A Splitting Theorem for the Medvedev and Muchnik Lattices.

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Abstract

This is a contribution to the study of the Muchnik and Medvedev lattices of non-empty Π^0_1 subsets of 2^ω. In both these lattices, any non-minimum element can be split, i.e. it is the non-trivial join of two other elements. In fact, in the Medvedev case, if $P >_M Q$, then P can be split above Q. Both of these facts are then generalised to the embedding of arbitrary finite distributive lattices. A consequence of this is that both lattices have decidible $\exists$-theories.

1 Introduction

After the concept of Turing reducibility between subsets of ω has been encountered, it is natural to try to extend this idea to subsets of ω^ω. Perhaps the two most obvious such extensions are the notions of Medvedev and Muchnik reducibility. Let X and Y be subsets of ω^ω. X is said to be *Muchnik reducible to* Y (denoted $Y \geqslant_w X$) if, for every $f \in Y$, there is a $g \in X$ such that $f \geqslant_T g$. Medvedev reducibility is the uniform version: X is *Medvedev reducible to* Y ($Y \geqslant_M X$) if there is a recursive functional mapping Y into X. In [8] §13.7, Rogers discusses Medvedev reduciblity in terms of *mass problems* (subsets of ω^ω representing solutions to "problems"). We write $P \equiv_w Q$ if and only if $P \geqslant_w Q$ and $Q \geqslant_w P$ and similarly for $\equiv_M$.

This work owes a lot to discussions with Stephen Simpson and to the referee for suggestions.

Both these reducibilities are pre-orders on the class of subsets of ω^ω and degree structures are induced in the same way as the r.e degrees, that is,

$$\deg_w(X) = \{Y : Y \equiv_w X\}$$

and similarly for $\deg_M(X)$. A canonical partial order on the degrees is then defined by

$$\deg_w(X) \geqslant \deg_w(Y) \text{ if and only if } X \geqslant_w Y$$

(likewise for the Medvedev degrees).

Recently, it has been suggested by Simpson ([4] Aug 13 1999) that the class of non-empty Π^0_1 subsets of 2^ω under Medvedev and Muchnik reducibility is a natural object of study - the comparison being made to the r.e. degrees under Turing reducibility. Rogers has also suggested something similar ([8] §15.1 pg 343). The idea has been investigated by Cenzer and Hinman [3] as well as Binns and Simpson [2], [1], [11] and Slaman [12]. This paper should be seen as a continuation of this project. Including this Introduction, it has three sections. The second concerns the structure of the Medvedev and Muchnik lattices - proving splitting and embedding theorems in both. The third proves a model-theoretic consequence of these theorems - namely the decidibility of the $\exists$-theories of the lattices. This result can be stated in more generality as it will be true of any distributive lattice with a maximum and minimum element with the embedding property of Theorem 9 and a non-branching minimum.

Let $\mathcal{P}_M$ and $\mathcal{P}_w$ denote the degree structures of the non-empty Π^0_1 subsets of 2^ω under Medvedev and Muchnik reducibility respectively. $\mathcal{P}_M$ and $\mathcal{P}_w$ form distributive lattices with maximum and minimum elements. If P and Q are non-empty Π^0_1 subsets of 2^ω, the join and meet of their degrees in both of these lattices are the respective degrees of:

$$P \vee Q = \{f \oplus g : f \in P \text{ and } g \in Q\},$$

and,

$$P \wedge Q = \{0^\frown f : f \in P\} \cup \{1^\frown f : f \in Q\},$$

where,

$$i^\frown f(n) = \begin{cases} i & \text{if } n = 0, \\ f(n-1) & \text{otherwise,} \end{cases}$$

The operations $\vee$ and $\wedge$ are applicable to any pairs of subsets of ω^ω and have been referred to elsewhere as $\times$ and $+$ respectively [1] [2].

If A and B are any two subsets of ω, then the *separating class* of A and B, denoted $\mathcal{S}(A,B)$, is the set $\{X : X \supseteq A, \text{ and } X \cap B = \emptyset\}$. If A and B are r.e. then $\mathcal{S}(A,B)$ is a Π^0_1 class. In both lattices, the separating class of $\{n : \{n\}(n)\downarrow = 0\}$ and $\{n : \{n\}(n)\downarrow = 1\}$ has maximum degree [10]. Any subset of 2^ω with a recursive element is a representative of the minimum degree. A *special* Π^0_1 class is one that is non-empty and has no recursive element. Any recursively bounded Π^0_1 subset of ω^ω is recursively homeomorphic to (and therefore Medvedev and Muchnik equivalent to) a Π^0_1 subset of 2^ω, so everything that follows can be generalised to recursively bounded Π^0_1 subsets of ω^ω.

2 Splitting Theorems

Theorem 1. *Let P be any special Π^0_1 subset of 2^ω. Then there exist two other (necessarily special) Π^0_1 subsets of 2^ω, P^0 and P^1, such that:*

i. $P^0, P^1 <_w P$,
ii. $P^0 \vee P^1 \equiv_w P$.

The above also holds for the same P^0 and P^1 with $<_M$ and $\equiv_M$ replacing $<_w$ and $\equiv_w$.

The essence of the theorem is contained in the following lemma. The proof of Theorem 1 will come after the proof of the lemma.

Lemma 2. *Let P be any special Π^0_1 subset of 2^ω and A be any r.e. set. Then there exist r.e. sets, A^0 and A^1, such that:*

i. $A^0 \cup A^1 = A, \quad A^0 \cap A^1 = \emptyset$,
ii. for each $i \in \{0,1\}$ *and* $f \in P, \quad A^i \not\geq_T f$.

Letting $\langle .,. \rangle : \omega^2 \to \omega$ be a recursive coding bijection, we will explicitly construct each A^i to satisfy all of the following requirements:

$$\mathcal{R}_{\langle e,i\rangle} \equiv \{e\}^{A^i} \notin P.$$

Notation and Conventions:

- If $P \subseteq 2^\omega$ is a given non-empty Π^0_1 class, $\langle P_s \rangle_{s\in\omega}$ will be a recursive sequence of nested clopen subsets of 2^ω such that $P = \bigcap_s P_s$.

