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# All Proper Normal Extensions of S5–square have the Polynomial Size Model Property

**Abstract.** We show that every proper normal extension of the bi-modal system $\mathbf{S5}^2$ has the poly-size model property. In fact, to every proper normal extension $L$ of $\mathbf{S5}^2$ corresponds a natural number $b(L)$ – the bound of $L$. For every $L$, there exists a polynomial $P(\cdot)$ of degree $b(L)+1$ such that every $L$–consistent formula $\varphi$ is satisfiable on an $L$-frame whose universe is bounded by $P(|\varphi|)$, where $|\varphi|$ denotes the number of subformulas of $\varphi$. It is shown that this bound is optimal.

## Introduction

It follows from [5] that the bi-modal system $\mathbf{S5}^2$ has the expontential size model property. In [1] it is proved that every normal extension of the bi-modal system $\mathbf{S5}^2$ has the finite model property. Using this result we show that in fact every proper normal extension of $\mathbf{S5}^2$ has the poly-size model property.

To every proper normal extension $L$ of $\mathbf{S5}^2$ we correspond a natural number $b(L)$ – the bound of $L$. We show that for every $L$, there exists a polynomial $P(\cdot)$ of degree $b(L)+1$ such that every $L$–consistent formula $\varphi$ is satisfiable on an $L$-frame whose universe is bounded by $P(|\varphi|)$, where $|\varphi|$ is the number of subformulas of $\varphi$. At the end of the paper we show that this bound is optimal.

The modal logic $\mathbf{S5}^2$ is widely studied, under a variety of names. Its algebraic counterpart is the variety $\mathsf{Df}_2 = \mathsf{RDf}_2$ of (representable) diagonal-free cylindric algebras of dimension two [7]. Segerberg [17] discusses an expansion under the name of two–dimensional modal logic. In Gabbay and Shehtman [2], $\mathbf{S5}^2$ is studied as a special case of taking products of modal logics.

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One of the sources of interest for $\mathbf{S5}^2$ is that it corresponds to a "clean" equality free fragment of first order logic with two variables. Our work is best motivated from this angle.

There is a wide variety of proofs available for the decidability of first-order logic with two variables and without function symbols. Equivalent results were stated and proved using quite different methods in first-order, modal and algebraic logic. We present a short historic overview.

Decidability of validity of equality free first-order sentences in two variables is due to Scott [15]. The proof uses a reduction to the set of prenex formulas of the form $\exists^2\forall^n\varphi$, whose validity is decidable by [3]. The result was stated with equality in the language, because at that time it was still believed that the $\exists^2\forall^n$ class of formulas containing equality is decidable for validity. This was however refuted in [4]. Scott's result was extended by Mortimer [12] by including equality in the language and showing that such sentences cannot force infinite models, obtaining decidability as a corollary. A simpler proof was provided in [5] and showed that any formula which can be satisfied, can actually be satisfied in a model whose size is single exponential in the length of the formula. Adding two unary function symbols to the language with only one variable leads to undecidability, as shown in [6].

Segerberg [17] proves decidability for a so-called "two-dimensional modal logic" which is essentially the equality free first-order logic with two variables. The modal technique of filtration provides the decidability result plus the finite model property. For an algebraic proof see [7] (Theorem 3.2.66 and Corollary 4.2.9).

Explicit bounds on the size of finite models are known. Every $\mathbf{S5}$-consistent formula $\phi$ is satisfiable in a model of size $|\phi| + 1$ [9]. For $\mathbf{S5}^2$ the models need to be much larger. Every $\mathbf{S5}^2$-consistent formula $\phi$ can be satisfied in a product model of size $2^{O(|\phi|)}$ [5]. Both bounds are optimal.

To these results we add that in extensions $L$ of $\mathbf{S5}^2$, every $L$-consistent formula $\phi$ is satisfiable in an $L$ model whose size is bounded by a polynomial function in $|\phi|$.

## Preliminaries

Recall that the language of $\mathbf{S5}^2$ is the propositional language equipped with two existential modalities $\Diamond_1$ and $\Diamond_2$ and their duals $\Box_1$ and $\Box_2$. Recall also that a set of formulas $L$ is called a *logic* if it contains all tautologies of the classical propositional calculus and is closed under the rule of modus ponens. A modal logic is called *normal* if it contains axiom schema 2) (see below) and is closed under the rule of necessitation. A logic $L_1$ is an extension of $L_2$ if $L_2 \subseteq L_1$.

Let $\mathbf{S5}^2$ be the set of formulas generated by the following axiom schemas, for $i = 1, 2$:

1) All tautologies of the classical propositional calculus;
2) $\Box_i(\varphi \to \psi) \to (\Box_i\varphi \to \Box_i\psi)$;
3) $\Box_i\varphi \to \varphi$;
4) $\Box_i\varphi \to \Box_i\Box_i\varphi$;
5) $\Diamond_i\Box_i\varphi \to \varphi$;
6) $\Box_1\Box_2\varphi \leftrightarrow \Box_2\Box_1\varphi$;
7) $\Box_i\varphi \leftrightarrow \neg\Diamond_i\neg\varphi$,

and closed under the following rules of inference:

Modus Ponens (MP): from $\varphi$ and $\varphi \to \psi$ infer $\psi$;
Necessitation $(\mathrm{N})_i$: from $\varphi$ infer $\Box_i\varphi$.

(Since $\Box_i\varphi \leftrightarrow \neg\Diamond_i\neg\varphi$, are axioms of $\mathbf{S5}^2$ for $i = 1, 2$, we will subsequently assume that $\Box_i$ do not appear explicitly in the formulas.)

We consider two classes of models for this logic. Call a triple $\mathcal{F} = (W, E_1, E_2)$ an $\mathbf{S5}^2$-frame if $W$ is a non-empty set, and $E_1$ and $E_2$ are equivalence relations on $W$ such that $E_1 \circ E_2 = E_2 \circ E_1$. The variety generated by the full complex algebras of $\mathbf{S5}^2$-frames is denoted by $\mathsf{Df}_2$ in the algebraic literature [7].

