David C. Makinson

Friendliness and sympathy in logic

Book section

Original citation:
Originally published in Beziau, Jean-Yves, *Logica universalis.* Basel, Switzerland: Birkhauser Verlag, 2005, pp. 191-206.

Friendliness and Sympathy in Logic†

David Makinson

Abstract. We define and examine a notion of *logical friendliness*, which is a broadening of the familiar notion of classical consequence. The concept is studied first in its simplest form, and then in a syntax-independent version, which we call *sympathy*. We also draw attention to the surprising number of familiar notions and operations with which it makes contact, providing a new light in which they may be seen.

1. Friendliness

1.1. Rationale, Definition, Notation

Recall the definition of classical consequence in propositional logic. Let A be any set of formulae, and x any individual formula. Then x is said to be a classical consequence of A, written $A \vdash x$, iff for every valuation v on all letters of the language, if $v(A) = 1$ then $v(x) = 1$.

Trivially, the only letters that count here are those occurring in A or in x. So the definition may be rephrased as: $A \vdash x$ iff for every partial valuation v on $E(A{,}x)$, if $v(A) = 1$ then $v(x) = 1$. Equivalently again, $A \vdash x$ iff for every partial valuation v on $E(A)$, if $v(A) = 1$ then $v^+(x) = 1$ for every extension v^+ to $E(A, x)$.

Expressed in this last way, classical consequence is a $\forall\forall$ concept. It is natural to ask: what does the corresponding $\forall\exists$ concept look like, and how does it behave? This simple question, born of no more than curiosity, is the starting point of our investigation.

The definition is straightforward:

† This paper revises and extends the version that appeared in the first edition of *Logica Universalis*. Specifically, it adds several new sections (1.8-1.10, 3.5-3.6, and all of part 2) as well as additional material in other sections (notably the axiomatization of friendliness in 1.5, a much stronger version of compactness in 1.6, more information about interpolant formulae in 1.7 and 3.5, and counterexamples to proof by exhaustion and to compactness for sympathy in 3.2). The present version also appeared as part of a festschrift for Dov Gabbay, see [Makinson 2005a].

- We say that A is *friendly* to x and write $A \mid\!\approx x$, iff every partial valuation v on $E(A)$ with $v(A) = 1$ may be extended to a partial valuation v^+ on $E(A, x)$ with $v^+(x) = 1$.
- Equivalently: iff for every partial valuation v on $E(A)$ with $v(A) = 1$ there is a partial valuation w on $E(x)$ agreeing with v on letters in $E(A) \cap E(x)$, with $w(x) = 1$.
- Equivalently: iff for every valuation v on the set E of all elementary letters of the language with $v(A) = 1$ there is a valuation w (on all letters) agreeing with v on letters in $E(A)$, with $w(x) = 1$.

The notation used in these definitions is fairly straightforward, but we state it explicitly for reference. We use lower case $a, b, \ldots, x, y, \ldots$, to range over *formulae* of classical propositional logic. It will be convenient to include the zero-ary falsum $\bot$ among the primitive connectives. *Sets of formulae* are denoted by upper case letters $A, B, \ldots, X, Y, \ldots$, reserving L for the set of all formulae, E for the set of all elementary letters, and $F, G, \ldots$ for subsets of the elementary letters. For any formula a, we write $E(a)$ to mean the set of all *elementary letters* occurring in a. Similarly for sets A of formulae. For any set A of formulae, L_A stands for the sub-language generated by $E(A)$, i.e. the set of all formulae y with $E(y) \subseteq E(A)$. Thus $L_A = L_{E(A)}$.

Classical consequence is written as $\vdash$ when treated as a relation, *Cn* when viewed as an operation. The relation of *classical equivalence* is written $\dashv\vdash$. When we speak of a *valuation*, we always mean a Boolean valuation, i.e. a function into {0,1} defined on the entire set E of elementary letters of the language and extended to cover all formulae in the usual way. A *partial valuation* is a restriction of a valuation to a subset of E.

To lighten notation, we follow the common convention of usually writing A, x for $A \cup \{x\}$. $A \vdash B$ is short for '$A \vdash b$ for all $b \in B$'. Also, $v(A) = 1$ is short for '$v(a) = 1$ for all $a \in A$', while $v(A) = 0$ is short for '$v(a) = 0$ for some $a \in A$'.

1.2. Remarks on the Definition

Of the three equivalent ways of defining friendliness, we will usually be working with the first. Thus throughout the paper (except for the appendix) we will be talking about *partial* valuations rather than full ones. In this context, it is essential to keep in mind some fine distinctions, which are easy to overlook because they are without much significance for classical consequence.

- $E(a)$ is the set of all elementary letters *actually occurring* in a, rather than the least set of letters needed to get a formula classically equivalent to a. For example, if $a = p \wedge (q \vee \neg q)$ then $E(a)$ is $\{p, q\}$, not $\{p\}$. We will look at least letter-sets and a corresponding notion of *sympathy* later, in section 3.
- When we speak of a partial valuation v on a set F of elementary letters, we mean one with *exactly* F as domain. Any valuation on a proper superset F^+ of F, agreeing with v over F, will be called an *extension* of v.

It will sometimes shorten formulations to apply the notion of friendliness to partial valuations themselves. Let F be any set of elementary letters, and let v be any partial valuation on F. Let x be any formula. We say that v is *friendly* to x iff it may be extended to a partial valuation v^+ on $F \cup E(x)$ with $v^+(x) = 1$. Clearly, whenever a partial valuation is friendly to a formula then so too are all its restrictions. In other words, whenever a partial valuation is not friendly to a formula, none of its extensions are friendly to it. The first definition of $A \mathrel{|\!\!\approx} x$ may thus be expressed concisely as follows:

- $A \mathrel{|\!\!\approx} x$ iff every partial valuation v on $E(A)$ with $v(A) = 1$ is friendly to x.

Similar definitions of friendliness may be made for first-order logic, speaking of (partial) models rather than partial valuations. It should be noted, however, that in the first-order case there are several ways of understanding the notion of an extension of a model, which give rise to variant concepts of friendliness. On the one hand, we could require that when we extend a partial model the domain of discourse must remain fixed, as well as the interpretations into it of the already given predicate letters; in the literature this is usually called an 'expansion'. On the other hand, we may allow the domain to increase. In this case we have sub-options to choose from, according to whether we keep the interpretations of the already given predicate letters fixed, or allow them to flow out into the enlarged domain in some way.

But for simplicity, in this paper *we will remain within the propositional context*. We will not discuss the question of what would be the most interesting way of generalizing the definition of friendliness to the first-order context. Nor, apart from some passing negative observations, will we tabulate which among our results for the propositional context carry over to which among the first-order notions.

In section 2 we discuss links between the notion of friendliness and several other operations and concepts in the literature. Readers of a historical bent may prefer to start there and return, but we begin by clarifying the behaviour of the friendliness relation itself.

1.3. Properties that Fail

At first sight, the relation of friendliness seems to be hopelessly ill behaved. It fails many familiar features of classical consequence. In particular:

- It is not closed under substitution for elementary letters. Example: $p \mathrel{|\!\!\approx} p \wedge q$ where p, q are (here and always) distinct elementary letters, but $p \not\mathrel{|\!\!\approx} p \wedge \neg p$.
- It fails monotony and left strengthening. Example: $p \mathrel{|\!\!\approx} p \wedge q$, but $\{p, \neg q\} \not\mathrel{|\!\!\approx} p \wedge q$ and similarly $p \wedge \neg q \not\mathrel{|\!\!\approx} p \wedge q$.
- It fails cautious monotony and cautious left strengthening. Example: $p \mathrel{|\!\!\approx} q$ and $p \mathrel{|\!\!\approx} \neg q$, but $\{p, q\} \not\mathrel{|\!\!\approx} \neg q$ and likewise $p \wedge q \not\mathrel{|\!\!\approx} \neg q$.
- It fails left classical equivalence. Example: $p \mathrel{|\!\!\approx} p \wedge q$ but $p \wedge (q \vee \neg q) \not\mathrel{|\!\!\approx} p \wedge q$.
- It fails conjunction in the conclusion. Example: $p \mathrel{|\!\!\approx} q, p \mathrel{|\!\!\approx} \neg q$, but $p \not\mathrel{|\!\!\approx} q \wedge \neg q$.
- For essentially the same reason, it fails a general form of cumulative transitivity. Example: $p \mathrel{|\!\!\approx} q, p \mathrel{|\!\!\approx} \neg q$, and $p \wedge q \wedge \neg q \mathrel{|\!\!\approx} \neg p$, but $p \not\mathrel{|\!\!\approx} \neg p$.

- It fails plain transitivity. Example: $p \mathrel{|\!\!\approx} q, q \mathrel{|\!\!\approx} \neg p$, but $p \mathrel{|\!\!\not\approx} \neg p$.
- It fails disjunction in the premisses. Example: $p \mathrel{|\!\!\approx} p \leftrightarrow q, q \mathrel{|\!\!\approx} p \leftrightarrow q$, but $p \vee q \mathrel{|\!\!\not\approx} p \leftrightarrow q$.

Nevertheless, friendliness does have positive properties including 'local' versions of some of the above, which we now describe.

1.4. Relationship of Friendliness to Classical Consequence

We begin by clarifying the relation of friendliness to classical consequence.
Supraclassicality. Whenever $A \vdash x$ then $A \mathrel{|\!\!\approx} x$. Briefly: $\vdash \subseteq \mathrel{|\!\!\approx}$

Verification. Immediate from the definition of $\mathrel{|\!\!\approx}$. □

The inclusion is proper; for example, when p, q are distinct elementary letters then $p \mathrel{|\!\!\approx} q$ but not $p \vdash q$. Friendliness is not the trivial relation over the language; for example, when a is a tautology and x a contradiction, $a \mathrel{|\!\!\not\approx} x$. For a less extreme example, $p \vee q \mathrel{|\!\!\not\approx} p \wedge q$ where p, q are distinct elementary letters.

However, there are special cases where friendliness collapses into classical consequence, and others where it collapses into non-consequence of the negation.
First Reduction case. Whenever $E(x) \subseteq E(A)$ then $A \mathrel{|\!\!\approx} x$ iff $A \vdash x$.

Verification. Right to left is given unconditionally by supraclassicality, so we need only show left to right. Suppose $E(x) \subseteq E(A)$ and $A \mathrel{|\!\!\approx} x$. Let v be any partial valuation on $E(A)$ with $v(A) = 1$. We need to show that $v^+(x) = 1$ for every extension v^+ of v to $E(A, x)$. Since $A \mathrel{|\!\!\approx} x, v^+(x) = 1$ for some extension v^+ of v to $E(A, x)$. But since $E(x) \subseteq E(A), E(A, x) = E(A)$, so the unique extension of v to $E(A, x)$ is v itself. Thus $v(x) = 1$ and indeed $v^+(x) = 1$ for every extension v^+ of v to $E(A, x)$. □

Second Reduction Case. Suppose A is consistent and for each elementary letter $p \in E(A)$, either $A \vdash p$ or $A \vdash \neg p$. Then $A \mathrel{|\!\!\approx} x$ iff $A \nvdash \neg x$.

Verification. Under the hypotheses, suppose first that $A \mathrel{|\!\!\approx} x$. Since A is consistent, there is some partial valuation v on $E(A)$ with $v(A) = 1$. Choose any one such v. Since $A \mathrel{|\!\!\approx} x$, we have $v^+(x) = 1$ for some extension v^+ of v to $E(A, x)$. Thus $v^+(\neg x) = 0$ while $v^+(A) = 1$, so $A \nvdash \neg x$.

For the converse, suppose $A \nvdash \neg x$. Then there a partial valuation v on $E(A)$ with $v(A) = 1$ that can be extended to a partial valuation v^+ on $E(A, x)$ with $v^+(x) = 1$. Since either $A \vdash p$ or $A \nvdash \neg p$, for each elementary letter $p \in E(A)$, v is the only partial valuation on $E(A)$ with $v(A) = 1$. Hence every partial valuation w on $E(A)$ with $w(A) = 1$ can be extended to a partial valuation w^+ on $E(A, x)$ with $w^+(x) = 1$. □

We also have the following important characterization of friendliness in terms of classical consistency.