• If P is a Π^0_1 class, let T_P be a fixed recursive binary tree such that P is exactly the set of paths through T_P. We write $T_{P,s}$ for T_{P_s}.

• $u(A; i, m, s)$ is the maximum use made of $A \subseteq \omega$ in the computation $\{i\}^A_s(m)$. If $f \in 2^\omega$ then $u(A; A \oplus f, i, m, s)$ is the maximum use made of A in the computation $\{i\}^{A\oplus f}_s(m)$.

• $[n]$ is the set $\{0, 1, 2, \dots n-1\}$ and $\{i\}[n]$ is a partial sequence of length n. To say $\{i\}[n] \in T_P$ is to say that for all $m < n$, $\{i\}(m)\downarrow$ and

$$\langle \{i\}(0), \{i\}(1), \dots \{i\}(n-1)\rangle \in T_P.$$

• $f|_u = f$ restricted to $[u]$. $A|_u = \chi_A|_u$.

• If $\tau \in 2^{<\omega}$, then $|\tau|$ is the length of τ.

The method we will use is very similar to that used to prove Sacks' Splitting Theorem for the r.e. degrees, and we will closely follow the exposition in Soare ([13] Theorem VII.3.2). Lemma 2 may also be seen as a strengthening of Theorem 2 in [6].

Construction: Let P, A and i be as in Lemma 2 and we fix a recursive enumeration of A such that $A_{s+1} \setminus A_s$ has exactly one element for each s. For each i we will define a recursive sequence of finite sets, $\langle A^i_s\rangle_{s\in\omega}$, and A^i will then be $\bigcup_s A^i_s$.

Stage 0: $A^i_0 = \emptyset$.

Stage s+1: Assume A^i_s has been defined. We can then make the following definitions:

Length-of-agreement functions:

$l_s(e, i) := \max\{y : \{e\}^{A^i_s}_s[y] \in T_P\}$.

Restraint functions:

$r_s(e, i) := \max\{u(A^i_s; e, x, s) : x \leqslant l_s(e, i)\}$.

Injury sets:

$I_{\langle e,i\rangle} := \{x : \exists s\ x \in A^i_{s+1} \setminus A^i_s \text{ and } x \leqslant r_s(e, i)\}$.

If $x \in A^i_{s+1} \setminus A^i_s$ and $x \leqslant r_s(e, i)$, we say $\mathcal{R}_{\langle e,i\rangle}$ *is injured at stage* $s+1$.

Let x be the unique element of $A_{s+1} \setminus A_s$. Choose the least $\langle e, i\rangle < s$ such that $x \leqslant r_s(e, i)$ and enumerate x into A^{1-i}_{s+1}. That is, let $A^{1-i}_{s+1} =$

$A_s^{1-i} \cup \{x\}$. Set $A_{s+1}^i = A_s^i$ and say $\mathcal{R}_{\langle e,i\rangle}$ *receives attention at stage* $s+1$.

If there is no such $\langle e, i\rangle$, then enumerate x into A_{s+1}^0 and leave A_s^1 unchanged.

Lemma 3. *If* $\{e\}^{A^i} \in P$*, then* $\lim_s l_s(e,i) = \infty$.

Proof. Suppose $\{e\}^{A^i} \in P$ and let $n \in \omega$ be arbitrary. Then let $u = \max\{u(A^i; e, m) : m < n\}$ and now take s' so large that both the following hold:

i. $A_{s'}^i|_u = A^i|_u$,

ii. $\forall m < n\ \{e\}_{s'}^{A^i}(m)\downarrow$.

Then $\{e\}_s^{A_s^i}[n] = \{e\}^{A^i}[n] \in T_P$ and $l_s(e,i) \geqslant n$ for all $s \geqslant s'$. As n was arbitrary, the result follows.

□

Lemma 4. *For all* $e \in \omega$ *and* $i \in \{0,1\}$,

I. $I_{\langle e,i\rangle}$ *is finite,*
II. $\{e\}^{A^i} \notin P$,
III. $r(e,i) := \lim_s r_s(e,i)$ *exists and is finite.*

Proof. Take any $e \in \omega$ and $i \in \{0,1\}$. As induction hypothesis assume I., II., and III. hold for all $\langle e', i'\rangle < \langle e, i\rangle$.

I. By III. we can choose t and r such that for all $\langle e', i'\rangle < \langle e, i\rangle$ and $s \geqslant t$, $r_s(e', i') = r(e', i')$ and $r > r(e', i')$. Now take $v > t$ such that $A_v|_r = A|_r$. So $\mathcal{R}_{\langle e,i\rangle}$ cannot be injured after stage v and I. holds for $\langle e, i\rangle$.

II. Assume $\{e\}^{A^i} \in P$. To get a contradiction we will construct a recursive path $f \in P$. Let s' be such that $\mathcal{R}_{\langle e,i\rangle}$ is never injured after stage s'. Fix any $n \in \omega$ and we will recursively compute $f(n)$. Using I., $\lim_s\ l_s(e,i) = \infty$ so choose the least $s = s(n) > s'$ such that $l_s(e,i) > n$. If x is enumerated into A^i after stage s, then it must be greater than $u(A_s^i; e, n, s)$. So $\{e\}_s^{A_s^i}(n) = \{e\}^{A^i}(n)$. Set $f(n)$ equal to $\{e\}_s^{A_s^i}(n)$ for all $n \in \omega$. s is clearly a recursive function of n, so f itself is recursive and an element of P.

III. Let n be maximum such that $\{e\}^{A^i}[n] \in T_P$. Choose s' so large that for all $s \geqslant s'$,

i. $\{e\}_s^{A_s^i}[n] = \{e\}^{A^i}[n]$,
ii. $A_s^i|_u = A^i|_u$ where $u = \max\{u(A^i; e, m) : m < n\}$,
iii. $\mathcal{R}_{\langle e,i\rangle}$ is not injured at stage s.

