Relative Kolmogorov Complexity and Geometry

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Abstract

We use the connection of Hausdorff dimension and Kolmogorov complexity to describe a geometry on the Cantor set - including concepts of angle, projections and scalar multiplication. A question related to compressibility is addressed using these geometrical ideas.

1 Introduction

The aim of this paper is to investigate the role of geometric ideas in the study of Kolmogorov complexity. The basic concept is that of the *effective dimension* of reals - elements of the Cantor space $2^{\mathbb{N}}$. If σ is a finite binary string, $C(\sigma)$ will be the plain Kolmogorov complexity of σ and the *effective Hausdorff dimension of* X is defined here to be

$$\dim_H X = \liminf_n \frac{C(X\restriction n)}{n}.$$

The dual notion

$$\dim_p X = \limsup_n \frac{C(X\restriction n)}{n}$$

is the *effective packing dimension of* X. We will be concerned primarily with those $X \in 2^{\mathbb{N}}$ where these two quantities are equal - the so-called *regular reals* [6] - and for these we define the *effective dimension of* X to be

$$\dim X = \lim_n \frac{C(X\restriction n)}{n}.$$

The foundation for the geometrical ideas is formed by the function $d : 2^{\mathbb{N}} \times 2^{\mathbb{N}} \to [0,1]$ defined by

$$d(X \to Y) = \limsup_n \frac{C(Y\restriction n|X\restriction n)}{n}$$

where $C(Y \upharpoonright n | X \upharpoonright n)$ is the Kolmogorov complexity of $Y \upharpoonright n$ given $X \upharpoonright n$. The function d obeys the triangle inequality in the direction of the arrow, that is

$$d(X \to Y) + d(Y \to Z) \geqslant d(X \to Z)$$

and which we refer to as a *directed pseudometric*. A metric can be easily formed from d by defining

$$d(X, Y) = \max\{d(X \to Y, Y \to X)\}$$

and by identifying reals that are distance 0 from one another. We write $X \simeq_d Y$ if $d(X, Y) = 0$.

The paper is about transforming and combining regular reals and analysing the results using this metric.

In the Section 2 we look at the effect on a regular real X of diluting X with 0s. That is for a given $r \in [0, 1]$ we construct a real rX that consists of bits of X interspersed with 0s. The proportion of bits of X to these padding 0s is $r/(1 - r)$.

We show that for any $r_1, r_2 \in [0, 1]$

$$d(r_1 X \to r_2 X) = \max\{0, r_2 - r_1\} \cdot \dim X$$

and that for any two regular reals X_1, X_2 there is a continuous function φ from $[0, 1]$ to the set of regular reals such that $\varphi(0) = X_1$ and $\varphi(1) = X_2$. That is we show that the set of regular reals forms a path connected topological space under d.

In section 3 we generalise the procedure introduced in the previous section by defining $r[AB]$ to be the result of interspersing bits of A with bits of B in the proportion r to $1 - r$. The requirement for the coherence of this operation is that $\lim_n \frac{C(\alpha A, \beta B)}{n}$ exists for all $\alpha, \beta \in [0, 1]$. This we refer to as A and B's being *mutually regular*. For example, any two mutually random reals are mutually regular.

The set of all elements of the form $r[(\alpha A)(\alpha B)]$ we call the *hull of A and B* (denoted $\mathcal{H}_{AB}$) and the geometry of this set induced by the directed pseudometric d is the subject of the rest of the paper, and we begin this in section 4. First we describe a directed metric space we call $\mathcal{T}$ with directed metric δ. $\mathcal{T}$ is similar to the taxicab metric but defined on a unit equilateral triangle with a triangular coordinate system and a distinguished element O at one vertex as in Figure 2. The distance between two points is the (Euclidean) length of the shortest piecewise linear path between the two points if the components of the path are restricted to being parallel to the sides of the triangle (See Figure 3). Furthermore, the directed metric on $\mathcal{T}$ requires that linear paths in the direction of the origin parallel to a side have length 0. We also introduce an operation of scalar multiplication on elements of $\mathcal{T}$ where for any $r \in [0, 1]$ and $X \in \mathcal{T}$, rX is the point on $\overline{OX}$ at distance $r\delta(O \to X)$ from the origin.

The geometry of $\mathcal{H}_{AB}$ is studied by attempting to find linear isometries from $\mathcal{H}_{AB}$ into $\mathcal{T}$ - where $\varphi : \mathcal{H}_{AB} \hookrightarrow \mathcal{T}$ is a linear isometry if it preserves the metric on $\mathcal{H}_{AB}$ and for all $X \in \mathcal{H}_{AB}$ and $r \in [0, 1]$, $\varphi(rX) = r\varphi(X)$.

The most basic example is when A and B are mutually random reals, in which case there is a linear isomorphism (bijective linear isometry) from $\mathcal{H}_{AB}$ onto $\mathcal{T}$. We also give an example of two mutually regular reals A and B (both randoms) for which, not only is $\mathcal{H}_{AB}$ not linearly isomorphic to $\mathcal{T}$ but for which there exists no linear isometry at all of $\mathcal{H}_{AB}$ into $\mathcal{T}$.

To make this argument we define notions of *angle* and *projection* in $\mathcal{T}$ and $\mathcal{H}_{AB}$ and show that these notions are preserved by linear isometries. The definitions are made in analogy to those in Euclidean space and we hope are natural enough to the reader. The angle between mutually random reals is maximal (equal to 1) and we take this concept to be a generalisation of 'mutually random" just as having maximal dimension is a generalisation of randomness. In $\mathcal{T}$ the definitions of $\angle XOY$, $\mathrm{Proj}_X Y$, and $\mathrm{Proj}_Y X$ (angles and projections respectively) are related by Equation 4.6 which we interpret as being the statement that the space $\mathcal{T}$ is *flat*. As a final result we show that given two mutually regular reals A, B, Equation 4.6 is a necessary and sufficient condition for there to be a linear isometry from $\mathcal{H}_{AB}$ into $\mathcal{T}$.

We want to think of the existence of linear isometries as a framework for answering compression/dilution type questions. For example, in [5] Reimann asks the question if every real of positive Hausdorff dimension is created by "diluting" a random real :

> If A has positive effective Hausdorff dimension, is there a random real $B \leqslant_T A$?

The question is answered negatively in [1] for positive effective packing dimension. Here we might ask a similar question thus:

> For any regular real Y of dimension r, does there exist a regular real X of dimension 1 such that $Y \simeq_d rX$.

We can also ask a two-dimensional version of this question:

> Given X, Y mutually regular, does there exist mutually regular A, B such that $\angle AB = 1$ and $\dim A = \dim B = 1$ and $X', Y' \in \mathcal{H}_{AB}$ such that $X \simeq_d X'$ and $Y \simeq_d Y'$?

A related question is:

> Given X, Y mutually regular, does there exist mutually regular A, B such that $\angle AB = 1$ and $\dim A = \dim B = 1$ and a linear isometry from $\mathcal{H}_{XY}$ into $\mathcal{H}_{AB}$ (equivalently into $\mathcal{T}$)?

We answer this last question in the negative by describing such an X and Y for which no such isometry exists and using very simple geometric arguments to establish this.

2 Basic Definitions and Notation

Undefined terminology regarding Kolmogorov complexity follows [4].

Definition 2.1. Let $A, B \in 2^{\mathbb{N}}$, define

$$d(A \to B) = \limsup_n \frac{C(B\restriction n \mid A\restriction n)}{n}.$$

Theorem 2.2. *$d(A \to B)$ is a directed pseudometric on $2^{\mathbb{N}}$. That is, for all $A, B, C \in 2^{\mathbb{N}}$,*

1. *$d(A \to B) \geqslant 0$,*
2. *$d(A \to A) = 0$,*
3. *$d(A \to C) \leqslant d(A \to B) + d(B \to C)$.*

Proof. 1 and 2 are immediate. To prove 3, notice that in order to describe $C\restriction n$ given $A\restriction n$ it is sufficient to be given a description of $B\restriction n$ given $A\restriction n$, a description of $C\restriction n$ given $B\restriction n$, and enough extra bits to distinguish these two descriptions from each other. That is, for all n

$$C(B\restriction n \mid A\restriction n) + C(C\restriction n \mid B\restriction n) + \mathcal{O}\big(\log C(C\restriction n \mid B\restriction n)\big) \geqslant C(C\restriction n \mid A\restriction n).$$

So

$$\begin{aligned}
d(A \to B) + d(B \to C) &= \limsup_n \frac{C(B\restriction n \mid A\restriction n)}{n} + \limsup_n \frac{C(C\restriction n \mid B\restriction n)}{n} \\
&\geqslant \limsup_n \frac{C(B\restriction n \mid A\restriction n) + C(C\restriction n \mid B\restriction n)}{n} \\
&\geqslant \limsup_n \frac{C(C\restriction n \mid A\restriction n) - \mathcal{O}\big(\log C(C\restriction n \mid B\restriction n)\big)}{n} \\
&= d(A \to C)
\end{aligned}$$

□

We now can create a metric from d in the standard way:

Definition 2.3. Let

$$d(A, B) = \max\{d(A \to B), d(B \to A)\}.$$

We write $A \simeq_d B$ if $d(A, B) = 0$, and denote $\{X : X \simeq_d A\}$ by $[A]_d$ - the *d-equivalence class of A*. A metric is induced by d on these equivalence classes.