Characterization in terms of consistency. $A \mathrel{|\!\!\approx} x$ iff every set B of formulae in L_A that is consistent with A, is consistent with x.

Verification. Suppose first that $A \mathrel{|\!\!\approx} x$. Let B be any set of formulae in L_A that is consistent with A. Then there is a partial valuation v on $E(A)$ with $v(A) = 1, v(B) = 1$. From the supposition, v may be extended to a partial valuation v^+ on $E(A, x)$ with $v^+(x) = 1$. Since v^+ extends v and $v(B) = 1$ we have $v^+(B) = 1$. Hence B is consistent with x, as desired.

For the converse, suppose that $A \not\mathrel{|\!\!\approx} x$. Then there is a partial valuation v on $E(A)$ with $v(A) = 1$, such that $v^+(x) = 0$ for every extension v^+ of v to $E(A, x)$. Put B to be the state-description (set of literals) in L_A that corresponds to v; in the limiting case that $E(A) = \emptyset$ put $B = \{\top\}$.

We complete the verification by showing that B is consistent with A but not consistent with x. The former is immediate from the fact that $v(A) = 1$ and by construction also $v(B) = 1$. For the latter, we observe that by construction, v is the only partial valuation on $E(B) = E(A)$ with $v(B) = 1$, and by hypothesis $v^+(x) = 0$ for every extension v^+ of v to $E(A, x)$. Thus there is no partial valuation w on $E(B, x) = E(A, x)$ with $w(B) = 1$ and $w(x) = 1$. In other words, B is inconsistent with x. $\square$

This characterization can be refined. Our first refinement says, in effect, that in the characterization individual formulae c can do all the work of sets B of formulae.

First Refinement. $A \mathrel{|\!\!\approx} x$ iff $A \vdash c$ for every $c \in L_A$ with $x \vdash c$.

Verification. Suppose first $A \mathrel{|\!\!\approx} x$. Applying the characterization from left to right, we have that every formula in L_A that is consistent with A, is consistent with x. Contrapositively, whenever $c \in L_A$ and $x \vdash c$ then $A \vdash c$.

In the other direction, suppose $A \not\mathrel{|\!\!\approx} x$. Applying the characterization from right to left, there is a set B of formulae in L_A that is consistent with A, but is not consistent with x. Since B is not consistent with x, compactness tells us that is has a finite subset C that is not consistent with x. Then $x \vdash c$, where $c = \neg \wedge C$. But $A \nvdash c$, since A is consistent with B and so with its subset C. $\square$

A second refinement will be useful for proving compactness for friendliness. In effect, in the characterization it suffices to consider only formulae $c \in L_A \cap L_x$, i.e. with $E(c) \subseteq E(A) \cap E(x)$.

Second Refinement. $A \mathrel{|\!\!\approx} x$ iff $A \vdash c$ for every $c \in L_A \cap L_x$ with $x \vdash c$.

Verification. Left to right is immediate from the first corollary. For the converse, suppose $A \not\mathrel{|\!\!\approx} x$. Then by the first corollary, there is a $d \in L_A$ with $x \vdash d$ but $A \nvdash d$. Since $x \vdash d$, classical interpolation tells us that there is a $c \in L_d \cap L_x \subseteq L_A \cap L_x$ with $x \vdash c \vdash d$. Since $c \vdash d$ and $A \nvdash d$ we have $A \nvdash c$ as desired. $\square$

We note in passing that in the first-order context, if we define friendliness in terms of expansions (see section 1.2), then the second reduction case, the characterization in terms of consistency, and its two refinements, all fail in their right-to-left part. A single example serves for the three. Consider the language L with just one unary predicate letter P (no equality symbol, no individual constants), and put $\Gamma = Cn(\forall x(Px))$ to be the complete and consistent theory in that language. Let φ be the formula $\exists x\exists y(Rxy\wedge\neg Ryx)$, containing the additional letter R not available in L. On the one hand $\Gamma \nvdash \neg\varphi$; also every set Δ of formulae in L that is consistent with Γ, is consistent with φ. On the other hand, there is a model that satisfies Γ which has no expansion satisfying φ. Take any model with a singleton domain interpreting P as the whole domain. This satisfies Γ, but it cannot be expanded to a model satisfying Γ,φ, which would require two elements in the domain.

1.5. Closure Properties of Friendliness

We now see which among the familiar properties of classical consequence remain for friendliness. We begin with two that carry over without restriction.

Right weakening. Whenever $A \mathrel{|\!\approx} x \vdash y$ then $A \mathrel{|\!\approx} y$.

Verification. Immediate from the definition of $\mathrel{|\!\approx}$. □

It follows from this, of course, that the relation is syntax-independent in its right argument, i.e. satisfies right classical equivalence: whenever $x \dashv\vdash y$ then $A \mathrel{|\!\approx} x$ iff $A \mathrel{|\!\approx} y$. This contrasts with the already noted syntax-dependence on the left.

Singleton cumulative transitivity. Whenever $A \mathrel{|\!\approx} x$ and $A, x \mathrel{|\!\approx} y$ then $A \mathrel{|\!\approx} y$.

Verification. Suppose $A \mathrel{|\!\approx} x$ and A,$x \mathrel{|\!\approx} y$. Let v be any partial valuation on $E(A)$ with $v(A) = 1$. By the first hypothesis, v may be extended to a partial valuation v^+ on $E(A, x)$ with $v^+(x) = 1$, so also $v^+(A, x) = 1$. By the second hypothesis, v^+ may be extended to a partial valuation v^{++} on $E(A, x, y)$ with $v^{++}(y) = 1$. Restrict v^{++} to $E(A, y)$, call it v^{++-}. Then v^{++-} is still an extension of v with domain $E(A)$, and $v^{++-}(y) = 1$. □

We now formulate some properties that carry over in a restricted form only. The following are straightforward; compactness and interpolation are subtler and will be discussed in the following sections.

Local left strengthening. Suppose $E(B) \subseteq E(A)$. Then $B \vdash A \mathrel{|\!\approx} x$ implies $B \mathrel{|\!\approx} x$.

Verification. Suppose $B \vdash A \mathrel{|\!\approx} x$. Consider any partial valuation v on $E(B)$ with $v(B) = 1$; we need to show that v is friendly to x. Extend v to any partial valuation v^+ on $E(A) \supseteq E(B)$. Then $v^+(B) = v(B) = 1$, and so since $B \vdash A$ we have $v^+(A) = 1$. Since $A \mathrel{|\!\approx} x$, there is an extension v^{++} of v^+ to $E(A, x)$ with $v^{++}(x) = 1$. Restrict v^{++} to $E(B, x)$, call it v^{++-}. Then clearly $v^{++-}(x) = v^{++}(x) = 1$. But v^{++-} is still an extension of v with domain $E(B)$. Hence v is friendly to x, as desired. □

Local left equivalence. Suppose $E(B) \subseteq E(A)$. Then $A \approx\!\!\!| \; x$ and $A \dashv\vdash B$ together imply $B \approx\!\!\!| \; x$.

Verification. When $A \dashv\vdash B$ then $B \vdash A$ so we can apply local left strengthening. $\square$

Local monotony. Suppose $E(B) \subseteq E(A)$. If $A \approx\!\!\!| \; x$ and $A \subseteq B$ then $B \approx\!\!\!| \; x$.

Verification. When $A \subseteq B$ then $B \vdash A$; apply local left strengthening. $\square$

Local disjunction in the premisses. Suppose $E(b_2) \subseteq E(A,b_1)$ and $E(b_1) \subseteq E(A, b_2)$. Then $A, b_1 \approx\!\!\!| \; x$ and $A, b_2 \approx\!\!\!| \; x$ together imply $A, b_1 \vee b_2 \approx\!\!\!| \; x$.

Verification. Suppose $A, b_1 \vee b_2 \not\approx\!\!\!| \; x$. Then there is a partial valuation v on $E(A, b_1 \vee b_2)$ with $v(A, b_1 \vee b_2) = 1$ that is not friendly to x. By the hypotheses, $E(A, b_1 \vee b_2) = E(A, b_1) = E(A, b_2)$. Since $v(A, b_1 \vee b_2) = 1$ either $v(A, b_1) = 1$ or $v(A, b_2) = 1$. Hence either v is a partial valuation on $E(A, b_1)$ with $v(A, b_1) = 1$ but not friendly to x, or similarly with b_2. That is, either $A, b_1 \not\approx\!\!\!| \; x$ or $A, b_2 \not\approx\!\!\!| \; x$. $\square$

Proof by exhaustion. $A, b \approx\!\!\!| \; x$ and $A, \neg b \approx\!\!\!| \; x$ together imply $A \approx\!\!\!| \; x$.

Verification. Clearly $E(\neg b) = E(b) \subseteq E(A, b)$ and conversely $E(b) = E(\neg b) \subseteq E(A, \neg b)$ so we may apply local disjunction in the premisses to get $A, b \vee \neg b \approx\!\!\!| \; x$. Clearly also $E(A) \subseteq E(A, b \vee \neg b)$ and also $A \vdash (A, b \vee \neg b) \approx x$, so we may apply local left strengthening to get $A \approx\!\!\!| \; x$ as desired. $\square$

The properties obtained so far lead to another characterization. In a broad sense of the term, it can be seen as an axiomatization of the relation of friendliness, modulo classical consequence. 'A broad sense', since the right-hand side of the third condition is not closed under substitution.

Observation. Friendliness is the least relation R between sets of formulae and individual formulae that satisfies the following three conditions:

1. $\vdash \subseteq R$,
2. $\langle A, x\rangle \in R$ whenever $\langle A \cup \{b\}, x\rangle \in R$ and $\langle A \cup \{\neg b\}, x\rangle \in R$,
3. $\langle A, x\rangle \in R$ whenever $A \nvdash \neg x$ and for each elementary letter $p \in E(A)$, either $A \vdash p$ or $A \vdash \neg p$.

Verification. First observe that the total relation between sets of formulae and individual formulae satisfies these three conditions, and so there is at least one such relation. Further, the intersection of any non-empty set of such relations is itself such a relation (despite the negative term $A \nvdash \neg x$ in the third condition, which negates classical consequence rather than the relation R). Thus there is a unique least such relation R, call it R_0.

We already know that $\approx\!\!\!|$ satisfies all three conditions (supraclassicality, proof by exhaustion, second reduction case). Thus $R_0 \subseteq \; \approx\!\!\!|$.

For the converse, suppose $\langle A, x\rangle \notin R_0$; we need to show that $A \not\approx\!\!\!| \; x$. Let $p_1, \ldots, p_n$ be all the elementary letters in $E(A)$. Define sets $A_0, \ldots, A_n$ by setting

$A_0 = A$ and putting $A_{i+1} = A_i \cup \{p_{i+1}\}$ if $\langle A_i \cup \{p_{i+1}\}, x\rangle \notin R_0$ and otherwise $A_{i+1} = A_i \cup \{\neg p_{i+1}\}$. By hypothesis, $\langle A_0, x\rangle \notin R_0$ and an easy induction using condition (2) gives us $\langle A_n, x\rangle \notin R_0$. But for each elementary letter $p \in E(A)$, either $A_n \vdash p$ or $A_n \vdash \neg p$, so condition (3) tells us that $A_n \vdash \neg x$. Also, since $\langle A_n, x\rangle \notin R_0$, condition (1) tells us that A_n is consistent, so there is at least one partial valuation v on $E(A_n) = E(A)$ with $v(A_n) = 1$. Since $A_n \vdash \neg x$, we have $v^+(x) = 0$ for every extension v^+ of v to $E(A, x)$, so $A \not\approx x$ as desired. □

1.6. Compactness

In the context of friendliness, some care must be taken with the formulation of compactness. When the property is formulated in exactly the same way as in classical logic, it tells us very little. For suppose $A \models\!\approx x$. Then:

- On the one hand, in the limiting case that x is inconsistent the definition of $\models\!\approx$ implies that A must also be inconsistent, so by classical compactness there is a finite inconsistent subset $B \subseteq A$, so that by the definition of $\models\!\approx$ again, $B \models\!\approx x$.
- On the other hand, in the principal case that x is consistent, we have immediately that $\emptyset \models\!\approx x$. This leaves us hungry, for while the empty set is certainly finite we would like something more substantial.

