together in the following lemma.

Lemma 2.4. *Let X be an Esakia space.*

(1) *If F is a closed subset of X, then ${\uparrow}F$ and ${\downarrow}F$ are closed subsets of X.*
(2) *If F and G are closed subsets of X such that ${\uparrow}F \cap {\downarrow}G = \emptyset$, then there exists a clopen upset U of X such that $F \subseteq U$ and $G \subseteq X - U$.*
(3) *If F is a closed upset of X, then F is an Esakia space in the induced topology and order.*
(4) *Let F be a closed subset of X and let $\max(F)$ and $\min(F)$ denote the sets of maximal and minimal points of F, respectively. Then for each $x \in F$ there exist $y \in \max(F)$ and $z \in \min(F)$ such that $z \leq x \leq y$.*

With regard to Lemma 2.4(3), it is worth pointing out that not every closed subset F of an Esakia space X is an Esakia space in the induced topology and order. For an example see [2, Remark 3]. Obviously such an F can not be an upset of X.

3. Generalized Esakia duality

Let A and B be Heyting algebras and $h : A \to B$ a map. Even if h is not a Heyting algebra homomorphism, it may still preserve some of Heyting algebra operations.

Definition 3.1. Let A and B be Heyting algebras and $h : A \to B$ a map.

(1) We call h a $(\wedge, \to)$-*homomorphism* if $h(a \wedge b) = h(a) \wedge h(b)$ and $h(a \to b) = h(a) \to h(b)$ for each $a, b \in A$.
(2) We call h a $(\wedge, \to, 0)$-*homomorphism* if h is a $(\wedge, \to)$-homomorphism and $h(0) = 0$.
(3) We call h a $(\wedge, \to, \vee)$-*homomorphism* if h is a $(\wedge, \to)$-homomorphism and $h(a \vee b) = h(a) \vee h(b)$ for each $a, b \in A$.

Since in a Heyting algebra we always have $a \to a = 1$, for each $(\wedge, \to)$-homomorphism $h : A \to B$ we clearly have $h(1) = 1$. On the other hand, there exist $(\wedge, \to)$-homomorphisms which are neither $(\wedge, \to, 0)$-homomorphisms nor $(\wedge, \to, \vee)$-homomorphisms. Moreover, there exist $(\wedge, \to, 0)$-homomorphisms which are not $(\wedge, \to, \vee)$-homomorphisms and vice versa. As an immediate consequence of Lemma 2.1(2), we obtain the following lemma, which will be used subsequently.

Lemma 3.2. *Let A and B be Heyting algebras and $h : A \to B$ an onto $(\wedge, \to)$-homomorphism. Then $h(0) = 0$ and $h(a \vee b) = h(a) \vee h(b)$ for each $a, b \in A$. Consequently, h is an onto Heyting algebra homomorphism.*

The main goal of this section is to generalize Esakia duality and provide the dual descriptions of $(\wedge, \to)$-homomorphisms, $(\wedge, \to, 0)$-homomorphisms, and $(\wedge, \to, \vee)$-homomorphisms. There are several different (but equivalent) ways to do so. As was shown in [3], where Esakia duality was generalized to implicative semilattices (and Priestley duality was generalized to distributive semilattices), this can be done either by means of special binary relations or by means of special partial maps between Esakia spaces. We choose to work with partial maps since it is closer in spirit to Zakharyaschev's approach.

3.1. **Partial Esakia morphisms.**

Definition 3.3. Let $(X, \leq)$ and $(Y, \leq)$ be Esakia spaces and $f : X \to Y$ a partial map. We denote by $\mathrm{dom}(f)$ the domain of f. We call f a *partial Esakia morphism* if the following conditions are satisfied:

(1) If $x, z \in \mathrm{dom}(f)$ and $x \leq z$, then $f(x) \leq f(z)$.
(2) If $x \in \mathrm{dom}(f)$, $y \in Y$, and $f(x) \leq y$, then there exists $z \in \mathrm{dom}(f)$ such that $x \leq z$ and $f(z) = y$.
(3) For $x \in X$, we have $x \in \mathrm{dom}(f)$ iff there exists $y \in Y$ such that $f[{\uparrow}x] = {\uparrow}y$.
(4) $f[{\uparrow}x]$ is closed for each $x \in X$.
(5) If $U \in \mathrm{CpUp}(Y)$, then $X - {\downarrow}f^{-1}(Y - U) \in \mathrm{CpUp}(X)$.

Remark 3.4. Let $x \in X$. As follows from condition (3) of Definition 3.3, if there exists $y \in Y$ such that $f[{\uparrow}x] = {\uparrow}y$, then $x \in \mathrm{dom}(f)$. In fact, we have $f(x) = y$. To see this, as $y \in f[{\uparrow}x]$, there exists $z \in \mathrm{dom}(f)$ such that $x \leq z$ and $f(z) = y$. This, by condition (1) of Definition 3.3, implies that $f(x) \leq y$. On the other hand, from $x \in {\uparrow}x$ it follows that $f(x) \in f[{\uparrow}x] = {\uparrow}y$. Therefore, $y \leq f(x)$, and so $f(x) = y$. Consequently, if $x \in \mathrm{dom}(f)$, then $f[{\uparrow}x] = {\uparrow}f(x)$. We will use these facts frequently.

Lemma 3.5. *Let X and Y be Esakia spaces and $f : X \to Y$ a partial map. Then for each $x \in X$ and $U \subseteq Y$ we have $x \in X - {\downarrow}f^{-1}(Y - U)$ iff $f[{\uparrow}x] \subseteq U$.*

Proof. We have:

$$\begin{aligned} x \in X - {\downarrow}f^{-1}(Y - U) \quad &\text{iff} \quad x \notin {\downarrow}f^{-1}(Y - U) \\ &\text{iff} \quad {\uparrow}x \cap f^{-1}(Y - U) = \emptyset \\ &\text{iff} \quad f[{\uparrow}x] \cap (Y - U) = \emptyset \\ &\text{iff} \quad f[{\uparrow}x] \subseteq U. \end{aligned}$$

□

We show that if $f : X \to Y$ is a partial Esakia morphism, then $\mathrm{dom}(f)$ is a closed subset of X. This involves the concept of a *net*. In order to keep the paper self-contained, we recall the necessary definitions and facts. For more background on nets we refer to Willard [34, Sec. 11].

We recall that a poset $(I, \leq)$ is *directed* if for each $\alpha, \beta \in I$ there exists $\sigma \in I$ such that $\alpha, \beta \leq \sigma$. Given two directed posets $(I, \leq)$ and $(J, \leq)$, a map $f : I \to J$ is *cofinal* if for each $\lambda \in J$, there exists $\sigma \in I$ such that $\lambda \leq f(\sigma)$.

Let X be a topological space. A *net* is a map from a directed poset into the space X. We will denote nets by $N = \{x_\sigma : \sigma \in I\}$, where I is the directed poset corresponding to the net N and x_σ is the point assigned to $\sigma \in I$. A net $M = \{y_\lambda : \lambda \in J\}$ is a *subnet* of a net $N = \{x_\sigma : \sigma \in I\}$ if there is an order-preserving cofinal map $f : J \to I$ such that $y_\lambda = x_{f(\lambda)}$ for all $\lambda \in J$.

Let $N = \{x_\sigma : \sigma \in I\}$ be a net and $A \subseteq X$. Then N *is in* A if $x_\sigma \in A$ for all $\sigma \in I$; N is *eventually* in A if there exists $\sigma_0 \in I$ such that $x_\sigma \in A$ for all $\sigma \geq \sigma_0$; and N is *cofinally* in A if for each $\sigma \in I$ there exists $\lambda \geq \sigma$ such that $x_\lambda \in A$. A set of the form $\{x_\sigma \in N : \sigma \geq \sigma_0 \in I\}$ is called a *tail* of N. We say that N *converges* to $x \in X$, or x is a *limit point* of N, if N is eventually in U for each open neighborhood U of x. A point $x \in X$ is a *cluster point* of N if N is cofinally in U for each open neighborhood U of x.

Lemma 3.6. [34, Sec. 11] *Let X be a topological space, $x \in X$, and $A \subseteq X$.*

1. *$x \in \overline{A}$ iff there exists a net N in A converging to x.*
2. *A net N converges to x iff every subnet of N converges to x.*
3. *x is a cluster point of a net N iff there is a subnet of N converging to x.*
4. *The set of all cluster points of a net is closed.*
5. *If X is compact, then each net has a cluster point.*

Lemma 3.7. *Let X and Y be Esakia spaces and $f : X \to Y$ a partial Esakia morphism. Then $\mathrm{dom}(f)$ is a closed subset of X.*

Proof. Let $x \in \overline{\mathrm{dom}(f)}$. We show that $x \in \mathrm{dom}(f)$. For this, by condition (3) of Definition 3.3, it is sufficient to show that $f[{\uparrow}x]$ has a least element. Since $x \in \overline{\mathrm{dom}(f)}$, by Lemma 3.6(1), there exists a net N in $\mathrm{dom}(f)$ converging to x. As N is a net in X, then $K = \{f(x_\sigma) : x_\sigma \in N\}$ is a net in Y. Let C be the set of cluster points of K. By conditions (4) and (5) of Lemma 3.6, C is a nonempty closed set. We show that $C \cap f[{\uparrow}x] \neq \emptyset$.

Suppose that $C \cap f[\uparrow x] = \emptyset$. By condition (2) of Definition 3.3, $f[\uparrow x]$ is an upset. Therefore, $\downarrow C \cap f[\uparrow x] = \emptyset$. Moreover, $f[\uparrow x]$ is closed by condition (4) of Definition 3.3 and $\downarrow C$ is closed by Lemma 2.4(1). Thus, by Lemma 2.4(2), there exists a clopen downset U of Y such that $\downarrow C \subseteq U$ and $U \cap f[\uparrow x] = \emptyset$. Since C is the set of cluster points of K, K is cofinally in U. Therefore, N is cofinally in $\downarrow f^{-1}(U)$. This implies that N has a subnet S which is contained in $\downarrow f^{-1}(U)$. Since N converges to x and S is a subnet of N, by Lemma 3.6(2), S converges to x. By condition (5) of Definition 3.3, $\downarrow f^{-1}(U)$ is a clopen subset of X. Thus, $x \in \downarrow f^{-1}(U)$, and so there exists $z \in f^{-1}(U)$ such that $x \leq z$. This means that $z \in \uparrow x \cap f^{-1}(U)$, implying that $f(z) \in f[\uparrow x] \cap U$. The obtained contradiction proves that $C \cap f[\uparrow x] \neq \emptyset$.

Let $y \in C \cap f[\uparrow x]$. We show that y is the least element of $f[\uparrow x]$. If not, then there exists $z \in f[\uparrow x]$ such that $y \not\leq z$. By Lemma 2.4(2), there is a clopen downset V of Y such that $z \in V$ and $y \notin V$. Because y is a cluster point of K, by Lemma 3.6(3), there is a subnet M of K converging to y. Then there is a subnet S of N such that $f(S) = M$. By Lemma 3.6(2), S converges to x. As M converges to y, no tail of M is contained in V. Therefore, no tail of S is contained in $f^{-1}(V)$. Since V is a downset of Y, we have that $f^{-1}(V)$ is a downset of $\mathrm{dom}(f)$. Thus, $f^{-1}(V) = \downarrow f^{-1}(V) \cap \mathrm{dom}(f)$. Because $S \subseteq \mathrm{dom}(f)$, this implies that no tail of S is contained in $\downarrow f^{-1}(V)$. On the other hand, $z \in f[\uparrow x]$ implies $x \in \downarrow f^{-1}(V)$, which is a contradiction as S converges to x. Consequently, y is the least element of $f[\uparrow x]$, and so $x \in \mathrm{dom}(f)$. As a result, we obtain that $\overline{\mathrm{dom}(f)} = \mathrm{dom}(f)$, which means that $\mathrm{dom}(f)$ is closed. □

It follows that $\mathrm{dom}(f)$ is a Stone space in the subspace topology. We show that the restriction of f to $\mathrm{dom}(f)$ is a continuous (total) function.

Lemma 3.8. *Let X and Y be Esakia spaces, $f : X \to Y$ a partial Esakia morphism, and U a clopen upset of Y. Then $f^{-1}(U) = \mathrm{dom}(f) \cap (X - \downarrow f^{-1}(Y - U))$.*

Proof. We have $x \in \mathrm{dom}(f) \cap (X - \downarrow f^{-1}(Y - U))$ iff $x \in \mathrm{dom}(f)$ and $f[\uparrow x] \subseteq U$. Since $x \in \mathrm{dom}(f)$, then $f[\uparrow x] = \uparrow f(x)$. Therefore, $f[\uparrow x] \subseteq U$ is equivalent to $\uparrow f(x) \subseteq U$, which is equivalent to $f(x) \in U$, or $x \in f^{-1}(U)$. Thus, $x \in \mathrm{dom}(f) \cap (X - \downarrow f^{-1}(Y - U))$ iff $x \in f^{-1}(U)$, and so $f^{-1}(U) = \mathrm{dom}(f) \cap (X - \downarrow f^{-1}(Y - U))$. □

Lemma 3.9. *Let X and Y be Esakia spaces and $f : X \to Y$ a partial Esakia morphism. Then the restriction of f to $\mathrm{dom}(f)$ is a continuous function.*

Proof. Let U be a clopen subset of Y. Then $U = \bigcup_{i=1}^{n}(U_i - V_i)$ for some $U_i, V_i \in \mathrm{CpUp}(Y)$. Therefore, $f^{-1}(U) = \bigcup_{i=1}^{n}(f^{-1}(U_i) - f^{-1}(V_i))$. By Lemma 3.8, $f^{-1}(U_i) = \mathrm{dom}(f) \cap (X - \downarrow f^{-1}(Y - U))$ and $f^{-1}(V_i) = \mathrm{dom}(f) \cap (X - \downarrow f^{-1}(Y - V_i))$. By condition (5) of Definition 3.3, $X - \downarrow f^{-1}(Y - U_i)$ and $X - \downarrow f^{-1}(Y - V_i)$ are clopen upsets of X. Therefore, $\mathrm{dom}(f) \cap (X - \downarrow f^{-1}(Y - U_i))$ and $\mathrm{dom}(f) \cap (X - \downarrow f^{-1}(Y - V_i))$ are clopen upsets of $\mathrm{dom}(f)$. Thus, $f^{-1}(U)$ is a clopen subset of $\mathrm{dom}(f)$, and so the restriction of f to $\mathrm{dom}(f)$ is a continuous function. □

Corollary 3.10. *Let X and Y be Esakia spaces and $f : X \to Y$ a partial Esakia morphism. If A is a closed subset of X, then $f(A)$ is a closed subset of Y.*

Proof. Let A be a closed subset of X. Then $A \cap \mathrm{dom}(f)$ is a closed subset of $\mathrm{dom}(f)$. As the restriction of f to $\mathrm{dom}(f)$ is a continuous function between Stone spaces, $f(A) = f(A \cap \mathrm{dom}(f))$ is a closed subset of Y. □

3.2. **Partial Esakia morphisms and $(\wedge, \to)$-homomorphisms.** For Heyting algebras A and B and a $(\wedge, \to)$-homomorphism $h : A \to B$, we define $h_* : B_* \to A_*$ as follows:

$$\mathrm{dom}(h_*) = \{x \in B_* : h^{-1}(x) \in A_*\}$$

and for $x \in \mathrm{dom}(h_*)$ we set

$$h_*(x) = h^{-1}(x).$$

Lemma 3.11. *Let A and B be Heyting algebras, $h : A \to B$ a $(\wedge, \to)$-homomorphism, F a filter of B, and y a prime filter of A. If $h^{-1}(F) \subseteq y$, then there exists $x \in \mathrm{dom}(h_*)$ such that $F \subseteq x$ and $h_*(x) = y$.*

Proof. Let F be a filter of B, $y \in A_*$, and $h^{-1}(F) \subseteq y$. Consider the filter G of B generated by $F \cup h[y]$. If G is not proper, then there exist $a \in F$ and $b \in y$ such that $a \wedge h(b) = 0$. Therefore, $a \leq \neg h(b)$, and as $a \in F$, we have $\neg h(b) \in F$. Since h is a $(\wedge, \to)$-homomorphism, $h(\neg b) = h(b \to 0) = h(b) \to h(0) \geq h(b) \to 0 = \neg h(b)$. Thus, $\neg h(b) \leq h(\neg b)$, so $h(\neg b) \in F$, and so $\neg b \in h^{-1}(F) \subseteq y$. This implies that $b, \neg b \in y$, so $0 \in y$, which is a contradiction. Consequently, G is a proper filter.

