Probabilistic Logic under Coherence, Model-Theoretic Probabilistic Logic, and Default Reasoning

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Abstract. We study probabilistic logic under the viewpoint of the coherence principle of de Finetti. In detail, we explore the relationship between coherence-based and model-theoretic probabilistic logic. Interestingly, we show that the notions of g-coherence and of g-coherent entailment can be expressed by combining notions in model-theoretic probabilistic logic with concepts from default reasoning. Crucially, we even show that probabilistic reasoning under coherence is a probabilistic generalization of default reasoning in system P. That is, we provide a new probabilistic semantics for system P, which is neither based on infinitesimal probabilities nor on atomic-bound (or also big-stepped) probabilities. These results also give new insight into default reasoning with conditional objects.

1 Introduction

The probabilistic treatment of uncertainty plays an important role in many applications of knowledge representation and reasoning. Often, we need to reason with uncertain information under partial knowledge and then the use of precise probabilistic assessments seems unrealistic. Moreover, the family of uncertain quantities at hand has often no particular algebraic structure.

In such cases, a general approach is obtained by using (conditional and/or unconditional) probabilistic constraints, based on the coherence principle of de Finetti and suitable generalizations of it [5, 9, 15, 16]. Two important aspects in dealing with uncertainty are: (i) checking the consistency of a probabilistic assessment; and (ii) the propagation of a given assessment to further uncertain quantities.

Another approach for handling probabilistic constraints is model-theoretic probabilistic logic, whose roots go back to Boole's book of 1854 "The Laws of Thought" [8]. There is a wide spectrum of formal languages that have been explored in probabilistic logic, which ranges from constraints for unconditional and conditional events [2, 13, 19, 20, 22, 23] to rich languages that specify linear inequalities over events [12]. The main problems related to model-theoretic probabilistic logic are checking satisfiability, deciding logical entailment, and computing tight logically entailed intervals.

Coherence-based and model-theoretic probabilistic reasoning have been explored quite independently from each other by two different research communities. For this reason, the relationship between the two areas has not been studied in depth so far. The current paper and our work in [7] aim at filling this gap. More precisely, our research is essentially guided by the following two questions:

- Which is the semantic relationship between coherence-based and model-theoretic probabilistic reasoning?
- Can algorithms that have been developed for efficient reasoning in one area also be used in the other area?

Interestingly, it turns out that the answers to these two questions are closely related to default reasoning from conditional knowledge bases in system P.

The literature contains several different proposals for default reasoning and extensive work on its desired properties. The core of these properties are the rationality postulates of system P proposed by Kraus, Lehmann, and Magidor [18]. It turned out that these rationality postulates constitute a sound and complete axiom system for several classical model-theoretic entailment relations under uncertainty measures on worlds. More precisely, they characterize classical model-theoretic entailment under preferential structures [25, 18], infinitesimal probabilities [1, 24], possibility measures [10], and world rankings. They also characterize an entailment relation based on conditional objects [11]. A survey of all these relationships is given in [3].

Roughly speaking, coherence-based probabilistic reasoning is reducible to model-theoretic probabilistic reasoning using concepts from default reasoning. Crucially, it even turns out that *coherence-based probabilistic reasoning is a probabilistic generalization of default reasoning in system P*. That is, we provide a *new probabilistic semantics for system P*, which is neither based on infinitesimal probabilities nor on atomic-bound (or also big-stepped) probabilities [4, 26].

The current paper deals with the semantic aspects of these findings, while [7] focuses on its algorithmic implications for coherence-based probabilistic reasoning.

The main contributions of the current paper can be summarized as follows:

- We define a coherence-based probabilistic logic. We define a formal language of logical and conditional constraints, which are defined on arbitrary families of conditional events. We then define the notions of generalized coherence (or *g-coherence*), g-coherent consequence, and tight g-coherent consequence for this language.
- We explore the relationship between g-coherence and g-coherent entailment, on the one hand, and satisfiability and logical entailment, on the other hand.
- We show that probabilistic reasoning under coherence is a probabilistic generalization of default reasoning from conditional knowledge bases in system P.
- We show that this relationship reveals new insight into Dubois and Prade's approach to default reasoning with conditional objects [11, 3].

Note that detailed proofs of all results are given in the extended paper [6].

2 Probabilistic Logic under Coherence

In this section, we first introduce some technical preliminaries. We then briefly describe precise and imprecise probability assessments under coherence. We finally define our coherence-based probabilistic logic and give an illustrating example.

2.1 Preliminaries

We assume a nonempty set of *basic events* Φ. We use $\bot$ and $\top$ to denote *false* and *true*, respectively. The set of *events* is the closure of $\Phi \cup \{\bot, \top\}$ under the Boolean operations $\neg$ and $\wedge$. That is, each element of $\Phi \cup \{\bot, \top\}$ is an event, and if ϕ and ψ are events, then also $(\phi \wedge \psi)$ and $\neg\phi$. We use $(\phi \vee \psi)$ and $(\psi \Leftarrow \phi)$ to abbreviate $\neg(\neg\phi \wedge \neg\psi)$ and $\neg(\phi \wedge \neg\psi)$, respectively, and adopt the usual conventions to eliminate parentheses. We often denote by $\bar{\phi}$ the negation $\neg\phi$, and by $\phi\psi$ the conjunction $\phi \wedge \psi$. A *logical constraint* is an event of the form $\psi \Leftarrow \phi$. Note that $\bot \Leftarrow \alpha$ is equivalent to $\neg\alpha$.