This motivates the following strengthened formulation. Bearing in mind that friendliness does not satisfy monotony, it is quite strong.

Compactness. Let A be a non-empty set with $A \models\!\approx x$. Then there is a finite subset $B \subseteq A$ such that $C \models\!\approx x$ for every C with $B \subseteq C \subseteq A$.

Proof. Suppose $A \models\!\approx x$. By the second refinement of the characterization of friendliness in terms of consistency, whenever $c \in L_A \cap L_x$ and $x \vdash c$ then $A \vdash c$. Hence by compactness for classical consequence, for every $c \in L_A \cap L_x$ with $x \vdash c$ there is a finite subset $B_c \subseteq A$ with $B_c \vdash c$. Since x is an individual formula, there are only finitely many $c \in L_A \cap L_x \subseteq L_x$ up to classical equivalence. Taking the finite union of the corresponding sets B_c, we conclude that there is a finite subset $B \subseteq A$ such that $B \vdash c$ for every $c \in L_A \cap L_x$ with $x \vdash c$.

Now let C be any set with $B \subseteq C \subseteq A$. We need to show that $C \models\!\approx x$. Since $B \subseteq C$, monotony for classical consequence gives us $C \vdash c$ for every $c \in L_A \cap L_x$ with $x \vdash c$. Also, since $C \subseteq A$, we have $L_C \subseteq L_A$ and so $C \vdash c$ for every $c \in L_C \cap L_x$ with $x \vdash c$. Applying again the second refinement of the characterization of friendliness, we have $C \models\!\approx x$ as desired. □

1.7. Interpolation

As in the case of compactness, interpolation for friendliness is trivial when formulated in the way customary in classical logic. For suppose $A \models\!\approx x$; we want to show that there is a formula b with $E(b) \subseteq E(A) \cap E(x)$ such that both $A \models\!\approx b$ and $b \models\!\approx x$. On the one hand, if A is inconsistent, we can put $b = \perp$ giving us $A \vdash b \vdash x$ so $A \models\!\approx b \models\!\approx x$. On the other hand, if A is consistent then since $A \models\!\approx x, x$ must also be consistent, so we can put $b = \top$, so that $A \vdash b$ and thus $A \mathrel{|\!\!\approx} b$, and also $b \mathrel{|\!\!\approx} x$ using the consistency of x.

The following formulation strengthens the property by guaranteeing that in suitable conditions, b can be chosen more informatively.

Interpolation. Whenever $A \mathrel{|\!\!\approx} x$ there is a finite set $F \subseteq E(A) \cap E(x)$ of elementary letters such that for every finite set G of elementary letters with $F \subseteq G \subseteq E(A)$ there is a formula b with the following properties:

1. $E(b) = G$
2. $A \mathrel{|\!\!\approx} b$ (indeed $A \vdash b$)
3. $b \mathrel{|\!\!\approx} x$
4. b is consistent, provided A is consistent
5. b is not a tautology, provided there is a non-tautology $y \in L_A \cap L_x$ with $A \vdash y$.

Remark. Before giving the proof, we note that the rather odd proviso in property (5) cannot be weakened to, say: A and x are not tautologous. Example: $A = p \vee q, x = q \vee r$. Then $A \mathrel{|\!\!\approx} x$, but the only formulae b with $E(b) \subseteq E(A) \cap E(x) = \{q\}$ and both $A \mathrel{|\!\!\approx} b$ and $b \mathrel{|\!\!\approx} x$ are the tautologies containing at most the letter q.

Proof. Suppose $A \mathrel{|\!\!\approx} x$. Since x is a single formula, $E(x)$ is finite, and thus so too is $E(A) \cap E(x)$. Hence, up to classical equivalence, there is a strongest formula a with $E(a) \subseteq E(A) \cap E(x)$ and $A \vdash a$. Take any such a and put $F = E(a)$, which is clearly finite. Let G be any finite set of letters with $F \subseteq G \subseteq E(A)$. Form b by conjoining with a the disjunctions $q \vee \neg q$ for the finitely many letters q in $G \setminus F$. We claim that b fulfils all requirements.

Property (1) is immediate by construction. Also by construction $A \vdash a \dashv\vdash b$ and so by supraclassicality, $A \mathrel{|\!\!\approx} b$, giving (2). For property (4), if A is consistent then since $A \vdash b$, b is also consistent. For (5), suppose there is a non-tautology $y \in L_A \cap L_x$ with $A \vdash y$. Then by its construction, a is not a tautology, and so since $a \dashv\vdash b$, b is not a tautology.

It remains to show (3). Suppose $b \mathrel{|\!\!\not\approx} x$; we derive a contradiction. Since $b \mathrel{|\!\!\not\approx} x$ there is a partial valuation v on $E(b) = G \subseteq E(A)$ with $v(b) = 1$, which is not friendly to x, i.e. such that $v^+(x) = 0$ for every extension v^+ of v to $E(b, x)$. Fix such a v for the remainder of the proof.

Write k for the state-description formula in L_b that corresponds to v. Then clearly $v(k) = 1$ and also $k \vdash \neg x$. Put $b^* = b \wedge \neg k$. We show that b^* is a formula in L_A with $A \vdash b^*$ and $b \nvdash b^*$, thus contradicting the construction of b.

For $b^* \in L_A$: This is immediate since both $b, \neg k \in L_A$.

For $b \nvdash b^*$: It suffices to show $b \nvdash \neg k$, i.e. that $k \nvdash \neg b$. We have by its construction that $k \vdash b$; and since $v(k) = 1, k$ is satisfiable, so $b \nvdash \neg k$ as desired.

For $A \vdash b^*$: Since $A \vdash b$ it suffices to show $A \vdash \neg k$. As a preliminary observation, we show that there is no extension w of v to $E(A)$ with $w(A) = 1$. For let w be such an extension. Since by hypothesis $A \mathrel{|\!\!\approx} x$, there is an extension w^+ of w to $E(A, x)$ with $w^+(x) = 1$. Clearly, w^+ is also an extension of v to

$E(A, x)$. Now restrict w^+ to $E(b, x)$, which is possible since $E(b) \subseteq E(A)$ so that $E(b, x) \subseteq E(A, x)$, and call it w^{+-}. Clearly $w^{+-}(x) = 1$ and also w^{+-} is still an extension of v, which has domain $E(b)$. But this contradicts the fact that v is not friendly to x. This completes the preliminary step of showing that there is no extension w of v to $E(A)$ with $w(A) = 1$.

Now let w be any partial valuation on $E(A) \supseteq E(b) = E(k)$ with $w(\neg k) = 0$, i.e. $w(k) = 1$. It remains to show that $w(A) = 0$. Restrict w to $E(k) = E(b)$ = domain(v), call it w^-. Clearly $w^-(k) = 1$. Hence by the construction of k as a state-description in L_b corresponding to v, $w^- = v$. Thus w is an extension of v to $E(A)$. So by the preliminary observation, $w(A) = 0$ as desired. □

1.8. Friendliness as an Operation

Up to now, we have treated friendliness as a relation between formulae (or sets of formulae) on the left and formulae on the right. But just as in the case of classical consequence and well-known nonmonotonic consequences, we can consider it as an operation, taking sets of formulae to sets of formulae, by defining $Fr(A) = \{x : A \mathrel{|\!\approx} x\}$.

However, this may not be a very useful perspective for friendliness, in contrast to the situation for the usual nonmonotonic consequence relations. The reason is that friendliness is much further from being a closure relation. It fails monotony but also, as we have seen in section 1.3, it fails both conjunction in the conclusion and general cumulative transitivity. Expressed as an operation, it also fails idempotence (the same counterexample can be used as for cumulative transitivity). These properties are all satisfied by the usual nonmonotonic consequence relations (see e.g. Makinson 2005), and their absence makes the operational notation much less convenient to use.

So, in this section we examine just one question regarding the operational version: when do we have $Fr(A) = Fr(B)$ for sets A, B of formulae?

Observation. $Fr(A) = Fr(B)$ iff either $A \dashv\vdash B$ and $E(A) = E(B)$ or else A, B are both contradictions.

Verification. In one direction, suppose RHS. We want to show $Fr(A) = Fr(B)$. In the limiting case that A,B are both contradictions, we have $Fr(A) = L = Fr(B)$ vacuously from the definition of friendliness. So consider the principal case that $A \dashv\vdash B$ and $E(A) = E(B)$. Then $Fr(A) = Fr(B)$ by two applications of local left equivalence (section 1.5).

For the other direction, suppose $Fr(A) = Fr(B)$. Suppose that A, B are not both contradictions. We need to show that $E(A) = E(B)$ and $A \dashv\vdash B$.

First, we observe that neither of A,B is a contradiction. For suppose A, say, is a contradiction. Then $A \vdash \perp$ and so by supraclassicality of friendliness, $A \mathrel{|\!\approx} \perp$ and so since $Fr(A) = Fr(B)$ we have $B \mathrel{|\!\approx} \perp$, so B is a contradiction.

Next, we show $E(A) = E(B)$. It suffices to show $E(A) \subseteq E(B)$; the converse is similar. Suppose $p \in E(A)$ but $p \notin E(B)$; we derive a contradiction. Since $p \notin E(B)$ clearly $B \mathrel{|\!\approx} p$ and also $B \mathrel{|\!\approx} \neg p$. Since $Fr(A) = Fr(B)$, this gives us

$A \mathrel{|\!\approx} p$ and also $A \mathrel{|\!\approx} \neg p$. Since $p \in E(A)$, the first reduction case for friendliness tells us that $A \vdash p$ and also $A \vdash \neg p$ so that A is inconsistent, contradicting what has been shown.

Finally, we show $A \dashv\vdash B$. It suffices to show $A \vdash B$; the converse is similar. Take any $b \in B$. we need to show $A \vdash b$. Now $B \vdash b$ so by supraclassicality $B \mathrel{|\!\approx} b$ so since $Fr(A) = Fr(B)$ we have $A \mathrel{|\!\approx} b$. Since $E(A) = E(B)$ and $b \in B$ we have $E(b) \subseteq E(A)$ so by the first reduction case for friendliness, $A \vdash b$ as desired, and the proof is complete. $\square$

1.9. Joint Friendliness: Two Notions

For classical consequence, we have followed the common convention of writing $A \vdash B$ to mean that $A \vdash b$ for all $b \in B$. For friendliness, it is tempting to write $A \mathrel{|\!\approx} B$ analogously. But care is needed, for there is an important distinction that does not arise in the classical case. We must distinguish between two relationships:

- $A \mathrel{|\!\approx}_{\forall\forall\exists} B$: for every partial valuation v on $E(A)$ with $v(A) = 1$ and every $b \in B$, there is an extension v^+ of v to $E(A, b)$ with $v^+(b) = 1$.
- $A \mathrel{|\!\approx}_{\forall\exists\forall} B$: For every partial valuation v on $E(A)$ with $v(A) = 1$ there is an extension v^+ of v to $E(A, B)$ with $v^+(B) = 1$, i.e. with $v^+(b) = 1$ for every $b \in B$.

The former says the same as $A \mathrel{|\!\approx} b$ for all $b \in B$. But the latter says more. For classical consequence, where conjunction in the conclusion is satisfied, no such distinction arose. We call $\mathrel{|\!\approx}_{\forall\forall\exists}$ *weak* joint friendliness, $\mathrel{|\!\approx}_{\forall\exists\forall}$ *strong*. When we refer to joint friendliness (sections 2.2 and 3.4), we will specify clearly which is intended.

1.10. Internalizing the Relation

It is natural to ask whether we can internalize the relation of friendliness as a conditional connective of the object language.