If $\{e\}_s^{A_s^i}(n)\uparrow$ for all $s \geqslant s'$, then $u(A_s^i; e, n, s) = 0$ and $r_s(e,i) = r_{s'}(e,i)$ for all $s \geqslant s'$. So $\lim_s r_s(e,i)$ exists. On the other hand, suppose $\{e\}_t^{A_t^i}(n)\downarrow$ for some $t \geqslant s'$. If $x \in A^i \setminus A_t^i$ then $x \in A_{v+1}^i \setminus A_v^i$ for some $v \geqslant t$. As $\mathcal{R}_{\langle e,i\rangle}$ is not injured at any stage $s \geqslant t$, $x > r_{v+1}(e,i)$. But $r_{v+1}(e,i) = r_t(e,i)$ by conditions i. and ii. above. So $x > u(A_t^i; e, n, t)$ and the computation $\{e\}_t^{A_t^i}(n)$ is preserved forever. Therefore, for all $s \geqslant t$,

$$\{e\}_s^{A_s^i}[n+1] = \{e\}^{A^i}[n+1] \notin T_P.$$

So $l_s(e,i) = l_t(e,i) = n$ and $u(A_s^i; e, x, s) = u(A_t^i; e, x, t)$ for all $x \leqslant n$ and $s \geqslant t$. $r(e,i)$ then exists by the definition of $r_s(e,i)$. $\square$

The construction makes it clear that $A = A^0 \cup A^1$ and $A^0 \cap A^1 = \emptyset$, so Lemma 2 follows immediately from Lemma 4. Now we are in a position to prove Theorem 1. We will prove the Medvedev and Muchnik cases simultaneously.

Proof. (Theorem 1). Let A and B be such that $S = \mathcal{S}(A,B)$ is Medvedev (and therefore Muchnik) complete. For example, let $A = \{n : \{n\}(n)\downarrow= 1\}$ and $B = \{n : \{n\}(n)\downarrow= 0\}$ (see [10]). Let A^0 and A^1 be as in Lemma 2 and let $S^i = \mathcal{S}(A^i, B)$ for each $i \in \{0,1\}$. Note that if $A^0 \subseteq X \subseteq \overline{B}$ and $A^1 \subseteq Y \subseteq \overline{B}$, then $A \subseteq X \cup Y \subseteq \overline{B}$, so it becomes clear that $S \leqslant_M S^0 \vee S^1$. Also, $S \subseteq S^0,\ S^1$ so $S \geqslant_M S^0, S^1$ and therefore $S \geqslant_M S^0 \vee S^1$. That is, $S \equiv_M S^0 \vee S^1$ (and $S \equiv_w S^0 \vee S^1$).

Set $P^i = P \wedge S^i$. It is immediate that $P^i \leqslant_M P$ and $P^i \leqslant_w P$ and because $A^i \in S^i$, item ii. of Lemma 2 implies $S^i \not\geqslant_w P$ (and $S^i \not\geqslant_M P$). So in fact, $P^i <_M P$ and $P^i <_w P$ for each $i \in \{0,1\}$. Finally we can make the following calculation:

$$\begin{aligned} P^0 \vee P^1 &= (P \wedge S^0) \vee (P \wedge S^1) \\ &\equiv_M P \wedge (P \vee S^0) \wedge (P \vee S^1) \wedge S \\ &\equiv_M P \wedge (P \vee S^0) \wedge (P \vee S^1). \end{aligned}$$

But,

$$P \geqslant_M P \wedge (P \vee S^0) \wedge (P \vee S^1) \equiv_M P \vee (P \wedge S^0 \wedge S^1) \geqslant_M P,$$

so $P^0 \vee P^1 \equiv_M P$ and $P^0 \vee P^1 \equiv_w P$. This gives us the required splitting.

$\square$

Lemma 2 is true even when P is taken to be a Π^0_1 subset of ω^ω. This can be seen in two ways. First, the assumption of recursive boundedness is never used in the proof, so the generalisation follows immediately from the proof of the lemma. Second, via a theorem of Jockusch and Soare (Corollary 1.3, [6]) which states that for any special Π^0_2 class, P, there is a special, recursively bounded Π^0_1 class, Q, such that

$$\{\deg_T(f) : f \in Q\} \supseteq \{\deg_T(f) : f \in P\}.$$

In this more general form, the lemma implies Sacks' Splitting Theorem. Let C be any non-recursive Δ^0_2 set. Then $\{C\}$ is a special Π^0_2 class. Take Q as above and then Lemma 2 easily implies Sacks' theorem.

In the Medvedev case, we can improve Theorem 1 considerably by proving the following refinement of Lemma 2:

Lemma 5. *Let P and Q be non-empty Π^0_1 subsets of 2^ω such that $P >_M Q$, and let A be any r.e. set. Then there exist r.e. sets, A^0 and A^1, such that:*

i. $A^0 \cup A^1 = A, \quad A^0 \cap A^1 = \emptyset$,
ii. for each $i \in \{0,1\}, \quad \{A^i\} \vee Q \not\geqslant_M P$.

We will use this lemma as we used Lemma 2 - this time to prove that P can be split above Q. This is in contrast to the r.e. degrees, where Lachlan's "monster" theorem [7] states that such dense splitting fails.

The requirements for the construction will be:

$$\mathcal{R}^*_{\langle e,i\rangle} \equiv \{e\} : \{A^i\} \vee Q \not\twoheadrightarrow P.$$

We will make similar definitions to before. The compactness of Π^0_1 subsets of 2^ω ensures that the following are well defined:

Length-of-agreement functions:
$l^*_s(e,i) := \max\{y : \text{ for all } f \in Q_s, \ \{e\}^{A^i_s \oplus f}_s[y] \in T_P\}.$

Restraint functions:
$r^*_s(e,i) := \max\{u(A^i_s; A^i_s \oplus f, e, x, s) : x \leqslant l^*_s(e,i), \ f \in Q_s\}.$

Injury sets:

$I^*_{\langle e,i\rangle} := \{x : \exists s\ x \in A^i_{s+1} \smallsetminus A^i_s \text{ and } x \leqslant r^*_s(e,i)\}$.

If $x \in A^i_{s+1} \smallsetminus A^i_s$ and $x \leqslant r^*_s(e,i)$, we say $\mathcal{R}^*_{\langle e,i\rangle}$ *is injured at stage* $s+1$.

Note that $l^*_s(e,i)$ and $r^*_s(e,i)$ are recursive in e, i and s.