We show that $h^{-1}(G) = y$. Clearly $y \subseteq h^{-1}(G)$ as $h[y] \subseteq G$. Conversely, if $a \in h^{-1}(G)$, then $h(a) \in G$, and so there exist $b \in F$ and $c \in y$ such that $b \wedge h(c) \leq h(a)$. Therefore, $b \leq h(c) \to h(a) = h(c \to a)$. This implies that $h(c \to a) \in F$, and so $c \to a \in h^{-1}(F) \subseteq y$. Thus, $c \wedge (c \to a) \in y$, and so $a \in y$, which implies that $h^{-1}(G) = y$.

Let x be the maximal filter of B with the property that $G \subseteq x$ and $h^{-1}(x) = y$. It exists by Zorn's lemma. Moreover, $F \subseteq x$. We show that $x \in B_*$. Let $a \vee b \in x$, M be the filter generated by a and x, and N be the filter generated by b and x. If y is properly contained in both $h^{-1}(M)$ and $h^{-1}(N)$, then there exist $c \in h^{-1}(M)$ and $d \in h^{-1}(N)$ such that $c, d \notin y$. Therefore, there exist $e, k \in x$ such that $a \wedge e \leq h(c)$ and $b \wedge k \leq h(d)$. Thus, $(a \wedge e) \vee (b \wedge k) = (a \vee b) \wedge (a \vee k) \wedge (e \vee b) \wedge (e \vee k) \leq h(c) \vee h(d) \leq h(c \vee d)$. Since $a \vee b \in x$, it follows that $(a \vee b) \wedge (a \vee k) \wedge (e \vee b) \wedge (e \vee k) \in x$, and so $h(c \vee d) \in x$. This implies that $c \vee d \in h^{-1}(x) = y$. As $y \in A_*$, we obtain that $c \in y$ or $d \in y$, a contradiction. Therefore, $y = h^{-1}(M)$ or $y = h^{-1}(N)$, so $M = x$ or $N = x$, and so $a \in x$ or $b \in x$. Thus, $x \in B_*$. Consequently, $x \in \mathrm{dom}(h_*)$, and so we have found $x \in \mathrm{dom}(h_*)$ such that $F \subseteq x$ and $h_*(x) = y$. □

Lemma 3.12. *Let A and B be Heyting algebras and $h : A \to B$ a $(\wedge, \to)$-homomorphism. For each $x \in B_*$ and $y \in A_*$, we have $y \in h_*[\uparrow x]$ iff $h^{-1}(x) \subseteq y$. Consequently, if $x \in \mathrm{dom}(h_*)$, then $h_*[\uparrow x] = \uparrow h_*(x)$.*

Proof. We have that $y \in h_*[\uparrow x]$ iff there exists $z \in \mathrm{dom}(h_*)$ such that $x \subseteq z$ and $h_*(z) = y$. Since $h_*(z) = h^{-1}(z)$, if the last condition holds, then clearly $h^{-1}(x) \subseteq y$. Conversely, suppose that $h^{-1}(x) \subseteq y$. Then, by Lemma 3.11, there exists $z \in \mathrm{dom}(h_*)$ such that $x \subseteq z$ and $h_*(z) = y$. Consequently, $y \in h_*[\uparrow x]$ iff $h^{-1}(x) \subseteq y$.

Now suppose that $x \in \mathrm{dom}(h_*)$. Then $y \in h_*[\uparrow x]$ iff $h^{-1}(x) \subseteq y$ iff $h_*(x) \subseteq y$ iff $y \in \uparrow h_*(x)$, and so $h_*[\uparrow x] = \uparrow h_*(x)$. □

Lemma 3.13. *Let A and B be Heyting algebras and $h : A \to B$ a $(\wedge, \to)$-homomorphism.*

1. *For each $a \in A$, we have $x \in \varphi(h(a))$ iff $h_*[\uparrow x] \subseteq \varphi(a)$.*
2. *For each $a \in A$, we have $\varphi(h(a)) = B_* - \downarrow h_*^{-1}(A_* - \varphi(a))$.*
3. *For each $x \in B_*$ and $y \in A_*$, we have $h_*[\uparrow x] = \uparrow y$ iff $h^{-1}(x) = y$.*

Proof. (1) First suppose $x \in \varphi(h(a))$ and show that $h_*[\uparrow x] \subseteq \varphi(a)$. Let $y \in h_*[\uparrow x]$. Then there exists $z \in \mathrm{dom}(h_*)$ such that $x \subseteq z$ and $h^{-1}(z) = y$. From $x \in \varphi(h(a))$ it follows that $h(a) \in x$, and so $a \in h^{-1}(x)$. Since $x \subseteq z$, then $h^{-1}(x) \subseteq h^{-1}(z) = y$. Therefore, $a \in y$, and so $y \in \varphi(a)$. Thus, $h_*[\uparrow x] \subseteq \varphi(a)$. Conversely, if $x \notin \varphi(h(a))$, then $a \notin h^{-1}(x)$. Therefore, there exists $y \in A_*$ such that $h^{-1}(x) \subseteq y$ and $a \notin y$. By Lemma 3.11, $h^{-1}(x) \subseteq y$ implies there exists $z \in \mathrm{dom}(h_*)$ such that $x \subseteq z$ and $h^{-1}(z) = y$. Thus, $y \in h_*[\uparrow x]$ and $y \notin \varphi(a)$, implying that $h_*[\uparrow x] \not\subseteq \varphi(a)$.

(2) is an immediate consequence of (1) as $x \in B_* - \downarrow h_*^{-1}(A_* - \varphi(a))$ iff $h_*[\uparrow x] \subseteq \varphi(a)$.

(3) First suppose that $h^{-1}(x) = y$. Then $x \in \mathrm{dom}(h_*)$. Therefore, by Lemma 3.12, $h_*[\uparrow x] = \uparrow h_*(x) = \uparrow y$. Now suppose that $h_*[\uparrow x] = \uparrow y$. Then $y \in h_*[\uparrow x]$. Thus, by Lemma 3.12, $h^{-1}(x) \subseteq y$. If $y \not\subseteq h^{-1}(x)$, then there exists $a \in A$ such that $a \in y$ and $a \notin h^{-1}(x)$. Therefore, $y \in \varphi(a)$ and $x \notin \varphi(h(a))$. By (1), $x \notin \varphi(h(a))$ implies $h_*[\uparrow x] \not\subseteq \varphi(a)$. But $h_*[\uparrow x] = \uparrow y$. Thus, $y \notin \varphi(a)$, a contradiction. Consequently, $y \subseteq h^{-1}(x)$ and so $h^{-1}(x) = y$. □

Theorem 3.14. *Let A and B be Heyting algebras. If $h : A \to B$ is a $(\wedge, \to)$-homomorphism, then $h_* : B_* \to A_*$ is a partial Esakia morphism.*

Proof. To see that h_* is a partial Esakia morphism, we need to show that h_* satisfies conditions (1)–(5) of Definition 3.3. If $x, z \in \mathrm{dom}(h_*)$ and $x \subseteq z$, then clearly $h_*(x) = h^{-1}(x) \subseteq h^{-1}(z) = h_*(z)$, and so condition (1) of Definition 3.3 is satisfied.

Let $x \in \mathrm{dom}(h_*)$, $y \in A_*$, and $h_*(x) \subseteq y$. Then $h^{-1}(x) \subseteq y$. By Lemma 3.11, there exists $z \in \mathrm{dom}(h_*)$ such that $x \subseteq z$ and $h_*(z) = y$. Therefore, condition (2) of Definition 3.3 is satisfied.

Let $x \in \mathrm{dom}(h_*)$. By Lemma 3.12, $h_*[\uparrow x] = \uparrow h_*(x)$, and so there exists $y \in Y$ $(y = h_*(x))$ such that $h_*[\uparrow x] = \uparrow y$. Conversely, suppose that there exists $y \in Y$ such that $h_*[\uparrow x] = \uparrow y$. Then, by Lemma 3.13(3), $h^{-1}(x) = y$. Therefore, $x \in \mathrm{dom}(h_*)$ and $h_*(x) = y$. Thus, condition (3) of Definition 3.3 is satisfied.

Let $y \notin h_*[\uparrow x]$. By Lemma 3.12, $h^{-1}(x) \not\subseteq y$. Therefore, there exists $a \in h^{-1}(x)$ such that $a \notin y$. Thus, $h(a) \in x$ and $a \notin y$. This means that $x \in \varphi(h(a))$ and $y \notin \varphi(a)$. Therefore, by Lemma 3.13(1), $h_*[\uparrow x] \subseteq \varphi(a)$ and $y \notin \varphi(a)$. Consequently, there exists $U \in \mathrm{Cp}(A_*)$ $(U = X - \varphi(a))$ such that $h_*[\uparrow x] \cap U = \emptyset$ and $y \in U$, so $h_*[\uparrow x]$ is closed, and so condition (4) of Definition 3.3 is satisfied.

Let $U \in \mathrm{CpUp}(A_*)$. Then there exists $a \in A$ such that $U = \varphi(a)$. By Lemma 3.13(2), $B_* - \downarrow h_*^{-1}(A_* - \varphi(a)) = \varphi(h(a)) \in \mathrm{CpUp}(B_*)$. Therefore, $B_* - \downarrow h_*^{-1}(A_* - U) \in \mathrm{CpUp}(B_*)$, so h_* satisfies condition (5) of Definition 3.3, and so h_* is a partial Esakia morphism. □

Let X and Y be Esakia spaces and $f : X \to Y$ a partial Esakia morphism. Define $f^* : \mathrm{CpUp}(Y) \to \mathrm{CpUp}(X)$ by

$$f^*(U) = X - \downarrow f^{-1}(Y - U)$$

for each $U \in \mathrm{CpUp}(Y)$. It follows from condition (5) of Definition 3.3 that f^* is well defined.

Theorem 3.15. *Let X and Y be Esakia spaces and $f : X \to Y$ a partial Esakia morphism. Then f^* is a $(\wedge, \to)$-homomorphism.*

Proof. Let $U, V \in \mathrm{CpUp}(Y)$. By Lemma 3.5,

$$\begin{aligned} x \in f^*(U\cap V) & \text{ iff } f[\uparrow x] \subseteq U \cap V \\ & \text{ iff } f[\uparrow x] \subseteq U \text{ and } f[\uparrow x] \subseteq V \\ & \text{ iff } x \in f^*(U) \text{ and } x \in f^*(V) \\ & \text{ iff } x \in f^*(U) \cap f^*(V). \end{aligned}$$

Therefore, $f^*(U\cap V) = f^*(U)\cap f^*(V)$. This implies that $f^*(U \to V) \subseteq f^*(U) \to f^*(V)$. Indeed, $f^*(U)\cap f^*(U\to V) = f^*(U\cap(U\to V)) \subseteq f^*(V)$, and so $f^*(U\to V)\subseteq f^*(U)\to f^*(V)$. Conversely, let $x \notin f^*(U \to V)$. Then there exists $z \in \mathrm{dom}(f)$ such that $x \le z$ and $U \cap \uparrow f(z) \not\subseteq V$. Therefore, there exists $y \in U$ such that $f(z) \le y$ and $y \notin V$. By condition (2) of Definition 3.3, there exists $u \in \mathrm{dom}(f)$ such that $z \le u$ and $f(u) = y$. Since $x \le z \le u$, then $x \le u$. As $f(u) = y \in U$, then $f[\uparrow u] = \uparrow f(u) = \uparrow y \subseteq U$. This implies that $u \in f^*(U)$. On the other hand, $f(u) = y \notin V$ implies that $u \notin f^*(V)$. Thus, there exists $u \in X$ such that $x \le u$, $u \in f^*(U)$, and $u \notin f^*(V)$, and so $x \notin f^*(U) \to f^*(V)$. Consequently, $f^*(U) \to f^*(V) \subseteq f^*(U \to V)$, so $f^*(U\to V) = f^*(U)\to f^*(V)$, and so f^* is a $(\wedge, \to)$-homomorphism. □

Lemma 3.16. *Let A and B be Heyting algebras and $h : A \to B$ a $(\wedge, \to)$-homomorphism. Then for each $a \in A$ we have $\varphi(h(a)) = {h_*}^*(\varphi(a))$.*

Proof. Let $x \in B_*$. Then, by Lemmas 3.5 and 3.12, we have:

$$\begin{aligned} x \in {h_*}^*(\varphi(a)) & \text{ iff } h_*[\uparrow x] \subseteq \varphi(a) \\ & \text{ iff } (\forall y \in A_*)(h^{-1}(x) \subseteq y \Rightarrow a \in y) \\ & \text{ iff } a \in h^{-1}(x) \\ & \text{ iff } h(a) \in x \\ & \text{ iff } x \in \varphi(h(a)). \end{aligned}$$

□

Lemma 3.17. *Let X and Y be Esakia spaces and $f : X \to Y$ a partial Esakia morphism. Then $x \in \mathrm{dom}(f)$ iff $\varepsilon(x) \in \mathrm{dom}({f^*}_*)$, and for each $x \in \mathrm{dom}(f)$, we have $\varepsilon(f(x)) = {f^*}_*(\varepsilon(x))$.*

Proof. First suppose that $x \in \mathrm{dom}(f)$. Then $f[\uparrow x] = \uparrow f(x)$. To see that $\varepsilon(x) \in \mathrm{dom}({f^*}_*)$ it is sufficient to show that $(f^*)^{-1}(\varepsilon(x))$ is a prime filter of Y^*. Since $f^*(U\cap V) = f^*(U)\cap f^*(V)$, we have that $(f^*)^{-1}(\varepsilon(x))$ is closed under $\cap$. As $f[\uparrow x] = \uparrow f(x) \neq \emptyset$, we also have $\emptyset \notin (f^*)^{-1}(\varepsilon(x))$. Therefore, it is left to be shown that $(f^*)^{-1}(\varepsilon(x))$ is closed under $\cup$. Let $U \cup V \in (f^*)^{-1}(\varepsilon(x))$. Then $x \in f^*(U \cup V)$. This yields $f[\uparrow x] \subseteq U \cup V$, which implies that $\uparrow f(x) \subseteq U\cup V$. Thus, $\uparrow f(x) \subseteq U$ or $\uparrow f(x) \subseteq V$, and so $f[\uparrow x] \subseteq U$ or $f[\uparrow x] \subseteq V$. Therefore, $x \in f^*(U)$ or $x \in f^*(V)$, and so $U \in (f^*)^{-1}(\varepsilon(x))$ or $V \in (f^*)^{-1}(\varepsilon(x))$. Consequently, $(f^*)^{-1}(\varepsilon(x)) \in {Y^*}_*$, which implies that $\varepsilon(x) \in \mathrm{dom}({f^*}_*)$.