It can be done quite trivially by adding an iterable two-place connective $\rightsquigarrow$ to the object language and adding to the familiar Boolean rules the following one. To bring the formulation as close as possible to standard ones for propositional connectives, we state it with v,w,u understood as full valuations, i.e. defined on the set E of all elementary letters.

$v(a \rightsquigarrow x) = 1$ iff for every full valuation w with $w(a) = 1$ there is a full valuation u that agrees with w on all elementary letters in $E(a)$ and such that $u(x) = 1$.

The same effect can be achieved by means of indexed unary modal operators. Consider a language with operators $\Box_a$ and $\Diamond_a$ for all formulae a. This is a little unusual, as the set of connectives is not fixed in advance, but is defined inductively along with the formulae in which they occur; but that is not a problem. We read these connectives by the following rules:

$v(\Box_a x) = 1$ iff for every valuation w that agrees with v on all elementary letters in $E(a)$, we have $w(x) = 1$.

$v(\Diamond_a x) = 1$ iff for some valuation w that agrees with v on all elementary letters in $E(a)$, we have $w(x) = 1$.

We may then identify plain $\Box$ and $\Diamond$ as $\Box_\top$ and $\Diamond_\top$ (or equivalently $\Box_\bot$ and $\Diamond_\bot$), giving us the familiar evaluation rules:

$v(\Box x) = v(\Box_\top x) = v(\Box_\bot x) = 1$ iff $w(x) = 1$ for every valuation w.

$v(\Diamond x) = v(\Diamond_\top x) = v(\Diamond_\bot x) = 1$ iff $w(x) = 1$ for some valuation w.

With this equipment, we may represent $a \mathrel{|\!\!\approx} x$ in the object language by the formula $\Box(a \to \Diamond_a x)$. Given the rules given above for evaluating indexed modal operators, this formula will satisfy the same evaluation condition that we gave for the trivial internalization. It will come out as true under one valuation iff it does so under all valuations, and that iff the relation $a \mathrel{|\!\!\approx} x$ holds.

However, it should be understood that when we internalize the relation of friendliness (whether directly or via indexed modal operators) the resulting system is rather unusual. The set of all valid formulae (defined as those formulae that are true under every valuation) is not closed under substitution, for the very same reason as the relation of friendliness was not so closed. The same example can be used to illustrate the failure. On the one hand, the formula $(p \to \Diamond_p(p \wedge q)$ is valid, while its substitution instance $(p \to \Diamond_p(p \wedge \neg p)$ is not.

Thus while internalization is perfectly possible, the propositional system that it gives us is unlike most modal and other non-classical propositional logics, for which the set of valid formulae is closed under substitution. In the author's view, this difference is not a disqualification — see e.g the discussion in Makinson (2005). But it not clear that internalization provides any insights that are not already available when friendliness is treated as a relation between formulae.

2. Links with Familiar Notions

Friendliness has many friends: several other notions familiar from the literature are connected with it. Roughly speaking, the links are of two main kinds.

- Certain well-known operations from the history of logic, distant and recent, can be seen as *instances* of friendliness.
- There are also more general *conceptual links*, notably with Ramsey eliminability and related notions that have been studied in the context of first-order logic.

We begin with some instances of friendliness.

2.1. Forgetting Letters from Formulae

Consider any formula a and any subset F of its elementary letters, i.e. $F \subseteq E(a)$. Let $\sigma_1, \ldots, \sigma_k$ be the $k = 2^n$ substitutions of $\bot, \top$ for the n letters in F. Following Weber (1987) and later papers such as Lin and Reiter (1994) and Lang, Liberatore, Marquis (2003), we may define $f_F(a)$, the result of *forgetting* the letters in F from a, as $\sigma_1(a) \vee \ldots \vee \sigma_k(a)$. Equivalently, in recursive form, $f_\emptyset(a) = a$, and

$f_{F,q}(a) = \sigma_\bot(f_F(a)) \vee \sigma_\top(f_F(a))$, where the functions $\sigma_\bot$ and $\sigma_\top$ substitute $\bot, \top$ for the letter q.

As is well known, $a \vdash f_F(a)$. The converse fails, i.e. $f_F(a) \nvdash a$; for example $f_F(p) = \bot \vee \top \nvdash p$. However, $f_F(a)$ is easily shown to be the strongest formula b in the language generated by $E(a) \setminus F$ such that $a \vdash b$.

In fact, the notion goes back to Boole, whose focus was however rather different. From his point of view, the central logical relation was equality, coresponding to classical equivalence. Accordingly, the most important fact for him about what we now call forgetting was the equality that he introduced under the name of 'development' in Boole (1847): $a \dashv\vdash (\neg p \wedge \sigma_\bot(a)) \vee (p \wedge \sigma_\top(a))$. The consequence $a \vdash \sigma_\bot(a) \vee \sigma_\top(a) = f_a(a)$ is however implicit (in dual form) in the discussion of the 'elimination' of a term in an equation, in Boole (1854).

Observation. $f_F(a) \mathrel{|\!\approx} a$.

Verification. Let v be any partial valuation on $E(f_F(a)) = E(a) \setminus F$ and suppose $v(f_F(a)) = 1$. Then $v(\sigma_i(a)) = 1$ for some $i \leq k$. Extend v to v^+ on $E(a)$ by putting $v^+(q) = 0{,}1$ according as $\sigma_i(q)) = \bot, \top$ for each $q \in F$. Then clearly by induction on length of formulae, $v^+(a) = v(\sigma_i(a)) = 1$ and we are done. $\square$

2.2. Ejective Substitution

It is natural to ask whether this observation can be extended to a more general result linking friendliness and substitution. It cannot cover all substitutions, for we do not always have $\sigma(a) \mathrel{|\!\approx} a$, even when σ is a one-one correspondence on letters. Consider for example the formula $a = p \wedge \neg q$ and the substitution σ that simply interchanges the two letters, putting $\sigma(p) = q$ and $\sigma(q) = p$ so that $\sigma(a) = q \wedge \neg p \mathrel{|\!\not\approx} a = p \wedge \neg q$ (witness the only partial valuation that makes the premiss true).

Nevertheless, we do have a positive result for a certain class of substitutions. Let σ be any substitution on the set E of all elementary letters, and let A be any set of formulae. We call σ *ejective for* A iff for every letter $p \in E(A)$, either $\sigma(p) = p$ or $p \notin E(\sigma(A))$.

Observation. Let a be any formula, and let σ be any substitution that is ejective for a. Then $\sigma(a) \mathrel{|\!\approx} a$. More generally, when A is a set of formulae and σ is ejective for A then $\sigma(A) \mathrel{|\!\approx}_{\forall\exists\forall} A$.

Verification. The notation $\mathrel{|\!\approx}_{\forall\exists\forall}$ for strong joint friendliness is explained in section 1.9. Consider any partial valuation v on $E(\sigma(A))$ with $v(\sigma(A)) = 1$. We extend v to v^+ on $E(\sigma(A), A)$ by putting $v^+(q) = v(\sigma(q))$ for each letter q in $E(A) \setminus E(\sigma(A))$. We want to show that $v^+(A) = v(\sigma(A)) = 1$. It suffices to show by induction that for every subformula b of any formula in $A, v^+(b) = v(\sigma(b))$.

For the basis, if b is a letter p then either $\sigma(p) = p$ or $p \notin E(\sigma(A))$. In the former case $p \in E(\sigma(A))$, so $v(p)$ is defined, so since v^+ extends v we have $v^+(p) = v(p) = v(\sigma(p))$ as desired. In the latter case, $p \in E(A) \setminus E(\sigma(A))$, so that $v^+(p) = v(\sigma(p))$ by definition.

The induction step is then routine using the definitions of a substitution and of a Boolean valuation. □

This observation covers the 'friendly forgetfulness' property $f_F(a) \mid\!\approx a$ as a special case. For when a function σ substitutes $\bot, \top$ for some of the elementary letters in a (and is the identity on all other letters) then it is ejective for a. Indeed, it is ejective *tout court*, in the stronger sense that for every letter p, either $\sigma(p) = p$ or $p \notin E(\sigma(L)) = E(\sigma(E))$. Thus we have $f_F(a) = \sigma_1(a) \vee \ldots \vee \sigma_k(a)$ where each substitution σ_i is ejective, so that each $\sigma_i(a) \mid\!\approx a$. But $E(\sigma_i(a)) = E(a) \setminus F = E(\sigma_j(a))$ for all $i, j \leq k$ and so we may apply local disjunction in the premisses (section 1.5) putting $A = \emptyset$ to conclude that $\sigma_1(a) \vee \ldots \vee \sigma_k(a) \mid\!\approx a$ as desired.

2.3. Identifying Letters

The above observation has a further corollary. By an *identification of letters* we mean a substitution σ on E into E such that for every letter p, either $\sigma(p) = p$ or $p \neq \sigma(q)$ for all letters q. Equivalently: such that whenever $p = \sigma(q)$ for some letter q then $\sigma(p) = p$. Equivalently: such that for some partition of E and some choice function γ on that partition, $\sigma(p) = \gamma(|p|)$.

Corollary. $\sigma(A) \mid\!\approx_{\forall\exists\forall} A$ for any identification σ of letters. In particular, when a is an individual formula and σ is an identification of letters, then $\sigma(a) \mid\!\approx a$.

Verification. By the observation in section 2.2, it suffices to observe that every identification of letters is ejective *tout court*, and so ejective for A. Let σ be any identification of letters. Suppose $p \in E(A)$ and $\sigma(p) \neq p$. Since σ is an identification of letters, this gives us $p \neq \sigma(q)$ for all letters q. Since σ takes E into E this implies that $p \notin E(\sigma(E)) = E(\sigma(L))$. □

2.4. Existential Quantification

The concept of forgetting can also be expressed in the language of quantified Boolean formulae. Put $g_F(a) = \exists p_1 \ldots \exists p_n(a)$ where $F = \{p_1, \ldots, p_n\}$. Then under the standard semantics for quantified Boolean formulae, $g_F(a)$ has exactly the same truth conditions as $f_F(a)$. So, with the notion of friendliness suitably enlarged to cover such formulae (rather than just unquantified Boolean formulae, as in this paper), we can say that $g_F(a)$ is friendly to a.

More generally, it is clear that in any language admitting existential quantifiers over a syntactic category of items, the existential quantification $\exists i_1 \ldots \exists i_n(a)$ over selected variables from that category will, under a natural enlargement of the notion, be friendly to a.

However, it should also be observed that the forgetting function $f_F(a)$, its quantified Boolean analogue $g_F(a)$, and existentialization $\exists i_1 \ldots \exists i_n(a)$ all have a more intimate relation to their argument a than mere friendliness. For we have not only $f_F(a) \mid\!\approx a, g_F(a) \mid\!\approx a, \exists i_1 \ldots \exists i_n(a) \mid\!\approx a$ but also the classical consequences in the reverse direction: $a \vdash f_F(a), a \vdash g_F(a), a \vdash \exists i_1 \ldots \exists i_n(a)(a)$. This contrasts with the fact that for friendliness in general we may have $b \mid\!\approx a$ without $a \vdash b$: witness the example $p \mid\!\approx q$ but $q \nvdash p$ where p, q are distinct elementary letters.

2.5. Skolemization

The process of Skolemization of a formula of first-order logic manifests friendliness in a very special way. Taking for example the formula $\alpha = \forall x\exists y(Rxy)$, we can introduce a function letter f and consider both the formula $sk(\alpha) = \forall x(Rxf(x))$ and its existential quantification $\exists f(sk(\alpha)) = \exists f\forall x(Rxf(x))$. These formulae belong respectively to first-order logic with function letters, and second-order logic.

As Skolem observed, we have $sk(\alpha) \vdash \alpha$ in first-order logic, and also $\alpha \dashv\vdash \exists f(sk(\alpha))$ in second-order logic (assuming the axiom of choice in our metalanguage). The equivalence between α and $\exists f(sk(\alpha))$ means that the relation between these two is much tighter than for plain existentialization.