A couple $\mathcal{M} = (\mathcal{F}, \models)$, is called a *model* if $\mathcal{F}$ is an $\mathbf{S5}^2$-frame, and $\models$ is a binary relation on $W \times \Phi$ (where $\Phi$ is the set of formulas) such that for any $w \in W$ and $\varphi, \psi \in \Phi$:

$$\begin{array}{lcl}
w \models \phi \wedge \psi & \Leftrightarrow & w \models \phi \;\&\; w \models \psi, \\
w \models \phi \vee \psi & \Leftrightarrow & w \models \phi \;\mathit{or}\; w \models \psi, \\
w \models \neg\psi & \Leftrightarrow & w \not\models \psi, \\
w \models \Box_i\psi & \Leftrightarrow & \forall v \in W(wE_iv \Rightarrow v \models \psi), \\
w \models \Diamond_i\psi & \Leftrightarrow & \exists v \in W(wE_iv \;\&\; v \models \psi).
\end{array}$$

A formula $\varphi$ is *satisfiable* in an $\mathbf{S5}^2$-frame $\mathcal{F} = (W, E_1, E_2)$ if there exists a model $\mathcal{M} = (\mathcal{F}, \models)$ and a point $w \in W$ such that $w \models \varphi$. A formula $\varphi$ is *valid* in $\mathcal{F}$ if for any model $\mathcal{M} = (\mathcal{F}, \models)$, and any point $w \in W$, we have $w \models \varphi$.

For any equivalence relation $E$ on $W$, let $E(w) = \{v \in W : vEw\}$ and $E(V) = \bigcup_{w \in V} E(w)$ for $w \in W$ and $V \subseteq W$. An especially interesting subclass of the class of $\mathbf{S5}^2$-frames is the class of frames which satisfy an additional condition: $E_1(w) \cap E_2(w) = \{w\}$ for any $w \in W$. These frames can be represented as products of two $\mathbf{S5}$-frames. Recall that for given two $\mathbf{S5}$-frames $\mathcal{F} = (W, E)$ and $\mathcal{F}' = (W', E')$, the *product* $\mathcal{F} \times \mathcal{F}'$ is defined as the triple $(W \times W', E_1, E_2)$, where

$$\begin{array}{lll} (w, w')E_1(v, v') & \text{iff} & w = v \text{ and } w'E'v' \\ (w, w')E_2(v, v') & \text{iff} & wEv \text{ and } w' = v'. \end{array}$$

Call an $\mathbf{S5}^2$-frame $(W, E_1, E_2)$ *rooted* if $E_1E_2(w) = W$ for any $w \in W$. $\mathbf{S5}^2$ has the fmp with respect to both classes of rooted frames (cf., [7] or [2]), in other words for every formula $\varphi$ of the language of $\mathbf{S5}^2$ we have:

$$\begin{array}{ll} \varphi \in \mathbf{S5}^2 & \text{iff } \varphi \text{ is valid in every finite rooted } \mathbf{S5}^2\text{-frame} \\ & \text{iff } \varphi \text{ is valid in every finite rooted product frame.} \end{array}$$

(Note that in [7] $\mathsf{RDf}_2$ denotes the variety generated by the full complex algebras of product frames.) Below we assume that every $\mathbf{S5}^2$-frame is rooted.

One of the main reasons for studying $\mathbf{S5}^2$ is that $\mathbf{S5}^2$ axiomatizes the two-variable substitution-free fragment of the first order classical logic, $FOL$ for short. Indeed, consider the following translation of the formulas of the language of $\mathbf{S5}^2$ to the formulas of the language of $FOL$:

$p^t = P(x_1, x_2)$,
$(\cdot)^t$ is a homomorphism for the Booleans,
$(\Diamond_1\varphi)^t = \exists x_1 \varphi^t$,
$(\Diamond_2\varphi)^t = \exists x_2 \varphi^t$.

Then one can show (see e.g. [7]) that this translation is faithful, for all formulas $\varphi$ we have $\varphi \in \mathbf{S5}^2$ iff $FOL \vdash \varphi^t$.

Similarly to $\mathbf{S5}^2$, one can show that the logic $\mathbf{S5}^n$ of $n$-ary products of $\mathbf{S5}$-frames is a subfragment of the $n$-variable $FOL$ when the translation is defined in the same way as for $\mathbf{S5}^2$ above.

It is relevant here to recall that $\mathbf{S5}$ axiomatizes the one-variable fragment of $FOL$ (see [18]). The lattice of (normal) extensions of $\mathbf{S5}$ is rather easy to describe, it is a $(\omega + 1)$-chain; every (normal) extension of $\mathbf{S5}$ is semantically characterized by the $\mathbf{S5}$-frame shown in Figure 1; all of them are finitely axiomatizable, enjoy the finite model property and are decidable (see Scroggs [16]). Actually, every (normal) extension of $\mathbf{S5}$ has the linear-size model property, and every proper (normal) extension of $\mathbf{S5}$ is tabular.

Let $\underbrace{[\mathbf{S5}, \mathbf{S5}, \ldots, \mathbf{S5}]}_{n-times}$ denote the modal logic of $n$ commuting $\mathbf{S5}$ diamonds (meaning that for all $i, j$, $\Diamond_i\Diamond_j\phi \leftrightarrow \Diamond_j\Diamond_i\phi$ is an axiom of $\underbrace{[\mathbf{S5}, \mathbf{S5}, \ldots, \mathbf{S5}]}_{n-times}$).

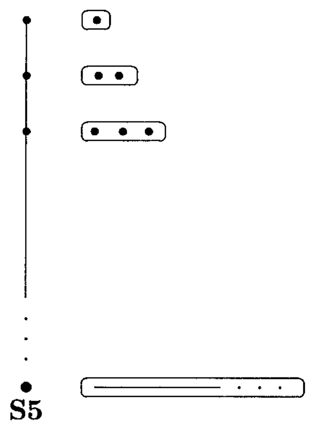

Figure 1. The proper extensions of **S5**.

However, unlike **S5**, in the case of $\underbrace{[\mathbf{S5}, \mathbf{S5}, \dots, \mathbf{S5}]}_{n-times}$, for $n \geq 3$, the lattice of extensions of $\underbrace{[\mathbf{S5}, \mathbf{S5}, \dots, \mathbf{S5}]}_{n-times}$ is much more complicated. It has been shown by Maddux [10] that every logic between $[\mathbf{S5}, \mathbf{S5}, \mathbf{S5}]$ and $\mathbf{S5}^3$ is undecidable. Kurucz [8] strengthened this by showing that fmp also fails for all logics in this interval.