Conversely, let $\varepsilon(x) \in \mathrm{dom}({f^*}_*)$. Then $(f^*)^{-1}(\varepsilon(x))$ is a prime filter of Y^*. We show that there exists $y \in Y$ such that $f[\uparrow x] = \uparrow y$, which, by condition (3) of Definition 3.3, implies that $x \in \mathrm{dom}(f)$. If $\min f[\uparrow x]$ consists of at least two points, then an argument similar to [1, Thm. 2.7(1)] produces $U, V \in Y^*$ such that $f[\uparrow x] \subseteq U \cup V$, but $f[\uparrow x] \not\subseteq U$ and $f[\uparrow x] \not\subseteq V$.[1] Note that $W \in (f^*)^{-1}(\varepsilon(x))$ iff $x \in f^*(W)$ iff $f[\uparrow x] \subseteq W$. Therefore, $U \cup V \in (f^*)^{-1}(\varepsilon(x))$,

[1]In order to keep the proof self-contained, we reproduce the argument. By Lemma 2.4(4), for each closed upset U of an Esakia space, we have $U = \uparrow\min(U)$. Therefore, $f[\uparrow x] = \uparrow\min f[\uparrow x]$. Let y and z be two distinct points of $\min f[\uparrow x]$. Obviously for each $w \in \min f[\uparrow x]$ with $y \neq w$ we have $\uparrow w \cap \downarrow y = \emptyset$. By Lemma 2.4(2), there exists a clopen upset U_w of Y such that $w \in U_w$ and $y \notin U_w$. Also, $y \not\le z$ implies there exists a clopen upset U_y of Y such that $y \in U_y$ and $z \notin U_y$. Then $\min f[\uparrow x] \subseteq U_y \cup \bigcup\{U_w : w \in \min f[\uparrow x] \text{ and } w \neq y\}$,

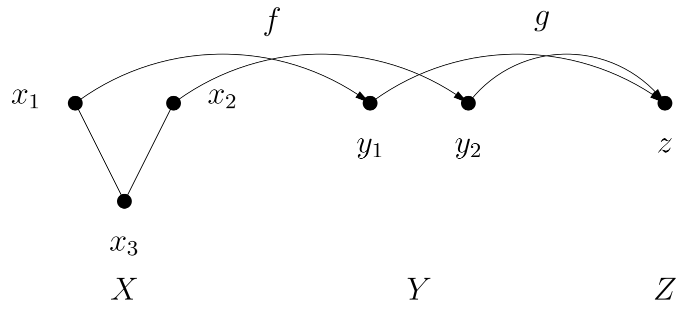


FIGURE 1

but neither $U \in (f^*)^{-1}(\varepsilon(x))$ nor $V \in (f^*)^{-1}(\varepsilon(x))$, which means that $(f^*)^{-1}(\varepsilon(x))$ is not prime. The obtained contradiction proves that $\min f[\uparrow x]$ consists of a single point. Thus, there exists $y \in Y$ such that $f[\uparrow x] = \uparrow y$, and so $x \in \mathrm{dom}(f)$.

Now suppose that $x \in \mathrm{dom}(f)$. We have $U \in \varepsilon(f(x))$ iff $f(x) \in U$, and $U \in {f^*}_*(\varepsilon(x))$ iff $f^*(U) \in \varepsilon(x)$ iff $x \in f^*(U)$ iff $f[\uparrow x] \subseteq U$. But as $x \in \mathrm{dom}(f)$, we have $f[\uparrow x] = \uparrow f(x)$. Therefore, $f[\uparrow x] \subseteq U$ iff $\uparrow f(x) \subseteq U$ iff $f(x) \in U$. Thus, $\varepsilon(f(x)) = {f^*}_*(\varepsilon(x))$. □

3.3. **Composition of partial Esakia morphisms.** In general, the composition of partial Esakia morphisms may not be a partial Esakia morphism as follows from the following example.

Example 3.18. Let X, Y, and Z be the finite Esakia spaces (that is, the finite posets) and let $f : X \to Y$ and $g : Y \to Z$ be the partial Esakia morphisms shown in Figure 1. Note that g is in fact an Esakia morphism. Let $g \circ f : X \to Z$ denote the composition of f and g. Then $\mathrm{dom}(g \circ f) = \mathrm{dom}(f) = \{x_1, x_2\}$. Moreover, $(g \circ f)[\uparrow x_3] = g(\{y_1, y_2\}) = \{z\} = \uparrow z$. But $x_3 \notin \mathrm{dom}(g \circ f)$. Therefore, condition (3) of Definition 3.3 is not satisfied, and so $g \circ f$ is not a partial Esakia morphism.

This indicates that the composition of two partial Esakia morphisms needs to be defined in a slightly different fashion. Let X, Y, and Z be Esakia spaces and let $f : X \to Y$ and $g : Y \to Z$ be partial Esakia morphisms. We define $g * f : X \to Y$ as follows. We set

$$\mathrm{dom}(g * f) = \{x \in X : g(f[\uparrow x]) = \uparrow z \text{ for some } z \in Z\}$$

and for each $x \in \mathrm{dom}(g * f)$ we set $(g * f)(x) = z$, where $g(f[\uparrow x]) = \uparrow z$.

Remark 3.19. It follows from the definition of $\mathrm{dom}(g * f)$ that $\{x \in \mathrm{dom}(f) : f(x) \in \mathrm{dom}(g)\}$ is a subset of $\mathrm{dom}(g * f)$ as if $x \in \mathrm{dom}(f)$ and $f(x) \in \mathrm{dom}(g)$, then $g(f[\uparrow x]) = \uparrow g(f(x))$ and $(g * f)(x) = g(f(x))$. On the other hand, Example 3.18 shows that $\{x \in \mathrm{dom}(f) : f(x) \in \mathrm{dom}(g)\}$ may be a proper subset of $\mathrm{dom}(g * f)$.

It is our goal to show that $g * f$ is a partial Esakia morphism that is dual to $f^* \circ g^*$.

Lemma 3.20. *Let X, Y, and Z be Esakia spaces and let $f : X \to Y$ and $g : Y \to Z$ be partial Esakia morphisms. Then $(g * f)[\uparrow x] = g(f[\uparrow x])$.*

---

and so $f[\uparrow x] = \uparrow \min f[\uparrow x] \subseteq U_y \cup \bigcup\{U_w : w \in \min f[\uparrow x] \text{ and } w \neq y\}$. Since $f[\uparrow x]$ is compact, there exist $U_{w_1}, \dots, U_{w_n}$ such that $f[\uparrow x] \subseteq U_y \cup U_{w_1} \cup \dots \cup U_{w_n}$. Let $U = U_y$ and $V = U_{w_1} \cup \dots \cup U_{w_n}$. Then $f[\uparrow x] \subseteq U \cup V$, but $f[\uparrow x] \not\subseteq U$ and $f[\uparrow x] \not\subseteq V$.

Proof. Let $z \in (g*f)[\uparrow x]$. Then there exists $u \in \mathrm{dom}(g*f)$ such that $x \leq u$ and $(g*f)(u) = z$. This by the definition of $\mathrm{dom}(g * f)$ means that $g(f[\uparrow u]) = \uparrow z$. Since $x \leq u$, it follows that $g(f[\uparrow u]) \subseteq g(f[\uparrow x])$. Thus, $z \in g(f[\uparrow x])$, and so $(g * f)[\uparrow x] \subseteq g(f[\uparrow x])$. Conversely, let $z \in g(f[\uparrow x])$. Then there exists $u \in \mathrm{dom}(f)$ such that $x \leq u$, $f(u) \in \mathrm{dom}(g)$, and $g(f(u)) = z$. Therefore, $gf[\uparrow u] = g[\uparrow f(u)] = \uparrow gf(u)$, which means that $u \in \mathrm{dom}(g * f)$ and $z = (g * f)(u)$. Thus, $z \in (g * f)[\uparrow x]$, and so $g(f[\uparrow x]) \subseteq (g * f)[\uparrow x]$. Consequently, $(g * f)[\uparrow x] = g(f[\uparrow x])$. □

Lemma 3.21. *Let X, Y, and Z be Esakia spaces and let $f : X \to Y$ and $g : Y \to Z$ be partial Esakia morphisms. Then $g * f : X \to Y$ is a partial Esakia morphism.*

Proof. Let $x, z \in \mathrm{dom}(g * f)$ and $x \leq z$. Then $\uparrow z \subseteq \uparrow x$, and so $g(f[\uparrow z]) \subseteq g(f[\uparrow x])$. Since $x, z \in \mathrm{dom}(g*f)$, there exist $u, v \in Z$ such that $g(f[\uparrow x]) = \uparrow u$ and $g(f[\uparrow z]) = \uparrow v$. Therefore, $\uparrow v \subseteq \uparrow u$, and so $u \leq v$. As $u = (g*f)(x)$ and $v = (g*f)(z)$, we obtain $(g*f)(x) \leq (g*f)(z)$. Thus, condition (1) of Definition 3.3 is satisfied.

Next let $x \in \mathrm{dom}(g * f)$, $y \in Z$, and $(g * f)(x) \leq y$. Then there exists $u \in Z$ such that $g(f[\uparrow x]) = \uparrow u$ and $(g * f)(x) = u$. Therefore, $y \in g(f[\uparrow(x)])$, and so there exists $z \in \mathrm{dom}(f)$ such that $x \leq z$, $f(z) \in \mathrm{dom}(g)$, and $g(f(z)) = y$. But then $g(f[\uparrow(z)]) = g[\uparrow f(z)] = \uparrow gf(z)$, so $z \in \mathrm{dom}(g * f)$ and $(g * f)(z) = g(f(z))$. Thus, there exists $z \in \mathrm{dom}(g * f)$ such that $x \leq z$ and $(g * f)(z) = y$, and so condition (2) of Definition 3.3 is satisfied.

That $g * f$ satisfies condition (3) of Definition 3.3 follows from the definition of $g * f$ and Lemma 3.20.

To see that $g * f$ satisfies condition (4) of Definition 3.3, let $x \in X$. Since f is a partial Esakia morphism, $f[\uparrow x]$ is a closed subset of Y. As g is a partial Esakia morphism, by Corollary 3.10, $g(f[\uparrow x])$ is a closed subset of Z. By Lemma 3.20, $g(f[\uparrow x]) = (g * f)[\uparrow x]$. Therefore, $(g * f)[\uparrow x]$ is a closed subset of Z.

Finally, to see that $g*f$ satisfies condition (5) of Definition 3.3, let $U \in \mathrm{CpUp}(Z)$. We show that $X - {\downarrow}(g*f)^{-1}(Z-U) = X - {\downarrow}f^{-1}({\downarrow}g^{-1}(Z-U))$. We have $x \in X - {\downarrow}(g*f)^{-1}(Z-U)$ iff $(g*f)[\uparrow x] \subseteq U$. By Lemma 3.20, $(g*f)[\uparrow x] = g(f[\uparrow x])$. Therefore, $x \in X - {\downarrow}(g*f)^{-1}(Z-U)$ iff $g(f[\uparrow x]) \subseteq U$. On the other hand, $x \in X - {\downarrow}f^{-1}({\downarrow}g^{-1}(Z-U))$ iff $f[\uparrow x] \cap {\downarrow}g^{-1}(Z-U) = \emptyset$. Since $f[\uparrow x]$ is an upset, the last condition is equivalent to $f[\uparrow x] \cap g^{-1}(Z - U) = \emptyset$, which is equivalent to $g(f[\uparrow x]) \subseteq U$. Thus, $x \in X - {\downarrow}(g*f)^{-1}(Z-U)$ iff $x \in X - {\downarrow}f^{-1}({\downarrow}g^{-1}(Z-U))$, and so $X - {\downarrow}(g*f)^{-1}(Z-U) = X - {\downarrow}f^{-1}({\downarrow}g^{-1}(Z-U))$. As g is a partial Esakia morphism, ${\downarrow}g^{-1}(Z - U)$ is a clopen downset of Y, and because f is a partial Esakia morphism, $X - {\downarrow}f^{-1}({\downarrow}g^{-1}(Z-U))$ is a clopen upset of X. Consequently, $X - {\downarrow}(g*f)^{-1}(Z-U) \in \mathrm{CpUp}(X)$, so condition (5) of Definition 3.3 is satisfied, and so $g * f$ is a partial Esakia morphism. □

Lemma 3.22. *Let X, Y, and Z be Esakia spaces and let $f : X \to Y$ and $g : Y \to Z$ be partial Esakia morphisms. For each $x \in X$ and a clopen upset U of Z, we have $x \in (f^* \circ g^*)(U)$ iff $g(f[\uparrow x]) \subseteq U$.*

Proof. By Lemma 3.5, we have:

$$\begin{aligned} x \in (f^* \circ g^*)(U) \quad &\text{iff} \quad x \in f^*(g^*(U)) \\ &\text{iff} \quad f[\uparrow x] \subseteq g^*(U) \\ &\text{iff} \quad (\forall y \in f[\uparrow x])(y \in g^*(U)) \\ &\text{iff} \quad (\forall y \in f[\uparrow x])(g[\uparrow y] \subseteq U) \\ &\text{iff} \quad g(f[\uparrow x]) \subseteq U. \end{aligned}$$

□

Putting Lemmas 3.5, 3.20, and 3.22 together we obtain:

Lemma 3.23. *Let X, Y, and Z be Esakia spaces and let $f : X \to Y$ and $g : Y \to Z$ be partial Esakia morphisms. Then $(g * f)^* = f^* \circ g^*$.*

Proof. It is sufficient to show that $(g * f)^*(U) = (f^* \circ g^*)(U)$ for each clopen upset U of Z. We have:

$$\begin{array}{rll} x \in (g * f)^*(U) & \text{iff} & (g * f)[\uparrow x] \subseteq U \\ & \text{iff} & g(f[\uparrow x]) \subseteq U \\ & \text{iff} & x \in (f^* \circ g^*)(U). \end{array}$$

Therefore, $(g * f)^*(U) = (f^* \circ g^*)(U)$, and so $(g * f)^* = f^* \circ g^*$. □

Lemma 3.24. *Let X, Y, Z, and W be Esakia spaces and let $f : X \to Y$, $g : Y \to Z$, and $h : Z \to W$ be partial Esakia morphisms. Then $h * (g * f) = (h * g) * f$.*

Proof. This follows easily from $f^* \circ (g^* \circ h^*) = (f^* \circ g^*) \circ h^*$ and Lemma 3.23. □

It is also clear that the identity map on an Esakia space is a partial Esakia morphism. Therefore, Esakia spaces and partial Esakia morphisms with the composition $*$ form a category we denote by $\mathbf{Esa^P}$. Let also $\mathbf{Heyt}^{(\wedge,\to)}$ denote the category of Heyting algebras and $(\wedge, \to)$-homomorphisms.

3.4. **Generalized Esakia duality.** It is our goal to show that $\mathbf{Heyt}^{(\wedge,\to)}$ is dually equivalent to $\mathbf{Esa^P}$, thus generalizing Esakia duality.

Let A, B, and C be Heyting algebras and let $h : A \to B$ and $k : B \to C$ be $(\wedge, \to)$-homomorphisms. We show that $(k \circ h)_* = h_* * k_*$.