A *world* I is a truth assignment to the basic events in Φ (that is, a mapping $I\colon \Phi \rightarrow \{\mathbf{false}, \mathbf{true}\}$), which is extended to all events as usual (that is, $(\phi \wedge \psi)$ is true in I iff ϕ and ψ are true in I, and $\neg\phi$ is true in I iff ϕ is not true in I). We use $\mathcal{I}_\Phi$ to denote the set of all worlds for Φ. A world I *satisfies* an event ϕ, or I is a *model* of ϕ, denoted $I \models \phi$, iff $I(\phi) = \mathbf{true}$. I *satisfies* a set of events L, or I is a *model* of L, denoted $I \models L$, iff I is a model of all $\phi \in L$. An event ϕ (resp., a set of events L) is *satisfiable* iff a model of ϕ (resp., L) exists. An event ψ is a *logical consequence* of ϕ (resp., L), denoted $\phi \models \psi$ (resp., $L \models \psi$), iff each model of ϕ (resp., L) is also a model of ψ. We use $\phi \not\models \psi$ (resp., $L \not\models \psi$) to denote that $\phi \models \psi$ (resp., $L \models \psi$) does not hold.

2.2 Probability Assessments

A *conditional event* is an expression $\psi|\phi$ with events ψ and $\phi \neq \bot$. It can be looked at as a three-valued logical entity, with values $\mathbf{true}$, or $\mathbf{false}$, or $\mathbf{indeterminate}$, according to whether ψ and ϕ are true, or ψ is false and ϕ is true, or ϕ is false, respectively. That is, we extend worlds I to conditional events $\psi|\phi$ by $I(\psi|\phi) = \mathbf{true}$ iff $I \models \psi \wedge \phi$, $I(\psi|\phi) = \mathbf{false}$ iff $I \models \neg\psi \wedge \phi$, and $I(\psi|\phi) = \mathbf{indeterminate}$ iff $I \models \neg\phi$. Note that $\psi|\phi$ coincides with $\psi{\wedge}\phi|\phi$. More generally, $\psi_1|\phi_1$ and $\psi_2|\phi_2$ coincide iff $\psi_1 \wedge \phi_1 = \psi_2 \wedge \phi_2$ and $\phi_1 = \phi_2$.

A *probability assessment* (L, A) on a set of conditional events $\mathcal{E}$ consists of a set of logical constraints L, and a mapping A that assigns each $\varepsilon \in \mathcal{E}$ a real number in $[0, 1]$. Informally, L describes logical relationships, while A represents probabilistic knowledge. For $\{\psi_1|\phi_1, \ldots, \psi_n|\phi_n\} \subseteq \mathcal{E}$ with $n \geq 1$ and n real numbers $s_1, \ldots, s_n$, let the mapping $G\colon \mathcal{I}_\Phi \rightarrow \mathbf{R}$ be defined as follows. For every $I \in \mathcal{I}_\Phi$:

$$G(I) \;=\; \sum_{i=1}^{n} s_i \cdot I(\phi_i) \cdot (I(\psi_i) - A(\psi_i|\phi_i))\,.$$

In the previous formula, we identify the truth values $\mathbf{false}$ and $\mathbf{true}$ with the real numbers 0 and 1, respectively. Intuitively, G can be interpreted as the random gain corresponding to a combination of n bets of amounts $s_1 \cdot A(\psi_1|\phi_1), \ldots, s_n \cdot A(\psi_n|\phi_n)$ on $\psi_1|\phi_1, \ldots, \psi_n|\phi_n$ with stakes $s_1, \ldots, s_n$. In detail, to bet on $\psi_i|\phi_i$, one pays an amount of $s_i \cdot A(\psi_i|\phi_i)$, and one gets back the amount of s_i, 0, and $s_i \cdot A(\psi_i|\phi_i)$, when $\psi_i \wedge \phi_i$,

$\neg\psi_i \wedge \phi_i$, and $\neg\phi_i$, respectively, turns out to be true. The following notion of *coherence* now assures that it is impossible (for both the gambler and the bookmaker) to have *uniform loss*.

A probability assessment (L, A) on a set of conditional events $\mathcal{E}$ is *coherent* iff for every $\{\psi_1|\phi_1, \ldots, \psi_n|\phi_n\} \subseteq \mathcal{E}$ with $n \geq 1$ and for all real numbers $s_1, \ldots, s_n$, it holds $\max\{G(I) \mid I \in \mathcal{I}_\Phi,\ I \models L,\ I \models \phi_1 \vee \cdots \vee \phi_n\} \geq 0$.

An *imprecise probability assessment* (L, A) on a set of conditional events $\mathcal{E}$ consists of a set of logical constraints L and a mapping A that assigns each $\varepsilon \in \mathcal{E}$ an interval $[l, u] \subseteq [0, 1]$ with $l \leq u$. We say (L, A) is *g-coherent* iff there exists a coherent precise probability assessment $(L, A^\star)$ on $\mathcal{E}$ such that $A^\star(\varepsilon) \in A(\varepsilon)$ for all $\varepsilon \in \mathcal{E}$.

Let (L, A) be a g-coherent imprecise probability assessment on a set of conditional events $\mathcal{E}$. The imprecise probability assessment $[l, u]$ on a conditional event γ is called a *g-coherent consequence* of (L, A) iff $A^\star(\gamma) \in [l, u]$ for every g-coherent precise probability assessment $A^\star$ on $\mathcal{E} \cup \{\gamma\}$ such that $A^\star(\varepsilon) \in A(\varepsilon)$ for all $\varepsilon \in \mathcal{E}$. It is a *tight g-coherent consequence* of (L, A) iff l (resp., u) is the infimum (resp., supremum) of $A^\star(\gamma)$ subject to all g-coherent precise probability assessments $A^\star$ on $\mathcal{E} \cup \{\gamma\}$ such that $A^\star(\varepsilon) \in A(\varepsilon)$ for all $\varepsilon \in \mathcal{E}$.