However, the situation improves in the two-dimensional case. Indeed, we have the following consequence of [1]:

THEOREM 1. *Every normal extension $L$ of* $\mathbf{S5}^2$ *enjoys the finite model property.*

PROOF. It is shown in [1], that every subvariety of the variety $\mathsf{Df}_2$ of the two-dimensional diagonal-free cylindric algebras is generated by its finite members. Since there exists a lattice anti-isomorphism between the lattice of subvarieties of $\mathsf{Df}_2$ and the lattice of normal extensions of $\mathbf{S5}^2$ (see e.g.[7] or [13]), and since for any finitely approximable variety, the corresponding logical system has the finite model property, we conclude that every normal extension of $\mathbf{S5}^2$ enjoys the finite model property. ∎

For every formula $\varphi$, let $Sub(\varphi)$ denote the set of all subformulas of $\varphi$, and let $|\varphi|$ denote the cardinality of the set $Sub(\varphi)$. A logic $L$ is said to have the *poly-size model property* if there exist a polynomial $P(\cdot)$ such that every $L$-consistent formula $\varphi$ (that is $\perp \notin L \cup \{\varphi\}$) is satisfiable in an $L$–frame containing at most $P(|\varphi|)$ points. If $P(\cdot)$ is a linear function, then $L$ is said to have the *linear-size model property.*

We will show that all proper normal extensions of $\mathbf{S5}^2$ have the poly-size model property.

Finally, let us mention that our notation is slightly different from the standard one in [7]: instead of $c_0$ and $c_1$ we use $\Diamond_1$ and $\Diamond_2$, and instead of $T_0$ and $T_1$ we use $E_1$ and $E_2$ respectively.

## Normal extensions of S5$^2$

In this section we prove a more specific version of Theorem 1. In particular, for each proper normal extension of $\mathbf{S5}^2$ we will describe the class of its finite frames in terms of the depth of $E_1$ and $E_2$ equivalence classes. For this we introduce the following terminology.

For a given $\mathbf{S5}^2$-frame $\mathcal{F} = (W, E_1, E_2)$, $w \in W$ and $i = 1, 2$ we call the sets of the form $E_i(w)$, *$E_i$-clusters.* We also call the sets of the form $E_1(w) \cap E_2(w)$ *$E_0$-clusters* and denote them by $E_0(w)$. (It should be clear that $E_0 = E_1 \cap E_2$ is also an equivalence relation.)

We denote $E_1$-clusters of $\mathcal{F}$ by $C_i^1$, $E_2$-clusters by $C_j^2$, and $E_0$-clusters $C_i^1 \cap C_j^2$ by $C_{i,j}^0$.

DEFINITION 2. A given $\mathbf{S5}^2$-frame $\mathcal{F}$ is said to be of *$E_i$-depth $n$* ($i = 1, 2$ and $n \in \omega$), written as $d_i(\mathcal{F}) = n$, if it contains precisely $n$ $E_i$-clusters.

$\mathcal{F}$ is said to be of an infinite *$E_i$-depth* (i=1,2), written as $d_i(\mathcal{F}) = \omega$, if the number of $E_i$-clusters of $\mathcal{F}$ is infinite.

A class $\mathbf{F}$ of $\mathbf{S5}^2$-frames is said to be of *$E_i$-depth $n$* ($i = 1, 2$ and $n \in \omega$), written as $d_i(\mathbf{F}) = n$, if there is a member of $\mathbf{F}$ of $E_i$-depth $n$, and $E_i$-depth of every other member of $\mathbf{F}$ is less than or equal to $n$.

$\mathbf{F}$ is said to be of *$E_i$-depth $\omega$*, written as $d_i(F) = \omega$, if $E_i$-depth of members of $\mathbf{F}$ is not restricted to any natural $n$.

For every normal extension $L$ of $\mathbf{S5^2}$, we say that an $\mathbf{S5^2}$-frame $\mathcal{F}$ is an *$L$-frame* if $\mathcal{F}$ validates all the theorems of $L$. Let $\mathbf{F}_L$ denote the class of all finite $L$-frames. It follows from Theorem 1 that every normal extension $L$ of $\mathbf{S5}^2$ is complete with respect to $\mathbf{F}_L$. We say that a logic $L$ is of *$E_i$-depth $n$* ($i = 1, 2$ and $n \in \omega$), written as $d_i(L) = n$, if $\mathbf{F}_L$ is of $E_i$-depth $n$. $L$ is said to be of *$E_i$-depth $\omega$*, written as $d_i(L) = \omega$, if $d(\mathbf{F}_L) = \omega$.

THEOREM 3. *For every proper normal extension $L$ of $\mathbf{S5}^2$ there exists a natural number $n$ such that $\mathbf{F}_L$ can be divided into three disjoint classes*

$\mathbf{F}_L = \mathbf{F}_1 \sqcup \mathbf{F}_2 \sqcup \mathbf{F}_3$, *where* $d_2(\mathbf{F}_1)$, $d_1(\mathbf{F}_2)$, $d_1(\mathbf{F}_3)$, $d_2(\mathbf{F}_3) \leq n$. *(Note that any two of the classes* $\mathbf{F}_1, \mathbf{F}_2$ *and* $\mathbf{F}_3$ *may be empty.)*

The proof of this theorem depends on two lemmas. The first one will also be used in the next section. It states that certain frames can be compressed and still remain frames of the logic. The compression is a standard technique in modal logic: define an equivalence relation, let the new states be the equivalence classes, define the accessibility relations minimally and show that the function which sends states to their equivalence classes is a $p$-morphism.[1] The proof is left to the reader.

LEMMA 4. *Let* $\mathcal{F} = (W, E_1, E_2)$ *be a finite* $\mathbf{S5}^2$*-frame and* $B$ *an equivalence relation on* $W$. *Let* $\mathcal{F}_B = (W/B, E_1', E_2')$, *where*

$$B(w)E_i'B(v) \text{ iff there exist } w' \in B(w) \text{ and } v' \in B(v) \text{ with } w'E_iv',$$

*for* $i = 1, 2$. *Let the function* $f_B : W \to W/B$ *be defined by* $f_B(w) = B(w)$ *for any* $w \in W$. *If any of the following three cases* $(1), (2a), (2b)$ *holds, then* $f_B$ *is a p-morphism from* $\mathcal{F}$ *onto* $\mathcal{F}_B$.

(1) $B \subseteq E_0$ *(that is,* $B$ *identifies only points from* $E_0$*-clusters).*

$(2a)$ $B \subseteq E_2$ *and* $uBv$ *implies that for every* $u' \in E_1(u)$ *there exists some* $v' \in E_1(v)$ *with* $u'Bv'$.

$(2b)$ $B \subseteq E_1$ *and* $uBv$ *implies that for every* $u' \in E_2(u)$ *there exists some* $v' \in E_2(v)$ *with* $u'Bv'$.