Let x be the unique element of $A_{s+1} \smallsetminus A_s$. Choose the least $\langle e,i\rangle < s$ such that $x \leqslant r^*_s(e,i)$ and enumerate x into A^{1-i}_{s+1}.

If there is no such $\langle e,i\rangle$, then enumerate x into A^0_{s+1}.

Lemma 6. *If* $\{e\} : \{A^i\} \vee Q \to P$*, then* $\lim_s l^*_s(e,i) = \infty$.

Proof. Suppose $\{e\} : \{A^i\} \vee Q \to P$ and let $n \in \omega$ be arbitrary. Then let:

$u = \max\{u(f; A^i \oplus f, e, m) : m < n,\ f \in Q\}$,
(again this exists by compactness)
$v = \max\{u(A^i; A^i \oplus f|_{u+1}, e, m) : m < n,\ f \in Q\}$,
$w =$ least k, $A^i_k|_{v+1} = A^i|_{v+1}$,
$t =$ least k, $\{\tau \in T_{Q,k} :\ |\tau| = u+1\} = \{\tau \in T_Q :\ |\tau| = u+1\}$.

Then for all $s \geqslant \max\{w,t\}$ such that $\{e\}^{A^i_s \oplus f}_s(m)\downarrow$ for all $m < n$, we have, $\{e\}^{A^i_s \oplus f}_s[n] = \{e\}^{A^i \oplus f}[n] \in T_P$ for all $f \in Q_s$. That is $l^*_s(e,i) \geqslant n$ and, as n was arbitrary, $\lim_s l_s(e,i) = \infty$.

□

Lemma 7. *For all* $e \in \omega$ *and* $i \in \{0,1\}$,

I. $I^*_{\langle e,i\rangle}$ *is finite,*
II. $\{e\} : \{A^i\} \vee Q \nrightarrow P$,
III. $r^*(e,i) := \lim_s r^*_s(e,i)$ *exists and is finite.*

Proof. Take any $e \in \omega$ and $i \in \{0,1\}$. As induction hypothesis assume I., II., and III. hold for all $\langle e',i'\rangle < \langle e,i\rangle$.

I. By III. we can choose t and r such that for all $\langle e',i'\rangle < \langle e,i\rangle$ and $s \geqslant t$, $r_s(e',i') = r(e',i')$ and $r > r(e',i')$. Now take $v > t$ such that $A_v|_r = A|_r$. So $\mathcal{R}^*_{\langle e,i\rangle}$ cannot be injured after stage v and I. holds for $\langle e,i\rangle$.

II. Assume $\{e\}^{A^i \oplus f} \in P$ for all $f \in Q$. Fix any $n \in \omega$. Using I., let s' be such that $\mathcal{R}^*_{\langle e,i\rangle}$ is never injured after stage s'. $\lim_s l^*_s(e,i) = \infty$, so choose the least $s = s(n) > s'$ such that $l^*_s(e,i) > n$. If x is enumerated into A^i after stage s, then it must be greater than $u(A^i_s; A^i_s \oplus f, e, n, s)$ for all $f \in Q$. So $\{e\}^{A^i_s \oplus f}_s(n) = \{e\}^{A^i \oplus f}(n)$ for all $f \in Q$. s is a recursive function of n, so $f \mapsto \{e\}^{A^i_s \oplus f}_s$ describes a recursive functional from Q into P, contradicting the fact that $P >_M Q$.

III. Let n be maximum such that for all $f \in Q$, $\{e\}^{A^i \oplus f}[n] \in T_P$. Using the compactness of Q, choose s' so large that for all $s \geqslant s'$,

i. $\{e\}_s^{A_s^i \oplus f}[n] = \{e\}^{A^i \oplus f}[n]$, for all $f \in Q$,
ii. $A_s^i|_u = A^i|_u$ where $u = \max\{u(A^i; A^i \oplus f, e, m) : m < n, f \in Q\}$
iii. $\mathcal{R}^*_{\langle e,i\rangle}$ is not injured at stage s.

If $\{e\}_s^{A_s^i \oplus f}(n)\uparrow$ for all $s \geqslant s'$ and $f \in Q$, then $u(A_s^i; A_s^i \oplus f, e, n, s) = 0$ and $r_s^*(e,i) = r_{s'}^*(e,i)$ for all $s \geqslant s'$. So $\lim_s r_s^*(e,i)$ exists. On the other hand, suppose $\{e\}_t^{A_t^i \oplus f}(n)\downarrow$ for some $t \geqslant s'$ and $f \in Q$. As before, $\mathcal{R}^*_{\langle e,i\rangle}$ is not injured at any stage $\geqslant s'$, so the computation is preserved forever. Therefore $l_s^*(e,i) = n$ for all $s \geqslant t$ also as before.

By compactness, there is a v such that for all $f \in Q$, $x \leqslant n$ and $s \geqslant t$,

$$\begin{aligned} \{e\}_s^{A_s^i \oplus f}(x) &\simeq \{e\}_t^{A_t^i \oplus f}(x) \\ &\simeq \{e\}_t^{A_t^i \oplus f|_v}(x). \end{aligned}$$

Let $k \geqslant t$ be a stage when $\{f|_v : f \in Q_k\} = \{f|_v : f \in Q\}$ and then for all $s \geqslant k$, $f \in Q_s$ and $x \leqslant n$, $u(A_s^i; A_s^i \oplus f, e, x, s) = u(A_k^i; A_k^i \oplus f, e, x, k)$ and $l_s^*(e,i) = n$. Finally we have, for all $s \geqslant k$,

$$\begin{aligned} r_s^*(e,i) &= \max\{u(A_s^i; A_s^i \oplus f, e, x, s) : x \leqslant l_s^*(e,i),\ f \in Q_s\} \\ &= \max\{u(A_k^i; A_k^i \oplus \tau, e, x, k) : x \leqslant n,\ \tau \in T_Q,\ |\tau| = v\} \end{aligned}$$

which is the maximum of a fixed finite set. Therefore $\lim_s r_s^*(e,i)$ exists and is finite.

$\square$

This also concludes the proof of Lemma 5, the main purpose of which is to prove the following "dense splitting" theorem.