Lemma 3.25. *Let A, B, and C be Heyting algebras, let $h : A \to B$ and $k : B \to C$ be $(\wedge, \to)$-homomorphisms, $x \in C_*$, and $z \in A_*$.*

1. *$z \in h_*(k_*[\uparrow x])$ iff $h^{-1}(k^{-1}(x)) \subseteq z$.*
2. *$h_*(k_*[\uparrow x]) = \uparrow z$ iff $h^{-1}(k^{-1}(x)) = z$.*

Proof. (1) First suppose that $z \in h_*(k_*[\uparrow x])$. Then there exists $y \in \mathrm{dom}(h_*)$ such that $y \in k_*[\uparrow x]$ and $h_*(y) = z$. Since $y \in k_*[\uparrow x]$, there exists $u \in \mathrm{dom}(k_*)$ such that $x \subseteq u$ and $k_*(u) = y$. But then $h^{-1}(k^{-1}(x)) \subseteq h^{-1}(k^{-1}(u))$ and $h^{-1}(k^{-1}(u)) = h_*(k_*(u)) = h_*(y) = z$. Therefore, $h^{-1}(k^{-1}(x)) \subseteq z$. Conversely, suppose that $h^{-1}(k^{-1}(x)) \subseteq z$. By Lemma 3.11, there exists $y \in \mathrm{dom}(h_*)$ such that $k^{-1}(x) \subseteq y$ and $h_*(y) = z$. Using Lemma 3.11 again produces $u \in \mathrm{dom}(k_*)$ such that $x \subseteq u$ and $k_*(u) = y$. But then $h_*(k_*(u)) = z$, and as $u \in \uparrow x$, we obtain $z \in h_*(k_*[\uparrow x])$.

(2) First suppose that $h_*(k_*[\uparrow x]) = \uparrow z$. Then $z \in h_*(k_*[\uparrow x])$, and by (1), $h^{-1}(k^{-1}(x)) \subseteq z$. If $z \not\subseteq h^{-1}(k^{-1}(x))$, then there exists $a \in z$ such that $k(h(a)) \notin x$. Therefore, $z \in \varphi(a)$ and $x \notin \varphi(k(h(a)))$. By Lemma 3.13(1), $x \notin \varphi(k(h(a)))$ implies $k_*[\uparrow x] \not\subseteq \varphi(h(a))$. Therefore, there exists $y \in k_*[\uparrow x]$ such that $y \notin \varphi(h(a))$. Applying Lemma 3.13(1) again, we obtain $h_*[\uparrow y] \not\subseteq \varphi(a)$. Thus, $h_*(k_*[\uparrow x]) \not\subseteq \varphi(a)$. But $h_*(k_*[\uparrow x]) = \uparrow z$. This implies that $z \notin \varphi(a)$, which is a contradiction. Consequently, $z \subseteq h^{-1}(k^{-1}(x))$, and so $h^{-1}(k^{-1}(x)) = z$.

Conversely, suppose that $h^{-1}(k^{-1}(x)) = z$. Then $h^{-1}(k^{-1}(x)) \subseteq z$, which by (1) means that $z \in h_*(k_*[\uparrow x])$. Therefore, $\uparrow z \subseteq h_*(k_*[\uparrow x])$. Let $w \in h_*(k_*[\uparrow x])$. Then there exists $u \in \mathrm{dom}(k_*)$ such that $x \subseteq u$, $k_*(u) \in \mathrm{dom}(h_*)$, and $w = h_*(k_*(u))$. Thus, $w = h^{-1}(k^{-1}(u)) \supseteq h^{-1}(k^{-1}(x)) = z$. This implies that $w \in \uparrow z$, so $h_*(k_*[\uparrow x]) \subseteq \uparrow z$, and so $h_*(k_*[\uparrow x]) = \uparrow z$. □

Lemma 3.26. *Let A, B, and C be Heyting algebras and let $h : A \to B$ and $k : B \to C$ be $(\wedge, \to)$-homomorphisms. Then $(k \circ h)_* = h_* * k_*$.*

Proof. By Lemma 3.25(2), we have:

$$\begin{aligned} x \in \mathrm{dom}(k \circ h)_* \quad &\text{iff} \quad (k \circ h)^{-1}(x) \in A_* \\ &\text{iff} \quad h^{-1}(k^{-1}(x)) \in A_* \\ &\text{iff} \quad h_*(k_*[\uparrow x]) = \uparrow z \text{ for some } z \in A_* \\ &\text{iff} \quad x \in \mathrm{dom}(h_* * k_*). \end{aligned}$$

Therefore, $\mathrm{dom}(k \circ h)_* = \mathrm{dom}(h_* * k_*)$, and for each $x \in \mathrm{dom}(k \circ h)_*$, we have

$$(k \circ h)_*(x) = (k \circ h)^{-1}(x) = h^{-1}(k^{-1}(x)) = h_*(k_*(x)) = (h_* * k_*)(x).$$

Thus, $(k \circ h)_* = h_* * k_*$. □

Theorem 3.27. *The categories* $\mathbf{Heyt}^{(\wedge,\to)}$ *and* $\mathbf{Esa^P}$ *are dually equivalent.*

Proof. By Theorem 3.14 and Lemma 3.26, $(-)_* : \mathbf{Heyt}^{(\wedge,\to)} \to \mathbf{Esa^P}$ is a well-defined functor. Also, by Theorem 3.15 and Lemma 3.23, $(-)^* : \mathbf{Esa^P} \to \mathbf{Heyt}^{(\wedge,\to)}$ is a well-defined functor. Lastly, Lemmas 3.16 and 3.17 imply that the functors $(-)_*$ and $(-)^*$ establish a dual equivalence of $\mathbf{Heyt}^{(\wedge,\to)}$ and $\mathbf{Esa^P}$. □

As a consequence of Theorem 3.27, we give the dual description of 1-1 and onto $(\wedge, \to)$-homomorphisms. This will play an important role in Section 5.

Definition 3.28. Let X and Y be Esakia spaces and $f : X \to Y$ a partial Esakia morphism. We call f *onto* if the restriction of f to $\mathrm{dom}(f)$ is an onto map.

Lemma 3.29. *Let A and B be Heyting algebras and $h : A \to B$ a $(\wedge, \to)$-homomorphism. Then h is 1-1 iff $h_* : B_* \to A_*$ is onto.*

Proof. First suppose that h is 1-1. For $y \in A_*$ let F be the filter of B generated by $h[y]$ and I be the ideal of B generated by $h[A - y]$. If there exists $b \in F \cap I$, then there exist $a \in y$ and $c_1, \dots, c_n \in A - y$ such that $h(a) \le b \le h(c_1) \vee \dots \vee h(c_n) \le h(c_1 \vee \dots \vee c_n)$. Since h is 1-1, the last inequality implies $a \le c_1 \vee \dots \vee c_n$. As $A - y$ is an ideal of A, we obtain $a \in A - y$. The obtained contradiction proves that $F \cap I = \emptyset$. Therefore, there exists a prime filter x of B such that $F \subseteq x$ and $x \cap I = \emptyset$. From $F \subseteq x$ it follows that $h[y] \subseteq x$, and from $x \cap I = \emptyset$ it follows that $a \notin y$ implies $h(a) \notin x$. Thus, $y \subseteq h^{-1}(x)$ and $h^{-1}(x) \subseteq y$, implying that $y = h^{-1}(x)$. Consequently, $x \in \mathrm{dom}(h_*)$ and $h_*(x) = y$, which means that h_* is onto.

Now suppose that h_* is onto. To see that h is 1-1, it is sufficient to show that $a \not\le b$ implies $h(a) \not\le h(b)$ for each $a, b \in A$. From $a \not\le b$ it follows that there exists a prime filter y of A such that $a \in y$ and $b \notin y$. Since h_* is onto, there exists $x \in \mathrm{dom}(h_*)$ such that $h_*(x) = y$. Therefore, $h^{-1}(x) = y$, and so $h(a) \in x$ and $h(b) \notin x$. But then $h(a) \not\le h(b)$, implying that h is 1-1. □

In order to give the dual description of onto $(\wedge, \to)$-homomorphisms, we recall that by Lemma 2.1(2), onto $(\wedge, \to)$-homomorphisms are characterized by filters of Heyting algebras, and hence are simply onto Heyting algebra homomorphisms. It is well known that in Esakia duality onto Heyting algebra homomorphisms are characterized by 1-1 Esakia morphisms. Consequently, if A and B are Heyting algebras and $h : A \to B$ is a $(\wedge, \to)$-homomorphism, then h is onto iff h is an onto Heyting algebra homomorphism, which is equivalent to h_* being a 1-1 Esakia morphism. Since 1-1 Esakia morphisms into an Esakia space X correspond to closed upsets of X, we obtain that onto $(\wedge, \to)$-homomorphisms from a Heyting algebra A dually correspond to closed upsets of A_*.

3.5. **Well partial Esakia morphisms.**

Definition 3.30. Let $(X, \leq)$ and $(Y, \leq)$ be Esakia spaces and $f : X \to Y$ a partial Esakia morphism. We call f a *well partial Esakia morphism* if for each $x \in X$ there exists $z \in \mathrm{dom}(f)$ such that $x \leq z$.

Lemma 3.31. *Let A and B be Heyting algebras. If $h : A \to B$ is a $(\wedge, \to, 0)$-homomorphism, then $h_* : B_* \to A_*$ is a well partial Esakia morphism.*

Proof. It follows from Theorem 3.14 that h_* is a partial Esakia morphism. Let $x \in B_*$. Then $0 \notin x$. Since $h(0) = 0$, then $0 \notin h^{-1}(x)$. Therefore, there exists $y \in A_*$ such that $h^{-1}(x) \subseteq y$. By Lemma 3.11, there exists $z \in \mathrm{dom}(h_*)$ such that $x \subseteq z$ and $h_*(z) = y$. Thus, h_* is a well partial Esakia morphism. □

Lemma 3.32. *Let X and Y be Esakia spaces and $f : X \to Y$ a well partial Esakia morphism. Then f^* is a $(\wedge, \to, 0)$-homomorphism.*

Proof. It follows from Theorem 3.15 that f^* is a $(\wedge, \to)$-homomorphism. Since f is well, then ${\downarrow} f^{-1}(Y) = X$. Therefore,

$$f^*(\emptyset) = X - {\downarrow} f^{-1}(Y - \emptyset) = X - {\downarrow} f^{-1}(Y) = X - X = \emptyset,$$

and so f^* is a $(\wedge, \to, 0)$-homomorphism. □

It is also clear that the identity map on an Esakia space is a well partial Esakia morphism, and if f and g are well partial Esakia morphisms, then so is $g * f$. Thus, Esakia spaces and well partial Esakia morphisms form a category we denote by $\mathbf{Esa}^{\mathbf{W}}$. Clearly, $\mathbf{Esa}^{\mathbf{W}}$ is a proper subcategory of $\mathbf{Esa}^{\mathbf{P}}$. Let also $\mathbf{Heyt}^{(\wedge,\to,0)}$ denote the category of Heyting algebras and $(\wedge, \to, 0)$-homomorphisms. Clearly $\mathbf{Heyt}^{(\wedge,\to,0)}$ is a proper subcategory of $\mathbf{Heyt}^{(\wedge,\to)}$. As an immediate consequence of Theorem 3.27 and Lemmas 3.31 and 3.32, we obtain:

Theorem 3.33. *The categories $\mathbf{Heyt}^{(\wedge,\to,0)}$ and $\mathbf{Esa}^{\mathbf{W}}$ are dually equivalent.*

3.6. **Strong partial Esakia morphisms.**

Definition 3.34. Let X and Y be Esakia spaces and $f : X \to Y$ a partial Esakia morphism. We call f *strong* if for each $x \in X$, whenever $f[{\uparrow} x] \neq \emptyset$, then $x \in \mathrm{dom}(f)$.

Lemma 3.35. *Let A and B be Heyting algebras. If $h : A \to B$ is a $(\wedge, \to, \vee)$-homomorphism, then $h_* : B_* \to A_*$ is a strong partial Esakia morphism.*

Proof. It follows from Theorem 3.14 that h_* is a partial Esakia morphism. Let $x \in B_*$. By Lemma 3.12, $y \in h_*[{\uparrow} x]$ iff $h^{-1}(x) \subseteq y$. We have that either $0 \in h^{-1}(x)$ or $0 \notin h^{-1}(x)$. If $0 \in h^{-1}(x)$, then $h^{-1}(x) = A$. Therefore, there is no $y \in A_*$ such that $h^{-1}(x) \subseteq y$, and so $h_*[{\uparrow} x] = \emptyset$. On the other hand, if $0 \notin h^{-1}(x)$, then as h is a $(\wedge, \to, \vee)$-homomorphism, $h^{-1}(x) \in A_*$. This yields that $x \in \mathrm{dom}(h_*)$. Thus, if $h_*[{\uparrow} x] \neq \emptyset$, then $x \in \mathrm{dom}(h_*)$, and so h_* is a strong partial Esakia morphism. □

Lemma 3.36. *Let X and Y be Esakia spaces and $f : X \to Y$ a strong partial Esakia morphism. Then $f^* : Y^* \to X^*$ is a $(\wedge, \to, \vee)$-homomorphism.*

Proof. Since f is a partial Esakia morphism, by Theorem 3.15, f^* is a $(\wedge, \to)$-homomorphism. We show that $f^*(U \cup V) = f^*(U) \cup f^*(V)$ for each $U, V \in \mathrm{CpUp}(X)$. It is obvious that $f^*(U) \cup f^*(V) \subseteq f^*(U \cup V)$. Let $x \in f^*(U \cup V)$. Then $f[\uparrow x] \subseteq U \cup V$. If $f[\uparrow x] = \emptyset$, then it is clear that $f[\uparrow x] \subseteq U$, so $x \in f^*(U)$, and so $x \in f^*(U) \cup f^*(V)$. Suppose that $f[\uparrow x] \neq \emptyset$. Then, as f is a strong partial Esakia morphism, $x \in \mathrm{dom}(f)$. Therefore, $f[\uparrow x] = \uparrow f(x)$. This implies that $\uparrow f(x) \subseteq U \cup V$, and so $\uparrow f(x) \subseteq U$ or $\uparrow f(x) \subseteq V$. Thus, $f[\uparrow x] \subseteq U$ or $f[\uparrow x] \subseteq V$, so $x \in f^*(U)$ or $x \in f^*(V)$, and so $x \in f^*(U) \cup f^*(V)$. Consequently, f^* is a $(\wedge, \to, \vee)$-homomorphism. □

It is also clear that the identity map on an Esakia space is a strong partial Esakia morphism, and if f and g are strong partial Esakia morphisms, then so is $g * f$. Thus, Esakia spaces and strong partial Esakia morphisms form a category we denote by $\mathbf{Esa^S}$. Clearly $\mathbf{Esa^S}$ is a proper subcategory of $\mathbf{Esa^P}$. Let also $\mathbf{Heyt}^{(\wedge,\to,\vee)}$ denote the category of Heyting algebras and $(\wedge, \to, \vee)$-homomorphisms. Clearly $\mathbf{Heyt}^{(\wedge,\to,\vee)}$ is a proper subcategory of $\mathbf{Heyt}^{(\wedge,\to)}$. As an immediate consequence of Theorem 3.27 and Lemmas 3.35 and 3.36, we obtain:

Theorem 3.37. *The categories* $\mathbf{Heyt}^{(\wedge,\to,\vee)}$ *and* $\mathbf{Esa^S}$ *are dually equivalent.*

Now Esakia duality is an easy consequence of Theorems 3.33 and 3.37. Let A and B be Heyting algebras and $h : A \to B$ a Heyting algebra homomorphism. Then h is both a $(\wedge, \to, 0)$-homomorphism and a $(\wedge, \to, \vee)$-homomorphism. Therefore, $h_* : B_* \to A_*$ is a partial Esakia morphism which is both well and strong. Since h_* is well, $h_*[\uparrow x] \neq \emptyset$ for each $x \in X$, and as h_* is strong, $x \in \mathrm{dom}(h_*)$ for each $x \in X$. Therefore, h_* is a total function. But then h_* is an Esakia morphism. Conversely, let X and Y be Esakia spaces and $f : X \to Y$ an Esakia morphism. Then f is a total function. Therefore, if D is a downset of Y, then $f^{-1}(D)$ is a downset of X. Thus, for $U \in \mathrm{CpUp}(Y)$, we have $f^*(U) = X - \downarrow f^{-1}(Y - U) = X - f^{-1}(Y - U) = X - (X - f^{-1}(U)) = f^{-1}(U)$, and so $f^* : Y^* \to X^*$ is a Heyting algebra homomorphism. Moreover, for Esakia morphisms $f : X \to Y$ and $g : Y \to Z$, the composition $g * f$ coincides with the usual set-theoretic composition $g \circ f$. Consequently, we obtain that **Heyt** is dually equivalent to **Esa**.