Denote by $\mathbf{n}$ the $\mathbf{S5}$-frame $(W_n, E)$, where $W_n$ is an $n$-element set and $E = W_n \times W_n$. Also denote by $\mathbf{n} \times \mathbf{m}$ the product of $\mathbf{n}$ and $\mathbf{m}$. Obviously $\mathbf{n} \times \mathbf{m}$ is an $\mathbf{S5}^2$-frame.

LEMMA 5. *(i) Every* $\mathbf{S5}^2$*-frame* $\mathcal{F} = (W, E_1, E_2)$ *with* $d_1(\mathcal{F}) = k$ *and* $d_2(\mathcal{F}) = m$ *can be p-morphically maped onto the product frame* $\mathbf{k} \times \mathbf{m}$.

*(ii) If* $k \geq k'$ *and* $m \geq m'$, *then* $\mathbf{k} \times \mathbf{m}$ *can be p-morphically maped onto* $\mathbf{k}' \times \mathbf{m}'$.

PROOF. (i) Consider $\mathcal{F}_{E_0} = (W/E_0, E_1', E_2')$. Define $f_{E_0} : W \to W/E_0$ by setting $f_{E_0}(w) = E_0(w)$ for any $w \in W$. By Lemma 4 (1) is a $p$-morphism and it is obvious that $\mathcal{F}_{E_0}$ is isomorphic to the product frame $\mathbf{k} \times \mathbf{m}$.

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[1] Given two $\mathbf{S5}^2$-frames $\mathcal{F}_1 = (W, E_1, E_2)$ and $\mathcal{F}_2 = (W', E_1', E_2')$, a function $f : W \to W'$ is said to be a *p-morphism*, if, for any $w \in W, w' \in W'$ and $i = 1, 2$, $f(w)E_i'w'$ iff $(\exists v \in W)(wE_iv \ \& \ f(v) = w')$. $\mathcal{F}_2$ is said to be a *p-morphic image* of $\mathcal{F}_1$ if there is a $p$-morphism from $W$ onto $W'$. It is well-known that $p$-morphic images preserve validity of formulas.

(ii) Fix $k' - 1$ different $E_1$-clusters $C_1^1, \dots, C_{k'-1}^1$ of $\mathbf{k} \times \mathbf{m}$. Let $Y = \bigcup_{i=1}^{k'-1} C_i^1$. Define an equivalence relation $B_1$ on $\mathbf{k} \times \mathbf{m}$ by putting:

$wB_1v$ iff $w = v$ for any $w, v \in Y$,
$wB_1v$ iff $wE_2v$ for any $w, v \in (\mathbf{k} \times \mathbf{m}) - Y$.

Now define $f_{B_1}$ from $\mathbf{k} \times \mathbf{m}$ onto $\mathbf{k} \times \mathbf{m}/B_1$ by setting $f_{B_1}(w) = B_1(w)$ for any $w \in \mathbf{k} \times \mathbf{m}$. By Lemma 4 (2$a$) it follows directly that $f_{B_1}$ is a $p$-morphism. It is straightforward to check that $\mathbf{k} \times \mathbf{m}/B_1$ is isomorphic to $\mathbf{k}' \times \mathbf{m}$.

Now fix $m' - 1$ different $E_2$-clusters $C_1^2, \dots, C_{m'-1}^2$ of $\mathbf{k}' \times \mathbf{m}$. Let $Z = \bigcup_{j=1}^{m'-1} C_j^2$. Define an equivalence relation $B_2$ on $\mathbf{k}' \times \mathbf{m}$ by putting:

$wB_2v$ iff $w = v$ for any $w, v \in Z$,
$wB_2v$ iff $wE_1v$ for any $w, v \in (\mathbf{k}' \times \mathbf{m}) - Z$.

Define $f_{B_2}$ from $\mathbf{k}' \times \mathbf{m}$ onto $\mathbf{k}' \times \mathbf{m}/B_2$ by setting $f_{B_2}(w) = B_2(w)$ for any $w \in \mathbf{k}' \times \mathbf{m}$. From Lemma 4 (2$b$) it follows directly that $f_{B_2}$ is a $p$-morphism and it is straightforward to check that $\mathbf{k}' \times \mathbf{m}/B_2$ is isomorphic to $\mathbf{k}' \times \mathbf{m}'$. Therefore, $\mathbf{k}' \times \mathbf{m}'$ is a $p$-morphic image of $\mathbf{k} \times \mathbf{m}$. ■

Now we are ready to prove Theorem 3.

PROOF OF THEOREM 3. Suppose $L$ is a proper normal extension of $\mathbf{S5}^2$. Because $\mathbf{S5}^2$ is complete with respect to the class of all finite product frames of the form $\mathbf{n} \times \mathbf{n}$ (see e.g. [17]), there exists a product frame $\mathbf{n} \times \mathbf{n}$ such that $\mathbf{n} \times \mathbf{n} \notin \mathbf{F}_L$. Let $n$ be the minimal number such that $\mathbf{n} \times \mathbf{n} \notin \mathbf{F}_L$. Consider three subclasses of $\mathbf{F}_L$: $\mathbf{F}_1 = \{\mathcal{F} \in \mathbf{F}_L : d_1(\mathcal{F}) > n\}$, $\mathbf{F}_2 = \{\mathcal{F} \in \mathbf{F}_L : d_2(\mathcal{F}) > n\}$ and $\mathbf{F}_3 = \{\mathcal{F} \in \mathbf{F}_L : d_1(\mathcal{F}), d_2(\mathcal{F}) \leq n\}$.

Let us show that if $\mathcal{F} \in \mathbf{F}_1$, then $d_2(\mathcal{F}) \leq n$ and if $\mathcal{F} \in \mathbf{F}_2$, then $d_1(\mathcal{F}) \leq n$. Indeed, suppose $\mathcal{F} \in \mathbf{F}_1 \sqcup \mathbf{F}_2$, $d_1(\mathcal{F}) = k$, $d_2(\mathcal{F}) = m$ and both $k, m > n$. By Lemma 5 (i) $\mathbf{k} \times \mathbf{m}$ is a $p$-morphic image of $\mathcal{F}$ and by Lemma 5 (ii) $\mathbf{n} \times \mathbf{n}$ is a $p$-morphic image of $\mathbf{k} \times \mathbf{m}$. So, $\mathbf{n} \times \mathbf{n}$ is a $p$-morphic image of $\mathcal{F}$ and hence $\mathbf{n} \times \mathbf{n}$ belongs to $\mathbf{F}_L$ which is a contradiction. So, if $\mathcal{F} \in \mathbf{F}_1$, then $d_1(\mathcal{F}) > n$ and $d_2(\mathcal{F}) \leq n$, if $\mathcal{F} \in \mathbf{F}_2$, then $d_1(\mathcal{F}) \leq n$ and $d_2(\mathcal{F}) > n$ and if $\mathcal{F} \in \mathbf{F}_3$, then $d_1(\mathcal{F}), d_2(\mathcal{F}) \leq n$. This shows that all three classes $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ are disjoint. ■

## Poly-size model property

In this section we prove the main result of the paper, that every proper normal extension of $\mathbf{S5}^2$ has the poly-size model property. First we introduce some terminology.