Definition 2.4.

The quantity $\limsup_n C(A\restriction n)/n$ is referred to as the *effective packing dimension of A*. The dual quantity,

$$\liminf_n \frac{C(A\restriction n)}{n},$$

is the *effective Hausdorff dimension of A.* For a detailed discussion of the packing and Hausdorff dimension, see for example [3]. If these two dimensions are equal then we will simply refer to the *dimension of A*, and denote this $\dim A$.

Definition 2.5. $A \in 2^{\mathbb{N}}$ is *regular* if

$$\limsup_n \frac{C(A\restriction n)}{n} = \liminf_n \frac{C(A\restriction n)}{n}.$$

That is, if $\lim_n C(A\restriction n)/n$ exists.

If $\mathbf{0}$ is the infinite sequence of 0s (or equivalently any computable sequence) then it is immediate that:

Observation 2.6. *For all $A \in 2^{\mathbb{N}}$,*

$$d(\mathbf{0} \to A) = \text{ the effective packing dimension of } A,$$

and if A is regular, then

$$d(\mathbf{0} \to A) = \dim A.$$

Definition 2.7. Let **REG** be the class of regular elements of $2^{\mathbb{N}}$ equipped with the directed pseudometric d defined above.

The main result of this section is that **REG** is a path-connected topological space. Given any $A \in \mathbf{REG}$, we explicitly construct a continuous function $\varphi : [0,1] \to \mathbf{REG}$ such that $\varphi(1) = A$ and $\varphi(0) = \mathbf{0}$. Furthermore, the mapping φ will also have the property that for all $\alpha, \beta \in [0,1]$

$$d(\varphi(\alpha), \varphi(\beta)) = |\alpha - \beta| \dim A.$$

Concatenation of paths then allows us to connect any two $A, B \in \mathbf{REG}$.

First we define φ and prove some lemmas.

Definition 2.8. Let $\alpha \in [0,1]$ and $n \in \mathbb{N}$. Then let $p_n(\alpha)$ be the least natural number x that minimises $\left|\, \alpha n - x \,\right|$. We then have that

$$\alpha n - 1/2 \leqslant p_n(\alpha) \leqslant \alpha n + 1/2$$

and that $\lim_n p_n(\alpha)/n = \alpha$.

Definition 2.9. Let $A \in \mathbf{REG}$ and let $\alpha \in [0,1]$. Let $\varphi(\alpha)$ be

$$\sigma_1 0^{a_1} \sigma_2 0^{a_2} \sigma_3 0^{a_3} \dots \sigma_i 0^{a_i} \dots$$

where

1. $A = \sigma_1\sigma_2\sigma_3\dots$
2. $|\sigma_i| = p_i(\alpha)$
3. $|\sigma_i 0^{a_i}| = i$.

Notation. Note that $|\sigma_1 0^{a_1} \sigma_2 0^{a_2} \dots \sigma_n 0^{a_n}| = n(n+1)/2$. To make the calculations more readable, we let

- $N := n(n+1)/2$,
- $P_n(\alpha) := \sum_{i=1}^{n} p_i(\alpha)$ and
- $\alpha A := \varphi(\alpha)$.

This notation will be used throughout the paper. We will also refer to the string $\sigma_1\sigma_2\sigma_3\dots\sigma_n$ above as *the bits of A in $\alpha A \restriction N$*, and to the added 0s as *the padding bits.*

Lemma 2.10. *If $A \in \mathbf{REG}$, then $\alpha A \in \mathbf{REG}$ for all $\alpha \in [0,1]$. Furthermore,* $\dim(\alpha A) = \alpha \dim(A)$.

Proof. Let $n \in \mathbb{N}^+$ and consider $\alpha A \restriction n$. Let $m = m(n)$ be the the largest positive integer such that $m(m+1)/2 \leqslant n$. Then $\alpha A \restriction n$ is of the form

$$\sigma_1 0^{a_1} \sigma_2 0^{a_2} \sigma_3 0^{a_3} \dots \sigma_m 0^{a_m} \tau$$

where $|\tau| < m+1$. To describe $\alpha A \restriction n$ it is sufficient to know $A \restriction P_m(\alpha)$, the values of $p_i(\alpha)$ for all $i \leqslant m$, and the string τ. Each $p_i(\alpha)$ is bounded by i, and there are m of them so we need no more than $\mathcal{O}(m \log m)$ bits to describe them. The length of τ is bounded by m so we need at most $\mathcal{O}(m)$ bits to describe τ. Therefore

$$C(\alpha A \restriction n) \leqslant C(A \restriction P_m(\alpha)) + \mathcal{O}(m \log m).$$

Conversely, to describe $A \restriction P_m(\alpha)$, it is sufficient to describe $\alpha A \restriction n$ and to distinguish in $\alpha A \restriction n$ the padding bits from the bits of A. To do this it is sufficient to know the values of $p_i(\alpha)$ for all $i \leqslant m$. Thus

$$C(A \restriction P_m(\alpha)) \leqslant C(\alpha A \restriction n) + \mathcal{O}(m \log m),$$

and consequently

$$C(A\restriction P_m(\alpha)) = C(\alpha A\restriction n) \pm \mathcal{O}(m\log m). \tag{1}$$

But $n \geqslant m(m+1)/2$ so $\sqrt{2n} \geqslant m$, and therefore

$$\begin{aligned}
\limsup_n \frac{C(\alpha A\restriction n)}{n} &\leqslant \limsup_n \frac{C(A\restriction P_m(\alpha)) + \mathcal{O}(m\log m)}{n} \\
&\leqslant \limsup_n \frac{C(A\restriction P_m(\alpha)) + \mathcal{O}(\sqrt{n}\log\sqrt{n})}{n} \\
&\leqslant \limsup_n \frac{P_m(\alpha)}{n} \limsup_n \frac{C(A\restriction P_m(\alpha))}{P_m(\alpha)} \qquad (2)\\
&= \liminf_n \frac{P_m(\alpha)}{n} \liminf_n \frac{C(A\restriction P_m(\alpha))}{P_m(\alpha)} \\
&\text{(as } \lim_n \frac{P_m(\alpha)}{n} \text{ exists and } A \text{ is regular)} \\
&\leqslant \liminf_n \frac{C(A\restriction P_m(\alpha))}{n} \\
&\leqslant \liminf_n \frac{C(\alpha A\restriction n) + \mathcal{O}(m\log m)}{n} \\
&= \liminf_n \frac{C(\alpha A\restriction n)}{n}
\end{aligned}$$

Thus αA is regular. From line (2) it is now straightforward to show $\dim(\alpha A) = \alpha\dim(A)$.

□

Lemma 2.11.

$$\alpha(\beta A) \simeq_d (\alpha\beta)A,$$

and we can thus write $\alpha\beta A$ with only marginal ambiguity.

Proof. (*Sketch.*) For large values of n, the number of bits of A in $\alpha(\beta A)$ is approximately equal to the number of bits of A in $(\alpha\beta)A$ (when compared to N). So to describe $\alpha(\beta A)$ from $(\alpha\beta)A$ one only needs to know the values of $p_i(\alpha), p_i(\beta)$, and $p_i(\alpha\beta)$. As before this requires at most $\mathcal{O}(n\log n)$ bits. This term disappears in the limit, so $d((\alpha\beta)A \to \alpha(\beta A)) = 0$. A symmetrical argument shows $d(\alpha(\beta A) \to (\alpha\beta)A) = 0$.