2.3 Probabilistic Logic under Coherence

In the rest of this paper, we assume that Φ is finite. A *conditional constraint* is an expression $(\psi|\phi)[l, u]$ with real numbers $l, u \in [0, 1]$ and events ψ and ϕ. A probabilistic knowledge base $KB = (L, P)$ consists of a finite set of logical constraints L, and a finite set of conditional constraints P such that (i) $l \leq u$ for all $(\psi|\phi)[l, u] \in P$, and (ii) $\psi_1|\phi_1 \neq \psi_2|\phi_2$ for all distinct $(\psi_1|\phi_1)[l_1, u_1]$, $(\psi_2|\phi_2)[l_2, u_2] \in P$.

Every imprecise probability assessment $IP = (L, A)$ with finite L on a finite set of conditional events $\mathcal{E}$ can be represented by the following probabilistic knowledge base:

$$KB_{IP} = (L, \{(\psi|\phi)[l, u] \mid \psi|\phi \in \mathcal{E},\ A(\psi|\phi) = [l, u]\})\,.$$

Conversely, every probabilistic knowledge base $KB = (L, P)$ can expressed by the following imprecise probability assessment $IP_{KB} = (L, A_{KB})$ on $\mathcal{E}_{KB}$:

$$\begin{aligned} A_{KB} &= \{(\psi|\phi, [l, u]) \mid (\psi|\phi)[l, u] \in KB\}\,, \\ \mathcal{E}_{KB} &= \{\psi|\phi \mid \exists\, l, u \in [0, 1] \colon (\psi|\phi)[l, u] \in KB\}\,. \end{aligned}$$

A probabilistic knowledge base KB is said *g-coherent* iff IP_{KB} is g-coherent. For g-coherent KB and conditional constraints $(\psi|\phi)[l, u]$, we say $(\psi|\phi)[l, u]$ is a *g-coherent consequence* of KB, denoted $KB \mathrel{|\!\sim} (\psi|\phi)[l, u]$, iff $\{(\psi|\phi, [l, u])\}$ is a g-coherent consequence of IP_{KB}. It is a *tight g-coherent consequence* of KB, denoted $KB \mathrel{|\!\sim}_{tight} (\psi|\phi)[l, u]$, iff $\{(\psi|\phi, [l, u])\}$ is a tight g-coherent consequence of IP_{KB}.

Example 2.1. The logical knowledge "all penguins are birds" and the probabilistic knowledge "birds have legs with a probability of at least 0.95", "birds fly with a probability between 0.9 and 0.95", and "penguins fly with a probability of at most 0.05" can be expressed by the following probabilistic knowledge base $KB = (\{\mathsf{bird} \Leftarrow \mathsf{penguin}\}$, $\{(\mathsf{legs}|\mathsf{bird})[.95, 1], (\mathsf{fly}|\mathsf{bird})[.9, .95], (\mathsf{fly}|\mathsf{penguin})[0, .05]\})$.

It is easy to see that KB is g-coherent and that $(\mathsf{legs}|\mathsf{bird})[.95,1]$, $(\mathsf{legs}|\mathsf{penguin})[0,1]$, $(\mathsf{fly}|\mathsf{bird})[.9, .95]$, and $(\mathsf{fly}|\mathsf{penguin})[0, .05]$ are tight g-coherent consequences of KB.

3 Relationship to Model-Theoretic Probabilistic Logic

In this section, we characterize the notions of g-coherence and of g-coherent entailment in terms of the notions of satisfiability and of logical entailment.

3.1 Model-Theoretic Probabilistic Logic

A *probabilistic interpretation* Pr is a probability function on $\mathcal{I}_\Phi$ (that is, a mapping $Pr\colon \mathcal{I}_\Phi \to [0,1]$ such that all $Pr(I)$ with $I \in \mathcal{I}_\Phi$ sum up to 1). The *probability* of an event ϕ in the probabilistic interpretation Pr, denoted $Pr(\phi)$, is defined as the sum of all $Pr(I)$ such that $I \in \mathcal{I}_\Phi$ and $I \models \phi$. For events ϕ and ψ with $Pr(\phi) > 0$, we use $Pr(\psi|\phi)$ to abbreviate $Pr(\psi \wedge \phi)\ /\ Pr(\phi)$. The *truth* of logical and conditional constraints F in a probabilistic interpretation Pr, denoted $Pr \models F$, is defined as follows:

- $Pr \models \psi \Leftarrow \phi$ iff $Pr(\psi \wedge \phi) = Pr(\phi)$.
- $Pr \models (\psi|\phi)[l,u]$ iff $Pr(\phi) = 0$ or $Pr(\psi|\phi) \in [l,u]$.

We say Pr *satisfies* a logical or conditional constraint F, or Pr is a *model* of F, iff $Pr \models F$. We say Pr *satisfies* a set of logical and conditional constraints $\mathcal{F}$, or Pr is a *model* of $\mathcal{F}$, denoted $Pr \models \mathcal{F}$, iff Pr is a model of all $F \in \mathcal{F}$. We say that $\mathcal{F}$ is *satisfiable* iff a model of $\mathcal{F}$ exists.

We next define the notion of logical entailment. A conditional constraint $F = (\psi|\phi)[l,u]$ is a *logical consequence* of a set of logical and conditional constraints $\mathcal{F}$, denoted $\mathcal{F} \models F$, iff each model of $\mathcal{F}$ is also a model of F. It is a *tight logical consequence* of $\mathcal{F}$, denoted $\mathcal{F} \models_{tight} F$, iff l (resp., u) is the infimum (resp., supremum) of $Pr(\psi|\phi)$ subject to all models Pr of $\mathcal{F}$ with $Pr(\phi) > 0$. Note that we define $l = 1$ and $u = 0$, when $\mathcal{F} \models (\phi|\top)[0,0]$. A probabilistic knowledge bases $KB = (L,P)$ is *satisfiable* iff $L \cup P$ is satisfiable. A conditional constraint $(\psi|\phi)[l,u]$ is a *logical consequence* of KB, denoted $KB \models (\psi|\phi)[l,u]$, iff $L \cup P \models (\psi|\phi)[l,u]$. It is a *tight logical consequence* of KB, denoted $KB \models_{tight} (\psi|\phi)[l,u]$, iff $L \cup P \models_{tight} (\psi|\phi)[l,u]$.