For every proper normal extension $L$ of $\mathbf{S5}^2$, we introduce the following parameters: for $i \in \{1,2\}$, $k \in \{1,2,3\}$, $p_i^k = d_i(\mathbf{F}_k)$. The parameter $p_i^k$ gives the $E_i$ depth of the class $\mathbf{F}_k$. Call a parameter finite, if it is not $\omega$. Note that the only parameters which may be infinite are $p_1^1$ and $p_2^2$. Denote by $b(L)$ the maximum between all the finite parameters of $L$, and call it the *bound* of $L$. Note that if $p_1^1$ and $p_2^2$ are $\omega$, then $b(L) = n$, where $n$ is the minimal natural number such that $\mathbf{n} \times \mathbf{n} \notin \mathbf{F}_L$.

Let $|\varphi|$ denote the *modal size* of $\varphi$, that is the number of subformulas of $\varphi$ of the form $\Diamond_1\psi$ and $\Diamond_2\chi$. Recall that a polynomial $P(n)$ is said to be of *degree* $k$ if $n^k$ occurs in $P(n)$ and $n^m$ does not occur in $P(n)$ for any $m > k$.

THEOREM 6. *Let $L$ be a proper normal extension of $\mathbf{S5}^2$ with bound $b(L)$. Then every $L$–satisfiable formula $\varphi$ is satisfiable in an $L$-frame of size $P(|\varphi|)$, for $P(|\varphi|)$ a polynomial of degree $b(L)+1$. Moreover, if all the parameters of $L$ are finite, then $P(|\varphi|)$ is just linear in $|\varphi|$.*

In the proof we create small models from large ones taking care that 1) the frame of the small model is still a frame of the logic, and 2) certain formulas are still satisfied in the small model. For the first part we can use Lemma 4, for the latter part we use next lemma.

LEMMA 7. *For any proper normal extension $L$ of $\mathbf{S5}^2$, if $\varphi$ is $L$-satisfiable, then it is satisfiable in an $L$-frame $\mathcal{F} = (W, E_1, E_2)$ such that*

$$|W| \leq d_1(\mathcal{F})|\varphi| + d_2(\mathcal{F})|\varphi| + d_1(\mathcal{F}) \cdot d_2(\mathcal{F}) + 1.$$

*Moreover, the size of any $E_0$-cluster in $\mathcal{F}$ is at most $|\varphi|$.*

PROOF. Let $\mathcal{F} = (W, E_1, E_2)$ be an $L$-frame satisfying formula $\varphi$. Then there exists a valuation $\models$ on $\mathcal{F}$ and a point $w \in W$, such that $w \models \varphi$.

CLAIM 8. *Let $\mathcal{M} = (\mathcal{F}, \models)$ be a model on some $\mathbf{S5}^2$-frame $\mathcal{F} = (W, E_1, E_2)$. Let $W' \subseteq W$, $\mathcal{F}' = (W', E_1|_{W'}, E_2|_{W'})$, ($E_i|_{W'}$ is the restriction of $E_i$ on $W'$) and $\mathcal{M}' = (\mathcal{F}', \models')$ be such that $v \models' p$ iff $v \models p$, for all variables $p$ and $v \in W'$. Suppose $W'$ is such that*

(i) *for all $\Diamond_1\psi \in Sub(\varphi)$ and $E_1$-cluster $C_i^1$ in $\mathcal{F}$, if there exists $x \in C_i^1$ with $x \models \psi$ then there exists $y \in C_i^1 \cap W'$ with $y \models \psi$, and*
(ii) *for all $\Diamond_2\psi \in Sub(\varphi)$ and $E_2$-cluster $C_j^2$ in $\mathcal{F}$, if there exists $x \in C_j^2$ with $x \models \psi$ then there exists $y \in C_j^2 \cap W'$ with $y \models \psi$.*

*Then for all $v \in W'$ and $\psi \in Sub(\varphi)$, $v \models \psi$ iff $v \models' \psi$.*

PROOF. By induction on the size of $\psi \in Sub(\varphi)$. The Boolean clauses are trivial. Let $\psi = \Diamond_i\chi$, $i = 1, 2$. Then $v \models' \Diamond_i\chi$ implies that there exists $v' \in W'$ such that $vE_iv'$ and $v' \models' \chi$. But then by the induction hypothesis $v' \models \chi$, and hence $v \models \Diamond_i\chi$. Conversely, from $v \models \Diamond_i\chi$ it follows that $\chi$ is satisfied in $E_i(v)$. From (i) and (ii) it follows that there exists $y \in W'$ such that $vE_iy$ and $y \models \chi$. But then by the induction hypothesis $y \models' \chi$, and hence $v \models' \Diamond_i\chi$. ∎

Now we will create a small satisfiable model from $\mathcal{F}$. For every $E_1$-cluster $C_i^1$ $(1 \leq i \leq d_1(\mathcal{F}))$ and every $\Diamond_1\psi \in Sub(\varphi)$, we choose a point $x \in C_i^1$ such that $x \models \psi$ (if such a point exists at all). We do the same for $E_2$-clusters and $\Diamond_2\psi \in Sub(\varphi)$. Moreover, if there are $E_0$-clusters of $W$ which do not contain any selected points, we choose one point from each of them. Denote by $W'$ the set of all selected points plus $w$. (Note that if $\mathcal{F}$ is a product-frame, then $W = W'$.) Define the relation $B$ on $W$ as follows. By the definition of $W'$, for each $E_0$-cluster $C_{i,j}^0$ of $\mathcal{F}$ we have chosen at least one point $witness(C_{i,j}^0) \in C_{i,j}^0$ to be in $W'$. Now let $B$ be the smallest equivalence relation which identifies the points from $C_{i,j}^0 - W'$ with $witness(C_{i,j}^0)$ and define $f_B : \mathcal{F} \longrightarrow \mathcal{F}_B$ by putting $f_B(w) = B(w)$ for any $w \in W$. Then the frame $\mathcal{F}_B$ is isomorphic to $\mathcal{F}'$, and $B \subseteq E_0$. Therefore by Lemma 4 (1), $\mathcal{F}'$ is a $p$-morphic image of $\mathcal{F}$, and hence is an $L$-frame.