While $sk(\alpha) \vdash \alpha$, the converse fails: $\alpha \nvdash sk(\alpha)$. But we do have $\alpha \models\!\approx sk(\alpha)$ where $\models\!\approx$ is the friendliness in the first-order context, understood in terms of expansions (section 1.2). For every (partial) model interpreting the predicate letter R in a domain, if that model satisfies α then it has an expansion also interpreting the function letter f in the same domain that satisfies $sk(\alpha)$.

Here again there is an especially close relationship. As is well known, a and $sk(\alpha)$ are equivalent for logical truth, i.e. a is true in all first-order models iff $sk(\alpha)$ is. This does not hold for friendliness in general. In our base territory of classical propositional logic, $p \vee \neg p \models\!\approx q$ but the left is a tautology while the right is not.

As is well known, the passage from α to $sk(\alpha)$ also contrasts with existentialization in this regard. For example $\exists x(\exists x(Px) \to Px)$ is friendly to $\exists x(Px) \to Px$, but the left is a logical truth while the right is not.

2.6. Ramsey Eliminability

As well as the above particular instances of friendliness, there are also more general connections with concepts that have arisen elsewhere. Of these, the closest is with Ramsey eliminability of a predicate or other term in a theory.

This notion takes its origin in the philosophy of science, and more specifically in discussions concerning the relation between the observational and theoretical components of empirical scientific theories. It was first sketched in rough terms by F. P. Ramsey in notes of 1929, published in the posthumous collection Ramsey (1931, chapter 'Theories'). It was taken up and given its name by Sneed (1971, chapter 3); and subsequently discussed in a number of books and papers including van Benthem (1978) and Rantala (1991). All of these are expressed in the context of first-order languages.

Formulations differ in subtle but significant respects. What they all have in common is that every model of one set Γ of (first-order) formulae should be capable of expansion to a model of a larger set Δ that possibly contains further letters (individual constants, predicates, or function signs). We recall that by an *expansion* of a model is meant another model with the same domain, same interpretations of the letters that were interpreted in the first model, plus interpretations of whatever new letters are concerned.

Where the formulations differ is in what Γ and Δ are taken to be; which of them is taken to be an arbitrary set of formulae while the other is taken as a function of it. The story is as follows.

- For Rantala (1991, pages 150–151): Γ is taken to be an arbitrary set of first-order formulae, and Δ is put as $\Gamma \cup \{\varphi\}$ where φ is a (likewise first-order) formula. Rantala focusses on the case that this formula has just one new letter beyond those occurring in Γ, thought of as a candidate for reduction; however the definition is meaningful without that restriction. The concept is envisaged as expressing a property of the new letter(s) in φ modulo the set $\Gamma \cup \{\varphi\}$, rather than a relation between Γ and $\Delta = \Gamma \cup \{\varphi\}$.
- By contrast, for van Benthem (1978, page 325), it is Δ that is is taken to be an arbitrary set of first-order formulae, while Γ is taken to be $Cn(\Delta) \cap L_0$, where L_0 is an arbitrarily chosen sublanguage of the language L of Δ. Again, the concept is envisaged as expressing a property of the omitted letter set in $L \setminus L_0$ modulo the formula set Δ.

Typically, L_0 will be made up of all the letters in L except for one, which is thought of as a candidate for reduction. In that case, we have exactly a notion introduced by de Bouvère (1959, chapter II.2). He used the failure of this property of the omitted letter (say, a predicate P) modulo a theory Δ, as a method for showing that P is not explicitly definable in Δ. This contrasts with the better-known technique going back to Padoa (1901), which proceeds by showing that some model of Γ can be expanded in two distinct ways to a model of Δ. Unlike de Bouvère's method, that of Padoa is complete for the task, as shown in a celebrated theorem of Beth (1956).

As is well known, the formulations of Rantala and van Benthem are not equivalent. On the one hand, when $\Delta = \Gamma \cup \{\varphi\}$ and L_0 is the language of Γ, then $\Gamma \subseteq Cn(\Delta) \cap L_0$. Hence, if every model of Γ can be expanded to a model of Δ, then every model of $Cn(\Delta) \cap L_0$ can too. In other words, Ramsey eliminability in the sense of Rantala implies the same in the sense of van Benthem. But in general, Γ may be a proper subset of $Cn(\Delta) \cap L_0$. So it may happen that whilst every model of $Cn(\Delta) \cap L_0$ can be expanded to one of Δ, there is some model of Γ (but not satisfying $Cn(\Delta) \cap L_0$) that cannot be so expanded. Thus Ramsey eliminability in the sense of van Benthem does not imply the same in the sense of Rantala. Specific examples have been given in the literature.

To compare these two concepts with friendliness as studied in this paper, we extract the purely propositional content, and write it in the notation that we have been using. We write $L_{E(B)\setminus F}$ for the language generated by the letters that are in $E(B) \setminus F$.

- From Rantala: The letters in $E(x) \setminus E(A)$ are Ramsey eliminable from a set A, x of formulae iff every partial valuation v on $E(A)$ with $v(A) = 1$ can be extended to a partial valuation v^+ on $E(A, x)$ with $v^+(A, x) = 1$.
- From van Benthem: Consider any set B of formulae and any set F of elementary letters with $F \subseteq E(B)$. The letters in F are Ramsey eliminable from B

iff every partial valuation v on $E(B) \setminus F$ with $v(Cn(B) \cap L_{E(B)\setminus F}) = 1$ can be extended to a partial valuation v^+ on $E(B)$ with $v^+(B) = 1$.

Of these, the Rantala-style concept is equivalent to friendliness of A to x, as defined and studied in this paper.

Observation. Let A be any set of propositional formulae and x a propositional formula. Then A is friendly to x iff the letters in $E(x)\setminus E(A)$ are Ramsey eliminable from A, x in the sense of Rantala.

Verification. The only difference between the definition of friendliness and the propositional reduction of Rantala's version of Ramsey eliminability is that whereas the former requires the extension v^+ to satisfy x, the latter requires it to satisfy A, x. But these are equivalent when v^+ extends v and $v(A) = 1$. □

We have already remarked that even in the first-order context, the formulation of van Benthem is weaker than that of Rantala. Indeed, it is very much weaker since, as is well-known, every *finite* model of $Cn(\Delta) \cap L_0$ can be expanded to a model of Δ. In the purely propositional context, it becomes so much weaker that it always holds, as we now show.

Observation. Let B be any set of propositional formulae and $F \subseteq E(B)$ any subset of its elementary letters. Then the letters in F are Ramsey eliminable from B in the sense of van Benthem.

Proof. We need to show that every partial valuation v on $E(B)\setminus F$ with $v(Cn(B)\cap L_{E(B)\setminus F}) = 1$ can be extended to a partial valuation v^+ on $E(B)$ with $v^+(B) = 1$.

Let v be a partial valuation on $E(B) \setminus F$ with $v(Cn(B) \cap L_{E(B)\setminus F}) = 1$. Suppose for reductio ad absurdum that v cannot be extended to a partial valuation v^+ on $E(B)$ with $v^+(B) = 1$. Let S be the state-description corresponding to v, i.e. the set of all literals in $L_{E(B)\setminus F}$ that are true under v. In the limiting case that $F = E(B)$ so that $E(B) \setminus F = \emptyset$, put $S = \{\top\}$.

We note first that $S \cup B$ is inconsistent. Reason: For any partial valuation w on $E(S \cup B) = E(B)$ with $w(S \cup B) = 1$ we have $w(S) = 1$ so w must must agree with v over F, so w is an extension of v to $E(B)$. Also $w(B) = 1$, contrary to the supposition.

Since $S \cup B$ is inconsistent, compactness tells us that there is a formula s that is the conjunction of finitely many elements of S, such that $\neg s \in Cn(B)$. But also by construction, $\neg s \in L_{E(B)\setminus F}$. Hence $\neg s \in Cn(B) \cap L_{E(B)\setminus F}$ and so by hypothesis $v(\neg s) = 1$, contradicting the fact that by the construction of S we have $v(s) = 1$. □

This argument is along much the same lines as that for the characterization of friendliness in terms of consistency in section 1.4. Like that characterization, it does not carry over to first-order contexts; indeed, the counterexample given in section 1.4 also serves here.

Corollary. De Bouvère's method can never be used in purely propositional logic as a way of showing that an elementary letter is not explicitly definable given a set A of propositional formulae.

Verification. Apply the observation with F chosen to be a singleton subset of $E(B)$. □

2.7. Leśniewski's Criterion of Conservativity

Friendliness is also closely related to the criterion of conservativity (alias non-creativity) in the theory of definition.

In lectures of the early 1920s, Leśniewski articulated two criteria that we usually want definitions to satisfy: eliminability and conservativity. A published account was given in Leśniewski (1931), with an easily accessible exposition in Suppes (1957, chapter 8). It is conservativity that connects with friendliness. The concept is usually formulated in the context of first-order logic. To clarify the link with friendliness, we again extract the purely propositional content.

Let A be any set of propositional formulae and let x be a formula. A,x is said to be a *conservative extension* of A iff $Cn(A, x) \cap L_A \subseteq Cn(A)$, i.e. iff $A \vdash c$ for every $c \in L_A$ such that $A, x \vdash c$.

Observation. In the propositional context: $A \mathrel{|\!\approx} x$ iff A,x is a conservative extension of A.

Proof. We already know from the first refinement of the characterization of friendliness in terms of consistency, in section 1.4, that $A \mathrel{|\!\approx} x$ iff (1) $A \vdash c$ for every $c \in L_A$ with $x \vdash c$. So we need only show the equivalence of this with (2) $A \vdash c$ for every $c \in L_A$ with $A, x \vdash c$.

One direction is immediate: by the monotony of classical consequence, (2) clearly implies (1). For the converse, suppose (1). Suppose $c \in L_A$ and $A, x \vdash c$; we need to show $A \vdash c$. Since $A, x \vdash c$ compactness tells us that $a, x \vdash c$ where a is the conjunction of some finite subset of A, and so also $x \vdash a \to c$. Clearly since $c \in L_A$ we also have $a \to c \in L_A$ So we may apply (1) to get $A \vdash a \to c$, and so since $A \vdash a$ we have $A \vdash c$ as desired. □

Corollary. On the level of propositional logic: A, x is a conservative extension of A iff the letters in $E(x) \setminus E(A)$ are Ramsey eliminable from A, x in the sense of Rantala.

Verification. By the observation just established, A, x is a conservative extension of A iff $A \mathrel{|\!\approx} x$. By the first observation of section 2.6, $A \mathrel{|\!\approx} x$ iff the letters in $E(x) \setminus E(A)$ are Ramsey eliminable from A, x in the sense of Rantala. □

Again this corollary is known to fail in the first-order context, where only the right-to-left half holds. An equivalence does hold, but it is between the left and a weaker version of the right: Γ,φ is a conservative extension of Γ iff every model of Γ is elementary equivalent to (i.e. satisfies the same first-order formulae as) some model of Γ that can be expanded to a model of Γ, φ.

2.8. Information-Preserving Paraconsistent Consequence

A less intimate connection with friendliness can be found in the construction of a certain paraconsistent consequence relation, effected in Pietruszczak (2004). This relation, which is a subrelation of classical consequence, is defined by Pietruszczak using a notion of preservation of information. But he also gives it an alternative characterization (his theorem 6.1) that makes contact with friendliness, or more precisely, with its syntax-independent counterpart sympathy, which we will define below in section 3.1.

Specifically, Pietruszczak's relation of information-preserving consequence holds between a formula a and a formula x iff four conditions hold: a classically entails x; a is classically consistent; x is not a tautology; and a further condition, formulated in terms of valuations, also holds. This further condition is not given a name, but is exactly the relation of sympathy, holding in the reverse direction from x to a.

Thus, roughly speaking, the syntax-independent version of friendliness has been used as one of the ingredients to construct a certain kind of paraconsistent subrelation of classical consequence. We have, in other words, an application of the relation.

The present author would comment, however, that the paraconsistent consequence so defined has a rather mixed bag of properties. As well as failing certain consequences that the paraconsistent logician desperately seeks to avoid (e.g. implication from $a \wedge \neg a$ to any proposition whatsoever, and from any proposition to $x \vee \neg x$), and failing others that some are willing to lose in order to achieve this (e.g. from a to $a \vee x$ for any x) the relation fails certain other properties that few paraconsistent logicians would be happy to see depart.