□

Lemma 2.12. *Let $f \leqslant g \leqslant h$ be functions from $\mathbb{N}^+$ to $\mathbb{N}^+$ with $f(n)$ and $g(n)$ nondecreasing, $\lim_n f(n) = \lim_n g(n) = \infty$, and h strictly increasing. If $A \in 2^{\mathbb{N}}$ is regular and $\lim_n \frac{g(n)}{h(n)}$ and $\lim_n \frac{f(n)}{h(n)}$ both exist, then*

$$\lim_n \frac{C(A\restriction g(n) \mid A\restriction f(n))}{h(n)} = \lim_n \frac{g(n) - f(n)}{h(n)} \cdot \dim A.$$

Proof. It is well known (see for example [2]) that for all $\sigma, \tau \in 2^{<\mathbb{N}}$,

$$C(\sigma|\tau) + C(\tau) = C(\sigma, \tau) + \mathcal{O}\big(\log C(\sigma, \tau)\big).$$

This equality is usually referred to as *the symmetry of information.*

If $\tau \preceq \sigma$, then this implies

$$C(\sigma) \leqslant C(\sigma|\tau) + C(\tau) + \mathcal{O}\big(\log C(\tau)\big) \leqslant C(\sigma) + \mathcal{O}\big(\log |\tau|\big) + \mathcal{O}\big(\log C(\sigma, \tau)\big).$$

Taking σ and τ to be $A \restriction g(n)$ and $A \restriction f(n)$ respectively gives the two inequalities

$$C(A \restriction g(n)) - C(A \restriction f(n)) \leqslant C(A \restriction g(n) \bigm| A \restriction f(n)) + \mathcal{O}\big(\log C(A \restriction f(n))\big),$$

and

$$C(A \restriction g(n) \bigm| A \restriction f(n)) \leqslant C(A \restriction g(n)) - C(A \restriction f(n)) + \mathcal{O}(\log f(n)) + \mathcal{O}\big(\log C(A \restriction g(n), A \restriction f(n))\big).$$

Dividing by $h(n)$ and taking limit suprema of both sides of the first inequality gives

$$\begin{aligned} \limsup_n \frac{C(A \restriction g(n) \bigm| A \restriction f(n))}{h(n)} &\geqslant \limsup_n \frac{C(A \restriction g(n)) - C(A \restriction f(n))}{h(n)} \\ &= \limsup_n \frac{g(n)}{h(n)} \frac{C(A \restriction g(n))}{g(n)} - \frac{f(n)}{h(n)} \frac{C(A \restriction f(n))}{f(n)} \end{aligned}$$

As A is regular, the limit of the RHS exists, so

$$\lim_n \frac{C(A \restriction g(n) \bigm| A \restriction f(n))}{h(n)} \geqslant \lim_n \frac{g(n) - f(n)}{h(n)} \cdot \dim A.$$

The same calculation on the second inequality will give

$$\lim_n \frac{C(A \restriction g(n) \bigm| A \restriction f(n))}{h(n)} \leqslant \lim_n \frac{g(n) - f(n)}{h(n)} \cdot \dim A,$$

when one observes that

$$\lim_n \frac{\mathcal{O}(\log f(n)) + \mathcal{O}\big(\log C(A \restriction g(n), A \restriction f(n))\big)}{h(n)} = 0.$$

□

Theorem 2.13. *If* $\alpha, \beta \in [0, 1]$, *then*

$$d(\alpha A \to \beta A) = \max\{0, (\beta - \alpha) \dim A\},$$

and thus $d(\alpha A, \beta B) = |\beta - \alpha| \dim A.$

Proof. If $\alpha \geqslant \beta$, then $C(\beta A \mid \alpha A) \leqslant \mathcal{O}(n \log(n))$ which approaches 0 after being divided by N. So $d(\alpha A \to \beta B) = 0$. If $\alpha \leqslant \beta$, then by an argument similar to that in Lemma 2.10, we show that

$$C(\alpha A \restriction N, \beta A \restriction N) = C(A \restriction P_n(\alpha), A \restriction P_n(\beta)) \pm \mathcal{O}(n \log(n)),$$

and so

$$\begin{aligned}
\lim_n \frac{C(\beta A \restriction N \mid \alpha A \restriction N)}{N} &= \lim_n \frac{C(\alpha A \restriction N, \beta A \restriction N) - C(\alpha A \restriction N)}{N} \\
&= \lim_n \frac{C(A \restriction P_n(\alpha), A \restriction P_n(\beta)) - C(A \restriction P_n(\alpha))}{N} \\
&= \lim_n \frac{C(A \restriction P_n(\beta) \mid A \restriction P_n(\alpha))}{N} \\
&= (\beta - \alpha) \dim(A) \qquad \text{(by Lemma 2.12)}
\end{aligned}$$

□

We have established the following with φ given in Definition 2.8.

Theorem 2.14. **REG** *is a path connected topological space. That is, for any* $A, B \in$ **REG**, *there is a continuous mapping* φ *from* $[0, 1]$ *into* **REG** *with* $\varphi(0) = A$ *and* $\varphi(1) = B$.

Proof. The continuous map φ defined above connects any regular X to $\mathbf{0}$. Concatenation of paths gives the result.

□

3 Mutually Regular Reals

Definition 3.1. We call any pair of reals, A and B, *mutually regular* if for all $\alpha, \beta \in [0, 1]$,

$$\lim_n \frac{C(\alpha A \restriction n, \beta B \restriction n)}{n}$$

exists.

By taking $\alpha = 0$ and $\beta = 1$ (or vice versa) we see immediately that mutually regular reals are regular.

Lemma 3.2. *Mutually regular reals exist.*

Proof. If R_1 and R_2 are such that $R_1 \oplus R_2$ is random, then we claim that R_1 and R_2 are mutually regular. Let $\alpha, \beta \in [0,1]$. By the symmetry of information, it is sufficient to show that $\lim_n \frac{C(\beta R_2\restriction n|\alpha R_1\restriction n)}{n}$ exists, which is what we do.

$$\begin{aligned}
\beta &= \limsup_n \frac{C(\beta R_2\restriction n)}{n} && \textit{cf.}\ (1) \text{ and } (2)\\
&\geqslant \limsup_n \frac{C(\beta R_2\restriction n \bigm| \alpha R_1\restriction n)}{n}\\
&\geqslant \liminf_n \frac{C(\beta R_2\restriction n \bigm| \alpha R_1\restriction n)}{n}\\
&= \liminf_n \frac{C(R_2\restriction P_n(\beta) \bigm| R_1\restriction P_n(\alpha))}{N} && \textit{cf.}\ (1)\\
&\geqslant \liminf_n \frac{C^{R_1}(R_2\restriction P_n(\beta))}{N} && (3)\\
&= \liminf_n \frac{C(R_2\restriction P_n(\beta))}{N} && (4)\\
&= \beta
\end{aligned}$$

Line 4 follows from the fact that R_2 is random relative to R_1 and hence has R_2-effective Hausdorff dimension 1. Line 3 follows from the previous line because if $f \in 2^{\mathbb{N}}$ and $\sigma, \tau \in 2^{<\mathbb{N}}$ such that $f \succ \tau$, then

$$C(\sigma|\tau) + \mathcal{O}(C(|\tau|)) \geqslant C^f(\sigma).$$

□

Lemma 3.3. *If A and B are mutually regular and* $\dim A = \dim B$, *then* $d(A \to B) = d(B \to A) = d(A,B)$

Proof. Using again the symmetry of information:

$$\begin{aligned}
d(B \to A) &= \lim_n \frac{C(A\restriction n \bigm| B\restriction n)}{n}\\
&= \lim_n \frac{C(A\restriction n, B\restriction n) - C(B\restriction n)}{n}\\
&= \lim_n \frac{C(A\restriction n, B\restriction n) - C(A\restriction n)}{n} && \text{as } \dim A = \dim B\\
&= \lim_n \frac{C(B\restriction n \bigm| A\restriction n)}{n}\\
&= d(A \to B)
\end{aligned}$$

□

The expression $\lim_n \frac{C(A\restriction n, B\restriction n)}{n}$ is used so often in the following that we introduce the following notation:

Definition 3.4. Let A and B be mutually regular reals. Then

$$C^*(A,B) := \lim_n \frac{C(A\restriction n, B\restriction n)}{n}.$$

We will also find use for the generalisation:

$$C^*(A_1, A_2 \dots, A_k) := \lim_n \frac{C(A_1\restriction n, A_2\restriction n \dots, A_k\restriction n)}{n},$$

if this limit exists.

Lemma 3.5. *If A and B are mutually regular and $\alpha \in [0,1]$, then*

$$C^*(\alpha A, \alpha B\big) = \alpha C^*\big(A, B\big),$$

and hence $d\big(\alpha A \to \alpha B\big) = \alpha d\big(A \to B\big)$.

Proof.

$$\begin{aligned} C^*\big(\alpha A, \alpha B\big) &= \lim_n \frac{C\big(\alpha A\restriction N, \alpha B\restriction N\big)}{N} \\ &= \lim_n \frac{P_n(\alpha)}{N} \cdot \frac{C\big(A\restriction P_n(\alpha), B\restriction P_n(\alpha)\big)}{P_n(\alpha)} \\ &= \alpha C^*\big(A, B\big). \end{aligned}$$

Then $d\big(\alpha A \to \alpha B) = C^*\big(\alpha A, \alpha B\big) - \alpha \dim(A) = \alpha d(A \to B)$. □

Definition 3.6. If $\gamma \in [0,1]$ and A and B are mutually regular reals, then let $\gamma[AB] \in 2^{\mathbb{N}}$ be defined as follows.