3.2 G-Coherence in Model-Theoretic Probabilistic Logic

The following theorem shows how g-coherence can be expressed through the existence of probabilistic interpretations. This result follows from a characterization of g-coherence in [15]. It shows that $KB = (L,P)$ is g-coherent iff every nonempty $P' \subseteq P$ has a model Pr such that $Pr \models L$ and that $Pr(\phi) > 0$ for at least one $(\psi|\phi)[l,u] \in P'$.

Theorem 3.1. *Let $KB = (L,P)$ be a probabilistic knowledge base. Then, KB is g-coherent iff for every nonempty $P_n = \{(\psi_1|\phi_1)[l_1,u_1], \ldots, (\psi_n|\phi_n)[l_n,u_n]\} \subseteq P$, there exists a model Pr of $L \cup P_n$ such that $Pr(\phi_1 \vee \cdots \vee \phi_n) > 0$.*

It then follows that g-coherence has a characterization similar to p-consistency in default reasoning. To formulate this result, we adopt the following terminology from default reasoning from conditional knowledge bases [3]. A probabilistic interpretation Pr *verifies* a conditional constraint $(\psi|\phi)[l,u]$, iff $Pr(\phi) > 0$ and $Pr \models (\psi|\phi)[l,u]$. A set of conditional constraints P *tolerates* a conditional constraint F under a set of logical constraints L, iff there exists a model of $L \cup P$ that verifies F. We say P is *under L in conflict* with F, iff no model of $L \cup P$ verifies F.

We are now ready to characterize g-coherence in a way similar to p-consistency by Goldszmidt and Pearl [17]. Note that in [7] we use this characterization to provide a new algorithm for deciding g-coherence, which is essentially a reformulation of a previous algorithm by Gilio [15] using terminology from default reasoning, and which is closely related to an algorithm for checking p-consistency given in [17].[1]

Theorem 3.2. *A probabilistic knowledge base $KB = (L, P)$ is g-coherent iff there exists an ordered partition $(P_0, \ldots, P_k)$ of P such that either*

(a) every P_i, $0 \leq i \leq k$, is the set of all $F \in \bigcup_{j=i}^{k} P_j$ tolerated under L by $\bigcup_{j=i}^{k} P_j$, or
(b) for every i, $0 \leq i \leq k$, each $F \in P_i$ is tolerated under L by $\bigcup_{j=i}^{k} P_j$.

3.3 G-Coherent Entailment in Model-Theoretic Probabilistic Logic

We next show how g-coherent entailment can be reduced to logical entailment.

For probabilistic knowledge bases $KB = (L, P)$ and events α, let $P_\alpha(KB)$ denote the set of all subsets $P_n = \{(\psi_1|\phi_1)[l_1, u_1], \ldots, (\psi_n|\phi_n)[l_n, u_n]\}$ of P such that every model Pr of $L \cup P_n$ with $Pr(\phi_1 \vee \cdots \vee \phi_n \vee \alpha) > 0$ satisfies $Pr(\alpha) > 0$.

The following theorem shows that the tight interval concluded under coherence can be expressed as the intersection of some logically entailed tight intervals.

Theorem 3.3. *Let $KB = (L, P)$ be a g-coherent probabilistic knowledge base, and let $\beta|\alpha$ be a conditional event. Then, $KB \mathrel{|\!\!\sim}_{tight} (\beta|\alpha)[l, u]$, where*

$$[l, u] = \bigcap \{[c, d] \mid L \cup P' \models_{tight} (\beta|\alpha)[c, d] \text{ for some } P' \in P_\alpha(KB)\}\,.$$

Clearly, this reduction of g-coherent entailment to logical entailment is computationally expensive, as we have to compute a tight logically entailed interval for each member of $P_\alpha(KB)$. In the following, we show that we can restrict our attention to the unique greatest element in $P_\alpha(KB)$. The following lemma shows that $P_\alpha(KB)$ contains indeed a unique greatest element with respect to set inclusion. This result can be proved by showing that $P_\alpha(KB)$ is nonempty and closed under set union.

Lemma 3.4. *Let $KB = (L, P)$ be a g-coherent probabilistic knowledge base, and let α be an event. Then, $P_\alpha(KB)$ contains a unique greatest element.*

The next theorem now shows the crucial result that *g-coherent entailment from KB can be reduced to logical entailment from the greatest element in $P_\alpha(KB)$.*

Theorem 3.5. *Let $KB = (L, P)$ be a g-coherent probabilistic knowledge base, and let $F = (\beta|\alpha)[l, u]$ be a conditional constraint. Let $KB^\star = (L, P^\star)$, where $P^\star$ is the greatest element in $P_\alpha(KB)$. Then,*

(a) $KB \mathrel{|\!\!\sim} F$ iff $KB^\star \models F$.
(b) $KB \mathrel{|\!\!\sim}_{tight} F$ iff $KB^\star \models_{tight} F$.

[1] Note that the relationship between the algorithms in [15] and [17] was suggested first by Didier Dubois (personal communication).

Thus, computing tight g-coherent consequences can be reduced to computing tight logical consequences from the greatest element $P^\star$ in $P_\alpha(KB)$. The following theorem shows how $P^\star$ can be characterized and thus computed. More precisely, it specifies some $P^\star$ by two conditions (i) and (ii). It can be shown that (i) implies that every member of $P_\alpha(KB)$ is a subset of $P^\star$, and that (ii) implies that $P^\star$ belongs to $P_\alpha(KB)$. In summary, this proves that the specified $P^\star$ is the greatest element in $P_\alpha(KB)$.