One of these is closure of the consequence relation under uniform substitution (of arbitrary formulae for elementary letters). Others are implication from $p \wedge q$ to any of $p \vee q, p \leftrightarrow q, p \rightarrow q, q \rightarrow p$, and likewise from $p \leftrightarrow q$ to either of $p \rightarrow q, q \rightarrow p$. Verification of all these failures is straightforward: none of the right formulae is friendly to the left one.

2.9. Coupled Semantic Decomposition Trees

Finally, we mention a connection with the theory of semantic decomposition trees (alias semantic tableaux) in classical logic. Developed by Beth, Hintikka and others, these trees entered the arena of textbooks with Jeffrey [1967]. Designed to test formulae for satisfiablility, the trees can of course be used to test an inference for invalidity by checking the satisfiability of the set (or conjunction) consisting of the premisses and negation of the conclusion. But Jeffrey also suggested another technique for the purpose, which he called 'coupled trees'.

Roughly speaking, he constructed a (signed) tree for the premisses, and another one for the conclusion. If every open branch of the former tree contains all the signed elementary letters (alias literals) that occur on some open branch of the latter one, then the inference is valid. However, as Jeffrey noted, the converse is not true without qualification. This is due to the possible absence of elementary letters in branches of the first tree, as for the inference from p to $q \vee \neg q$, likewise from p to $(p \wedge q) \vee (p \wedge \neg q)$. For this reason, he introduced an additional rule allowing the introduction of new elementary letters (by branching to an arbitrary formula and to its negation) when constructing a tree.

In the revised version of the textbook, published in 1981, Jeffrey omitted the technique of 'coupled trees' altogether, presumably because of the inelegance of the additional rule. In the meantime, Dunn [1976] showed that it could be adapted neatly to the so-called first-degree entailments of relevance logics. One simply requires that every branch (even closed) of the former tree contains all the signed elementary letters that occur on some branch (even closed) of the latter tree. This characterizes first-degree entailment without the need for any additional rules.

We remark that the technique of 'coupled trees' is even more naturally suited to determining whether a set A of formulae is friendly to another formula x. Construct the two (signed) trees as before. Call two branches *compatible* iff they do not contain any elementary letter with opposite signs. To test whether A is friendly to x, we simply check whether every open branch of the tree for A is compatible with some open branch of the tree for x. This characterizes friendliness without additional rules. We omit the straightforward verification.

3. From Friendliness to Sympathy

3.1. Definitions

We now consider a normalized version of friendliness that is syntax-independent on the left as well as on the right.

It is well known that for any finite set A of Boolean formulae, there is a unique least set F of elementary letters such that A is classically equivalent to some set of formulae in the language generated by F.

Although this is usually stated and proven for finite sets A only, it also holds for infinite ones. More specifically, let A be any set of formulae:

- Put $E!(A)$ to be the set of all letters p that are *essential for* A, in the sense that there are two valuations v, w, on the set E of all elementary letters of the language, that agree on all letters other than p but disagree in the value they give to A. Clearly $E!(A) \subseteq E(A)$.
- Put A^* to be the set of all formulae x with both $A \vdash x$ and $E(x) \subseteq E!(A)$. Clearly $E(A^*) = E!(A)$.

Clearly, whenever $A \dashv\vdash B$ then $E!(A) = E!(B)$ and also $A^* = B^*$. Moreover, as we show in the Appendix:

Least letter-set theorem. $A \dashv\vdash A^*$, and for every set B of formulae with $A \dashv\vdash B, E(A^*) \subseteq E(B)$.

We say that a set A of formulae is *sympathetic* to x and write $A \;|\!\sim\; x$, iff $A^* \,|\!\approx x$. This notion can be seen as a normalized version of friendliness, making it syntax-independent in the left argument.

Unrestricted left classical equivalence for $|\!\sim$. Whenever $A \dashv\vdash B$, then $A \;|\!\sim\; x$ iff $B \;|\!\sim\; x$.

Verification. Whenever $A \dashv\vdash B$ then as noted $A^* = B^*$, so $A^* \,|\!\approx x$ iff $B^* \,|\!\approx x$, i.e. $A \;|\!\sim\; x$ iff $B \;|\!\sim\; x$. □

From the least letter-set theorem we have immediately the following useful criterion for membership in $E!(A)$.

Criterion for membership in $E!(A)$. Let p be any elementary letter. Then $p \in E!(A)$ iff $p \in E(B)$ for every set B of formulae with $B \dashv\vdash A$.

We also have the following four criteria for sympathy.

Criteria for sympathy. Each of the following is equivalent to $A \;|\!\sim\; x$:

(a) $B \,|\!\approx x$ for every B with $A \dashv\vdash B$ and $E(B) = E!(A)$
(b) $A^* \,|\!\approx x$
(c) $B \,|\!\approx x$ for some B with $A \dashv\vdash B$ and $E(B) = E!(A)$
(d) $B \,|\!\approx x$ for some B with $A \dashv\vdash B$.

Verification. $A \;|\!\sim\; x$ is defined as (b), and immediately (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d). So we need only show (d) $\Rightarrow$ (a). Suppose $B \,|\!\approx x$ for some B with $A \dashv\vdash B$. Let $A \dashv\vdash C$ and $E(C) = E!(A)$. We need to show $C \,|\!\approx x$. Let v be any partial valuation on $E(C)$ with $v(C) = 1$. We need to find an extension v^+ of v to $E(C, x)$ with $v^+(x) = 1$. Since $E(C) = E!(A) = E(A^*) \subseteq E(B)$ by the least letter-set theorem, we may fix an arbitrary extension w of v to $E(B)$. Since $C \dashv\vdash A \dashv\vdash B$, we have $w(B) = 1$. Since $B \,|\!\approx x$ there is an extension w^+ of w to $E(B, x)$ with $w^+(x) = 1$. Then w^+ is an extension of v to $E(B, x)$. Since $E(C) \subseteq E(B)$ we also have $E(C, x) \subseteq E(B, x)$, so we may restrict w^+ to $E(C, x)$, call it w^{+-}. Clearly w^{+-} is still an extension of v and also $w^{+-}(x) = w^+(x) = 1$, so we may put $v^+ = w^{+-}$ and it has the desired properties. □

Corollary: broadening. Whenever $A \,|\!\approx x$ then $A \;|\!\sim\; x$.

Verification. By criterion (d). □

Evidently, the inclusion converse to broadening fails. Example: $p\wedge(q\vee\neg q) \not|\!\approx p\wedge q$ but $(p \wedge (q \vee \neg q)) \;|\!\sim\; p \wedge q$ since $(p \wedge (q \vee \neg q)) \dashv\vdash p \,|\!\approx p \wedge q$.

3.2. Property Failures for Sympathy: Inherited and New

All of the property failures that we bulleted for $|\!\approx$ in section 1.3 are also failures for $|\!\sim$. We can take the same counterexamples and observe that for each premiss a, $E!(a) = E(a)$. On the other hand and perhaps surprisingly, there are two important properties that succeeded for $|\!\approx$ but fail for $|\!\sim$: local disjunction in the premisses and compactness.

The following example, due to Pavlos Peppas (personal communication) illustrates the failure of local disjunction in the premisses.

Counterexample to local disjunction in the premisses. Put $a = p \vee r$, $b_1 = p \wedge q, b_2 = \neg q$, and $x = \neg q \vee \neg r$. Then $E(b_2) \subseteq E(a, b_1); E(b_1) \subseteq E(a, b_2); a, b_1 \mid\!\sim x; a, b_2 \mid\!\sim x$; but $a, b_1 \vee b_2 \not\mid\!\sim x$.

Verification. Clearly $E(b_2) \subseteq E(a, b_1)$ and indeed $E!(b_2) \subseteq E!(a, b_1)$. Also $E(b_1) \subseteq E(a, b_2)$ and indeed $E!(b_1) \subseteq E!(a, b_2)$. Also $a, b_1 \mid\!\sim x$ since $\{a, b_1\} \dashv\vdash b_1 \mid\!\approx x$, applying criterion (d) for sympathy. Also $a \wedge b_2 \vdash x$ so that $a \wedge b_2 \mid\!\approx x$ and thus $a, b_2 \mid\!\sim x$. But $a, (b_1 \vee b_2) \not\mid\!\sim x$.

To check the last, note that $a \wedge (b_1 \vee b_2) = (p \vee r) \wedge ((p \wedge q) \vee \neg q) \dashv\vdash p \vee (r \wedge \neg q)$ so that $E!(a, (b_1 \vee b_2)) = \{p, q, r\}$. So by criterion (a) for sympathy, it suffices to check that $p \vee (r \wedge \neg q) \not\mid\!\approx \neg q \vee \neg r$. Since every letter on the right already occurs on the left, it suffices to show $p \vee (r \wedge \neg q) \nvdash \neg q \vee \neg r$ by the reduction case for friendliness (section 1.4). But this is clear putting $v(p) = v(q) = v(r) = 1$. □

By suitably tweaking this example, we can turn it into one that illustrates the failure, for sympathy, of the closely related rule of proof by exhaustion.

Counterexample to proof by exhaustion. Put $a = p \vee \neg q \vee r$, $b = p \wedge q; x = \neg q \vee \neg r$. Then $a, b \mid\!\sim x; a, \neg b \mid\!\sim x$; but $a \not\mid\!\sim x$.

Verification. Similar to that of the preceding example, but we give the details. Again we have $a, b \mid\!\sim x$ since $\{a, b\} \dashv\vdash b \mid\!\approx x$, applying criterion (d) for sympathy. Also $a, \neg b \mid\!\sim x$ since $a, \neg b \vdash x$. But $a \not\mid\!\sim x$ since $E!(a) = \{p, q, r\}$, so by criterion (a) for sympathy, it suffices to check that $p \vee \neg q \vee r \not\mid\!\approx \neg q \vee \neg r$. Since every letter on the right already occurs on the left, it suffices to show $p \vee \neg q \vee r \nvdash \neg q \vee \neg r$ by the reduction case for friendliness. But this is clear putting $v(p) = v(q) = v(r) = 1$. □

The next example illustrates the failure of compactness for sympathy. Consider a language with countably many elementary letters $q, p_1, p_2, \ldots$.

Counterexample to compactness. Put A to be the set of all formulae a_n that are of the form $(p_1 \wedge \ldots \wedge p_n) \vee q$ for odd $n \geq 1$, or of the form $(p_1 \wedge \ldots \wedge p_n) \vee \neg q$ for even $n \geq 1$. Then $A \mid\!\sim q$ but $B \not\mid\!\sim q$ for every finite non-empty subset $B \subseteq A$.

Verification. To show $A \mid\!\sim q$ it suffices, by criterion (d) for sympathy, to find an $X \dashv\vdash A$ with $X \mid\!\approx q$. Putting $X = \{p_i : i \geq 0\}$ we clearly have the former, and since q does not occur in any formula in X we also have the latter.

Now let B be any finite non-empty subset of A. To complete the verification of the example, we need to show that $B \not\mid\!\sim q$, i.e. that $B^* \not\mid\!\approx q$.

First, we show that q is essential to B. Consider the largest n such that $a_n \in B$; this exists because B is finite and non-empty. We examine the case that n is odd, so that $a_n = (p_1 \wedge \ldots \wedge p_n) \vee q$; the case for even n is similar. Put $v(p_i) = w(p_i) = 1$ for all $i < n, v(p_n) = w(p_n) = 0$, and $v(q) = 1$ while $w(q) = 0$. Then $w(a_n) = 0$ so that $w(B) = 0$. On the other hand, $v(a_n) = 1$ (since $v(q) = 1$) and also $v(a_i) = 1$ for all $i < n$ (since p_n does not occur in any such a_i) so that

$v(B) = 1$. Since v, w agree on all p_i for all $i \leq n$ while disagreeing on B, this shows that q is essential to B, as desired.