Theorem 8. *For any two non-empty Π^0_1 subsets of 2^ω, $P >_M Q$, there exist two other Π^0_1 subsets of 2^ω, P^0 and P^1 such that:*
i. $P^0, P^1 <_M P$,
ii. $P^0 \vee P^1 \equiv_M P$,
iii. $P^0, P^1 >_M Q$.

Proof. Let $A = \{n : \{n\}(n)\downarrow = 0\}$, $B = \{n : \{n\}(n)\downarrow = 1\}$ so that $S = \mathcal{S}(A,B)$ is Medvedev complete. Take A^0 and A^1 to be as in Lemma 5, and $S^i = \mathcal{S}(A^i, B)$ for $i \in \{0,1\}$. Then set $P^i = P \wedge (S^i \vee Q)$.

$P^i \leqslant_M P$ and as $A^i \in S^i$, Lemma 5 implies $S^i \vee Q \not\geqslant_M P$. So $P^i <_M P$. Also,

$$\begin{array}{rcl} P^0 \vee P^1 & = & (P \wedge (Q \vee S^0)) \vee (P \wedge (Q \vee S^1)) \\ & \equiv_M & P \wedge (Q \vee S^0 \vee S^1) \\ & \equiv_M & P \end{array}$$

As P^0 and P^1 must be Medvedev incomparable, the theorem follows. □

Theorem 8 implies immediately the density of $\mathcal{P}_M$. The proof given here, however, is significantly different from the ones given in [3] and [1].

Theorems 1 and 8 can be extended even further to a "generalised splitting" theorem and a "generalised dense splitting" theorem respectively:

Theorem 9. *Let P be any special Π^0_1 subset of 2^ω and $\mathcal{L}$ be any finite distributive lattice. Then there is a lattice embedding of $\mathcal{L}$ into $\mathcal{P}_w$ sending the maximum element of $\mathcal{L}$ to the Muchnik degree of P.*

Theorem 10. *Given Π^0_1 subsets of 2^ω, $P >_M Q$, and any finite distributive lattice, $\mathcal{L}$, there is a lattice embedding of $\mathcal{L}$ into $\mathcal{P}_M$ between P and Q taking the maximum element of $\mathcal{L}$ to the Medvedev degree of P.*

These theorems then have Theorems 1 and 8 as corollaries if $\mathcal{L}$ is taken to be the four element diamond lattice. The proofs will use the following lattice-theoretic lemma.

Lemma 11. *Every finite distributive lattice can be lattice-embedded into a free finite distributive lattice, in a way that preserves the maximum element.*

Proof. Let $FD(m)$ be the free distributive lattice with m generators and let $\mathcal{B}_n$ denote the lattice of subsets of $N = \{0, 1, 2, \ldots, n-1\}$ under $\cup$ and $\cap$. Let $\mathcal{L}$ be a distributive lattice with operations $\vee$ and $\wedge$.

First observe that, using a representation theorem for finite distributive lattices (Theorem II.1.9 [5]), $\mathcal{L}$ can be represented as a sublattice of $\mathcal{B}_n$ for some n (in fact n is the number of join-irreducible elements of $\mathcal{L}$) and that the maximum element of $\mathcal{L}$ is represented by N - the maximum element of $\mathcal{B}_n$. So it is enough to embed $\mathcal{B}_n$

into $FD(n)$ preserving the maximum element. We will constuct an embedding, $\epsilon : \mathcal{B}_n \hookrightarrow FD(n)$, which preserves the least element of $\mathcal{B}_n$. As both $\mathcal{B}_n$ and $FD(n)$ are self dual, it is easy to convert this to an embedding that preserves the maximum.

Let $FD(n)$ be freely generated by $Y = \{y_0, y_1, \dots y_{n-1}\}$ and let $\widehat{y_i}$ denote $\bigwedge_{j \neq i} y_j$. If $Z \subseteq N$, we define,

$$\epsilon(Z) = \begin{cases} \bigvee\limits_{i \in Z} \widehat{y_i} & \text{if } Z \neq \emptyset \\ \bigwedge\limits_{i \in N} y_i & \text{if } Z = \emptyset \end{cases}$$

$\bigwedge_{i \in N} y_i$ is the minimum of $FD(n)$ so ϵ preserves the minimum. It is also clear that $\epsilon(Z_1 \cup Z_2) = \epsilon(Z_1) \vee \epsilon(Z_2)$. To see that ϵ preserves meets, note that $\widehat{y_i} \wedge \widehat{y_j} = \bigwedge_{i \in N} y_i$ if $i \neq j$ and that the distributive laws then give,

$$\bigvee_{i \in Z_1} \widehat{y_i} \wedge \bigvee_{i \in Z_2} \widehat{y_i} = \bigvee_{i \in Z_1 \cap Z_2} \widehat{y_i}.$$

The proof that ϵ is one-to-one is also straightforward - if $\epsilon(X) = \epsilon(Y)$ and $k \in X \smallsetminus Y$ then,

$$\widehat{y_k} \leqslant \bigvee_{i \in X} \widehat{y_i} = \bigvee_{i \in Y} \widehat{y_i} \leqslant y_k,$$

contradicting freeness (see Theorem II.2.3 in [5]).

$\square$

The proofs of Theorems 9 and 10 now proceed as before. First, analogues of Lemmas 2 and 5 are established (Lemmas 12 and 13) and then Theorems 9 and 10 follow.

Lemma 12. *Let P be any special Π^0_1 subset of 2^ω and A be any r.e. set. Then there exist r.e. sets, A^i, $0 \leqslant i \leqslant n-1$, such that:*

i. $\{A^i : 0 \leqslant i \leqslant n-1\}$ forms a partition of A,
ii. for each $i \in \{0, 1, \dots n-1\}$ and $f \in P$, $\bigoplus\limits_{j \neq i} A^j \not\geqslant_T f$.

Proof. (*sketch*)

The proof will be virtually the same as Lemma 2. The requirements will be:

$$\mathcal{R}_{\langle e,i \rangle} \equiv \{e\} : \{\bigoplus_{j \neq i} A^j\} \nrightarrow P,$$

and corresponding changes are made to the definitions of the length-of-agreement function, restraint function and injury set. To construct the partition, one takes the least $\langle e,i\rangle < s$ such that $x \leqslant r_s(e,i)$ and enumerates x into A^i_{s+1} (or A^0_{s+1} if no such $\langle e,i\rangle$ exists).