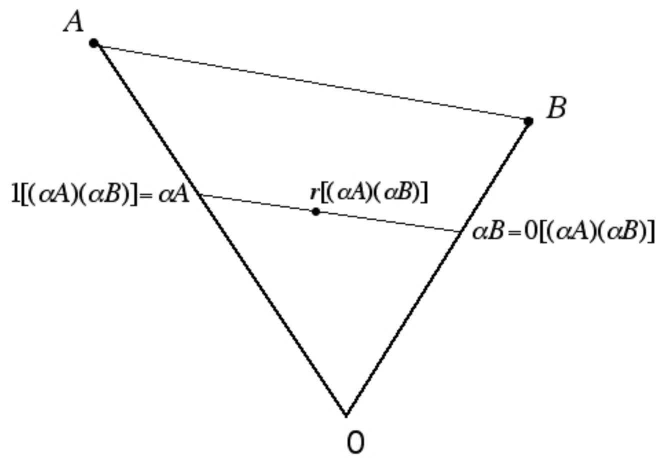
$$\gamma[AB] = \alpha_1\beta_1\alpha_2\beta_2 \dots$$

where

1. $|\alpha_i\beta_i| = i$
2. $|\alpha_i| = p_i(\gamma)$
3. $A = \alpha_1\alpha_2\alpha_3 \dots$
4. $B = \beta_1\beta_2\beta_3 \dots$

Notice that $\gamma[AB]$ is a generalisation of γA in the sense that $\gamma A = \gamma[A\mathbf{0}]$, but also notice that it is not in general true that $\alpha(\gamma[AB]) \simeq_d (\alpha\gamma)[AB]$.

Figure 1: $r[(\alpha A)(\alpha B)]$



Definition 3.7. Let $\mathcal{H}_{AB}$ be the space consisting of the set of reals

$$\{r[(\alpha A)(\alpha B)] : r, \alpha \in [0,1]\}$$

together with the directed pseudometric d. We refer to $\mathcal{H}_{AB}$ as *the hull of A and B*, and note that $A = 1[(1A)(1B)]$, $B = 0[(1A)(1B)]$, and $\mathbf{0} = 0[(1A)(1B)]$ are all elements of $\mathcal{H}_{AB}$. See Figure 1.

We will need to extend this to the space of all reals that are distance 0 from elements of $\mathcal{H}_{AB}$. Thus we define:

Definition 3.8. Let A and B be mutually regular reals. Then *the extended hull of A and B* is

$$\widehat{\mathcal{H}}_{AB} = \{Y \in 2^{\mathbb{N}} : \exists X \in \mathcal{H}_{AB}\ X \simeq_d Y\},$$

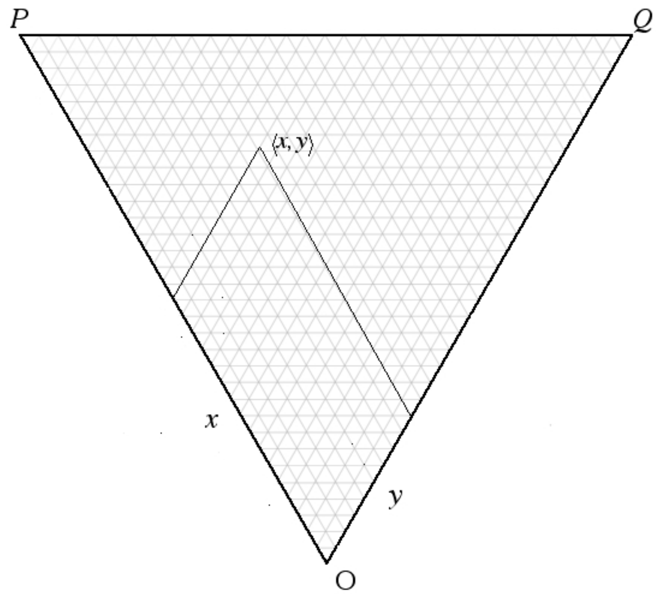
together with the directed pseudometric d.

Lemma 3.9. *If A and B are mutually regular, then any two elements of $\mathcal{H}_{AB}$ (and hence any two elements of $\widehat{\mathcal{H}}_{AB}$) are mutually regular.*

Proof. Sketch Let $r_1[(\alpha_1 A)(\alpha_1 B)]$, $r_2[(\alpha_2 A)(\alpha_2 B)] \in \mathcal{H}_{AB}$. By keeping a track of the number of bits of A and B in $r_1[(\alpha_1 A)(\alpha_1 B)]\restriction n$ and $r_2[(\alpha_2 A)(\alpha_2 B)]\restriction n$ we can see that

$$\begin{aligned}
&C\big(r_1[(\alpha_1 A)(\alpha_1 B)]\restriction n, r_2[(\alpha_2 A)(\alpha_2 B)]\restriction n\big)\\
&= C\big(r_1\alpha_1 A\restriction n, (1-r_1)\alpha_1 B\restriction n, r_2\alpha_2 A\restriction n, (1-r_2)\alpha_2 B\restriction n\big) \pm \mathcal{O}(n\log n)\\
&= C\big(\max\{r_1\alpha_1, r_2\alpha_2\}A\restriction n, \max\{(1-r_1)\alpha_1, (1-r_2)\alpha_2\}B\restriction n) \pm \mathcal{O}(n\log n)
\end{aligned}$$

Figure 2: Coordinate System for $\mathcal{T}_{PQ}$



and then use the fact that A and B are mutually regular to show that the relevant limit exists. Taking $\beta, \gamma \in [0,1]$ and repeating the argument on $\beta\big(r_1[(\alpha_1 A)(\alpha_1 B)]\big)$ and $\gamma\big(r_2[(\alpha_2 A)(\alpha_2 B)]\big)$ gives the result.

$\square$

4 The geometry of $\widehat{\mathcal{H}}_{AB}$

4.1 The metric space $\mathcal{T}$

We wish to investigate the space $\widehat{\mathcal{H}}_{AB}$ geometrically. To do this we first describe a geometrical space we call $\mathcal{T}$.

Let O, P and Q be the vertices of an equilateral triangle with unit sides. O is called *the origin*. We will define a directed metric δ (and hence an induced metric) on the triangular region bounded by $\triangle OPQ$. $\mathcal{T}$ will be this triangular region along with δ. To define δ we first coordinatise the points in the region as shown in Figure 2. We write simply $X = \langle x, y\rangle$ to mean X *has coordinates* $\langle x, y\rangle$. If $S = \langle x, y\rangle \in \mathcal{T}$ and $r \in [0,1]$, then by rS we mean $\langle rx, ry\rangle$. The directed metric δ is given by

$$\delta(\langle x_1, y_1\rangle \to \langle x_2, y_2\rangle) = \max\{0, x_2 - x_1\} + \max\{0, y_2 - y_1\}.$$

This gives rise to the metric

$$\delta(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle) = \max\{|x_2 - x_1|, |y_2 - y_1|, |(x_2 - x_1) + (y_2 - y_1)|\},$$

and is the Euclidean length of a path from $\langle x_1, y_1\rangle$ and $\langle x_2, y_2\rangle$ if one is restricted to moving along lines parallel to the sides of the triangle (compare the New York metric in rectangular coordinates) The directed metric is similar, with distances calculated by moving parallel to the sides of the triangle. However if one moves towards the origin parallel to a side, the distance is 0 - see Figure 3. The reader may confirm that $\delta(rX \to rY) = r\delta(X \to Y)$.

We also note that with this coordinate system we have the convenient fact that if $X = \langle x, y\rangle$, then $|\overline{OX}| = \delta(O \to X) = x + y$. We can thus extend the definition of $r\langle x, y\rangle$ to the case where $r > 1$ as long as $rx + ry \leqslant 1$.

Angles in $\mathcal{T}$ will be defined in analogy to Euclidean angles - the length of arc of a sector. Given R and S in $\mathcal{T}$ with $|OR| = r$ and $|OS| = s$, both nonzero,

$$\angle ROS := \delta(\frac{1}{r}R \to \frac{1}{s}S).$$

See Figure 4. The reader may confirm that the expected properties of angles hold. For example

1. $\angle ROS = \angle SOR$,
2. If $\overline{OS}$ is between $\overline{OR}$ and $\overline{OT}$, then
$$\angle ROS + \angle SOT = \angle ROT,$$
3. If $\alpha, \beta \in (0, 1]$, then
$$\angle(\alpha R)O(\beta S) = \angle ROS.$$

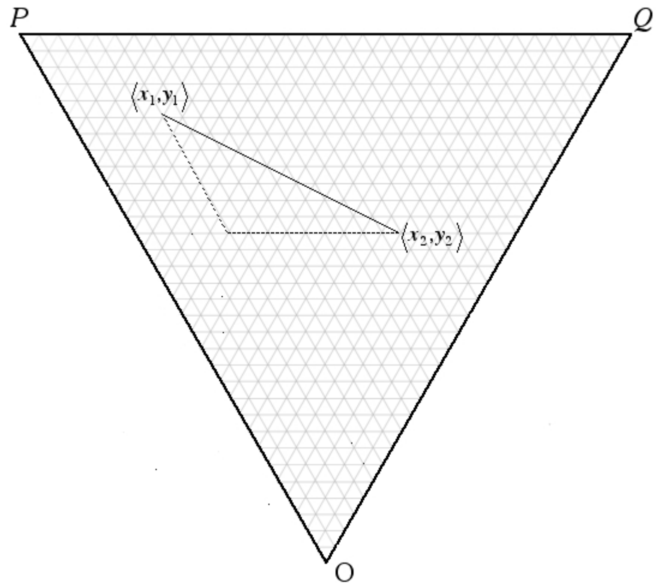
When we say we wish to determine the geometry of $\widehat{\mathcal{H}}_{AB}$ we mean we wish to find a map $\widehat{\varphi}$ from $\widehat{\mathcal{H}}_{AB}$ into $\mathcal{T}$ with the following properties:

1. For all $X \in \widehat{\mathcal{H}}_{AB}$ and $r \in [0, 1]$,
$$\widehat{\varphi}(rX) = r\widehat{\varphi}(X).$$
2. For all $X, Y \in \widehat{\mathcal{H}}_{AB}$,
$$\delta(\widehat{\varphi}(X) \to \widehat{\varphi}(Y)) = d(X \to Y).$$

We will refer to such a $\widehat{\varphi}$ as a *linear isometry*. If such a linear isometry is found for some extended hull $\widehat{\mathcal{H}}_{AB}$, we can pull back geometric properties from $\mathcal{T}$ and apply them to $\widehat{\mathcal{H}}_{AB}$. To show the usefulness of such techniques we will use them to answer the following question in the negative:

Question 4.1. *Given any pair A, B of mutually regular reals, does there exist a pair R_1, R_2 of mutually random reals such that $A, B \in \widehat{\mathcal{H}}_{R_1R_2}$?*

Figure 3:
Two points where $\delta(\langle x_1, y_1\rangle \to \langle x_2, y_2\rangle) = y_2 - y_1$ and $\delta(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle) = x_1 - x_2$



We can define a concept of angle in **REG** as well:

Definition 4.2. Suppose A, $B \in \mathbf{REG}$ with $\dim A = a$ and $\dim B = b$. Then let

$$\angle AB := \frac{1}{ab} d(bA \to aB).$$

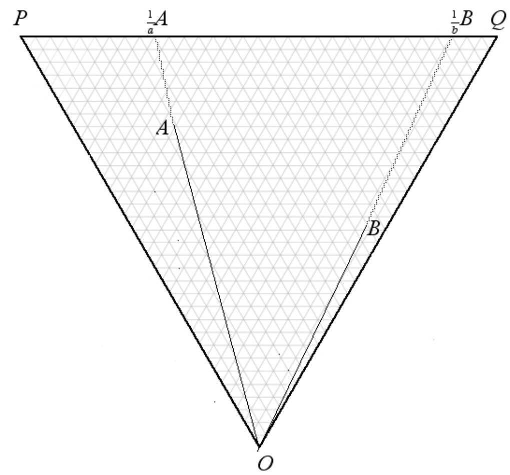
As $\dim aB = \dim bA$, $\angle AB = \angle BA = d(bA, aB)$. It may be thought clearer to define $\angle AB := d(\frac{1}{a}A \to \frac{1}{b}B)$ as in $\mathcal{T}$ but $\frac{1}{a}A$ and $\frac{1}{b}B$ are not defined when $a, b < 1$. The maximum angle between two regular reals is 1 and this is achieved by two mutually random reals. We intend $\angle AB = 1$ to be seen as a generalisation of A and B's being mutually random, and $\angle AB$ to be a measure of how independent A and B are. It is straightforward to show that linear isometries preserve angles.

We will now establish some results for mutually regular reals at maximum angle.

Lemma 4.3. *Let A, B be mutually regular with $\dim(A) = a \geqslant \dim(B) = b > 0$ and such that $\angle AB = 1$. If $\alpha, \beta \in [0, 1]$, then*

1. $d(\alpha \frac{b}{a} A \to \alpha B) = b\alpha = d(\alpha B \to \alpha \frac{b}{a} A)$
2. $d(\alpha A \to \beta B) = b\beta$.
3. $C^*\big(\alpha A, \beta B\big) = a\alpha + b\beta$

Figure 4: $\angle AB = \delta(\frac{1}{a}A \to \frac{1}{b}B)$



4. $\dim \gamma[(\alpha A)(\beta B)] = C^*\big(\gamma\alpha A, (1-\gamma)\beta B\big)$.

5. *Given any element* $Y \in \widehat{\mathcal{H}}_{AB}$*, there is a unique element* $X \in \mathcal{H}_{AB}$ *such that* $Y \simeq_d X$*.*

Proof. 1.

$$\begin{aligned} d(\alpha\tfrac{b}{a}A \to \alpha B) &= \tfrac{\alpha}{a} d(bA \to aB) \\ &= b\alpha \angle AB \\ &= b\alpha \end{aligned}$$

2. If $\alpha\frac{a}{b} \leqslant \beta$, then

$$\begin{aligned} d(\alpha A \to \beta B) &\leqslant d(\alpha\tfrac{a}{b}\cdot\tfrac{b}{a}A \to \alpha\tfrac{a}{b}B) + d(\alpha\tfrac{a}{b}B \to \beta B) \\ &= \alpha\tfrac{a}{b}\cdot b + (\beta - \alpha\tfrac{a}{b})b & \text{(using Part 1.)} \\ &= b\beta. \end{aligned}$$

But

$$\begin{aligned} b\beta &= d(\beta\tfrac{b}{a}A \to \beta B) & \text{(using Part 1.)} \\ &\leqslant d(\beta\tfrac{b}{a}A \to \alpha A) + d(\alpha A \to \beta B) \\ &= 0 + d(\alpha A \to \beta B) & (\text{as } \beta\tfrac{b}{a} \geqslant \alpha) \end{aligned}$$

So $d(\alpha A \to \beta B) = b\beta$. If $\alpha\frac{a}{b} \geqslant \beta$, then

$$\begin{aligned} d(\alpha A \to \beta B) &\leqslant d(\alpha A \to \beta\tfrac{b}{a}A) + d(\beta\tfrac{b}{a}A \to \beta B) \\ &= 0 + b\beta \end{aligned}$$

And

$$\begin{aligned} a\alpha &= d(\alpha A \to \alpha\tfrac{a}{b}B) \\ &\leqslant d(\alpha A \to \beta B) + d(\beta B \to \alpha\tfrac{a}{b}B) \\ &= d(\alpha A \to \beta B) + (\alpha\tfrac{a}{b} - \beta)b \end{aligned}$$

So in this case too, $d(\alpha A \to \beta B) = b\beta$.

3.

$$\begin{aligned} \lim_n \frac{C\big(\alpha A\restriction n, \beta B\restriction n\big)}{n} &= \lim_n \frac{C\big(\beta B\restriction n \;\big|\; \alpha A\restriction n\big)}{n} + \lim_n \frac{C\big(\alpha A\restriction n\big)}{n} \\ &= d(\alpha A \to \beta B) + \dim \alpha A \\ &= b\beta + a\alpha \end{aligned}$$

4.

$$\begin{aligned} \lim_n \frac{C\big(\gamma[(\alpha A)(\beta B)]\restriction n\big)}{n} &= \lim_n \frac{C\big(\alpha A\restriction P_n(\gamma), \beta B\restriction P_n(1-\gamma)\big)}{N} \\ &= \lim_n \frac{C\big(\gamma\alpha A\restriction n, (1-\gamma)\beta B\restriction n\big)}{n} \end{aligned}$$

and Part 3 completes the proof.

5. Suppose $X_1 = r_1[(\alpha_1 A)(\alpha_1 B)] \simeq_d Y \simeq_d r_2[(\alpha_2 A)(\alpha_2 B)] = X_2$. We prove that $r_1 = r_2$ and $\alpha_1 = \alpha_2$. The dimensions of X_1 and X_2 must be the same and so by 4,

$$\dim(X_i) = r_1\alpha_1 a + (1-r_1)\alpha_1 b = r_2\alpha_2 a + (1-r_2)\alpha_2 b. \tag{5}$$

Let $\gamma = \dim(X_i)$. Then if $\gamma \leqslant a$,

$$\begin{aligned} d(\tfrac{\gamma}{a}A \to X_i) &= C^*(\max\{\tfrac{\gamma}{a}, r_i\alpha_i\}A, (1-r_i)\alpha_i B) - \gamma \\ &= \max\{\gamma, r_i\alpha_i a\} + (1-r_i)\alpha_i b - \gamma && \text{(by 3)} \\ &= (1-r_i)\alpha_i b && \text{(as } \gamma \geqslant r_i\alpha_i a \text{ by (5))} \end{aligned}$$

Therefore $(1-r_1)\alpha_1 = (1-r_2)\alpha_2$ and with Equation (5) gives $r_1 = r_2$ and $\alpha_1 = \alpha_2$. Otherwise, if $a \leqslant \gamma$, we have

$$\begin{aligned} d(A \to \tfrac{a}{\gamma}X_i) &= C^*(A, \tfrac{a}{\gamma}(1-r_i)\alpha_i B) - a \\ &= a + \tfrac{a}{\gamma}(1-r_i)\alpha_i b - a \\ &= \tfrac{a}{\gamma}(1-r_i)\alpha_i b \end{aligned}$$

and so again $(1-r_1)\alpha_1 = (1-r_2)\alpha_2$, which gives the result with Equation (5).