Theorem 3.6. *Let $KB = (L, P)$ be a g-coherent probabilistic knowledge base and α be an event. Let $P^\star \subseteq P$ and $(P_0, \ldots, P_k)$ be an ordered partition of $P \backslash P^\star$ such that:*

(i) every P_i, $0 \leq i \leq k$, is the set of all elements in $P_i \cup \cdots \cup P_k \cup P^\star$ that are tolerated under $L \cup \{\bot \Leftarrow \alpha\}$ by $P_i \cup \cdots \cup P_k \cup P^\star$, and
(ii) no member of $P^\star$ is tolerated under $L \cup \{\bot \Leftarrow \alpha\}$ by $P^\star$.

Then, $P^\star$ is the greatest element in $P_\alpha(KB)$.

In summary, by Theorems 3.5 and 3.6, a tight interval under g-coherent entailment can be computed by first checking g-coherence, and then computing a tight interval under logical entailment [7]. Semantically, Theorems 3.5 and 3.6 show that g-coherent entailment coincides with logical entailment from a smaller knowledge base. That is, under g-coherent entailment, we simply cut away a part of the knowledge base. Roughly speaking, we remove all those conditional constraints $(\psi|\phi)[l, u] \in P$ where ϕ is "larger" than α. Intuitively, g-coherent entailment does not have the property of inheritance, neither for logical knowledge nor for probabilistic knowledge, while logical entailment shows inheritance of logical knowledge but not of probabilistic knowledge. The following example illustrates this difference.

Example 3.7. Consider the following probabilistic knowledge base:

$$KB = (\{\mathsf{bird} \Leftarrow \mathsf{penguin}\},\ \{(\mathsf{legs}|\mathsf{bird})[1,1],\ (\mathsf{wings}|\mathsf{bird})[.95,1]\})\,.$$

Notice that KB is g-coherent and satisfiable. Moreover, we have:

$$KB \mathrel{|\!\!\sim}_{tight} (\mathsf{legs}|\mathsf{penguin})[0,1] \text{ and } KB \models_{tight} (\mathsf{legs}|\mathsf{penguin})[1,1]\,,$$
$$KB \mathrel{|\!\!\sim}_{tight} (\mathsf{wings}|\mathsf{penguin})[0,1] \text{ and } KB \models_{tight} (\mathsf{wings}|\mathsf{penguin})[0,1]\,.$$

That is, under g-coherent entailment, neither the logical property of having legs nor the probabilistic one of having wings is inherited from birds to penguins. Under logical entailment, however, the logical property is inherited, while the probabilistic one is not.

3.4 Coherence-Based versus Model-Theoretic Probabilistic Logic

We now describe the rough relationship between g-coherence and satisfiability, and between g-coherent entailment and logical entailment. The following theorem shows that g-coherence implies satisfiability. This result is immediate by Theorem 3.1.

Theorem 3.8. *Every g-coherent probabilistic knowledge base KB is satisfiable.*

In fact, g-coherence is strictly stronger than satisfiability, as the next example shows.

Example 3.9. Consider the probabilistic knowledge base $KB = (\emptyset, \{(\mathsf{fly}|\mathsf{bird})[.9, 1],$ $(\neg\mathsf{fly}|\mathsf{bird})[.2, 1]\})$. It is easy to verify that KB is satisfiable, but not g-coherent.

The next theorem shows that logical entailment is stronger than g-coherent entailment. That is, g-coherent consequence implies logical consequence (or there are more conditional constraints logically entailed than entailed under g-coherence) and the tight intervals that are derived under logical entailment are subintervals of those derived under g-coherent entailment. This result follows immediately from Theorem 3.5.

Theorem 3.10. *Let $KB = (L, P)$ be a g-coherent probabilistic knowledge base, and let $(\beta|\alpha)[l, u]$ and $(\beta|\alpha)[r, s]$ be two conditional constraints. Then,*

(a) $KB \mathrel{|\!\sim} (\beta|\alpha)[l, u]$ *implies* $KB \models (\beta|\alpha)[l, u]$.
(b) $KB \mathrel{|\!\sim}_{tight} (\beta|\alpha)[l, u]$ *and* $KB \models_{tight} (\beta|\alpha)[r, s]$ *implies* $[l, u] \supseteq [r, s]$.

The following example now shows that logical entailment is in fact *strictly* stronger than g-coherent entailment (note that we identify $[1, 0]$ with the empty set).

Example 3.11. Consider the following probabilistic knowledge bases KB_1 and KB_2:

$$KB_1 = (\emptyset, \{(\mathsf{fly}|\mathsf{bird})[1,1], (\mathsf{mobile}|\mathsf{fly})[1,1]\}),$$
$$KB_2 = (\emptyset, \{(\mathsf{fly}|\mathsf{bird})[1,1], (\mathsf{bird}|\mathsf{penguin})[1,1], (\neg\mathsf{fly}|\mathsf{penguin})[1,1]\}).$$

Some tight g-coherent and tight logical consequences of KB_1 and KB_2 are given by:

$$KB_1 \mathrel{|\!\sim}_{tight} (\mathsf{mobile}|\mathsf{bird})[0,1] \text{ and } KB_1 \models_{tight} (\mathsf{mobile}|\mathsf{bird})[1,1],$$
$$KB_2 \mathrel{|\!\sim}_{tight} (\neg\mathsf{fly}|\mathsf{penguin})[1,1] \text{ and } KB_2 \models_{tight} (\neg\mathsf{fly}|\mathsf{penguin})[1,0].$$

4 Relationship to Default Reasoning in System P

In this section, we show that consistency and entailment in system P are special cases of g-coherence and of g-coherent entailment, respectively. That is, probabilistic logic under coherence gives a new probabilistic semantics for system P, which is neither based on infinitesimal probabilities nor on atomic-bound (or also big-stepped) probabilities.