□

Now Theorem 9 can be proved.

Proof. (Theorem 9) The lemma is sufficient to prove that $FD(n)$ can be embedded into $\mathcal{L}_w$ below P with the top element going to P. In fact we show that $\{P \wedge S^i : 0 \leqslant i \leqslant n-1\}$ freely generates $FD(n)$ where, as before, $S^i = \mathcal{S}(A^i, B)$. To do this, it is sufficient to show that for all non-empty $I \subsetneq \{0,1,2\ldots n-1\}$,

$$P \wedge \bigvee_{i\in I} S^i \ngeqslant_w P \wedge \bigwedge_{i\notin I} S^i,$$

(again use Theorem II.2.3 in [5]). Fix I as above. The requirements imply that $\left\{\bigoplus_{i\in I} A^i\right\} \ngeqslant_w P$ as I is a proper subset of $\{0,1,2\ldots n-1\}$. But if $\left\{\bigoplus_{i\in I} A^i\right\} \geqslant_w \bigwedge_{i\notin I} S^i$, then $\left\{\bigoplus_{i\in I} A^i\right\} \geqslant_w S^j$ for some $j \notin I$. This implies

$$\{\bigoplus_{i\neq j} A^i\} \geqslant_w \bigvee_{i<n} S^i \equiv_w \mathcal{S}(A,B) \geqslant_w P,$$

contradicting $\mathcal{R}_{\langle e,j\rangle}$. Therefore $\left\{\bigoplus_{i\in I} A^i\right\} \ngeqslant_w P \wedge \bigwedge_{i\notin I} S^i$ and so $P \wedge \bigvee_{i\in I} S^i \ngeqslant_w P \wedge \bigwedge_{i\notin I} S^i$, as required. The top element of $FD(n)$ is $P \wedge \bigvee_{i<n} S^i \equiv_w P$. Lemma 11 then completes the proof.

□

To prove Theorem 10 we need the following slightly more complex lemma.

Lemma 13. *Let P and Q be non-empty Π^0_1 subsets of 2^ω such that $P >_M Q$, and let A be any r.e. set. Then there exist r.e. sets, A^i, $0 \leqslant i \leqslant n-1$, such that:*

i. $\{A^i :\ 0 \leqslant i \leqslant n-1\}$ forms a partition of A,
ii. for each non-empty $J \subsetneq \{0,1,\ldots n-1\}$,

$$\{\bigoplus_{i\in J} A^i\} \vee Q \ngeqslant_M P \wedge \bigwedge_{i\notin J} S^i.$$

Proof. *(sketch)* Let $A^J = \bigoplus_{i \in J} A^i$ and $A^J_s = \bigoplus_{i \in J} A^i_s$. Let T^J be a recursive tree whose set of paths is $P \wedge \bigwedge_{i \notin J} S^i$ and T^J_s be a recursive tree whose set of paths is $P_s \wedge \bigwedge_{i \notin J} S^i_s$. The requirements for the construction are:

$$\mathcal{R}_{\langle e,J\rangle} \equiv \{e\} : \{A^J\} \vee Q \nrightarrow P \wedge \bigwedge_{i \notin J} S^i.$$

The length-of-agreement function, restraint function and injury sets are:

$$\begin{aligned} &l_s(e,J) := \max\{y : \text{ for all } f \in Q_s,\ \{e\}_s^{A^J_s \oplus f}[y] \in T^J_s\}, \\ &r_s(e,J) := \max\{u(A^i_s; A^J_s \oplus f, e, x, s) : i \in J,\ x \leqslant l_s(e,J),\ f \in Q_s\}, \\ &I_{\langle e,J\rangle} := \{x : \exists s\ \exists i \in J\ x \in A^i_{s+1} \setminus A^i_s \text{ and } x \leqslant r_s(e,J)\}. \end{aligned}$$

As before, to construct the partition, at stage s, one takes the least $\langle e,J\rangle < s$ such that $x \leqslant r_s(e,J)$ and the least $i \notin J$ and enumerates x into A^i_{s+1} (or into A^0_{s+1} if no such $\langle e,J\rangle$ exists). The equivalents of Lemmas 6 and 7 are then proved in the same way. □

Proof. *(Theorem 10.)* It will be shown that $\{(P \wedge S^i) \vee Q : 0 \leqslant i \leqslant n-1\}$ generates $FD(n)$ above Q. Straightforward manipulations show that P is the top element of this copy of $FD(n)$. Let J be a non-empty, proper subset of $\{0,1,2,\dots n-1\}$. Then,

$$\begin{array}{rrcl} & Q \vee \{A^J\} & \ngeqslant_M & P \wedge \bigwedge_{i \notin J} S^i \\ \Rightarrow & Q \vee \bigvee_{i \in J} S^i & \ngeqslant_M & P \wedge \bigwedge_{i \notin J} S^i \\ \Rightarrow & \bigvee_{i \in J} Q \vee S^i & \ngeqslant_M & \bigwedge_{i \notin J} P \wedge S^i \\ \Rightarrow & \bigvee_{i \in J}(P \wedge S^i) \vee Q & \ngeqslant_M & \bigwedge_{i \notin J}(P \wedge S^i) \vee Q. \end{array}$$

Applying Theorem II.2.3 in [5] again is then enough to finish the proof. □

3 The $\exists$-theories of $\mathcal{P}_w$ and $\mathcal{P}_M$

Definition 14. *If L' is a first-order language in the predicate calculus and $\mathcal{M}$ is an L'-structure, then the* $\exists$-theory *of $\mathcal{M}$ in L' is the set of all L'-sentences of the form $\exists x_1 x_2 \dots x_n \phi$ (where ϕ is a quantifier-free formula) that are true in $\mathcal{M}$. If $\mathcal{M} \models \exists x_1 x_2 \dots x_n \phi$, then ϕ is said to be* satisfiable *in $\mathcal{M}$. An $\exists$-theory is* decidable *if the set of Gödel numbers of its elements is recursive.*

The main theorem to be proved in this section is:

Theorem 15. *The $\exists$-theories of $\mathcal{P}_w$ and $\mathcal{P}_M$ in the language $\langle \wedge, \vee, \leqslant, =, \mathbf{0}, \mathbf{1}\rangle$ are identical and decidable.*

What follows is a proof only that the $\exists$-theory of $\mathcal{P}_w$ in the language $\langle \wedge, \vee, =, \mathbf{0}, \mathbf{1}\rangle$ is decidable. The proof of the $\mathcal{P}_M$ case will be the same and it will be clear that the decision procedure for the $\exists$-theory of $\mathcal{P}_M$ is identical to the decision procedure for the $\exists$-theory of $\mathcal{P}_w$ - implying that their $\exists$-theories are the same. $\leqslant$ can be defined in terms of $\wedge$ and $=$ so Theorem 15 will follow.