3.7. **The Closed Domain Condition.** Let A and B be Heyting algebras and $h : A \to B$ a $(\wedge, \to)$-homomorphism. Then h is obviously order-preserving, and so $h(a) \vee h(b) \leq h(a \vee b)$ for each $a, b \in A$. However, h may not preserve $\vee$. Nevertheless, even if h is not a $(\wedge, \to, \vee)$-homomorphism, there may still exist $a, b \in A$ such that $h(a \vee b) = h(a) \vee h(b)$. It turns out that whether or not $h(a \vee b) = h(a) \vee h(b)$ depends on the condition closely related to Zakharyaschev's Closed Domain Condition.

Let X and Y be Esakia spaces, $f : X \to Y$ a partial Esakia morphism, and $x \in X$. By condition (4) of Definition 3.3, $f[\uparrow x]$ is a closed subset of Y. Therefore, by Lemma 2.4(4), for each $y \in f[\uparrow x]$ there exists $z \in \min(f[\uparrow x])$ such that $z \leq y$.

Definition 3.38. Let X and Y be Esakia spaces and $f : X \to Y$ a partial Esakia morphism. Let also $\mathfrak{D}$ be a (possibly empty) set of anti-chains in Y. We say that f satisfies the *Closed Domain Condition (CDC) for* $\mathfrak{D}$ if:

$$x \notin \mathrm{dom}(f) \text{ implies } \min f[\uparrow x] \notin \mathfrak{D}.$$

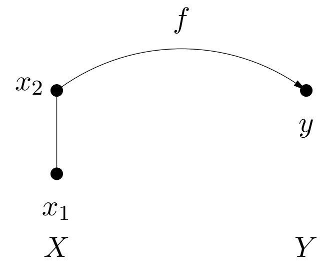


FIGURE 2

Lemma 3.39. *Let X and Y be Esakia spaces and $f : X \to Y$ a partial Esakia morphism. For $U, V \in \mathrm{CpUp}(Y)$, let*

$$\mathfrak{D}_{U,V} = \{\text{anti-chains } \mathfrak{d} \text{ in } U \cup V : \mathfrak{d} \cap (U - V) \neq \emptyset \text{ and } \mathfrak{d} \cap (V - U) \neq \emptyset\}.$$

Then the following two conditions are equivalent:

(1) $f^*(U \cup V) \subseteq f^*(U) \cup f^*(V)$.
(2) f *satisfies (CDC) for* $\mathfrak{D}_{U,V}$.

Proof. (1)⇒(2): Let $x \notin \mathrm{dom}(f)$. If $\mathrm{min} f[\uparrow x] \in \mathfrak{D}_{U,V}$, then $f[\uparrow x] = \uparrow \mathrm{min} f[\uparrow x] \subseteq U \cup V$, but neither $f[\uparrow x] \subseteq U$ nor $f[\uparrow x] \subseteq V$. Therefore, $x \in f^*(U \cup V)$, but $x \notin f^*(U)$ and $x \notin f^*(V)$. This contradicts to $f^*(U \cup V) \subseteq f^*(U) \cup f^*(V)$. Consequently, $\mathrm{min} f[\uparrow x] \notin \mathfrak{D}_{U,V}$, and so f satisfies (CDC) for $\mathfrak{D}_{U,V}$.

(2)⇒(1): Let $x \in f^*(U \cup V)$. Then $f[\uparrow x] \subseteq U \cup V$. We have that $x \in \mathrm{dom}(f)$ or $x \notin \mathrm{dom}(f)$. If $x \in \mathrm{dom}(f)$, then $f[\uparrow x] = \uparrow f(x)$. Therefore, $f[\uparrow x] \subseteq U \cup V$ implies $f[\uparrow x] \subseteq U$ or $f[\uparrow x] \subseteq V$. Thus, $x \in f^*(U)$ or $x \in f^*(V)$, and so $x \in f^*(U) \cup f^*(V)$. On the other hand, if $x \notin \mathrm{dom}(f)$, then as f satisfies (CDC) for $\mathfrak{D}_{U,V}$, we obtain that $\mathrm{min} f[\uparrow x] \notin \mathfrak{D}_{U,V}$. Therefore, $\mathrm{min} f[\uparrow x] \subseteq U$ or $\mathrm{min} f[\uparrow x] \subseteq V$. Thus, $f[\uparrow x] = \uparrow \mathrm{min} f[\uparrow x] \subseteq U$ or $f[\uparrow x] = \uparrow \mathrm{min} f[\uparrow x] \subseteq V$, which yields that $x \in f^*(U)$ or $x \in f^*(V)$. Consequently, $x \in f^*(U) \cup f^*(V)$, and so $f^*(U \cup V) \subseteq f^*(U) \cup f^*(V)$. □

As an immediate consequence of Lemma 3.39, we obtain:

Lemma 3.40. *Let A, B be Heyting algebras, $h : A \to B$ a $(\wedge, \to)$-homomorphism, and $a, b \in A$. Then $h(a \vee b) = h(a) \vee h(b)$ iff $h_* : B_* \to A_*$ satisfies (CDC) for $\mathfrak{D}_{\varphi(a),\varphi(b)}$.*

4. COMPARISON WITH ZAKHARYASCHEV'S APPROACH

In this section we compare our approach to that of Zakharyaschev. We show that partial Esakia morphisms, well partial Esakia morphisms, and strong partial morphisms provide sharpening of Zakharyaschev's subreductions, cofinal subreductions, and dense subreductions. We also show that our (CDC) sharpens Zakharyashev's (CDC). We will mostly follow [9, Sec. 9], which is a streamlined version of Zakharyaschev's earlier results. We point out that Zakharyaschev mostly works with intuitionistic general frames. An especially important subclass of the class of intuitionistic general frames is the class of intuitionistic descriptive frames, which correspond to Esakia spaces (see, e.g., [4, Sec. 2.3.3]). Consequently, instead of intuitionistic descriptive frames, we will work with Esakia spaces.

Definition 4.1. Let X and Y be Esakia spaces and $f : X \to Y$ a partial map.

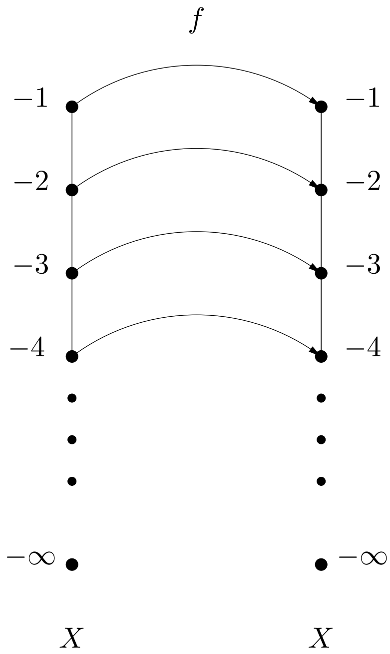


FIGURE 3

(1) [9, p. 289] We call f a *subreduction* if f satisfies conditions (1), (2), and (5) of Definition 3.3.
(2) [9, p. 295] We call f a *cofinal subreduction* if f is a subreduction and $x \in {\uparrow}\text{dom}(f)$ implies $x \in {\downarrow}\text{dom}(f)$.
(3) [9, p. 293] We call f a *dense subreduction* if f is a subreduction and ${\uparrow}\text{dom}(f) \cap {\downarrow}\text{dom}(f) = \text{dom}(f)$.

It follows from the definition that each partial Esakia morphism is a subreduction. However, the converse is not true as follows from the following example.

Example 4.2. We show that there exist subreductions which do not satisfy neither condition (3) nor condition (4) of Definition 3.3. Let X and Y be the finite Esakia spaces and $f : X \to Y$ the partial map shown in Figure 2. It is easy to verify that f is a subreduction. On the other hand, $f[{\uparrow}x_1] = \{y\} = {\uparrow}y$, but $x_1 \notin \text{dom}(f)$. Therefore, f does not satisfy condition (3) of Definition 3.3, thus it is not a partial Esakia morphism.

In order to exhibit a subreduction which does not satisfy condition (4) of Definition 3.3, we need to consider infinite Esakia spaces. Let X be the set of negative integers together with $-\infty$, with the usual order $\leq$ and with the topology in which each negative number is an isolated point and $-\infty$ is the limit of $X - \{-\infty\}$; that is, X is the one-point compactification of the discrete space $X - \{-\infty\}$. Let $f : X \to X$ be the identity map on $X - \{-\infty\}$ and undefined on $-\infty$ (see Figure 3). Then it is easy to see that f is a subreduction. On the other hand, $f[{\uparrow}(-\infty)] = f(X) = X - \{-\infty\}$, which is not a closed subset of X. Therefore, f does not satisfy condition (4) of Definition 3.3.

It follows from [9, Thm. 9.7 and Exercise 9.2] that if $f : X \to Y$ is a subreduction, then $f^* : \text{CpUp}(Y) \to \text{CpUp}(X)$ is a $(\wedge, \to)$-homomorphism, and that if $h : A \to B$ is a $(\wedge, \to)$-homomorphism, then $h_* : B_* \to A_*$ is a subreduction. On the other hand, as

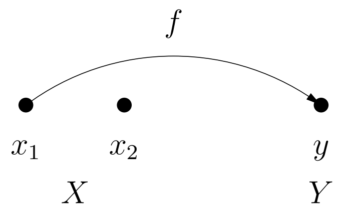


FIGURE 4

follows from Example 4.2, there is not a perfect balance between subreductions and $(\wedge, \to)$-homomorphisms. Therefore, in order to obtain duality for $\mathbf{Heyt}^{(\wedge,\to)}$, we need to work with partial Esakia morphisms instead of subreductions.

Let $f : X \to Y$ be a partial Esakia morphism. It is easy to see that if f is well, then f is cofinal. However, the converse is not true as follows from the following example.

Example 4.3. Let X and Y be the finite Esakia spaces and $f : X \to Y$ the partial Esakia morphism shown in Figure 4. It is obvious that ${\uparrow}\mathrm{dom}(f) = \{x_1\} = {\downarrow}\mathrm{dom}(f)$. Therefore, f is cofinal. On the other hand, for x_2 there is no $z \in \mathrm{dom}(f)$ such that $x_2 \le z$. Thus, f is not a well partial Esakia morphism.

Nevertheless, each cofinal partial Esakia morphism $f : X \to Y$ gives rise to a well partial Esakia morphism from a closed upset of X to Y.

Lemma 4.4. *Let X and Y be Esakia spaces and $f : X \to Y$ a partial Esakia morphism. Then ${\uparrow}\mathrm{dom}(f)$ is a closed upset of X. Consequently, ${\uparrow}\mathrm{dom}(f)$ is an Esakia space (in the induced topology and order).*

Proof. By Lemma 3.7, $\mathrm{dom}(f)$ is a closed subset of X. This, by Lemma 2.4(1), implies that ${\uparrow}\mathrm{dom}(f)$ is a closed upset of X. Consequently, by Lemma 2.4(3), ${\uparrow}\mathrm{dom}(f)$ is an Esakia space (in the induced topology and order). □

Lemma 4.5. *Let X and Y be Esakia spaces and $f : X \to Y$ a partial Esakia morphism. Then f is cofinal iff the restriction of f to ${\uparrow}\mathrm{dom}(f)$ is a well partial Esakia morphism.*

Proof. The right to left implication is straightforward. Conversely, by Lemma 4.4, ${\uparrow}\mathrm{dom}(f)$ is an Esakia space. Let g denote the restriction of f to ${\uparrow}\mathrm{dom}(f)$. Then $\mathrm{dom}(g) = \mathrm{dom}(f)$. Moreover, since f is a partial Esakia morphism, it is easy to verify that g is also a partial Esakia morphism. Suppose that f is cofinal. Then for each $x \in {\uparrow}\mathrm{dom}(f)$ we have $x \in {\downarrow}\mathrm{dom}(f)$. Therefore, for each $x \in {\uparrow}\mathrm{dom}(f)$ there exists $z \in \mathrm{dom}(f)$ such that $x \le z$. But then for each $x \in {\uparrow}\mathrm{dom}(f)$ there exists $z \in \mathrm{dom}(g)$ such that $x \le z$. Thus, $g : {\uparrow}\mathrm{dom}(f) \to Y$ is a well partial Esakia morphism. □

Let $f : X \to Y$ be a partial Esakia morphism. It is easy to see that if f is strong, then f is dense. However, the converse is not true as follows from the following example.

Example 4.6. Let X and Y be the finite Esakia spaces and $f : X \to Y$ the partial Esakia morphism shown in Figure 5. Then ${\uparrow}\mathrm{dom}(f) \cap {\downarrow}\mathrm{dom}(f) = \{x_1, x_2\} \cap X = \{x_1, x_2\} = \mathrm{dom}(f)$. Therefore, f is dense. On the other hand, $f[{\uparrow}x_3] = \{y_1, y_2\} \ne \emptyset$, but $x \notin \mathrm{dom}(f)$. Thus, f is not a strong partial Esakia morphism.

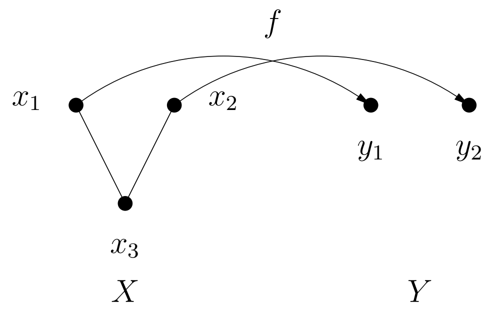


FIGURE 5

Nevertheless, similar to the case of cofinal partial Esakia morphisms, each dense partial Esakia morphism $f : X \to Y$ gives rise to a strong partial Esakia morphism from a closed upset of X to Y.