We can now show that $B^* \not\approx q$. Put $u(p_i) = 1$ for all $i \leq n$ and $u(q) = 0$. Then $u(a_i) = 1$ for all $i \leq n$ so that $u(B) = 1$ and hence $B \nvdash q$; so since $B \dashv\vdash B^*$ we have $B^* \nvdash q$. But since q is essential to B, q occurs in B^*. So by the first reduction case of section 1.4, since $B^* \nvdash q$ we have finally $B^* \not\approx q$ completing the verification of the example. $\square$

3.3. Property Successes for Sympathy

Apart from disjunction in the premisses and compactness, all of the other properties that we noted as satisfied by friendliness also hold for sympathy. We consider them one by one. Whenever possible, we derive the property for $\mid\!\sim$ from the one for $\approx$, rather than argue from scratch. Most of the verifications are straightforward; only singleton cumulative transitivity is rather tricky, needing some lemmas on least letter-sets.

Supraclassicality for $\mid\!\sim$. Whenever $A \vdash x$ then $A \mid\!\sim x$.

Verification. Suppose $A \vdash x$. Then $A \approx x$ by supraclassicality for $\approx$, so $A \mid\!\sim x$ by broadening. $\square$

Reduction case for $\mid\!\sim$. Whenever $E(x) \subseteq E!(A)$ then $A \mid\!\sim x$ iff $A \vdash x$.

Verification. Right to left is given by supraclassicality. For the converse, suppose $E(x) \subseteq E!(A)$. Suppose $A \mid\!\sim x$. By definition, $A^* \approx x$. Recalling that $E!(A) = E(A^*)$ so that $E(x) \subseteq E(A^*)$, the reduction case for friendliness tells us $A^* \vdash x$. Since $A \dashv\vdash A^*$ we have $A \vdash x$ as desired. $\square$

Characterization of $\mid\!\sim$ in terms of consistency. $A \mid\!\sim x$ iff every set of formulae in $L_{E!(A)}$ that is consistent with A, is consistent with x.

Verification. By definition, $A \mid\!\sim x$ iff $A^* \approx x$. Applying the corresponding consistency characterization of $\approx$ and the fact that $A^* \dashv\vdash A$, the desired equivalence follows. $\square$

Right weakening for $\mid\!\sim$. Whenever $A \mid\!\sim x \vdash y$ then $A \mid\!\sim y$

Verification. From the definition of $\mid\!\sim$ and right weakening for $\approx$. $\square$

This implies right classical equivalence for sympathy: whenever $x \dashv\vdash y$ then $A \mid\!\sim x$ iff $A \mid\!\sim y$. The relation $\mid\!\sim$ is thus syntax-independent on both left and right.

Local left strengthening for $\mid\!\sim$. Suppose $E!(B) \subseteq E!(A)$. If $B \vdash A \mid\!\sim x$ then $B \mid\!\sim x$.

Verification. Immediate from the corresponding property of $\approx$, the definition of $\mid\!\sim$, and the fact that $A^* \dashv\vdash A$. $\square$

Local monotony for $\mid\!\sim$. Suppose $E!(B) \subseteq E!(A)$. If $A \mid\!\sim x$ and $A \subseteq B$ then $B \mid\!\sim x$.

Verification. If $A \subseteq B$ then $B \vdash A$. □

Note that in these two 'local' properties, the locality condition concerns $E!(A), E!(B)$ rather than $E(A), E(B)$.

3.4. Singleton Cumulative Transitivity for Sympathy

We have postponed consideration of singleton cumulative transitivity because its proof requires two lemmas about least letter-sets.

Lemma. $E!(A, B) \subseteq E!(A) \cup E!(B) \subseteq E!(A) \cup E(B)$.

Verification. The right inclusion is immediate from $E!(B) \subseteq E(B)$. For the left inclusion, suppose $p \in E!(A, B)$. Then there are partial valuations v_0, v_1 on $E(A, B)$ that agree on all letters in this domain other than p, with $v_0(A, B) = 0$ and $v_1(A, B) = 1$. Since $v_0(A, B) = 0$, either $v_0(A) = 0$ or $v_0(B) = 0$.

Suppose the former; the argument for the latter is similar. Restrict v_0, v_1 to $E(A)$, call them v_0^-, v_1^-. Then $v_0^-(A) = 0$ whilst $v_1^-(A) = 1$, but v_0^-, v_1^- agree on all letters in their common domain other than p. Hence $p \in E!(A) \subseteq E!(A) \cup E!(B)$ as desired. □

Lemma. If $A \mid\!\approx x$ then $E!(A) \subseteq E!(A, x)$. Indeed, more generally: If $A \mid\!\approx_{\forall\exists\forall} B$ then $E!(A) \subseteq E!(A, B)$.

Verification. Suppose $A \mid\!\approx_{\forall\exists\forall} B$ (defined in section 1.9) and $p \in E!(A)$. From the latter, there are partial valuations v_0, v_1 on $E(A)$ that agree on all letters in this domain other than p, with $v_0(A) = 0$ and $v_1(A) = 1$. Since $A \mid\!\approx_{\forall\exists\forall} B, v_1$ can be extended to a valuation v_1^+ on $E(A, B)$ with $v_1^+(B) = 1$, so $v_1^+(A, B) = 1$. Now extend v_0 to $E(A, B)$ by putting $v_0^+(q) = v_1^+(q)$ for every letter $q \in E(A, B) \setminus E(A)$. Then clearly v_0^+, v_1^+ agree on all letters in their common domain except p, and disagree on A, B since $v_1^+(A, B) = 1$ while $v_0^+(A, B) = 0$ since $v_0(A) = 0$. Hence $p \in E!(A, B)$ as desired. □

Singleton cumulative transitivity for $\mid\!\sim$. Whenever $A \mid\!\sim x$ and $A, x \mid\!\sim y$ then $A \mid\!\sim y$.

Proof. Suppose $A \mid\!\sim x$ and $A, x \mid\!\sim y$. From the hypotheses we have $A^* \mid\!\approx x$ and $(A, x)^* \mid\!\approx y$. We need to show $A^* \mid\!\approx y$.

Let v be any partial valuation on $E(A^*) = E!(A)$ with $v(A^*) = 1$. We need to find an extension w of v to $E(A^*, y) = E!(A) \cup E(y)$ with $w(y) = 1$.

Since $A^* \mid\!\approx x$ and $v(A^*) = 1, v$ can be extended to a v^+ on $E(A^*, x) = E!(A) \cup E(x)$ with $v^+(x) = 1$. By the first lemma, we may restrict v^+ to the subset $E!(A, x)$ of its domain, call it v^{+-}. By the second lemma, since $A^* \mid\!\approx x$ we have $E(A^*) = E!(A) \subseteq E!(A, x)$, so v^{+-} is an extension of v. Also, $v^{+-}((A, x)^*) = v^+((A, x)^*) = v^+(A^*, x)$. Also $v^+(A^*) = v(A^*) = 1$ and $v^+(x) = 1$. Putting this together, $v^+(A^*, x) = 1$ so $v^{+-}((A, x)^*) = 1$.

Hence, since $(A, x)^* \mid\!\approx y, v^{+-}$ may be extended from $E!(A, x)$ to a valuation v^{+-+} on $E!(A, x) \cup E(y)$ with $v^{+-+}(y) = 1$. Since v^{+-} is an extension of v it follows that v^{+-+} is also an extension of v. Finally, restrict v^{+-+} to $E!(A) \cup E(y)$, which by the second lemma again is a subset of $E!(A, x) \cup E(y)$; call it v^{+-+-}. This is still an extension of v, defined on $E!(A)$, and also $v^{+-+-}(y)$ is well defined with $v^{+-+-}(y) = v^{+-+}(y) = 1$. Put $w = v^{+-+-}$ and the proof is complete. □

3.5. Interpolation for Sympathy

An interpolation property for sympathy follows readily from its counterpart for friendliness. We need to be careful, however, about where we can write A, versus A^*, in the formulation.

Interpolation for $\mid\!\sim$. Whenever $A \mid\!\sim x$ there is a finite set $F \subseteq E(A^*) \cap E(x) \subseteq E(A) \cap E(x)$ of elementary letters, such that for every finite set G of elementary letters with $F \subseteq G \subseteq E(A^*)$ there is a formula b with the following properties:

1. $E(b) = G$
2. $A \mid\!\sim b$ (indeed $A \vdash b$)
3. $b \mid\!\sim x$
4. b is consistent, provided A is consistent
5. b is not a tautology, provided there is a non-tautology $y \in L_A \cap L_x$ with $A \vdash y$.

Proof. Suppose $A \mid\!\sim x$. By definition, $A^* \mid\!\approx x$. So by interpolation for friendliness, we have the above but with A^* in place of A in properties (2), (4), (5). Since $A \dashv\vdash A^*$ we also have (2), (4) for A. It remains to check condition (5).

Suppose there is a non-tautology $y \in L_A \cap L_x$ with $A \vdash y$. We need to find a non-tautology $z \in L_{A^*} \cap L_x$ with $A^* \vdash z$. Consider the 2^k formulae that can be obtained from y by substituting $\top, \bot$ for the k letters ($k \geq 0$) in $E(y)$ that are not in $E(A^*)$. Since y is not a tautology, at least one of these 2^k formulae is not a tautology; choose one as z. Clearly $z \in L_{A*} \cap L_x$. Also, since $A \vdash y$ and $A \dashv\vdash A^*$ we have $A^* \vdash y$ and so since the substitution producing z is the identity on A^* we have $A^* \vdash z$ and the verification is complete. □

3.6. Further Remarks on the Concept of an Essential Letter

Karl Schlechta (personal communication) has observed that it is possible to generalize the notion of an essential letter, making it relative to an arbitrary *set of valuations* rather than to a *set of formulae*. In detail: let W be an arbitrary set of valuations. We say that a letter p is *essential to W* iff there are two valuations that agree on all letters other than p, but one in and the other outside W.

As is often the case when we pass to arbitrary sets of valuations in place of sets of formulae (which correspond to definable sets of valuations), we get an equivalent notion in the finite case, but a more general one in the infinite case with loss of some properties. Without following this through systematically, we give one example. When dealing with sets of formulae, we have the following:

Observation. Let A be any set of formulae. Then A is contingent (neither a tautology nor a contradiction) iff at least one of its elementary letters is essential to it.

Verification. Right to left is immediate from the definition of an essential letter. For the converse, suppose that A contingent. Then there are two partial valuations v, w on $E(A)$, with $v(A) = 1$ and $w(A) = 0$. From the latter, there is a formula $a \in A$ with $w(a) = 0$. Let v_w be the partial valuation on $E(A)$ defined by putting $v_w(p) = w(p)$ for all letters in $E(a)$, and $v_w(p) = v(p)$ for all other letters. Then v, v_w disagree on only finitely many letters, and we have $v(A) = 1$ while $v_w(a) = 0$ so that $v_w(A) = 0$.

Since v, w_v disagree on only finitely many letters, there is a finite chain $v_1, \ldots, v_n$ of partial valuations on $E(a)$ beginning with $v_1 = v$ and ending with $v_n = v_w$, each disagreeing with its predecessor on just one letter. Take the last v_k in the chain with $v_k(A) = 1$. Then $k < n$ and $v_{k+1}(A) = 0$. Thus v_k, v_{k+1} are partial valuations on $E(A)$ that agree on all letters except one, but give A different values, so that letter is essential to A. □

This argument goes through no matter what the cardinality of the set of the elementary letters, and independently of whether they can be well ordered. But the observation fails for its counterpart in terms of sets of valuations, even for a countable language.

The counterpart says: Let W be any subset of the set of all valuations; then W is proper and non-empty iff at least elementary letter is essential to it. Right to left does hold: if at least one letter is essential to W, then immediately from the definition W is neither empty nor the set of all valuations. But left to right fails. Example: put W to be the set of all valuations that make only finitely many elementary letters true. This is neither empty nor the set of all valuations. But when a valuation is in W, so is every valuation that differs from it at exactly one letter.