4.1 Default Reasoning in System P

We now describe the notions of consistency and of entailment in system P [18]. We define them in terms of world rankings.

A *conditional rule* (or *default*) is an expression of the form $\psi \leftarrow \phi$, where ϕ and ψ are events. A *conditional knowledge base* $KB = (L, D)$ consists of a finite set of logical constraints L and a finite set of defaults D.

A world I *satisfies* a default $\psi \leftarrow \phi$, or I is a *model* of $\psi \leftarrow \phi$, denoted $I \models \psi \leftarrow \phi$, iff $I \models \psi \Leftarrow \phi$. The world I *verifies* $\psi \leftarrow \phi$ iff $I \models \phi \wedge \psi$. The world I *falsifies* $\psi \leftarrow \phi$ iff $I \models \phi \wedge \neg\psi$ (that is, $I \not\models \psi \leftarrow \phi$). I *satisfies* a set of events and defaults K, or I is a *model* of K, denoted $I \models K$, iff I satisfies every member of K. We say K is *satisfiable* iff a model of K exists. A set of defaults D *tolerates* a default d *under* a set of classical formulas L iff $D \cup L$ has a model that verifies d. A set of defaults D is *under* L *in conflict* with a default $\psi \leftarrow \phi$ iff all models of $D \cup L \cup \{\phi\}$ satisfy $\neg\psi$.

A *world ranking* κ is a mapping $\kappa\colon \mathcal{I}_\Phi \to \{0, 1, \ldots\} \cup \{\infty\}$ such that $\kappa(I) = 0$ for at least one world I. It is extended to all events ϕ as follows. If ϕ is satisfiable, then $\kappa(\phi) = \min\{\kappa(I) \mid I \in \mathcal{I}_\Phi,\, I \models \phi\}$; otherwise, $\kappa(\phi) = \infty$. A world ranking κ

is *admissible* with a conditional knowledge base (L, D) iff $\kappa(\neg\phi) = \infty$ for all $\phi \in L$, and $\kappa(\phi) < \infty$ and $\kappa(\phi \wedge \psi) < \kappa(\phi \wedge \neg\psi)$ for all defaults $\psi \leftarrow \phi \in D$.

A conditional knowledge base KB is *p-consistent* iff there exists a world ranking that is admissible with KB. It is *p-inconsistent* iff no such a world ranking exists. We say KB *p-entails* a default $\psi \leftarrow \phi$ iff either $\kappa(\phi) = \infty$ (that is, ϕ is unsatisfiable) or $\kappa(\phi \wedge \psi) < \kappa(\phi \wedge \neg\psi)$ for all world rankings κ that are admissible with KB.

A *default ranking* σ on $KB = (L, D)$ maps each $d \in D$ to a nonnegative integer. It is *admissible* with KB iff each $D' \subseteq D$ that is under L in conflict with some $d \in D$ contains a default d' such that $\sigma(d') < \sigma(d)$.

4.2 G-Coherence and P-Consistency

We now show that g-coherence is a generalization of p-consistency.

Recall first that the characterization of p-consistency by Goldszmidt and Pearl [17] corresponds to the characterization of g-coherence given in Theorem 3.2.

The following well-known result (see especially [14]) shows that p-consistency is equivalent to the existence of admissible default rankings.

Theorem 4.1. *A conditional knowledge base KB is p-consistent iff there exists a default ranking on KB that is admissible with KB.*

A similar result holds for g-coherence, which is subsequently formulated using the following concepts. A *ranking* σ on $KB = (L, P)$ maps each element of P to a nonnegative integer. It is *admissible* with KB iff each $P' \subseteq P$ that is under L in conflict with some $F \in P$ contains a conditional constraint F' such that $\sigma(F') < \sigma(F)$.

Theorem 4.2. *A probabilistic knowledge base KB is g-coherent iff there exists a ranking on KB that is admissible with KB.*

The following theorem finally shows the important result that g-coherence is a generalization of p-consistency.

Theorem 4.3. *Let $KB = (L, \{(\psi_1|\phi_1)[1,1], \ldots, (\psi_n|\phi_n)[1,1]\})$ be a probabilistic knowledge base. Then, KB is g-coherent iff the conditional knowledge base $KB' = (L, \{\psi_1 \leftarrow \phi_1, \ldots, \psi_n \leftarrow \phi_n\})$ is p-consistent.*

4.3 G-Coherent Entailment and P-Entailment

We now show that g-coherent entailment is a generalization of p-entailment.

The following result is essentially due to Adams [1], who formulated it for $L = \emptyset$.

Theorem 4.4 (Adams [1]). *A conditional knowledge base $KB = (L, D)$ p-entails a default $\beta \leftarrow \alpha$ iff $(L, D \cup \{\neg\beta \leftarrow \alpha\})$ is p-inconsistent.*

The following theorem shows that a similar result holds for g-coherent consequence, which is an immediate implication of the definition of g-coherent entailment.

Theorem 4.5. *Let $KB = (L, P)$ be a g-coherent probabilistic knowledge base, and let $(\beta|\alpha)[l, u]$ be a conditional constraint. Then, $KB \mid\!\sim (\beta|\alpha)[l, u]$ iff $(L, P \cup \{(\beta|\alpha)[p, p]\})$ is not g-coherent for all $p \in [0, l) \cup (u, 1]$.*

The following related result for tight g-coherent consequence completes the picture.

Theorem 4.6. *Let $KB = (L, P)$ be a g-coherent probabilistic knowledge base, and let $(\beta|\alpha)[l, u]$ be a conditional constraint. Then, $KB \mathrel{|\!\sim}_{tight} (\beta|\alpha)[l, u]$ iff*

(i) $(L, P \cup \{(\beta|\alpha)[p, p]\})$ *is not g-coherent for all* $p \in [0, l) \cup (u, 1]$, *and*

(ii) $(L, P \cup \{(\beta|\alpha)[p, p]\})$ *is g-coherent for all* $p \in [l, u]$.