Lemma 4.7. *Let X and Y be Esakia spaces and $f : X \to Y$ a partial Esakia morphism. Then f is dense iff the restriction of f to $\uparrow\mathrm{dom}(f)$ is a strong partial Esakia morphism.*

Proof. The right to left implication is straightforward. Conversely, let $Z = \uparrow\mathrm{dom}(f)$ and g be the restriction of f to Z. By Lemma 4.4, Z is an Esakia space. Moreover, $\mathrm{dom}(g) = \mathrm{dom}(f)$, and as $f : X \to Y$ is a partial Esakia morphism, it is easy to verify that $g : Z \to Y$ is also a partial Esakia morphism. Suppose that f is dense. Let $x \in Z$ be such that $g[\uparrow x] \neq \emptyset$. Since $x \in Z$, there exists $y \in \mathrm{dom}(f)$ such that $y \leq z$. And as $g[\uparrow x] \neq \emptyset$, there exists $z \in \mathrm{dom}(g) = \mathrm{dom}(f)$ such that $x \leq z$. Therefore, $y \leq x \leq z$ with $y, z \in \mathrm{dom}(f)$, and so $x \in \uparrow\mathrm{dom}(f) \cap \downarrow\mathrm{dom}(f)$. Because f is dense, $\uparrow\mathrm{dom}(f) \cap \downarrow\mathrm{dom}(f) = \mathrm{dom}(f)$. Thus, $x \in \mathrm{dom}(f) = \mathrm{dom}(g)$, and so $g : Z \to Y$ is a strong partial Esakia morphism. □

As a result, we obtain that the notions of partial Esakia morphism, well partial Esakia morphism, and strong partial Esakia morphism sharpen the notions of subreduction, cofinal subreduction, and dense subreduction, respectively. In fact, as follows from Section 3, in order to obtain duality for $(\wedge, \to)$-homomorphisms, $(\wedge, \to, 0)$-homomorphisms, and $(\wedge, \to, \vee)$-homomorphisms, we have to work with partial Esakia morphisms, well partial Esakia morphisms, and strong partial Esakia morphisms rather than subreductions, cofinal subreductions, and dense subreductions.

We conclude this section by comparing our (CDC) with Zakharyaschev's (CDC). We point out that Zakharyaschev works with subreductions. However, we already saw that it is better to work with partial Esakia morphisms. Therefore, we adjust Zakharyaschev's definition and consider partial Esakia morphisms instead of subreductions. In addition, Zakharyaschev only considers subreductions into a finite poset. The main reason for this, of course, is that the canonical formulas he defines are associated with finite (rooted) posets rather than any Esakia space. On the other hand, our (CDC) applies to the infinite case as well (although the canonical formulas we will define will also be associated only with finite subdirectly irreducible Heyting algebras). Therefore, we will not assume that the target space is finite.

Definition 4.8. [9, p. 298] Let Y be an Esakia space and $\mathfrak{D}$ a (possibly empty) set of antichains in Y. We say that a partial Esakia morphism f from an Esakia space X to Y satisfies *Zakharyaschev's Closed Domain Condition (ZCDC) for* $\mathfrak{D}$ if:

$$x \in \uparrow\mathrm{dom}(f) \text{ and } f[\uparrow x] = \uparrow\mathfrak{d} \text{ for some } \mathfrak{d} \in \mathfrak{D} \text{ imply } x \in \mathrm{dom}(f).$$

Obviously (ZCDC) can be rewritten as

$$x \in {\uparrow}\mathrm{dom}(f) - \mathrm{dom}(f) \text{ implies there is no } \mathfrak{d} \in \mathfrak{D} \text{ such that } f[{\uparrow}x] = {\uparrow}\mathfrak{d}.$$

But $f[{\uparrow}x] = {\uparrow}(\min f[{\uparrow}x])$. Therefore, (ZCDC) can be rewritten as

$$x \in {\uparrow}\mathrm{dom}(f) - \mathrm{dom}(f) \text{ implies } \min f[{\uparrow}x] \notin \mathfrak{D}.$$

The last version of (ZCDC) makes it clear that (CDC) implies (ZCDC). However, the converse is not true in general. Nevertheless, similar to Lemmas 4.5 and 4.7, we have that (ZCDC) implies (CDC) for the restriction of f to ${\uparrow}\mathrm{dom}(f)$.

Corollary 4.9. *Let X and Y be Esakia spaces, f a partial Esakia morphism, and $\mathfrak{D}$ a (possibly empty) set of anti-chains in Y. Then f satisfies (ZCDC) for $\mathfrak{D}$ iff the restriction of f to ${\uparrow}\mathrm{dom}(f)$ is a partial Esakia morphism satisfying (CDC) for $\mathfrak{D}$.*

Proof. The right to left implication is straightforward. Conversely, we already saw that ${\uparrow}\mathrm{dom}(f)$ is an Esakia space and that the restriction of f to ${\uparrow}\mathrm{dom}(f)$ is a partial Esakia morphism. Suppose that f satisfies (ZCDC) for $\mathfrak{D}$, $x \in {\uparrow}\mathrm{dom}(f)$ and $x \notin \mathrm{dom}(f)$. Then $x \in {\uparrow}\mathrm{dom}(f) - \mathrm{dom}(f)$. By (ZCDC), $\min f[{\uparrow}x] \notin \mathfrak{D}$. Therefore, f satisfies (CDC) for $\mathfrak{D}$. □

5. Canonical formulas from an algebraic point of view

In this section we construct canonical formulas by purely algebraic means. Our approach generalizes Jankov's approach [18], which was described in detail by Wronski [35]. A similar approach was undertaken by Tomaszewski [31].

We show that each intermediate logic is axiomatizable by canonical formulas. As a consequence, we obtain that each intermediate logic axiomatizable by negation free formulas is axiomatizable by negation free canonical formulas, and show that Jankov formulas, subframe formulas, and cofinal subframe formulas are all particular instances of canonical formulas. Finally, we show that the algebraic approach of [2] to subframe formulas and cofinal subframe formulas is a particular case of our approach.

As was pointed out in the introduction, most of these results were obtained by Zakharyaschev using model-theoretic techniques. Our main contribution is in sharpening Zakharyaschev's technique, streamlining it as part of the generalized Esakia duality, and obtaining new and simplified proofs, which are algebraic in nature.

5.1. **Canonical formulas.** We recall that a Heyting algebra is *subdirectly irreducible* if it has the second largest element.

Definition 5.1. Let A be a finite subdirectly irreducible Heyting algebra, s the second largest element of A, and D a subset of A^2. For each $a \in A$ we introduce a new variable p_a and define the *canonical formula* $\alpha(A, D, \bot)$ associated with A and D as

$$\begin{aligned}\alpha(A, D, \bot) \quad = \quad & [\bigwedge\{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : a, b \in A\}\wedge \\ & \bigwedge\{p_{a\to b} \leftrightarrow p_a \to p_b : a, b \in A\}\wedge \\ & \bigwedge\{p_{\neg a} \leftrightarrow \neg p_a : a \in A\}\wedge \\ & \bigwedge\{p_{a\vee b} \leftrightarrow p_a \vee p_b : (a, b) \in D\}] \to p_s\end{aligned}$$

Lemma 5.2. *Let A be a finite subdirectly irreducible Heyting algebra, s the second largest element of A, and D a subset of A^2. Then $A \not\models \alpha(A, D, \bot)$.*

Proof. Define a valuation ν on A by $\nu(p_a) = a$ for each $a \in A$. Let Γ denote the antecedent of $\alpha(A, D, \bot)$. Then $\nu(\alpha(A, D, \bot)) = \nu(\Gamma \to p_s) = \nu(\Gamma) \to \nu(p_s) = 1 \to s = s$. Therefore, $A \not\models \alpha(A, D, \bot)$. □

Theorem 5.3. *Let A be a finite subdirectly irreducible Heyting algebra, $D \subseteq A^2$, and B a Heyting algebra. Then $B \not\models \alpha(A, D, \bot)$ iff there is a homomorphic image C of B and an $(\wedge, \to, 0)$-embedding $h : A \rightarrowtail C$ such that $h(a \vee b) = h(a) \vee h(b)$ for each $(a, b) \in D$.*

Proof. First assume that there is a homomorphic image C of B and an $(\wedge, \to, 0)$-embedding $h : A \rightarrowtail C$ such that $h(a \vee b) = h(a) \vee h(b)$ for each $(a, b) \in D$. By Lemma 5.2, there is a valuation ν on A refuting $\alpha(A, D, \bot)$. Clearly $h \circ \nu$ is a valuation on C. Since h is an $(\wedge, \to, 0)$-embedding of A in C such that $h(a \vee b) = h(a) \vee h(b)$ for each $(a, b) \in D$, it follows from the construction of $\alpha(A, D, \bot)$ that it is refuted by $h \circ \nu$. Now as C is a homomorphic image of B, we have that $B \not\models \alpha(A, D, \bot)$.

Conversely, assume that $B \not\models \alpha(A, D, \bot)$. It is well known (see, e.g., [35, Lemma 1]) that if $b \neq 1_B$, then there exists a subdirectly irreducible Heyting algebra C and an onto homomorphism $f : B \twoheadrightarrow C$ such that $f(b) = s_C$, where s_C is the second largest element of C. Therefore, if $B \not\models \alpha(A, D, \bot)$, then there exists a valuation μ on B such that $\mu(\alpha(A, D, \bot)) \neq 1_B$, and so there exists a subdirectly irreducible Heyting algebra C and an onto homomorphism $f : B \twoheadrightarrow C$ such that $f(\mu(\alpha(A, D, \bot))) = s_C$. Thus, $\nu = f \circ \mu$ is a valuation on C such that $\nu(\alpha(A, D, \bot)) = s_C$. Let Γ denote the antecedent of $\alpha(A, D, \bot)$. Then $\nu(\alpha(A, D, \bot)) = \nu(\Gamma \to p_s) = \nu(\Gamma) \to \nu(p_s) = s_C$. This obviously implies that $\nu(\Gamma) = 1_C$. We define $h : A \to C$ by $h(a) = \nu(p_a)$ for each $a \in A$ and show that h is an $(\wedge, \to, 0)$-embedding such that $h(a \vee b) = h(a) \vee h(b)$ for each $(a, b) \in D$.

To see that h is 1-1 it is sufficient to show that $a \not\leq b$ in A implies $h(a) \not\leq h(b)$ in C. If $a \not\leq b$, then $a \to b \neq 1$. Therefore, $(a \to b) \to s = 1$. Thus, $1_C = h(1) = h((a \to b) \to s) = h(a \to b) \to h(s) = (h(a) \to h(b)) \to \nu(p_s) = (h(a) \to h(b)) \to s_C$. It follows that $h(a) \to h(b) \leq s_C$, so $h(a) \to h(b) \neq 1_C$, and so $h(a) \not\leq h(b)$. Consequently, h is 1-1.

It is left to be shown that h is a $(\wedge, \to, 0)$-homomorphism such that $h(a \vee b) = h(a) \vee h(b)$ for each $(a, b) \in D$. Let $a, b \in A$. Since $\nu(\Gamma) = 1_C$ and $\nu(\Gamma) \leq \nu(p_{a\wedge b}) \leftrightarrow (\nu(p_a) \wedge \nu(p_b))$, we obtain that $\nu(p_{a\wedge b}) \leftrightarrow (\nu(p_a) \wedge \nu(p_b)) = 1_C$. Therefore, $\nu(p_{a\wedge b}) = \nu(p_a) \wedge \nu(p_b)$. By a similar argument, $\nu(p_{a\to b}) = \nu(p_a) \to \nu(p_b)$, $\nu(p_{\neg a}) = \neg\nu(p_a)$, and if $(a, b) \in D$, then $\nu(p_{a\vee b}) = \nu(p_a) \vee \nu(p_b)$. But $\nu(p_a) = h(a)$ for each $a \in A$. Therefore, for each $a, b \in A$, we have $h(a \wedge b) = h(a) \wedge h(b)$, $h(a \to b) = h(a) \to h(b)$, and $h(\neg a) = \neg h(a)$. Moreover, if $(a, b) \in D$, then $h(a \vee b) = h(a) \vee h(b)$. Thus, h is a $(\wedge, \to, 0)$-homomorphism such that $h(a\vee b) = h(a)\vee h(b)$ for each $(a, b) \in D$. Consequently, C is a homomorphic image of B and $h : A \rightarrowtail C$ is an $(\wedge, \to, 0)$-embedding such that $h(a\vee b) = h(a)\vee h(b)$ for each $(a, b) \in D$. □

For a Heyting algebra A, we recall that $\varphi(a) = \{x \in A_* : a \in x\}$.

Definition 5.4. Let A be a finite subdirectly irreducible Heyting algebra and $D \subseteq A^2$. For each $(a, b) \in D$ we set

$$\mathfrak{D}_{a,b} = \{\text{anti-chains } \mathfrak{d} \text{ in } \varphi(a) \cup \varphi(b) : \mathfrak{d} \cap (\varphi(a) - \varphi(b)) \neq \emptyset \text{ and } \mathfrak{d} \cap (\varphi(b) - \varphi(a)) \neq \emptyset\}.$$

For $D \subseteq A^2$ we let $\mathfrak{D} = \bigcup\{\mathfrak{D}_{a,b} : (a, b) \in D\}$, and call $\mathfrak{D}$ the *set of anti-chains associated with D*.

As an immediate consequence of Theorem 5.3 and the generalized Esakia duality, we obtain the following corollary, which corresponds to [9, Thm. 9.40(i)].

Corollary 5.5. *Let A be a finite subdirectly irreducible Heyting algebra, $D \subseteq A^2$, and $\mathfrak{D}$ the set of anti-chains of A_* associated with D. Then for each Esakia space X, we have $X \not\models \alpha(A, D, \bot)$ iff there is a closed upset Y of X and an onto well partial Esakia morphism $f : Y \twoheadrightarrow A_*$ such that f satisfies (CDC) for $\mathfrak{D}$.*

Proof. We have that $X \not\models \alpha(A, D, \bot)$ iff $X^* \not\models \alpha(A, D, \bot)$. By Theorem 5.3, $X^* \not\models \alpha(A, D, \bot)$ is equivalent to the existence of a homomorphic image C of X^* and an $(\wedge, \to, 0)$-embedding $h : A \rightarrowtail C$ such that $h(a \vee b) = h(a) \vee h(b)$ for each $(a, b) \in D$. By Esakia duality, C_* is a closed upset of X, and by the generalized Esakia duality, $h_* : C_* \twoheadrightarrow A_*$ is an onto well partial Esakia morphism. Moreover, by Lemma 3.40, $h(a \vee b) = h(a) \vee h(b)$ for each $(a, b) \in D$ iff h_* satisfies (CDC) for $\mathfrak{D}$. Putting all the pieces together, we obtain that $X \not\models \alpha(A, D, \bot)$ iff there is a closed upset Y of X and an onto well partial Esakia morphism $f : Y \twoheadrightarrow A_*$ such that f satisfies (CDC) for $\mathfrak{D}$. □

Remark 5.6. As we pointed out in the introduction, Zakharyschev's canonical formulas look quite different from our canonical formulas. Nevertheless, they serve the same purpose. To see this, let $\beta(A_*, \mathfrak{D}, \bot)$ be Zakharyaschev's canonical formula (for the definition, see [9, p. 311]). As follows from [9, Thm. 9.39(ii)], a general intuitionistic frame X refutes $\beta(A_*, \mathfrak{D}, \bot)$ iff there is a cofinal subreduction $f : X \to A_*$ satisfying (ZCDC). Now let X be an Esakia space. It follows from Lemma 4.5, Corollaries 4.9 and 5.5, and [9, Thm. 9.39(ii)] that if $X \not\models \alpha(A, D, \bot)$, then there exists a closed upset Y of X such that $Y \not\models \beta(A_*, \mathfrak{D}, \bot)$, and that if $X \not\models \beta(A_*, \mathfrak{D}, \bot)$, then $\uparrow\mathrm{dom}(f) \not\models \alpha(A, D, \bot)$. Since Y and $\uparrow\mathrm{dom}(f)$ are closed upsets of X, clearly $Y \not\models \beta(A_*, \mathfrak{D}, \bot)$ yields $X \not\models \beta(A_*, \mathfrak{D}, \bot)$ and $\uparrow\mathrm{dom}(f) \not\models \alpha(A, D, \bot)$ yields $X \not\models \alpha(A, D, \bot)$. Therefore, $X \not\models \alpha(A, D, \bot)$ iff $X \not\models \beta(A_*, \mathfrak{D}, \bot)$.