4. Open Questions

4.1. Specific Problems

- Can we give an axiomatic characterization of friendliness (or for sympathy) that is more traditional in style than the one at the end of section 1.5?
- What is the most interesting way of defining friendliness in a first-order context, and which of its properties carry over?
- Which properties of the notion of an essential letter carry over when that notion is understood modulo an arbitrary set of valuations, as in section 3.6, rather than modulo a set of formulae?

4.2. Open-Ended Questions

- How much of the theory of friendliness remains if we generalize from the classical two-valued context to a many-valued one?
- Is it helpful to characterize friendliness and sympathy using appropriate three-valued possible worlds structures, with a relation between possible worlds representing the extension of one partial valuation by another?
- Are there any interesting connections between the theory of friendliness and possible-worlds semantics for intuitionistic logic?

5. Appendix

5.1. Proof of Least Letter-Set Theorem

As remarked in the text, proofs of the least letter-set theorem usually cover only the finite case. Perhaps the most elegant such proof, given for example by Parikh (1999), uses interpolation for classical logic. We recall it briefly.

Let A be any finite set of Boolean formulae. Since A is finite, $E(A)$ is also finite, so there is at least one minimal subset $F \subseteq E(A)$ with the property that A is classically equivalent to some set of formulae in the language generated by F. So we need only show that F is unique. Let G be any other such minimal set of letters. Then there are sets B, C of formulae in L_F, L_G respectively with $B \dashv\vdash A \dashv\vdash C$ so $B \vdash C$ so by interpolation for classical logic there is a set X of formulae in $L_{F \cap G}$ with $B \vdash X \vdash C$ so $A \vdash B \vdash X \vdash C \vdash A$ so $A \dashv\vdash X$. But since F, G were both minimal, it follows that $F = F \cap G = G$ and we are done.

Unfortunately, this elegant argument is not available in the infinite case, as we cannot assume that there is a minimal F with the property. We give a different proof covering the infinite as well as the finite case. We have not been able to ascertain whether such a proof already occurs in the literature.

We recall from section 3.1 the definitions that will be needed.

- $E!(A)$ is the set of all letters p that are *essential for* A, in the sense that there are two valuations v, w, on the set E of all elementary letters of the language, that agree on all letters other than p but disagree in the value they give to A. Clearly $E!(A) \subseteq E(A)$, and whenever $A \dashv\vdash B$ then $E!(A) = E!(B)$.
- A^* is the set of all formulae x with both $A \vdash x$ and $E(x) \subseteq E!(A)$. Clearly $E(A^*) = E!(A)$. Clearly, whenever $A \dashv\vdash B$ then $A^* = B^*$.

Clearly, it would be equivalent to formulate the definition of $E!(A)$ in terms of partial valuations on $E(A)$ rather than full valuations on the entire set E of elementary letters, but working with full valuations here streamlines the argument.

We proceed via a lemma. Roughly speaking, it says that letters that are individually inessential to a set of formulae, are also jointly so.

Lemma. Let v, w be any two valuations on E that agree on $E!(A)$. Then $v(A) = 1$ iff $w(A) = 1$.

Proof. First we use induction to show that the lemma holds whenever v, w disagree on only finitely many letters. Then we use this to show that it holds when they disagree on infinitely many letters.

For the basis of the induction put $n = 0$, i.e. suppose that v, w disagree on no letters. Then $v = w$ and we are done. For the induction step, suppose that the lemma holds whenever two valuations disagree on just n letters. Suppose v,w disagree on just n+1 letters $p_1, \ldots, p_n, p_{n+1}$. Let w' be a valuation that is just like w except that $w'(p_{n+1}) = v(p_{n+1})$. Then w' disagrees with v on just n letters, and so by the induction hypothesis $v(A) = 1$ iff $w'(A) = 1$. But also w' disagrees with w on just the one letter p_{n+1}. Since v, w agree on $E!(A)$ while disagreeing on p_{n+1} we know that $p_{n+1} \notin E!(A)$, i.e. p_{n+1} is not essential for A. Hence since w, w' agree on every letter other than p_{n+1} we have by the definition of essential letters that $w(A) = 1$ iff $w'(A) = 1$. Putting these together, $v(A) = 1$ iff $w(A) = 1$ as desired. This completes the induction.

Now suppose that v, w are any two valuations on L that agree on $E!(A)$ but differ on infinitely many letters. We want to show that $v(A) = 1$ iff $w(A) = 1$. Suppose otherwise; we obtain a contradiction. Then either $v(A) = 1$ while $w(A) = 0$, or $w(A) = 1$ while $v(A) = 0$. Consider the former; the latter case is similar.

Since $w(A) = 0$, we have $w(a) = 0$ for some $a \in A$. Let v_w be the valuation like v except for the letters in a, where it is like w. Then v_w disagrees with v on just finitely many letters. Moreover, none of those letters are in $E!(A)$. For suppose $v_w(p) \neq v(p)$. Then the letter p occurs in a, so $v_w(p) = w(p)$ so $w(p) \neq v(p)$ and thus $p \notin E!(A)$ by the supposition that v, w agree on $E!(A)$. Hence the finite part of the lemma gives us $v(A) = 1$ iff $v_w(A) = 1$. By supposition, $v(A) = 1$ so we have $v_w(A) = 1$. Since $a \in A$ this gives $v_w(a) = 1$. But $w(a) = 0$ and by the construction of v_w we have $v_w(a) = w(a)$. Hence $v_w(a) = 0$ giving us the desired contradiction. $\square$

Least letter-set theorem. $A \dashv\vdash A^*$, and for every set B of formulae with $A \dashv\vdash B, E(A^*) \subseteq E(B)$.

Proof. We need to show (1) $E!(A) \subseteq E(B)$ for every B with $A \dashv\vdash B$, and (2) $A \dashv\vdash A^*$.

For (1), suppose $A \dashv\vdash B$, $p \in E!(A)$, but $p \notin E(B)$; we obtain a contradiction. The diagram illustrates the argument that follows.

$v(A)$	$\neq$	$w(A)$
$=$		$=$
$v(B)$	$=$	$w(B)$

Since $p \in E!(A)$ there are valuations v,w on L with $v(q) = w(q)$ for all letters q with $q \neq p$, but $v(A) \neq w(A)$ (top row). Since $p \notin E(B)$ this implies

$v(B) = w(B)$ (bottom row). But since $A \dashv\vdash B$ we have both $v(A) = v(B)$ and $w(A) = w(B)$ (side columns), giving a contradiction.

For (2), by construction, we have $A \vdash A^*$. Suppose $A^* \nvdash A$; we derive a contradiction. Since $A^* \nvdash A$ there is a valuation v with $v(A^*) = 1$ and $v(A) = 0$, i.e. $v(a) = 0$ for some $a \in A$. Let S be the set of all literals $\pm q$ with $q \in E(A^*)$ such that $v(\pm q) = 1$. Then clearly $S \vdash A^*$. We break the argument into two cases, deriving a contradiction in each.

Case 1. Suppose S is inconsistent with A. Then by classical compactness, some finite subset $S_f \subseteq S$ is inconsistent with A. Hence $A \vdash \neg \wedge S_f$. Since all letters in $\neg \wedge S_f$ are in $E(A^*)$ it follows that $\neg \wedge S_f \in A^*$, so since $v(A^*) = 1$ we have $v(\neg \wedge S_f) = 1$. But by the construction of S we also have $v(\wedge S_f) = 1$, giving us the desired contradiction.

Case 2. Suppose S is consistent with A. Then there is a valuation w with $w(S) = w(A) = 1$. Since $w(S) = 1$ it follows that w agrees with v on all letters in $E(A^*)$. So the lemma tells us that $v(A) = 1$ iff $w(A) = 1$. So since $w(A) = 1$ we have $v(A) = 1$. Since $a \in A$, this gives $v(a) = 1$, contradicting $v(a) = 0$ and completing the proof of (2). $\square$

References

[Beth,1953] Beth, Evert W. 1953. On Padoa's method in the theory of definition, *Nederl. Akad. Wetensch. Proc. Ser. A* 56: 330-339; also *Indagationes Mathematicae* 15: 330-339.

[Boole,1847] Boole, George. 1847. *The Mathematical Analysis of Logic*. Cambridge: Macmillan.

[Boole,1854] Boole, George. 1854. *An Investigation into the Laws of Thought*. London: Walton.

[de Bouvère,1959] de Bouvère, K.L. 1959. *A Method in Proofs of Undefinability*. Amsterdam: North Holland.

[Dunn,1976] Dunn, J.M. 1976. Intuitive semantics for first-degree entailments and coupled trees, *Philosophical Studies*, **29**, 149–168.

[Jeffrey,1967] Jeffrey, R.C. 1967. *Formal Logic: Its Scope and Limits* (second edition 1981). New York: McGraw-Hill.

[Lang *et al.*,2003] Lang, J., P. Liberatore, P. Marquis. 2003. Propositional independence: formula-variable independence and forgetting, *Journal of Artificial Intelligence Reseach*, 18: 391–443.

[Leśniewski,1931] Leśniewski, S. 1931. Über definitionen in der sogennanten Theorie der Deduktion. *Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie*, Classe 3, XXIV: 300-302.

[Lin and Reiter,1994] Lin, F. and R. Reiter (1994). Forget it!. In R. Greiner and D. Subramanian, eds. *Working Notes on AAAI Fall Symposium on Relevance*. Menlo Park: AAAI Press.

[Makinson,2005] Makinson, David. 2005. *Bridges from Classical to Nonmonotonic Logic*. London: College Publications. Series: Texts in Computing, vol 5.

[Makinson,2005a] Makinson, David. 2005. Friendliness for logicians. In Sergei N. Artemov, Howard Barringer, Artur S. d'Avila Garcez, Luis C. Lamb, and John Woods, editors, *We Will Show Them! Essays in Honour of Dov Gabbay, Volume Two*, pages 259-292. College Publications, 2005.

[Padoa,1901] Padoa, A. 1901. Essai d'une théorie algébrique des nombres entiers, précédé d'une introduction logique à une théorie deductive quelconque. *Bibliothèque du Congrès International de Philosophie, Paris 1900*, vol 3: 309-365. Paris: Armand Colin.

[Parikh,1999] Parikh, R. 1999. Beliefs, belief revision, and splitting languages. Pages 266–278 of L. Moss et al eds, *Logic, Language and Computation*, vol 2. CSLI Lecture Notes n ° 96: 266-278. California: CSLI Publications.

[Pietruszczak,2004] Pietruszczak, A. 2004. The consequence relation preserving logical information, *Logic and Logical Philosophy* 13: 89-120.

[Ramsey,1931] Ramsey, F.P. 1931. *The Foundations of Mathematics and Other Logical Essays* ed. R.B. Braithwaite. London: Kegan Paul, Trench, Trubner.

[Rantala,1991] Rantala, V. 1991. Definitions and definability, pages 135-159 of James H. Fetzer et al *Definitions and Definability: Philosophical Perspectives*. Dordrecht : Kluwer.

[Sneed,1971] Sneed, J.D. 1971. *The Logical Structure of Mathematical Physics*. Dordrecht: Reidel.

[Suppes,1957] Suppes, P. 1957. *Introduction to Logic*. Princeton: Van Nostrand.

[van Benthem,1978] Van Benthem, J.F.A.K. 1978. Ramsey eliminability, *Studia Logica* 37: 321-336.

[Weber,1987] Weber, A. 1987. Updating propositional formulae, pages 487–500 in L. Kerschberg, ed. *Proceedings of the First Conference on Expert Data Systems*. Benjamin Cummings.

Acknowledgments

Many friendly logicians helped in various ways. In particular, thanks to Pavlos Peppas for the counterexample to disjunction in the premiss for the relation of sympathy in section 3.2, Lloyd Humberstone for discussions on links in part 2, and Karl Schlechta for the concept of a letter essential to a set of valuations in section 3.6. Anatoli Degtyarev, Kurt Engesser, Maribel Fernández, Dov Gabbay, Jamie Gabbay, George Kourousias and Odinaldo Rodrigues also commented on various versions.

David Makinson
Dept. of Philosophy, Logic & Scientific Method, London School of Economics
Houghton Street, London WC2A 2AE
United Kingdom
e-mail: `david.makinson@gmail.com`