$\square$

Theorem 4.4. *Let $A, B \in 2^{\mathbb{N}}$ be mutually regular and such that $\angle AB = 1$ and let $\dim A = a, \dim B = b > 0$. If $\widehat{\mathcal{H}}_{AB}$ and $\mathcal{T}$ are defined as above, then there is a linear isometry $\widehat{\varphi} : \widehat{\mathcal{H}}_{AB} \to \mathcal{T}$. Furthermore, this isometry is unique up to the interchange of the x and y coordinates in $\mathcal{T}$.*

Consequently, if $A', B' \in 2^{\mathbb{N}}$ are mutually regular and have the same respective dimensions as A and B, and if $\angle A'B' = 1$, then $\widehat{\mathcal{H}}_{A'B'}$ is linearly isometrically isomorphic to $\widehat{\mathcal{H}}_{AB}$.

We can express this by saying that if $\angle AB = 1$, then the geometry of $\widehat{\mathcal{H}}_{AB}$ is determined completely by the respective dimensions of A and B.

Proof. We define $\widehat{\varphi}$ by first defining

$$\varphi(r[(\alpha A)(\alpha B)]) = \langle \alpha a r, \alpha b(1-r)\rangle.$$

Then we let $\widehat{\varphi}(Y) = \varphi(X)$ where X is the unique element of $\mathcal{H}_{AB}$ such that $X \simeq_d Y$ (see Lemma 4.3 Part 5). That this is a bijection on the d-equivalence classes of $\widehat{\mathcal{H}}_{AB}$ is easy to confirm as $\varphi^{-1}(\langle x, y\rangle) = r[(\alpha A)(\alpha B)]$ where

$$r = \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{bx}{ay+bx} & \text{otherwise} \end{cases} \qquad \text{and} \qquad \alpha = \frac{ay+bx}{ab}.$$

That it is linear follows from the fact that $s\big(r[(\alpha A)(\alpha B)]\big) \simeq_d r[(s\alpha A)(s\alpha B)]$ and Lemma 4.3 Part 5.

We now show that φ preserves the directed metric. That is, that

$$\delta\big(\langle \alpha_1 a r_1, \alpha_1 b(1-r_1)\rangle \to \langle \alpha_2 a r_2, \alpha_2 b(1-r_2)\rangle\big) = d\big(r_1[(\alpha_1 A)(\alpha_1 B)] \to r_2[(\alpha_2 A)(\alpha_2 B)]\big).$$

The left hand side of this equation is just

$$\begin{aligned} \text{LHS} &= \max\{0, \alpha_2 a r_2 - \alpha_1 a r_1\} + \max\{0, \alpha_2 b(1-r_2) - \alpha_1 b(1-r_1)\} \\ &= a\max\{0, \alpha_2 r_2 - \alpha_1 r_1\} + b\max\{0, \alpha_2(1-r_2) - \alpha_1(1-r_1)\} \end{aligned}$$

And the right hand side we calculate using Lemma 4.3 Part 3:

$$
\begin{aligned}
\text{RHS} &= \lim_n \frac{C\big(r_2[(\alpha_2 A)(\alpha_2 B)]\restriction n \mid r_1[(\alpha_1 A)(\alpha_1 B)]\restriction n\big)}{n} \\
&= C^*\big(r_2\alpha_2 A, (1-r_2)\alpha_2 B, r_1\alpha_1 A, (1-r_1)\alpha_1 B\big) - C^*\big(r_1\alpha_1 A, (1-r_1)\alpha_1 B\big) \\
&= a\max\{\alpha_2 r_2, \alpha_1 r_1\} + b\max\{\alpha_2(1-r_2), \alpha_1(1-r_1)\} - a\alpha_1 r_1 - b\alpha_1(1-r_1) \\
&= a\max\{0, \alpha_2 r_2 - \alpha_1 r_1\} + b\max\{0, \alpha_2(1-r_2) - \alpha_1(1-r_1)\} \\
&= \text{LHS}.
\end{aligned}
$$

To see that $\widehat{\varphi}$ is unique up to swapping x and y coordinates, notice that for any map ψ that preserves angles and dimensions $\psi(A) = aP$ and $\psi(B) = bQ$, or $\psi(A) = aQ$ and $\psi(B) = bP$. If $X \in \widehat{\mathcal{H}}_{AB}$, then $\angle AX$ and $\dim(X)$ are preserved by ψ and the result follows.

$\square$

4.2 The General Case.

We now address the more general situation where C and D are mutually regular reals and $\angle CD < 1$.We will be guided by the question - what properties of C and D determine the geometry of $\widehat{\mathcal{H}}_{CD}$? If, as in the previous section, $\angle CD = 1$, then only the dimensions of C and D would be relevant, and these determined an essentially unique linear isometry into $\mathcal{T}$. In contrast, however, if $\angle CD < 1$, one would suspect intuitively that the dimensions of C and D alone would not be sufficient to determine the geometry of $\widehat{\mathcal{H}}_{CD}$ because there are many distinct ways to embed C and D into $\mathcal{T}$ preserving their dimensions and the angle between them.

In a first attempt to answer the question, we make use of a notion of *projection*. If R and S are two points in $\mathcal{T}$ such that $\angle POR \leqslant \angle POS$ and a line drawn through R parallel to the side $|OP|$ that intersects the ray $\overrightarrow{OS}$ at T, then we define *the projection of R onto S* to be

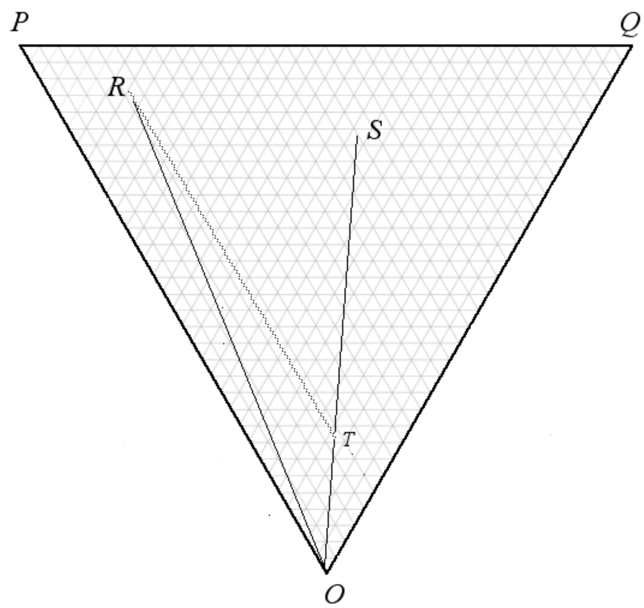
$$\mathrm{Proj}_S(R) := |OT|/|OS|.$$

See Figure 5. A similar definition gives the projection of S onto R. It is more useful to formally define projections in terms of the coordinates of R and S.

Definition 4.5. If $R, S \in \mathcal{T}$ with $R = \langle x_R, y_R \rangle$ and $Y = \langle x_S, y_S \rangle$ and such that (without losing generality) $\frac{y_R}{x_R} \leqslant \frac{y_S}{x_S}$ (equivalently $\angle POR \leqslant \angle POS$), then

$$\mathrm{Proj}_R(S) = \frac{x_S}{x_R},$$

Figure 5: $\mathrm{Proj}_S(R) := |OT|/|OS|$



and

$$\mathrm{Proj}_S(R) = \frac{y_R}{y_S},$$

Theorem 4.6. *For any two points $R,S \in \mathcal{T}$, the angle $\angle ROS$ is completely determined by the numbers $\mathrm{Proj}_R(S)$, $\mathrm{Proj}_S(R)$, $|OR|$, and $|OS|$, and given by the formula:*

$$\angle ROS = \frac{(s - \tau r)(r - \sigma s)}{rs(1 - \sigma\tau)},$$

where $r = |OR|$, $s = |OS|$, $\sigma = \mathrm{Proj}_S(R)$, and $\tau = \mathrm{Proj}_R(S)$.

Proof. This is a straightforward geometric calculation in $\mathcal{T}$. □

We can define a corresponding projection notion for any pair of mutually regular reals.

Definition 4.7. Let C, D be mutually regular such that $\dim C = c$ and $\dim D = d$. Then *the projection of C onto D* is defined to be

$$\mathrm{Proj}_D(C) = \max\{r : d\big(dC \to rdD\big) = 0\},$$

which exists by an elementary topological argument using the continuity of d.

At first it may seem preferable to define the projection of C onto D as $\max\{r : d\big(C \to rD\big) = 0\}$. This amounts to almost the same thing as $d(C \to rD) = 0$ if and only if

$d(dC \to rdD) = 0$. If this were the definition however, we would have to restrict the projection to having a value at most 1, which is not a restriction on the text definition (in which it may be arbitrarily large depending on the values of c and d and $\angle CD$). Similar comments of course apply to the projection of D onto C.

We have used the same notation and terminology for projections in $\mathcal{T}$ and $\widehat{\mathcal{H}}_{CD}$. This is justified by the next lemma.