Let $F(n)$ denote the free n-generated Heyting algebra and $g_1, \dots, g_n$ denote the generators of $F(n)$. The next theorem is an algebraic analogue of [9, Thm. 9.34 and 9.36].

Theorem 5.7. *If* **IPC** $\not\vdash \varphi(p_1, \dots, p_n)$*, then there exist* $(A_1, D_1), \dots, (A_m, D_m)$ *such that each* A_i *is a finite subdirectly irreducible Heyting algebra,* $D_i \subseteq A_i^2$*, and for each Heyting algebra* B *we have* $B \not\models \varphi(p_1, \dots, p_n)$ *iff there is* $i \leq m$*, a homomorphic image* C *of* B*, and an* $(\wedge, \to, 0)$*-embedding* $h : A_i \rightarrowtail C$ *such that* $h(a \vee b) = h(a) \vee h(b)$ *for each* $(a, b) \in D_i$*.*

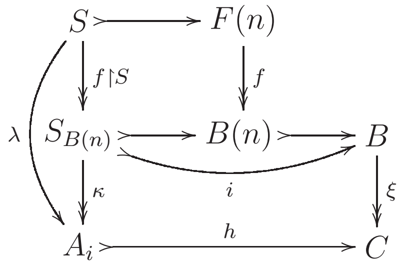
Proof. Let **IPC** $\not\vdash \varphi(p_1, \dots, p_n)$. Then $F(n) \not\models \varphi(p_1, \dots, p_n)$. Therefore, $\varphi(g_1, \dots, g_n) \neq 1$ in $F(n)$. Let $\mathrm{Sub}_{F(n)}(\varphi)$ denote the set of subpolynomials of $\varphi(g_1, \dots, g_n)$ in $F(n)$, and let S be the $(\wedge, \to, 0)$-subalgebra of $F(n)$ generated by $\mathrm{Sub}_{F(n)}(\varphi)$. By Lemma 2.1(1), S is finite. Therefore, $(S, \wedge, \to, 0, \dot\vee)$ is a finite Heyting algebra, where $a \mathbin{\dot\vee} b = \bigwedge\{s \in S : a, b \leq s\}$ for each $a, b \in S$. Clearly $a \vee b \leq a \mathbin{\dot\vee} b$ and $a \vee b = a \mathbin{\dot\vee} b$ whenever $a \vee b \in S$. Thus, $\varphi(g_1, \dots, g_n) \neq 1$ in S. We set $D = \{(a, b) \in [\mathrm{Sub}_{F(n)}(\varphi)]^2 : a \vee b \in \mathrm{Sub}_{F(n)}(\varphi)\}$.

Let $A_1, \dots, A_m$ be the list of subdirectly irreducible homomorphic images of S refuting φ and $h_i : S \twoheadrightarrow A_i$ the corresponding homomorphisms. We set $D_i = \{(h_i(a), h_i(b)) : (a, b) \in D\}$. Given a Heyting algebra B, we need to show that $B \not\models \varphi(p_1, \dots, p_n)$ iff there is $i \leq m$, a homomorphic image C of B, and an $(\wedge, \to, 0)$-embedding $h : A_i \rightarrowtail C$ such that $h(a \vee b) = h(a) \vee h(b)$ for each $(a, b) \in D_i$.

First suppose that there is $i \leq m$, a homomorphic image C of B, and an $(\wedge, \to, 0)$-embedding $h : A_i \rightarrowtail C$ such that $h(a \vee b) = h(a) \vee h(b)$ for each $(a, b) \in D_i$. Let $\mathrm{Sub}_{A_i}(\varphi)$ denote the set of subpolynomials of $\varphi(h_i(g_1), \dots, h_i(g_n))$ in A_i. Then $\mathrm{Sub}_{A_i}(\varphi) = h_i[\mathrm{Sub}_{F(n)}(\varphi)]$. Therefore, for each $a, b \in \mathrm{Sub}_{A_i}(\varphi)$, if $a \vee b \in \mathrm{Sub}_{A_i}(\varphi)$, then $(a, b) \in D_i$, and so $h(a \vee b) = h(a) \vee h(b)$. Thus, since $\varphi(h_i(g_1), \dots, h_i(g_n)) \neq 1$ in A_i, we also have $\varphi(h(h_i(g_1)), \dots, h(h_i(g_n))) \neq 1$ in C. Because C is a homomorphic image of B, there exist $b_1, \dots, b_n \in B$ such that $\varphi(b_1, \dots, b_n) \neq 1$ in B. Thus, $B \not\models \varphi(p_1, \dots, p_n)$.

Next suppose that $B \not\models \varphi(p_1, \dots, p_n)$. Then there exist $b_1, \dots, b_n \in B$ such that $\varphi(b_1, \dots, b_n) \neq 1$ in B. Let $B(n)$ denote the Heyting subalgebra of B generated by $b_1, \dots, b_n$. Clearly $B(n)$ is an n-generated Heyting algebra. Therefore, $B(n)$ is a homomorphic image of $F(n)$.

Let $f : F(n) \twoheadrightarrow B(n)$ be the corresponding homomorphism. Then $f(g_1) = b_1, \ldots, f(g_n) = b_n$. We let $\mathrm{Sub}_{B(n)}(\varphi)$ denote the set of subpolynomials of $\varphi(b_1, \ldots, b_n)$ in $B(n)$, and $S_{B(n)}$ denote the $(\wedge, \to, 0)$-subalgebra of $B(n)$ generated by $\mathrm{Sub}_{B(n)}(\varphi)$. Then $\mathrm{Sub}_{B(n)}(\varphi) = f[\mathrm{Sub}_{F(n)}(\varphi)]$ and the restriction of f to S is an onto $(\wedge, \to, 0)$-homomorphism. Since an onto $(\wedge, \to, 0)$-homomorphism is an onto Heyting algebra homomorphism (Lemma 3.2), we obtain that $S_{B(n)}$ is a homomorphic image of S. Moreover, $\varphi(f(g_1), \ldots, f(g_n)) \neq 1$ in $S_{B(n)}$, which implies that there exists a finite subdirectly irreducible Heyting algebra A_i and an onto Heyting algebra homomorphism $\kappa : S_{B(n)} \twoheadrightarrow A_i$ such that $\varphi(\kappa(f(g_1)), \ldots, \kappa(f(g_n))) \neq 1$ in A_i. Let $\lambda = \kappa \circ f$. Then λ is a Heyting algebra homomorphism from S onto A_i. Let also $\mathrm{Sub}_{A_i}(\varphi)$ denote the set of subpolynomials of $\varphi(\lambda(g_1), \ldots, \lambda(g_n))$. Then $\mathrm{Sub}_{A_i}(\varphi) = \lambda[\mathrm{Sub}_{F(n)}(\varphi)]$. We set $D_i = \{(\lambda(a), \lambda(b)) : (a, b) \in D\}$. Let $i : S_{B(n)} \to B$ be the identity map. Since $\kappa : S_{B(n)} \twoheadrightarrow A_i$ is an onto homomorphism and $i : S_{B(n)} \rightarrowtail B$ is an $(\wedge, \to, 0)$-embedding, by Lemma 2.1(3), there exists a homomorphic image C of B, with $\xi : B \twoheadrightarrow C$ the onto homomorphism, and an $(\wedge, \to, 0)$-embedding $h : A_i \rightarrowtail C$ such that $\xi \circ i = h \circ \kappa$.



Note that, by Lemma 3.2, ξ is a Heyting algebra homomorphism. Moreover, for each $(a, b) \in D_i$ there exists $(c, d) \in D$ such that $a = \lambda(c) = \kappa(f(c))$ and $b = \lambda(d) = \kappa(f(d))$. Therefore, since $i(f(c) \vee f(d)) = i(f(c)) \vee i(f(d))$, we obtain:

$$\begin{array}{rcl} h(a \vee b) & = & h[\kappa(f(c)) \vee \kappa(f(d))] \\ & = & (h \circ \kappa)[f(c) \vee f(d)] \\ & = & (\xi \circ i)(f(c) \vee f(d)) \\ & = & \xi[i(f(c)) \vee i(f(d))] \\ & = & \xi(i(f(c))) \vee \xi(i(f(d))) \\ & = & h(\kappa(f(c))) \vee h(\kappa(f(d))) \\ & = & h(a) \vee h(b). \end{array}$$

Thus, there is $i \leq m$, a homomorphic image C of B, and an $(\wedge, \to, 0)$-embedding $h : A_i \rightarrowtail C$ such that $h(a \vee b) = h(a) \vee h(b)$ for each $(a, b) \in D_i$. □

Remark 5.8. A similar result was also established in [31].

By the generalized Esakia duality, the dual reading of Theorem 5.7 is as follows:

Corollary 5.9. *If* **IPC** $\nvdash \varphi(p_1, \ldots, p_n)$*, then there exist* $(A_1, D_1), \ldots, (A_m, D_m)$ *such that each* A_i *is a finite subdirectly irreducible Heyting algebra,* $D_i \subseteq A_i^2$*,* $\mathfrak{D}_i$ *is the set of anti-chains of* $(A_i)_*$ *associated with* D*, and for each Esakia space* X *we have* $X \not\models \varphi(p_1, \ldots, p_n)$ *iff there is* $i \leq m$*, a closed upset* Y *of* X*, and an onto well partial Esakia morphism* $f : Y \twoheadrightarrow (A_i)_*$ *such that* f *satisfies (CDC) for* $\mathfrak{D}_i$*.*

Combining Theorems 5.3 and 5.7, we obtain:

Corollary 5.10. *If* **IPC** $\nvdash \varphi(p_1, \ldots, p_n)$*, then there exist* $(A_1, D_1), \ldots, (A_m, D_m)$ *such that each* A_i *is a finite subdirectly irreducible Heyting algebra,* $D_i \subseteq A_i^2$*, and for each Heyting algebra* B *we have:*

$$B \models \varphi(p_1, \ldots, p_n) \text{ iff } B \models \bigwedge_{i=1}^{m} \alpha(A_i, D_i, \bot).$$

The dual reading of Corollary 5.10 is as follows:

Corollary 5.11. *If* **IPC** $\nvdash \varphi(p_1, \ldots, p_n)$*, then there exist* $(A_1, D_1), \ldots, (A_m, D_m)$ *such that each* A_i *is a finite subdirectly irreducible Heyting algebra,* $D_i \subseteq A_i^2$*, and for each Esakia space* X *we have:*

$$X \models \varphi(p_1, \ldots, p_n) \text{ iff } X \models \bigwedge_{i=1}^{m} \alpha(A_i, D_i, \bot).$$

Remark 5.12. Corollary 5.9 corresponds to [9, Thm. 9.36(i)] and Corollary 5.11 corresponds to [9, Thm. 9.44(i)].

Zakharyaschev's theorem is now an immediate consequence of Corollary 5.10:

Corollary 5.13 (Zakharyaschev's Theorem)**.** *Each intermediate logic* L *is axiomatizable by canonical formulas. Moreover, if* L *is finitely axiomatizable, then* L *is axiomatizable by finitely many canonical formulas.*

Proof. Let L be an intermediate logic. Then L is obtained by adding $\{\varphi_i : i \in I\}$ to **IPC** as new axioms. Therefore, **IPC** $\nvdash \varphi_i$ for each $i \in I$. By Corollary 5.10, for each $i \in I$, there exist $(A_{i1}, D_{i1}), \ldots, (A_{im_i}, D_{im_i})$ such that A_{ij} is a finite subdirectly irreducible Heyting algebra, $D_{ij} \subseteq A_{ij}^2$, and for each Heyting algebra B we have $B \models \varphi_i$ iff $B \models \bigwedge_{j=1}^{m_i} \alpha(A_{ij}, D_{ij}, \bot)$. Thus, $B \models L$ iff $B \models \{\varphi_i : i \in I\}$, which happens iff $B \models \{\bigwedge_{j=1}^{m_i} \alpha(A_{ij}, D_{ij}, \bot) : i \in I\}$. Consequently, $L = \mathbf{IPC} + \{\bigwedge_{j=1}^{m_i} \alpha(A_{ij}, D_{ij}, \bot) : i \in I\}$, and so L is axiomatizable by canonical formulas. In particular, if L is finitely axiomatizable, then L is axiomatizable by finitely many canonical formulas. $\square$

5.2. **Negation free canonical formulas.** Now we consider intermediate logics axiomatizable by negation free formulas and construct negation free canonical formulas which axiomatize them. Negation free canonical formulas are obtained from canonical formulas by dropping the conjunct $\bigwedge\{p_{\neg a} \leftrightarrow \neg p_a : a \in A\}$ in the antecedent. Therefore, the proofs for the disjunction free case are obtained by the obvious simplifications of the general case. Because of this, we will only state the results without proofs.

Definition 5.14. Let A be a finite subdirectly irreducible Heyting algebra, s the second largest element of A, and D a subset of A^2. For each $a \in A$ we introduce a new variable p_a and define the *negation free canonical formula* $\alpha(A, D)$ associated with A and D by

$$\begin{aligned} \alpha(A, D) \quad = \quad & [\bigwedge\{p_{a \wedge b} \leftrightarrow p_a \wedge p_b : a, b \in A\} \wedge \\ & \bigwedge\{p_{a \to b} \leftrightarrow p_a \to p_b : a, b \in A\} \wedge \\ & \bigwedge\{p_{a \vee b} \leftrightarrow p_a \vee p_b : (a, b) \in D\}] \to p_s. \end{aligned}$$

That is, $\alpha(A, D)$ is obtained from $\alpha(A, D, \bot)$ by deleting the conjunct $\bigwedge\{p_{\neg a} \leftrightarrow \neg p_a : a \in A\}$ from the antecedent.

Corollary 5.15.

(1) *Let A be a finite subdirectly irreducible Heyting algebra, $D \subseteq A^2$, and B a Heyting algebra. Then $B \not\models \alpha(A, D)$ iff there is a homomorphic image C of B and an $(\wedge, \to)$-embedding $h : A \rightarrowtail C$ such that $h(a \vee b) = h(a) \vee h(b)$ for each $(a, b) \in D$.*

(2) *Let A be a finite subdirectly irreducible Heyting algebra, $D \subseteq A^2$, $\mathfrak{D}$ the set of anti-chains of A_* associated with D, and X an Esakia space. Then $X \not\models \alpha(A, D)$ iff there is a closed upset Y of X and an onto partial Esakia morphism $f : Y \twoheadrightarrow A_*$ such that f satisfies (CDC) for $\mathfrak{D}$.*

Corollary 5.16.

(1) *If* **IPC** $\not\vdash \varphi(p_1, \dots, p_n)$*, where $\varphi(p_1, \dots, p_n)$ is a negation free formula, then there exist $(A_1, D_1), \dots, (A_m, D_m)$ such that each A_i is a finite subdirectly irreducible Heyting algebra, $D_i \subseteq A_i^2$, and for each Heyting algebra B we have $B \not\models \varphi(p_1, \dots, p_n)$ iff there is $i \le m$, a homomorphic image C of B, and an $(\wedge, \to)$-embedding $h : A_i \rightarrowtail C$ such that $h(a \vee b) = h(a) \vee h(b)$ for each $(a, b) \in D_i$.*

(2) *If* **IPC** $\not\vdash \varphi(p_1, \dots, p_n)$*, , where $\varphi(p_1, \dots, p_n)$ is a negation free formula, then there exist $(A_1, D_1), \dots, (A_m, D_m)$ such that each A_i is a finite subdirectly irreducible Heyting algebra, $D_i \subseteq A_i^2$, $\mathfrak{D}_i$ is the set of anti-chains of $(A_i)_*$ associated with D, and for each Esakia space X we have $X \not\models \varphi(p_1, \dots, p_n)$ iff there is $i \le m$, a closed upset Y of X, and an onto partial Esakia morphism $f : Y \twoheadrightarrow (A_i)_*$ such that f satisfies (CDC) for $\mathfrak{D}_i$.*

Corollary 5.17.