In order to avoid confusion between propositional connectives and lattice operations we will use $\cdot$ and $+$ for the lattice operations $\wedge$ and $\vee$. $\prod$ and $\sum$ will be used to denote general products and sums.

Let L_{01} be the language $\langle \cdot, +, =, \mathbf{0}, \mathbf{1}\rangle$ with intended interpretation in $\mathcal{P}_w$ as $\wedge$, $\vee$, $=$ and the minimum and maximum elements of $\mathcal{P}_w$ respectively. The languages $L = \langle \cdot, +, =\rangle$ and $L_1 = \langle \cdot, +, =, \mathbf{1}\rangle$ will be restrictions of L_{01}. Two L_{01}-terms, σ and τ, with free variables among $x_1, x_2 \ldots x_n$ are said to be *equivalent (over $\mathcal{P}_w$)* if $\mathcal{P}_w \models \forall x_1 x_2 \ldots x_n(\tau = \sigma)$. Two formulas, ψ and ϕ, with free variables among $x_1, x_2, \ldots x_n$ are *equivalent (over $\mathcal{P}_w$)* if $\mathcal{P}_w \models \forall x_1 x_2 \ldots x_n(\phi \leftrightarrow \psi)$.

Lemma 16. *The $\exists$-theory of $\mathcal{P}_w$ in L is decidable.*

Proof. One can argue from Theorem 9 that a quantifier-free L-formula, ψ, is satisfiable in $\mathcal{P}_w$ if and only if it is satisfiable in some finite distributive lattice. As there are only finitely many distributive lattices of any given finite size, determining if ψ is satisfiable in a distibutive lattice of size $m \in \mathbb{N}$ is a finite task. To decide, then, if ψ is satisfiable in $\mathcal{P}_w$ it is enough to compute, uniformly in ψ, an m such that if ψ is satisfiable in some distributive lattice, it is satisfiable in a distributive lattice of size at most m. We do this now. m will depend only on the number of free variables in ψ.

Suppose ψ is as above with free variables x_1, x_2, $\ldots x_n$. Then ψ is equivalent to a formula of the form:

$$\bigvee_{i \in I}\Big[\bigwedge_{j \in J_i}(\tau_{ij} = \sigma_{ij}) \wedge \bigwedge_{\bar{j} \in \bar{J}_i}(\tau_{i\bar{j}} \neq \sigma_{i\bar{j}})\Big],$$

where $\tau_{ij}, \sigma_{ij}, \tau_{i\bar{j}}$ and $\sigma_{i\bar{j}}$ are L-terms and I, J_i and $\bar{J}_i$ are finite sets. If it is decidable whether or not each disjunct of ψ is satisfiable in $\mathcal{P}_w$, then it is decidable if ψ is satisfiable. So without losing generality, we can assume ψ is of the form:

$$\bigwedge_{j \in J} (\tau_j = \sigma_j) \wedge \bigwedge_{\bar{j} \in \bar{J}} (\tau_{\bar{j}} \neq \sigma_{\bar{j}}),$$

As before, let $FD(n)$ denote the free distributive lattice on n generators. If $\{\tau_k = \sigma_k : k \leqslant m\}$ is a finite set of lattice relations on $FD(n)$, then we can form the quotient lattice, $\{[\sigma] : \sigma \in FD(n)\}$, where $[\sigma] = [\tau]$ if and only if σ can be transformed formally into τ by applications of the axioms of distributive lattices and substitutions described by the relations. The lattice operations on the quotient lattice are then canonically induced. The claim is that if ψ is satisfiable in some lattice, then it is satifiable in the quotient of $FD(n)$ by $\{\tau_j = \sigma_j : j \in J\}$.

To see this, note that $\bigwedge_{j \in J} (\tau_j = \sigma_j)$ is satisfiable in this quotient lattice, and if some subformula of ψ of the form $\tau_{\bar{j}} \neq \sigma_{\bar{j}}$ were *not* satisfied in the quotient lattice, then $\tau_{\bar{j}}$ could be transformed into $\sigma_{\bar{j}}$ by applications of distributive laws and the relations $\{\tau_j = \sigma_j : j \in J\}$. But this could be done in *any* distributive lattice satisfying $\{\tau_j = \sigma_j : j \in J\}$ and so ψ would not be satisfiable in any distributive lattice. Therefore, if ψ is satisfiable in some distributive lattice, it is satisfiable in the quotient of $FD(n)$ by $\{\tau_j = \sigma_j : j \in J\}$.

The cardinality of the quotient lattice is less than the cardinality of $FD(n)$ which is bounded by $2^{2^n - 2}$ (Theorem II.2.1(iii) [5]). So this is the required m. $\square$

Lemma 17. *The $\exists$-theory of $\mathcal{P}_w$ in L_1 is decidable.*

Proof. Let ψ be a quantifier-free L_1-formula with $x_1, x_2, \ldots x_n$ its free variables. As above, we can assume ψ is of the form:

$$\bigwedge_{j \in J} (\tau_j = \sigma_j) \wedge \bigwedge_{\bar{j} \in \bar{J}} (\tau_{\bar{j}} \neq \sigma_{\bar{j}}).$$

Every such L_1-formula can be transformed using standard manipulations into an equivalent one of the form:

$$\bigwedge_{k \in K} (\nu_k = \mathbf{1}) \wedge \bigwedge_{\bar{k} \in \bar{K}} (\nu_{\bar{k}} \neq \mathbf{1}) \wedge \phi$$

where ϕ is a quantifier-free L-formula, ν_k and $\nu_{\bar{k}}$ are L-terms, and K and $\bar{K}$ are finite index sets. Let ψ^* be an L-formula formed from ψ by replacing every occurrence of $\mathbf{1}$ by $\sum_{i \leqslant n} x_i$. The claim is that ψ is satisfiable in $\mathcal{P}_w$ if and only if ψ^* is. Lemma 16 then gives the required result.