Finally, consider the model $\mathcal{M}' = (\mathcal{F}', \models')$, where $\mathcal{F}' = (W', E_1|_{W'}, E_2|_{W'})$ and $v \models' p$ iff $v \models p$, for every $v \in W'$ and every propositional letter $p$ occurring in $\varphi$. Then by Claim 8, $\mathcal{F}'$ also satisfies $\varphi$. Note, that $|W'| \leq d_1(\mathcal{F})|\varphi| + d_2(\mathcal{F})|\varphi| + d_1(\mathcal{F}) \cdot d_2(\mathcal{F}) + 1$. Indeed, there exist $d_1(\mathcal{F})$-many $E_1$-clusters and $d_2(\mathcal{F})$-many $E_2$-clusters of $W$. From every $E_i$-cluster, $i = 1, 2$, we select at most $|\varphi|$ points. So, we select $(d_1(\mathcal{F})|\varphi| + d_2(\mathcal{F})|\varphi|)$-many points, and then from every $E_0$-cluster which does not contain any selected point, we choose an additional point. Obviously there are $d_1(\mathcal{F}) \cdot d_2(\mathcal{F})$-many $E_0$-clusters in $W$, hence $|W'| \leq d_1(\mathcal{F})|\varphi| + d_2(\mathcal{F})|\varphi| + d_1(\mathcal{F}) \cdot d_2(\mathcal{F}) + 1$. ∎

Now we can quickly prove Theorem 6.

PROOF OF THEOREM 6. Let $L$ be as in the theorem with bound $b(L)$. Let $\varphi$ be $L$-satisfiable. Then there exist an $L$-frame $\mathcal{F} = (W, E_1, E_2)$, a valuation $\models$ on $\mathcal{F}$ and $w \in W$ such that $w \models \varphi$. By Lemma 7 we may assume that

$$|W| \leq d_1(\mathcal{F})|\varphi| + d_2(\mathcal{F})|\varphi| + d_1(\mathcal{F}) \cdot d_2(\mathcal{F}) + 1,$$

Moreover, the size of any $E_0$-cluster in $\mathcal{F}$ is at most $|\varphi|$. Hence, every $E_1$-cluster of $\mathcal{F}$ contains at most $d_2(\mathcal{F})|\varphi|$ and every $E_2$-cluster contains at most $d_1(\mathcal{F})|\varphi|$ points respectively. We split the proof in three cases.

**Case 1: [All parameters are finite or $\mathcal{F} \in \mathbf{F}_3$].** In this case, $d_1(\mathcal{F})$ and $d_2(\mathcal{F})$ are both smaller than $b(L)$, whence $\varphi$ is satisfied in a frame with at most $2b(L)|\varphi| + b(L)^2 + 1$ points, which is a linear function in $|\varphi|$.

**Case 2: [$\mathcal{F} \in \mathbf{F}_2$ and $d_2(\mathcal{F})$ is unbounded].** Because $\mathcal{F} \in \mathbf{F}_2$, $d_1(\mathcal{F}) \leq b(L)$, but $d_2(\mathcal{F})$ is unbounded, whence the frame might be too large. We make it smaller by defining an equivalence relation $B$ on $W$, and factorize $\mathcal{F}$ through it. To this end we say that two $E_2$-clusters $C^2_p$ and $C^2_q$ are *equivalent*, if

$$|C^1_i \cap C^2_p| = |C^1_i \cap C^2_q| \text{ for all } i \text{ between 1 and } d_1(\mathcal{F}).$$

Because the size of the $E_0$-clusters $C^1_i \cap C^2_j$ is bounded by $|\varphi|$, the number of non-equivalent $E_2$-clusters is bounded[2] by $|\varphi|^{d_1(\mathcal{F})}$.

Now we define a submodel of $\mathcal{M}$ which still satisfies $\phi$, its underlying frame is a p-morphic image of $\mathcal{F}$ and it is of the right (small) size.

For every $E_1$-cluster $C^1_i$ of $\mathcal{F}$ $(1 \leq i \leq d_1(\mathcal{F}))$ and every $\Diamond_1\psi \in Sub(\varphi)$, we choose a point $x \in C^1_i$ such that $x \models \psi$ (if such a point exists at all). Denote by $S$ the set of selected points plus $w$. It is easy to see that

$$|E_2(S)| \leq (d_1(\mathcal{F})|\varphi| + 1)d_1(\mathcal{F})|\varphi|$$

Indeed, from every $E_1$-cluster we select at most $|\varphi|$ points. There are $d_1(\mathcal{F})$ $E_1$-clusters in $\mathcal{F}$. So, we select points from at most $d_1(\mathcal{F})|\varphi| + 1$ different $E_2$-clusters and every $E_2$-cluster of $\mathcal{F}$ contains at most $d_1(\mathcal{F})|\varphi|$ points.

Now from every equivalence class let us choose one representative $C^2_p$ and let $W'$ be $E_2(S)$ plus this set of representatives. Put $\mathcal{F}' = (W', E_1|_{W'}, E_2|_{W'})$, and $\mathcal{M}' = (\mathcal{F}', \models')$ such that $v \models' p$ iff $v \models p$, for all $p$ in $v \in W'$. Then by Claim 8, $\mathcal{F}'$ satisfies $\varphi$. The number of points in $W'$ is bounded by

$$|E_2(S)| + (|\varphi|^{d_1(\mathcal{F})} \cdot d_1(\mathcal{F})|\varphi|) \leq b(L)^2|\varphi|^2 + b(L)|\varphi| + b(L)|\varphi|^{b(L)+1}.$$

Finally, almost the same construction as in Lemma 7 will provide us with a $p$-morphism from $\mathcal{F}$ to $\mathcal{F}'$. For every $E_2$-cluster $C^2_q \subseteq W - W'$, let $C^2_p \subseteq W'$ be a $E_2$-cluster which is equivalent to $C^2_q$. Then the $E_0$-clusters $|C^0_{i,p}|$ and $|C^0_{i,q}|$ contain the same number of points for every $i = 1, \ldots, d_1(\mathcal{F})$. Suppose $C^0_{i,p} = \{w_{i_1}, \ldots, w_{i_{n_i}}\}$ and $C^0_{i,q} = \{v_{i_1}, \ldots, v_{i_{n_i}}\}$. Let $B$ be the