The next result finally shows that g-coherent entailment generalizes p-entailment.

Theorem 4.7. *Let $KB = (L, \{(\psi_1|\phi_1)[1, 1], \ldots, (\psi_n|\phi_n)[1, 1]\})$ be a g-coherent probabilistic knowledge base. Then, $KB \mathrel{|\!\sim} (\beta|\alpha)[1, 1]$ iff the conditional knowledge base $(L, \{\psi_1 \leftarrow \phi_1, \ldots, \psi_n \leftarrow \phi_n\})$ p-entails $\beta \leftarrow \alpha$.*

5 Relationship to Default Reasoning with Conditional Objects

In this section, we relate coherence-based probabilistic reasoning to default reasoning with conditional objects, which goes back to Dubois and Prade [11, 3].

We associate with each set of defaults $D = \{\psi_1 \leftarrow \phi_1, \ldots, \psi_n \leftarrow \phi_n\}$, the set of conditional events $C_D = \{\psi_1|\phi_1, \ldots, \psi_n|\phi_n\}$. Given a nonempty set of conditional events $\mathcal{E} = \{\psi_1|\phi_1, \ldots, \psi_n|\phi_n\}$, the *quasi-conjunction* of $\mathcal{E}$, denoted $QC(\mathcal{E})$, is defined as the conditional event $(\psi_1 \Leftarrow \phi_1) \wedge \cdots \wedge (\psi_n \Leftarrow \phi_n) \mid \phi_1 \vee \cdots \vee \phi_n$.

We now define the notions of co-consistency and co-entailment as follows. A conditional knowledge base $KB = (L, D)$ is *co-consistent* iff, for every nonempty $D' \subseteq D$, there exists a model I of L such that $I(QC(C_{D'})) = \mathbf{true}$. We assume the total order $\mathbf{false} < \mathbf{indeterminate} < \mathbf{true}$. We say $KB = (L, D)$ *co-entails* a default $\beta \leftarrow \alpha$ iff either (i) $L \cup \{\alpha\} \models \beta$, or (ii) some nonempty $D' \subseteq D$ exists such that $I(QC(C_{D'})) \leq I(\beta|\alpha)$ for all models I of L.

The notions of co-consistency and co-entailment coincide with the notions of p-consistency and p-entailment, respectively [11, 3]. We now show that our results in Sections 3 and 4 are naturally related to default reasoning with conditional objects.

It is easy to verify that the following counterpart of Theorem 3.1 for p-consistency formulates the above notion of co-consistency. Note that the notion of satisfiability used in this theorem is defined as in Section 4.1.

Theorem 5.1. *A conditional knowledge base $KB = (L, D)$ is p-consistent iff $L \cup D' \cup \{\phi_1 \vee \cdots \vee \phi_n\}$ is satisfiable for every nonempty $D' = \{\psi_1 \leftarrow \phi_1, \ldots, \psi_n \leftarrow \phi_n\} \subseteq D$.*

For conditional knowledge bases $KB = (L, D)$ and events α, let $D_\alpha(KB)$ be the set of all $D' = \{\psi_1 \leftarrow \phi_1, \ldots, \psi_n \leftarrow \phi_n\} \subseteq D$ such that $L \cup D' \cup \{\phi_1 \vee \cdots \vee \phi_n \vee \alpha\} \models \alpha$. Observe now that for $D' = \{\psi_1 \leftarrow \phi_1, \ldots, \psi_n \leftarrow \phi_n\}$, condition (ii) in the definition of the notion of co-entailment is equivalent to $L \cup D' \cup \{\phi_1 \vee \cdots \vee \phi_n \vee \alpha\} \models \alpha$ and $L \cup D' \models \beta \Leftarrow \alpha$. Thus, the following counterpart of Theorem 3.3 for p-entailment formulates the above notion of co-entailment.

Theorem 5.2. *Let $KB = (L, D)$ be a p-consistent conditional knowledge base. Then, KB p-entails the default $\beta \leftarrow \alpha$, iff $L \cup D' \models \beta \Leftarrow \alpha$ for some $D' \in D_\alpha(KB)$.*

Crucially, we can now also formulate counterparts to Lemma 3.4 and Theorems 3.5 and 3.6. To our knowledge, these results for system P are unknown so far. The following result shows that $D_\alpha(KB)$ contains a unique greatest element.

Lemma 5.3. *Let $KB = (L, D)$ be a p-consistent conditional knowledge base, and let α be an event. Then, $D_\alpha(KB)$ contains a unique greatest element.*

The next result shows that p-entailment from KB coincides with logical entailment from the greatest element in $D_\alpha(KB)$. That is, we can replace item (ii) in the definition of *co*-entailment by (ii') $I(QC(C_{D^\star})) \leq I(\beta|\alpha)$ for the greatest $D^\star$ in $D_\alpha(KB)$.

Theorem 5.4. *Let $KB = (L, D)$ be a p-consistent conditional knowledge base, and let $\beta \leftarrow \alpha$ be a default. Let $D^\star$ denote the unique greatest element in $D_\alpha(KB)$. Then,*

$$KB \text{ } p\text{-entails } \beta \leftarrow \alpha \text{ iff } L \cup D^\star \models \beta \Leftarrow \alpha.$$

The following theorem shows how $D^\star$ can be characterized and thus computed.

Theorem 5.5. *Let $KB = (L, D)$ be a p-consistent conditional knowledge base and α be an event. Let $D^\star \subseteq D$ and $(D_0, \ldots, D_k)$ be an ordered partition of $D \setminus D^\star$ such that:*

(i) every D_i, $0 \leq i \leq k$, is the set of all elements in $D_i \cup \cdots \cup D_k \cup D^\star$ that are tolerated under $L \cup \{\bot \Leftarrow \alpha\}$ by $D_i \cup \cdots \cup D_k \cup D^\star$, and
(ii) no member of $D^\star$ is tolerated under $L \cup \{\bot \Leftarrow \alpha\}$ by $D^\star$.