Suppose ψ^* is satisfiable in $\mathcal{P}_w$. Then it is satisfiable in some quotient, $\mathcal{L}$, of $FD(n)$. The element $\sum_{i\leqslant n}[x_i]$ is the maximum of $\mathcal{L}$ and by Theorem 9 we can embed $\mathcal{L}$ into $\mathcal{P}_w$ with $\sum_{i\leqslant n}[x_i]$ mapping to $\mathbf{1}$. So $\psi^* \wedge \sum_{i\leqslant n} x_i = \mathbf{1}$ is satisfiable in $\mathcal{P}_w$ and therefore so is ψ.

Conversely, suppose ψ is satisfied in $\mathcal{P}_w$ by a given assignment of variables. There are two cases based on the form of ψ.

Case 1. $K = \emptyset$. Let ϕ be satisfiable in some finite distributive lattice, $\mathcal{L}$, and let $\mathbf{p}$ be an intermediate element of $\mathcal{P}_w$. Then $\mathcal{L}$ can be embedded into $\mathcal{P}_w$ below $\mathbf{p}$ (Theorem 9). Under the induced assignment of variables, $\nu_{\bar{k}} \neq \mathbf{1}$ is satisfied for all $\bar{k} \in \bar{K}$. So ψ^* is satisfiable.

Case 2. $K \neq \emptyset$. $\nu_k = \mathbf{1}$ formally implies $\sum_{i\leqslant n} x_i = \mathbf{1}$. So any assignment of variables that satisfies $\nu_k = \mathbf{1}$ will satisfy $\sum_{i\leqslant n} x_i = \mathbf{1}$. This also means that for all $\bar{k} \in \bar{K}$, $\nu_{\bar{k}} \neq \sum_{i\leqslant n} x_i$ under the given assignment. So ψ^* is satisfiable in $\mathcal{P}_w$. □

Theorem 18. *The $\exists$-theory of $\mathcal{P}_w$ in L_{01} is decidable.*

Proof. An effective procedure will be described that, given a quantifier-free formula, ψ, of L_{01}, will produce a quantifier-free formula, ψ_1, of L_1 which is satisfiable in $\mathcal{P}_w$ if and only if ψ is. Lemma 17 will then complete the proof.

Suppose ψ is as above with free variables $x_1, x_2, \ldots x_n$. As before, we can assume ψ is of the form:

$$\bigwedge_{j\in J} (\tau_j = \sigma_j) \wedge \bigwedge_{\bar{j}\in \bar{J}} (\tau_{\bar{j}} \neq \sigma_{\bar{j}}), \tag{1}$$

for some finite sets, J and $\bar{J}$. ψ is then equivalent to a formula of the form

$$\bigwedge_{k\in K} (\nu_k = \mathbf{0}) \wedge \bigwedge_{\bar{k}\in \bar{K}} (\nu_{\bar{k}} \neq \mathbf{0}) \wedge \phi, \tag{2}$$

where K and $\bar{K}$ and are finite sets, ϕ is a quantifier-free L_1-formula and ν_k and $\nu_{\bar{k}}$ are L-terms.

Case 1. $K = \emptyset$. Suppose ϕ is satisfiable in the finite lattice, $\mathcal{L}$. The proof of Lemma 9 describes an embedding of $\mathcal{L}$ into $\mathcal{P}_w$ strictly above

$\mathbf{0}$. So $\nu_{\bar{k}} \neq \mathbf{0}$ will be satisfied for all $\bar{k} \in \bar{K}$ by such an embedding. So ψ is satisfiable in $\mathcal{P}_w$ if and only if ϕ is.

Case 2. $K \neq \emptyset$. For each $k \in K$, ν_k is equivalent to $\sum_{s\in S} \prod_{t\in T_s} y_{st}$ where $y_{st} \in \{x_1, x_2, \dots x_n\}$ and T_s and S are some finite index sets. Using the fact that $\mathcal{P}_w \models \forall x, y[x \cdot y = \mathbf{0} \leftrightarrow (x = \mathbf{0} \vee y = \mathbf{0})]$, we can calculate that $\nu_k = \mathbf{0}$ is equivalent to $\bigwedge_{s\in S} \bigvee_{t\in T_s} (y_{st} = \mathbf{0})$. So ψ is equivalent to a formula of the form

$$\bigwedge_{m\in M} \bigvee_{p\in P_m} (y_{mp} = \mathbf{0}) \wedge \bigwedge_{\bar{k}\in\bar{K}} (\nu_{\bar{k}} \neq \mathbf{0}) \wedge \phi. \tag{3}$$

Putting this in disjunctive normal form, and re-indexing appropriately, we get something of the form

$$\bigvee_{u\in U} [\bigwedge_{v\in V_u} (y_{uv} = \mathbf{0}) \wedge \bigwedge_{\bar{k}\in\bar{K}} (\nu_{\bar{k}} \neq \mathbf{0}) \wedge \phi]. \tag{4}$$

Again it is enough to decide the satisfiablity of each disjunct, so we assume ψ is equivalent to a formula of the form

$$\bigwedge_{v\in V} (y_v = \mathbf{0}) \wedge \bigwedge_{\bar{k}\in\bar{K}} (\nu_{\bar{k}} \neq \mathbf{0}) \wedge \phi. \tag{5}$$

Let ψ^* be the formula obtained by replacing, for all $v \in V$, each occurrence of y_v with $\mathbf{0}$. ψ^* is satisfiable if and only if ψ is, and ψ^* is equivalent to a formula of the same form as Equation (1) but with strictly fewer variables.

By iterating the above process we get, finally, either $\mathbf{0} = \mathbf{0}$ or a formula to which Case 1 applies. $\square$

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