[2] Indeed, to every $E_2$-cluster $C^2_p$ from $\mathcal{F}$ corresponds the sequence of natural numbers $\overline{n} = (n_1, \ldots, n_{d_1(\mathcal{F})})$, where $n_1 = |C^0_{1,p}|, \ldots, n_{d_1(\mathcal{F})} = |C^0_{d_1(\mathcal{F}),p}|$. Obviously, $n_j \leq |\varphi|$ for $1 \leq j \leq d_1(\mathcal{F})$, and to equivalent $E_2$-clusters correspond the same sequences. Now since there exist only $|\varphi|^{d_1(\mathcal{F})}$-many different sequences $\overline{n} = (n_1, \ldots, n_{d_1(\mathcal{F})})$, there exist only $|\varphi|^{d_1(\mathcal{F})}$-many non-equivalent $E_2$-clusters.

smallest equivalence relation such that $w_{i_r} B v_{i_r}$ holds for all $r = 1, \ldots, n_i$ and $i = 1, \ldots, d_1(\mathcal{F})$. Then $B$ satisfies condition (2$b$) of Lemma 4. Thus by Lemma 4, $f_B$ is a $p$-morphism from $\mathcal{F}$ onto $\mathcal{F}_B$. But $\mathcal{F}_B$ is isomorphic to $\mathcal{F}'$, thus the latter is in $\mathbf{F}_L$.

Thus $\varphi$ is satisfiable in an $L$-frame containing at most $P(|\varphi|)$-many points, for $P(\cdot)$ a polynomial of degree $b(L) + 1$.

**Case 3: [$\mathcal{F} \in \mathbf{F}_1$ and $d_1(\mathcal{F})$ is unbounded].** This case is symmetrical to case 2. This finishes the proof of the theorem. ∎

COROLLARY 9. *Every proper normal extension of* $\mathbf{S5}^2$ *has the poly-size model property.*

**Proof**. Let $L$ be a proper normal extension of $\mathbf{S5}^2$ and $\varphi$ an $L$-consistent formula. Then $\neg\varphi \notin L$ and by Theorem 1 there is a finite $L$-frame $\mathcal{F}$ refuting $\neg\varphi$. Hence, $\mathcal{F}$ satisfies $\varphi$ and by Theorem 6 there exists an $L$-frame $\mathcal{F}'$ which satisfies $\varphi$ and the universe of $\mathcal{F}'$ is bounded by a polynomial of degree $b(L) + 1$ in $|\varphi|$. Therefore, $L$ has the poly-size model property. ∎

## Proper extensions of $\mathbf{S5}^2$ without linear-size model property

In the previous section we showed that proper normal extensions of $\mathbf{S5}^2$ have the poly-size model property. In this section we show that our bound is indeed optimal by constructing proper normal extensions $L_k$ of $\mathbf{S5}^2$ and formulas $\varphi_k^n$ such that the size of the smallest $L_k$-frame satisfying $\varphi_k^n$ is a polynomial of degree $b(L_k) + 1$ in $|\varphi_k^n|$. (Of course, the logics $L_k$ will have an infinite parameter, namely $p_2^2(L_k)$ will be $\omega$.)

Let a finite $\mathbf{S5}^2$-frame $\mathcal{F}$ be given and let $\{C_i^1\}_{i=1}^n$ and $\{C_j^2\}_{j=1}^m$ be the sets of $E_1$ and $E_2$-clusters of $\mathcal{F}$, respectively. Recall from the previous section that two distinct $E_2$-clusters $C_p^2$ and $C_q^2$ are *equivalent* if

$$|C_i^1 \cap C_p^2| = |C_i^1 \cap C_q^2| \text{ for all } i \text{ between 1 and } n.$$

Fix any natural number $k \geq 2$. For any natural number $n$, let $\mathcal{G}_k^n$ be an $\mathbf{S5}^2$-frame of $E_1$-depth $k$ such that every $E_2$-cluster of $\mathcal{G}_k^n$ contains exactly $k + n$ points and every two distinct $E_2$-clusters of $\mathcal{G}_k^n$ are not equivalent to each other. Note that $\mathcal{G}_k^n$ is not unique, since there are several (though finitely many) frames with this property. Let $\mathcal{F}_k^n$ be the maximal one with this property, that is $|\mathcal{G}_k^n| \leq |\mathcal{F}_k^n|$, for any $\mathcal{G}_k^n$.

Let $L_k = \bigcap_{n\in\omega} L(\mathcal{F}_k^n)$, where $L(\mathcal{F}_k^n)$ is the logic of the frame $\mathcal{F}_k^n$ for $n \in \omega$. Obviously, $p_2^2(L_k) = \omega$ and $b(L_k) = k$.

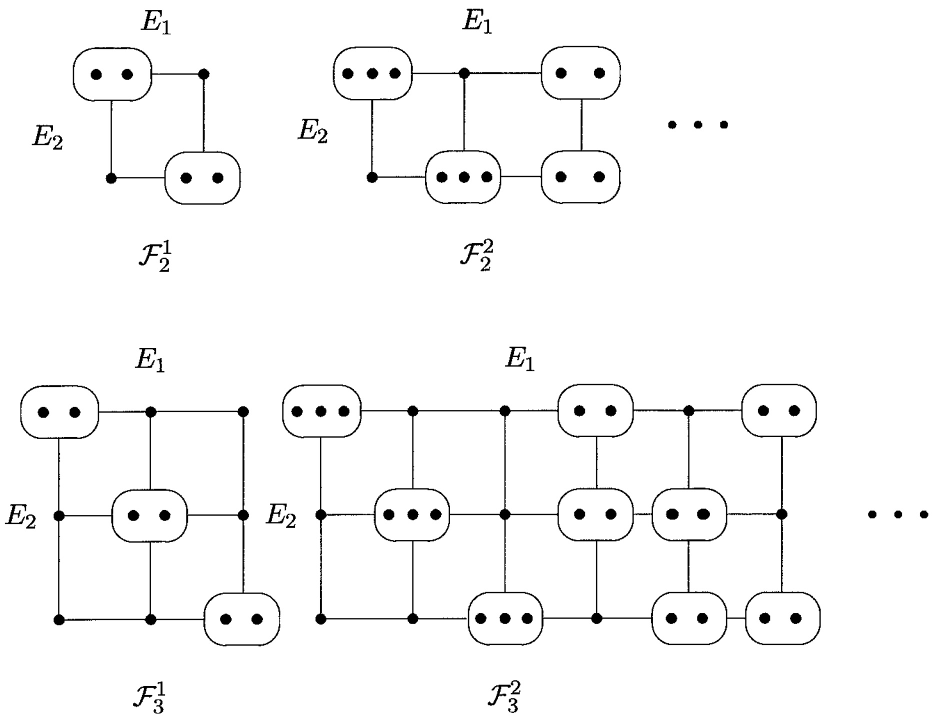


Figure 2. $\mathcal{F}_k^n$ frames for $k = 2$ and $k = 3$.