(1) *If* **IPC** $\not\vdash \varphi(p_1, \dots, p_n)$*, where $\varphi(p_1, \dots, p_n)$ is a negation free formula, then there exist $(A_1, D_1), \dots, (A_m, D_m)$ such that each A_i is a finite subdirectly irreducible Heyting algebra, $D_i \subseteq A_i^2$, and for each Heyting algebra B we have:*

$$B \models \varphi(p_1, \dots, p_n) \text{ iff } B \models \bigwedge_{i=1}^{m} \alpha(A_i, D_i).$$

(2) *If* **IPC** $\not\vdash \varphi(p_1, \dots, p_n)$*, where $\varphi(p_1, \dots, p_n)$ is a negation free formula, then there exist $(A_1, D_1), \dots, (A_m, D_m)$ such that each A_i is a finite subdirectly irreducible Heyting algebra, $D_i \subseteq A_i^2$, and for each Esakia space X we have:*

$$X \models \varphi(p_1, \dots, p_n) \text{ iff } X \models \bigwedge_{i=1}^{m} \alpha(A_i, D_i).$$

Remark 5.18. Corollary 5.15(2) corresponds to [9, Thm. 9.40(ii)], Corollary 5.16(2) corresponds to [9, Thm. 9.36(ii)], and Corollary 5.17(2) corresponds to [9, Thm. 9.44(ii)].

Zakharyaschev's theorem that each intermediate logic L axiomatizable by negation-free formulas is axiomatizable by negation-free canonical formulas is now an immediate consequence of the above.

Corollary 5.19. *Each intermediate logic L axiomatizable by negation-free formulas is axiomatizable by negation-free canonical formulas. Moreover, if L is axiomatizable by finitely many negation-free formulas, then L is axiomatizable by finitely many negation-free canonical formulas.*

5.3. **Jankov formulas.** Now we show that Jankov formulas are obtained from canonical formulas by taking $D = A^2$. Again, the proofs are straightforward from the general case and we skip most of the details.

Definition 5.20. Let A be a finite subdirectly irreducible Heyting algebra and $D = A^2$. We call $\alpha(A, D, \bot)$ the *Jankov formula of* A and denote it by $\chi(A)$; also, we call $\alpha(A, D)$ the *positive Jankov formula of* A and denote it by $P_\chi(A)$.

Corollary 5.21 (Jankov's Theorem)**.**

(1) *Let A be a finite subdirectly irreducible Heyting algebra and B a Heyting algebra.*
 (a) *$B \not\models P_\chi(A)$ iff there is a homomorphic image C of B and an $(\wedge, \to, \vee)$-embedding $h : A \rightarrowtail C$.*
 (b) *$B \not\models \chi(A)$ iff there is a homomorphic image C of B and a Heyting algebra embedding $h : A \rightarrowtail C$.*

(2) *Let A be a finite subdirectly irreducible Heyting algebra and X an Esakia space.*
 (a) *$X \not\models P_\chi(A)$ iff there is a closed upset Y of X and an onto strong partial Esakia morphism $f : Y \twoheadrightarrow A_*$.*
 (b) *$X \not\models \chi(A)$ iff there is a closed upset Y of X and an onto Esakia morphism $f : Y \twoheadrightarrow A_*$.*

We recall that an element s of a lattice A is a *splitting element* if there exists $t \in A$ such that for each $x \in A$ we have $s \le x$ or $x \le t$; that is the pair (s, t) *splits* the lattice A into ${\uparrow}s$ and ${\downarrow}t$. We also recall that an intermediate logic L is a *splitting logic* if it is a splitting element in the lattice of intermediate logics, and that L is *join-splitting* if L is a join of splitting logics. The next theorem goes back to Jankov [18].

Corollary 5.22. *Each join-splitting intermediate logic is axiomatizable by Jankov formulas.*

Proof. It is sufficient to show that each splitting intermediate logic is axiomatizable by a Jankov formula. Let L be a splitting intermediate logic. Then there exists an intermediate logic L' such that (L, L') splits the lattice of intermediate logics. Since the variety of Heyting algebras is congruence-distributive and it is generated by its finite algebras, it follows from a general result of McKenzie [24] that L' is the logic of a finite subdirectly irreducible Heyting algebra A. We show that $L = \mathbf{IPC} + \chi(A)$. Let B be a Heyting algebra. It is sufficient to show that $B \models L$ iff $B \models \chi(A)$. Let $B \models L$. If A is a homomorphic image of a subalgebra of B, then $A \models L$, which contradicts to (L, L') being a splitting pair. Therefore, A is not a homomorphic image of a subalgebra of B, which by Corollary 5.21, implies that $B \models \chi(A)$. Conversely, if $B \not\models L$, then as (L, L') is a splitting pair, the logic of B is contained in L'. Since L' is the logic of A and $A \not\models \chi(A)$, it follows that $B \not\models \chi(A)$. Consequently, $B \models L$ iff $B \models \chi(A)$, and so $\chi(A)$ axiomatizes L. □

5.4. **Subframe and cofinal subframe formulas.** Finally, we show that subframe formulas and cofinal subframe formulas are obtained from canonical formulas by taking $D = \emptyset$. Again we skip all the proofs because they are obtained easily from the general case.

Definition 5.23. Let A be a finite subdirectly irreducible Heyting algebra and $D = \emptyset$. We call $\alpha(A, D, \bot)$ the *cofinal subframe formula of* A and denote it by $\beta(A, \bot)$; also, we call $\alpha(A, D)$ the *subframe formula of* A and denote it by $\beta(A)$.

Corollary 5.24.

(1) *Let A be a finite subdirectly irreducible Heyting algebra and B a Heyting algebra.*
 (a) *$B \not\models \beta(A)$ iff there is a homomorphic image C of B and an $(\wedge, \to)$-embedding $h : A \rightarrowtail C$.*
 (b) *$B \not\models \beta(A, \bot)$ iff there is a homomorphic image C of B and an $(\wedge, \to, 0)$-embedding $h : A \rightarrowtail C$.*

(2) *Let A be a finite subdirectly irreducible Heyting algebra and X an Esakia space.*
  (a) $X \not\models \beta(A)$ *iff there is a closed upset* Y *of* X *and an onto partial Esakia morphism* $f : Y \twoheadrightarrow A_*$.
  (b) $X \not\models \beta(A, \bot)$ *iff there is a closed upset* Y *of* X *and an onto well partial Esakia morphism* $f : Y \twoheadrightarrow A_*$.

Remark 5.25. Corollary 5.24(2) corresponds to [9, Thm. 11.15].

Let X be an Esakia space. We recall (see [9, p. 289] and [2, p. 86]) that $Y \subseteq X$ is a *subframe* of X if (i) Y is a closed subset of X and (ii) U a clopen subset of Y (in the induced topology) implies $\downarrow U$ is a clopen subset of X. Equivalently, $Y \subseteq X$ is a subframe of X if Y is an Esakia space (in the induced topology and order) and the partial identity map $X \to Y$ is a partial Esakia morphism. We also recall (see [9, p. 295] and [2, p. 87]) that $Y \subseteq X$ is a *cofinal subframe* of X if Y is a subframe of X and $\uparrow Y \subseteq \downarrow Y$. Equivalently, $Y \subseteq X$ is a cofinal subframe of X if Y is an Esakia space (in the induced topology and order) and the partial identity map $X \to Y$ is a well partial Esakia morphism.

Let L be an intermediate logic. We recall that L is a *subframe logic* if for each Esakia space X and a subframe Y of X, from $X \models L$ it follows that $Y \models L$. We also recall that L is a *cofinal subframe logic* if for each Esakia space X and a cofinal subframe Y of X, from $X \models L$ it follows that $Y \models L$. As follows from [2], algebraically subframes are characterized by nuclei on Heyting algebras, and so subframe logics correspond to nuclear varieties, while cofinal subframe logics correspond to cofinal nuclear varieties. As was shown in [2, Thm. 41], nuclear varieties are axiomatizable by subframe formulas and cofinal nuclear varieties are axiomatizable by cofinal subframe formulas. Consequently, we arrive at the following:

Corollary 5.26. (Zakharyaschev [40, Thm. 5.7])**.** *Each subframe intermediate logic is axiomatizable by subframe formulas and each cofinal subframe intermediate logic is axiomatizable by cofinal subframe formulas.*

A different description of subframe and cofinal subframe formulas is given in [4, Sec 3.3.3], where a connection with the NNIL-formulas of [33] is also made.

Acknowledgments

The second author was partially supported by grant EP/F032102/1 from the UK EPSRC. We are thankful to Ian Hodkinson and Clemens Kupke for fruitful discussions, as well as to the referee for useful comments which improved the presentation of the paper.

References

[1] G. Bezhanishvili and N. Bezhanishvili. Profinite Heyting algebras. *Order*, 25(3):211–227, 2008.
[2] G. Bezhanishvili and S. Ghilardi. An algebraic approach to subframe logics. Intuitionistic case. *Ann. Pure Appl. Logic*, 147(1-2):84–100, 2007.
[3] G. Bezhanishvili and R. Jansana. Duality for distributive and implicative semi-lattices. Submitted. Available at `http://www.mat.ub.edu/~logica/docs/BeJa08-m.pdf`, 2008.
[4] N. Bezhanishvili. *Lattices of Intermediate and Cylindric Modal Logics*. PhD thesis, University of Amsterdam, 2006.
[5] N. Bezhanishvili. Frame based formulas for intermediate logics. *Studia Logica*, 90:139–159, 2008.
[6] W. Blok. *Varieties of Interior Algebras*. PhD thesis, University of Amsterdam, 1976.
[7] W. Blok. On the degree of incompleteness of modal logics. *Polish Acad. Sci. Inst. Philos. Sociol. Bull. Sect. Logic*, 7(4):167–175, 1978.
[8] W. Blok. The lattice of modal logics: an algebraic investigation. *J. Symbolic Logic*, 45(2):221–236, 1980.

[9] A. Chagrov and M. Zakharyaschev. *Modal Logic*, volume 35 of *Oxford Logic Guides*. The Clarendon Press Oxford University Press, New York, 1997.
[10] D. De Jongh. *Investigations on the Intuitionistic Propositional Calculus*. PhD thesis, University of Wisconsin, 1968.
[11] A. Diego. *Sur les Algèbres de Hilbert*. Translated from the Spanish by Luisa Iturrioz. Collection de Logique Mathématique, Sér. A, Fasc. XXI. Gauthier-Villars, Paris, 1966.
[12] L. Esakia. Topological Kripke models. *Soviet Math. Dokl.*, 15:147–151, 1974.
[13] L. Esakia. On the theory of modal and superintuitionistic systems. In *Logical inference (Moscow, 1974)*, pages 147–172. "Nauka", Moscow, 1979.
[14] L. Esakia. *Heyting Algebras I. Duality Theory (Russian)*. "Metsniereba", Tbilisi, 1985.
[15] K. Fine. An ascending chain of S4 logics. *Theoria*, 40(2):110–116, 1974.
[16] K. Fine. An incomplete logic containing S4. *Theoria*, 40(1):23–29, 1974.
[17] K. Fine. Logics containing K4. II. *J. Symbolic Logic*, 50(3):619–651, 1985.
[18] V. Jankov. On the relation between deducibility in intuitionistic propositional calculus and finite implicative structures. *Dokl. Akad. Nauk SSSR*, 151:1293–1294, 1963.
[19] V. Jankov. The construction of a sequence of strongly independent superintuitionistic propositional calculi. *Soviet Math. Dokl.*, 9:806–807, 1968.
[20] B. Jónsson. Algebras whose congruence lattices are distributive. *Math. Scand.*, 21:110–121 (1968), 1967.
[21] P. Köhler. Brouwerian semilattices. *Trans. Amer. Math. Soc.*, 268(1):103–126, 1981.
[22] T. Litak. A continuum of incomplete intermediate logics. *Rep. Math. Logic*, (36):131–141, 2002.
[23] C. McKay. The decidability of certain intermediate propositional logics. *J. Symbolic Logic*, 33:258–264, 1968.
[24] R. McKenzie. Equational bases and nonmodular lattice varieties. *Trans. Amer. Math. Soc.*, 174:1–43, 1972.
[25] P. Minari. Intermediate logics with the same disjunctionless fragment as intuitionistic logic. *Studia Logica*, 45(2):207–222, 1986.
[26] W. Nemitz. Implicative semi-lattices. *Trans. Amer. Math. Soc.*, 117:128–142, 1965.
[27] I. Nishimura. On formulas of one variable in intuitionistic propositional calculus. *Journal of Symbolic Logic*, 25:327–331, 1960.
[28] H. Rasiowa and R. Sikorski. *The Mathematics of Metamathematics*. Monografie Matematyczne, Tom 41. Państwowe Wydawnictwo Naukowe, Warsaw, 1963.
[29] L. Rieger. On the lattice theory of Brouwerian propositional logic. *Acta fac. rerum nat. Univ. Car.*, 189:1–40, 1949.
[30] V. Shehtman. Incomplete propositional logics. *Dokl. Akad. Nauk SSSR*, 235(3):542–545, 1977.
[31] E. Tomaszewski. *On Sufficiently Rich Sets of Formulas*. PhD thesis, Institute of Philosophy, Jagellonian University, Krakóv, 2003.
[32] T. Umezawa. On intermediate propositional logics. *J. Symb. Logic*, 24:20–36, 1959.
[33] A. Visser, D. de Jongh, J. van Benthem, and G. Renardel de Lavalette. NNIL a study in intuitionistic logic. In A. Ponse, M. de Rijke, and Y. Venema, editors, *Modal logics and Process Algebra: a bisimulation perspective*, pages 289–326, 1995.
[34] S. Willard. *General Topology*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1970.
[35] A. Wronski. Intermediate logics and the disjunction property. *Reports on Mathematical Logic*, 1:39–51, 1973.
[36] M. Zakharyaschev. On intermediate logics. *Soviet Math. Dokl.*, 27(2):274–277, 1983.
[37] M. Zakharyaschev. The disjunction property of superintuitionistic and modal logics. *Mat. Zametki*, 42(5):729–738, 763, 1987.
[38] M. Zakharyaschev. Syntax and semantics of superintuitionistic logics. *Algebra and Logic*, 28(4):262–282, 1989.
[39] M. Zakharyaschev. Canonical formulas for K4. I. Basic results. *J. Symbolic Logic*, 57(4):1377–1402, 1992.
[40] M. Zakharyaschev. Canonical formulas for K4. II. Cofinal subframe logics. *J. Symbolic Logic*, 61(2):421–449, 1996.

[41] M. Zakharyaschev. Canonical formulas for modal and superintuitionistic logics: a short outline. In *Advances in intensional logic*, volume 7 of *Appl. Log. Ser.*, pages 195–248. Kluwer Acad. Publ., Dordrecht, 1997.

Guram Bezhanishvili: Department of Mathematical Sciences, New Mexico State University, Las Cruces NM 88003, gbezhani@nmsu.edu

Nick Bezhanishvili: Department of Computing, Imperial College London, 180 Queen's Gate, London SW7 2AZ, UK, nbezhani@doc.ic.ac.uk.