Lemma 4.8. *Linear isometries preserve projections, that is for every pair of mutually regular reals C and D, and any linear isometry $\psi : \widehat{\mathcal{H}}_{CD} \to \mathcal{T}$,*

$$\mathrm{Proj}_C(D) = \mathrm{Proj}_{\psi(C)}(\psi(D)).$$

Proof. This is just a straightforward application of the definitions. □

The following fact is the essential point in answering question 4.1

Theorem 4.9. *The angle between two mutually regular reals C and D is not determined by the four numbers $\mathrm{Proj}_C(D), \mathrm{Proj}_D(C), \dim C,$ and $\dim D$. In particular, there are pairs of mutually regular reals X and Y for which the formula in Theorem 4.6 does not hold. As distances, angles and projections are preserved by linear isometries, this means there can be no linear isometry from $\widehat{\mathcal{H}}_{XY}$ into $\mathcal{T}$.*

Proof. Let $R = r_0 r_1 r_2 \dots r_n \dots$ be a random real and let

$$X = r_0 r_2 r_4 \dots r_{2n} \dots$$

$$Y = r_0 r_3 r_6 \dots r_{3n} \dots .$$

Then the reader can confirm that

1. X and Y are mutually regular
2. $\dim X = \dim Y = 1$
3. $\angle XY = d(Y \to X) = 2/3$
4. $\mathrm{Proj}_X(Y) = \mathrm{Proj}_Y(X) = 0$

and that these values violate the formula in Theorem 4.6. □

These two reals now give an answer to Question 4.1.

Corollary 4.10. *There is a pair of mutually regular reals X, Y for which there does not exist a pair of mutually random reals R_1, R_2 with $X, Y \in \widehat{\mathcal{H}}_{R_1 R_2}$.*

The fact that all linear isometries preserve angles and projections means that if there is a linear isometry from an extended hull $\widehat{\mathcal{H}}_{CD}$ into $\mathcal{T}$, then the formula in Theorem 4.6 is necessarily respected in $\widehat{\mathcal{H}}_{CD}$. We now show that this is also sufficient that this formula be respected for there to be a linear isometry from $\widehat{\mathcal{H}}_{CD}$ into $\mathcal{T}$.

Theorem 4.11. *If C and D are mutually regular and with nonzero dimensions and $\angle CD > 0$, then there is a linear isometry from $\widehat{\mathcal{H}}_{CD}$ into $\mathcal{T}$ if and only if*

$$\angle CD = \frac{(d - \tau c)(c - \sigma d)}{cd(1 - \sigma\tau)}, \tag{6}$$

where $c = \dim C$, $d = \dim D$, $\sigma = \mathrm{Proj}_D(C)$, and $\tau = \mathrm{Proj}_C(D)$.

Proof. The *only if* direction is immediate as linear isometries preserve angles and projections. For the *if* direction we define a linear isometry $\Phi : \widehat{\mathcal{H}}_{CD} \to \mathcal{T}$ directly. First let $\Phi(C) = \langle x_C, y_C \rangle$ and $\Phi(D) = \langle x_D, y_D \rangle$ where

$$x_C = \frac{\sigma(d - c\tau)}{1 - \sigma\tau} \qquad\qquad y_C = \frac{c - d\sigma}{1 - \sigma\tau}$$
$$x_D = \frac{d - c\tau}{1 - \sigma\tau} \qquad\qquad y_D = \frac{\tau(c - d\sigma)}{1 - \sigma\tau}$$

All angles distances and projections must be preserved by Φ and these are the only possible values for the images of C and D except for transposing the x and y coordinates. To eliminate this symmetry we assume, without losing generality, that $x_D/d > x_C/c$ and that consequently

$$y_C x_D > x_C y_D,\ \ y_C > 0,\ \ x_D > 0 \text{ and } \frac{x_D}{x_C + x_D} > \frac{y_D}{y_C + y_D}. \tag{7}$$

We now extend Φ to $\mathcal{H}_{CD}$ and from there to $\widehat{\mathcal{H}}_{CD}$. If $X = r[(aC)(aD)]$, we define

$$\Phi(X) \quad = \quad \big\langle a\max\{rx_C, (1-r)x_D\}, a\max\{ry_C, (1-r)y_D\}\big\rangle \tag{8}$$

$$= \begin{cases} \big\langle a(1-r)x_D, a(1-r)y_D \big\rangle & \text{if } r \leqslant \frac{y_D}{y_C + y_D} \\ \big\langle a(1-r)x_D, ary_C \big\rangle & \text{if } \frac{y_D}{y_C+y_D} \leqslant r \leqslant \frac{x_D}{x_C+x_D} \\ \big\langle arx_C, ary_C \big\rangle & \text{if } r \geqslant \frac{x_D}{x_C+x_D} \end{cases} \tag{9}$$

One can confirm that, as expected, this agrees with the values of $\Phi(C)$ and $\Phi(D)$ when $a = 1$ and r takes the values 1 and 0 respectively. If $Y \in \widehat{\mathcal{H}}_{CD}$ then we define

$\Phi(Y) = \Phi(X)$ where X is any element of $\mathcal{H}_{CD}$ with $Y \simeq_d$ X. For the definition of $\Phi(Y)$ to be coherent, we have to show that Φ is well-defined on the d-equivalence classes of $\mathcal{H}_{CD}$. This is not as straightforward as the earlier situation in Section 4.1 where we had $\angle CD = 1$, and in fact Lemma 4.3 Part 5 doesn't hold when $\angle CD < 1$.

Lemma 4.12. *Φ is well-defined on the equivalence classes of $\mathcal{H}_{CD}$.*

Proof. Let $X_i = r_i[(a_iC)(a_iD)]$ for $i \in \{1, 2\}$, and suppose $X_1 \simeq_d X_2$. We will break the proof into cases.

Case 1. $r_1 \leqslant \frac{y_D}{y_C+y_D}$ and $r_2 \geqslant \frac{x_D}{x_C+x_D}$ (or vice-versa). This contradicts the fact that $\angle CD > 0$ by the following calculation. Let $\gamma = \dim X_2$.

$$\begin{aligned} \angle CX_2 &= \tfrac{1}{c\gamma}d(\gamma C \to cX_2) \\ &= \tfrac{1}{c\gamma}C^*(\gamma C, cX_2) - 1 \\ &= \tfrac{1}{c\gamma}C^*(\max\{\gamma, ca_2r_2\}C, ca_2(1-r_2)D) - 1 \end{aligned}$$

But $\gamma \geqslant a_2r_2c$ as

$$\gamma = \dim(X_2) = C^*(a_2r_2C, a_2(1-r_2)D) \geqslant C^*(a_2r_2C) = a_2r_2c.$$

Therefore

$$\begin{aligned} \angle CX_2 &= \tfrac{1}{c\gamma}C^*(\gamma C, ca_2(1-r_2)D) - 1 \\ &= \tfrac{1}{c\gamma}d(\gamma C \to ca_2(1-r_2)D). \end{aligned}$$

But $\gamma \geqslant a_2r_2c$ and $r_2 \geqslant \frac{x_D}{x_C+x_D}$ together imply

$$ca_2(1-r_2) \leqslant \gamma\frac{1-r_2}{r_2} \leqslant \gamma\sigma$$

and so $\angle CX_2 = 0$ by the definition of σ. A similar argument shows that $\angle DX_1 = 0$ and as

$$\angle CD \leqslant \angle CX_2 + \angle X_2X_1 + \angle DX_1 = 0$$

we get the required contradiction.

Case 2. $r_1, r_2 \leqslant \frac{y_D}{y_C+y_D}$ or $r_1, r_2 \geqslant \frac{x_D}{x_C+x_D}$. We show only the former as the latter is symmetrical. We have

$$\begin{aligned} \dim X_i &= C^*(a_ir_iC, a_i(1-r_i)D) \\ &= a_i(1-r_i)d(D \to \tfrac{r_i}{1-r_i}C) + a_i(1-r_i)d. \end{aligned}$$

But as $r_i \leqslant \frac{y_D}{y_C+y_D}$, we have $\frac{r_i}{1-r_i} \leqslant \frac{y_D}{y_D} = \tau$. Thus $d(D \to \frac{r_i}{1-r_i}C) = 0$ by the definition of τ. Therefore $\dim X_i = a_i(1-r_i)d$. As $X_1 \simeq_d X_2$ and thus $\dim X_1 = \dim X_2$, we get $a_1(1-r_1) = a_2(1-r_2)$. Therefore $\Phi(X_1) = \Phi(X_2)$.