Then, $D^\star$ is the greatest element in $D_\alpha(KB)$.

6 Summary and Outlook

We explored the relationship between probabilistic logic under coherence, model-theoretic probabilistic logic, and default reasoning in system P. We showed that coherence-based probabilistic reasoning can be reduced to model-theoretic probabilistic reasoning by using concepts from default reasoning. Moreover, we showed that it is a probabilistic generalization of default reasoning in system P. That is, we gave a new probabilistic semantics for system P, which is neither based on infinitesimal probabilities nor on atomic-bound (or also big-stepped) probabilities. We finally showed that these results also give new insight into default reasoning with conditional objects.

Roughly speaking, the main difference between coherence-based and model-theoretic probabilistic reasoning is that the former generalizes default reasoning in system P, while the latter generalizes classical reasoning in propositional logic.

A very interesting topic of future research is to explore how other notions of coherence are related to model-theoretic probabilistic logic and to default reasoning. It would also be very interesting to develop coherence-based probabilistic extensions of notions of default reasoning different from system P (for example, in the spirit of [21]).

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References

1. E. W. Adams. *The Logic of Conditionals*, volume 86 of *Synthese Library*. D. Reidel, Dordrecht, Netherlands, 1975.

2. S. Amarger, D. Dubois, and H. Prade. Constraint propagation with imprecise conditional probabilities. In *Proceedings UAI-91*, pp. 26–34. 1991.
3. S. Benferhat, D. Dubois, and H. Prade. Nonmonotonic reasoning, conditional objects and possibility theory. *Artif. Intell.*, 92(1–2):259–276, 1997.
4. S. Benferhat, D. Dubois, and H. Prade. Possibilistic and standard probabilistic semantics of conditional knowledge bases. *J. Logic Computat.*, 9(6):873–895, 1999.
5. V. Biazzo and A. Gilio. A generalization of the fundamental theorem of de Finetti for imprecise conditional probability assessments. *Int. J. Approx. Reasoning*, 24:251–272, 2000.
6. V. Biazzo, A. Gilio, T. Lukasiewicz, and G. Sanfilippo. Probabilistic logic under coherence, model-theoretic probabilistic logic, and default reasoning. Technical Report INFSYS RR-1843-01-03, Institut für Informationssysteme, TU Wien, 2001.
7. V. Biazzo, A. Gilio, T. Lukasiewicz, and G. Sanfilippo. Probabilistic logic under coherence: Complexity and algorithms. In *Proceedings ISIPTA-01*, Ithaca, New York, USA, June 26–29, 2001. To appear.
8. G. Boole. *An Investigation of the Laws of Thought, on which are Founded the Mathematical Theories of Logic and Probabilities*. Walton and Maberley, London, 1854. (Reprint: Dover Publications, New York, 1958).
9. G. Coletti and R. Scozzafava. Conditioning and inference in intelligent systems. *Soft Computing*, 3(3):118–130, 1999.
10. D. Dubois and H. Prade. Possibilistic logic, preferential models, non-monotonicity and related issues. In *Proceedings IJCAI-91*, pp. 419–424. 1991.
11. D. Dubois and H. Prade. Conditional objects as nonmonotonic consequence relationships. *IEEE Trans. Syst. Man Cybern.*, 24(12):1724–1740, 1994.
12. R. Fagin, J. Y. Halpern, and N. Megiddo. A logic for reasoning about probabilities. *Inf. Comput.*, 87:78–128, 1990.
13. A. M. Frisch and P. Haddawy. Anytime deduction for probabilistic logic. *Artif. Intell.*, 69:93–122, 1994.
14. H. Geffner. *Default Reasoning: Causal and Conditional Theories*. MIT Press, 1992.
15. A. Gilio. Probabilistic consistency of conditional probability bounds. In *Advances in Intelligent Computing, LNCS* 945, pp. 200–209. Springer, 1995.
16. A. Gilio. Precise propagation of upper and lower probability bounds in system P. In *Proceedings of the 8th International Workshop on Non-monotonic Reasoning*, 2000.
17. M. Goldszmidt and J. Pearl. On the consistency of defeasible databases. *Artif. Intell.*, 52(2):121–149, 1991.
18. S. Kraus, D. Lehmann, and M. Magidor. Nonmonotonic reasoning, preferential models and cumulative logics. *Artif. Intell.*, 14(1):167–207, 1990.
19. T. Lukasiewicz. Local probabilistic deduction from taxonomic and probabilistic knowledge-bases over conjunctive events. *Int. J. Approx. Reasoning*, 21(1):23–61, 1999.
20. T. Lukasiewicz. Probabilistic deduction with conditional constraints over basic events. *J. Artif. Intell. Res.*, 10:199–241, 1999.
21. T. Lukasiewicz. Probabilistic default reasoning with conditional constraints. In *Proceedings of the 8th International Workshop on Non-monotonic Reasoning*, 2000.
22. T. Lukasiewicz. Probabilistic logic programming with conditional constraints. *ACM Trans. Computat. Logic*, 2(3):289–337, July 2001. To appear.
23. N. J. Nilsson. Probabilistic logic. *Artif. Intell.*, 28:71–88, 1986.
24. J. Pearl. Probabilistic semantics for nonmonotonic reasoning: A survey. In *Proceedings KR-89*, pp. 505–516. 1989.
25. Y. Shoham. A semantical approach to nonmonotonic logics. In *Proceedings of the 2nd IEEE Symposium on Logic in Computer Science*, pp. 275–279, 1987.
26. P. Snow. Diverse confidence levels in a probabilistic semantics for conditional logics. *Artif. Intell.*, 113:269–279, 1999.