The cases for $k = 2$ and $k = 3$ are shown in Figure 2.

Now for $n > k$, let $\varphi_k^n = Q_k \wedge \psi^n$, where

$$\begin{aligned} Q_k &= \bigwedge_{i=1}^k \Diamond_1 \Diamond_2 p_i \wedge \Box_1 \Box_2 [\bigwedge_{i=1}^k (\Diamond_1 p_i \leftrightarrow p_i) \wedge \bigwedge_{1 \le i \ne j \le k} \neg(p_i \wedge p_j)], \\ \psi^n &= \Box_1 [\bigwedge_{i=1}^n \Diamond_2 q_i \wedge \Box_2 (\bigwedge_{1 \le i \ne j \le n} \neg(q_i \wedge q_j))]. \end{aligned}$$

It is not difficult to show that

$$Q_k \text{ is satisfiable in } \mathcal{F} \text{ iff } \mathcal{F} \text{ contains at least } k\text{-many } E_1\text{-clusters;} \tag{1}$$

$$\begin{gathered} \psi^n \text{ is satisfiable in } \mathcal{F} \text{ iff} \\ \text{every } E_2\text{–cluster of } \mathcal{F} \text{ contains at least } n \text{ points.} \end{gathered} \tag{2}$$

Thus the formula $\varphi_k^n$ is satisfiable in the frame $\mathcal{F}_k^{n-k}$. The next claim states that in the logic $L_k$ we cannot do better.

$$\mathcal{F}_k^{n-k} \text{ is the smallest } L_k\text{-frame satisfying } \varphi_k^n. \tag{3}$$

To prove (3), suppose $\varphi_k^n$ is satisfiable in a finite $L_k$-frame $\mathcal{F}$. Then $\mathcal{F}$ is a $p$-morphic image of some $\mathcal{F}_k^i$, $i \in \omega$, that is, there is an onto $p$-morphism $f : \mathcal{F}_k^i \to \mathcal{F}$. As $\psi^n$ is satisfied in $\mathcal{F}$, by (2), $i \ge n - k$.

Let $i = n - k$. The argument when $i > n - k$ is similar. Since $Q_k$ is satisfiable in $\mathcal{F}$, (1) implies that $\mathcal{F}$ contains $k$-many $E_1$-clusters. Thus, $f$ cannot identify points from different $E_1$-clusters of $\mathcal{F}_k^{n-k}$. Also note that since $\psi^n$ is satisfiable in $\mathcal{F}$ and every $E_2$-cluster of $\mathcal{F}_k^{n-k}$ contains $n$ points, $f$ cannot identify points from the same $E_2$-cluster. Let us show that $f$ cannot identify points from different $E_2$-clusters either. Indeed, suppose there exist $w \in C_{i,p}^0$ and $v \in C_{i,q}^0$ such that $f(w) = f(v)$. Since $f$ is a $p$-morphism, for any $j = 1, \ldots, k$ and $w' \in C_{j,p}^0$ there exists $v' \in C_{j,q}^0$ such that $f(w') = f(v')$. Now since $C_p^2$ is not equivalent to $C_q^2$, at least two points from some $C_{j,p}^0$ will be identified by $f$. Hence the number of points of the $E_2$-cluster $f(C_p^2)$ of $\mathcal{F}$ is strictly less than $n$, which again contradicts the satisfiability of $\psi^n$ in $\mathcal{F}$. Therefore, $f$ should be the identity map, and $\mathcal{F} = \mathcal{F}_k^{n-k}$.

Now we compute the size of $\mathcal{F}_k^{n-k}$. As in Theorem 6, to every $E_2$-cluster $C_p^2$ of $\mathcal{F}_k^{n-k}$ we correspond the sequence of natural numbers $(m_1, \ldots, m_k)$, where $m_1 = |C_{1,p}^0|, \ldots, m_k = |C_{k,p}^0|$. From the definition of $\mathcal{F}_k^{n-k}$ it follows that $m_1 + \ldots + m_k = n$. But then the number of different sequences $(m_1, \ldots, m_k)$ will be

$$C_{n-1}^k = \frac{(n-1)!}{k!(n-(k+1))!} = \frac{(n-1)\ldots(n-k)}{k!} \geq \frac{(n-k)^k}{k!}.$$

Furthermore, every $E_2$-cluster of $\mathcal{F}_k^{n-k}$ contains precisely $n$ points. So the size of $\mathcal{F}_k^{n-k}$ is at least $\frac{n(n-k)^k}{k!}$, hence

$$\text{The size of } \mathcal{F}_k^{n-k} \text{ is a polynomial of degree } k+1 \text{ in } n. \qquad (4)$$

Putting (3) and (4) together yields

THEOREM 10. *There exist infinitely many proper normal extensions $L_k$ of $\mathbf{S5}^2$ and formulas $\varphi_k^n$ such that the size of the smallest $L_k$-frame satisfying $\varphi_k^n$ is a polynomial of degree $b(L_k) + 1$ in $|\varphi_k^n|$.* □

## Conclusions

In this paper we presented a generalization of results for the one-dimensional modal system to its two-dimensional analog. Indeed, the modal system **S5** has the linear size model property but all its proper extensions are tabular. Two-dimensional **S5**, the logic $\mathbf{S5}^2$, has the exponential size model property. However, as we show in this paper, proper normal extensions of $\mathbf{S5}^2$ are simpler - they all have the poly-size model property. One can ask which modal systems allow such kind of generalizations to the two-dimensional case. For instance, for the well known modal system **S4** it is still not known whether its two-dimensional analog is decidable or has a finite model property. On the other hand it is known that two-dimensional products of linear modal logics are undecidable (see [14]).

We did not discuss consequences of the results concerning the computational complexity of extensions of $\mathbf{S5}^2$, as well as the question of finite axiomatizability of extensions of $\mathbf{S5}^2$. One of the directions for further research would be to provide a finite axiomatization for every extension of $\mathbf{S5}^2$ (which is believed to be possible). As a consequence this would imply that the satisfiability problem of every proper extension of $\mathbf{S5}^2$ is NP-complete.[3]

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[3] Added in proof: The first author together with Ian Hodkinson and Szabolcs Mikulas has recently proven this conjecture.

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