Case 3. $r_1, r_2 \in (\frac{y_D}{y_C+y_D}, \frac{x_D}{x_C+x_D})$. Without losing generality suppose $a_1(1-r_1) \geqslant a_2(1-r_2)$. Consider the distance $d(a_2(1-r_2)C \to \sigma a_1(1-r_1)D)$:

$$
\begin{aligned}
d(a_2(1-r_2)C \to \sigma a_1(1-r_1)D) \quad &\leqslant \quad d(a_2(1-r_2)C \to \sigma X_2) + d(\sigma X_2 \to \sigma X_1) + \\
&\qquad\qquad\qquad\qquad d(\sigma X_1 \to \sigma a_1(1-r_1)D)
\end{aligned}
$$

But the last two terms on the RHS are 0 as $X_1 \simeq_d X_2$ and

$$d(X_1 \to a_1(1-r_1)D) = C^*(a_1 r_1 C, a_1(1-r_1)D) - \dim X_1 = 0.$$

Therefore

$$d(a_2(1-r_2)C \to \sigma a_1(1-r_1)D) \leqslant C^*(a_2 \max\{1-r_2, \sigma r_2\}C, \sigma a_2(1-r_2)D) - a_2(1-r_2)c.$$

But $r_2 \leqslant \frac{x_D}{x_C+x_D}$ implies that $r_2\sigma \leqslant 1-r_2$, and so

$$
\begin{aligned}
d(a_2(1-r_2)C \to \sigma a_1(1-r_1)D) \quad &\leqslant \quad a_2(1-r_2)C^*(C, \sigma D) - a_2(1-r_2)c \\
&= \quad a_2(1-r_2)d(C \to \sigma D) \\
&= \quad 0.
\end{aligned}
$$

But $d(a_2(1-r_2)C \to \sigma a_1(1-r_1)D) = 0$ if and only if $d(C \to \sigma \frac{a_1(1-r_1)}{a_2(1-r_2)} D) = 0$ if and only if $a_1(1-r_1) \leqslant a_2(1-r_2)$ by the definition of σ. Hence from our original assumption, $a_1(1-r_1) = a_2(1-r_2)$. An entirely symmetrical argument can be used to show that $a_1 r_1 = a_2 r_2$. First it is shown that $d(a_2 r_2 D \to \tau a_1 r_1 C) = 0$ and then using the definition of τ that $a_1 r_1 = a_2 r_2$. Therefore $X_1 = X_2$ and $\varphi(X_1) = \Phi(X_2)$.

$\square$

It is now left to prove that Φ is a linear isometry.

Lemma 4.13. *Let $X_1, X_2 \in \widehat{\mathcal{H}}_{CD}$. If $\Phi(X_1) = \Phi(X_2)$, then $X_1 \simeq_d X_2$.*

Proof. It is enough to prove this for $X_1, X_2 \in \mathcal{H}_{CD}$. We first deal with the situation where $\Phi(X_i) = \mathrm{O}$ with $X_i = r_i[(a_i C)(a_i D)]$ and $i \in \{1, 2\}$. We look at all three possible values for $\Phi(X_i)$ according to (9). If $r_i \leqslant \frac{y_D}{y_C+y_D}$, then $\big\langle a_i(1-r_i)x_D, a_i(1-r_i)y_D \big\rangle = \langle 0, 0\rangle$. But $r_i < 1$ as by (7) above $y_C > 0$. Thus $a_i = 0$ and $X_i = \mathbf{0}$. If $\frac{y_D}{y_C+y_D} \leqslant r_i \leqslant \frac{x_D}{x_C+x_D}$, then $\big\langle a_i(1-r_i)x_D, a_i r_i y_C \big\rangle = \langle 0, 0\rangle$ and thus $a_i = 0$ and $X_i = \mathbf{0}$. If $r_i \geqslant \frac{x_D}{x_C+x_D}$, then $\big\langle a_i r_i x_C, a_i r_i y_C \big\rangle = \langle 0, 0\rangle$. But $r_i \neq 0$ as $x_D \neq 0$ and so $a_i = 0$ and $X_i = \mathbf{0}$.

Now we deal the general situation. Let $X_i = r_i[(a_i C)(a_i D)]$ and $\Phi\big(X_1\big) = \Phi\big(X_2\big)$ with $i \in \{1, 2\}$. We divide into four cases. Every possible situation falls into one of these cases or is symmetrical to one of them.

Case 1. $\frac{y_D}{y_C+y_D} \leqslant r_1, r_2 \leqslant \frac{x_D}{x_C+x_D}$. As above $x_D \neq 0$ and $y_C \neq 0$. Thus

$$\Phi\big(X_1\big) = \langle a_1(1-r_1)x_D, a_1 r_1 y_C\rangle = \langle a_2(1-r_2)x_D, a_2 r_2 y_C\rangle = \Phi\big(X_2\big)$$

which implies $a_1 = a_2$ and $r_1 = r_2$ and $X_1 = X_2$.

Case 2. $r_1, r_2 \leqslant \frac{y_D}{y_C+y_D}$. Then

$$\Phi\big(X_1\big) = \langle a_1(1-r_1)x_D, a_1(1-r_1)y_C\rangle = \langle a_2(1-r_2)x_D, a_2(1-r_2)y_C\rangle = \Phi\big(X_2\big).$$

so we can conclude only that $a_1(1-r_1) = a_2(1-r_2)$. Without losing generality assume that $a_1 r_1 \leqslant a_2 r_2$. Now

$$\begin{aligned}
d\big(X_2 \to X_1\big) &= C^*\big(X_1, X_2\big) - \dim\big(X_2\big) \\
&= C^*\big(a_1 r_1 C, a_1(1-r_1)D, a_2 r_2 C, a_2(1-r_2)D\big) - C^*\big(a_2 r_2 C, a_2(1-r_2)D\big) \\
&= C^*\big(a_2 r_2 C, a_2(1-r_2)D\big) - C^*\big(a_2 r_2 C, a_2(1-r_2)D\big) \quad (\text{as } a_1 r_1 \leqslant a_2 r_2) \\
&= 0
\end{aligned}$$

To calculate $d\big(X_1 \to X_2\big)$ first note that for each X_i

$$\begin{aligned}
\dim(X_i) &= C^*\big(a_i r_i C, a_i(1-r_i)D\big) \\
&= d\big(a_i(1-r_i)D \to a_i r_i C\big) + a_i(1-r_i)d \\
&= a_i(1-r_i)d\big(D \to \frac{r_i}{(1-r_i)}C\big) + a_i(1-r_i)d
\end{aligned}$$

But as $r_i \leqslant \frac{y_D}{y_C+y_D} < 1$,

$$\frac{r_i}{(1-r_i)} \leqslant \frac{y_D}{y_C+y_D} \cdot \frac{y_C+y_D}{y_C} = \frac{y_D}{y_C} = \tau.$$

Thus by the definition of τ, $d\big(D \to \frac{r_i}{(1-r_i)}C\big) = 0$. So for each i, $\dim\big(X_i\big) = a_i(1-r_i)d$ and therefore $\dim\big(X_1\big) = \dim\big(X_2\big)$. As $d\big(X_1 \to X_2\big) = d\big(X_2 \to X_1\big) + \dim\big(X_2\big) - \dim\big(X_1\big)$ and $d\big(X_2 \to X_1\big) = 0$, the result follows.

Case 3. $r_1 \leqslant \frac{y_D}{y_C+y_D} \leqslant r_2 \leqslant \frac{x_D}{x_C+x_D}$. In this case

$$\Phi\big(X_1\big) = \langle a_1(1-r_1)x_D, a_1(1-r_1)y_D\rangle = \langle a_2(1-r_2)x_D, a_2 r_2 y_C\rangle = \Phi\big(X_2\big).$$

So that $a_1(1-r_1) = a_2(1-r_2)$ and $a_1(1-r_1)y_D = a_2 r_2 y_C$. But from these equations we deduce that $r_2 = \frac{y_D}{y_C+y_D}$ and we are actually in the previous case.

Case 4. $r_1 \leqslant \frac{y_D}{y_C+y_D} \leqslant \frac{x_D}{x_C+x_D} \leqslant r_2$. Then

$$\Phi\big(X_1\big) = \langle a_1(1-r_1)x_D, a_1(1-r_1)y_D\rangle = \langle a_2 r_2 x_C, a_2 r_2 y_C\rangle = \Phi\big(X_2\big).$$

But this implies that $y_C x_D = x_C y_D$, contradicting (7).

The rest of the cases can be dealt with by exchanging r with $1-r$. □

Lemma 4.14. Φ *is linear.*

Proof. Let $X = r[(aC)(aD)]$. If $\alpha \in [0,1]$, then $\alpha X \simeq_d r[(\alpha a C)(\alpha a D)]$. By inspection of the definition of $\Phi(\alpha X)$ and Lemma 4.13, this now follows. □

Lemma 4.15. *There is a positive number $\gamma \leqslant 1$ such that for all $\mu \leqslant \gamma$ and all $x \in [\frac{\mu x_C}{c}, \frac{\mu x_D}{d}]$,*

$$\langle x, \mu - x\rangle \in \text{range } \Phi.$$

That is, the triangular region bordered by the line segments $\overline{\mathbf{0}D}$, $\overline{\mathbf{0}C}$ and the horizontal line $\overline{\frac{\gamma}{d}D\frac{\gamma}{c}C}$ lies completely within the range of Φ.

Proof. Let $\gamma = \min\{\frac{c x_D}{x_C+x_D}, \frac{d y_C}{y_C+y_D}\}$ and $\mu \leqslant \gamma$. Given $x \in [\frac{\mu x_C}{c}, \frac{\mu x_D}{d}]$, let

$$r = \frac{(\mu - x)x_D}{x y_C + (\mu - x)x_D} \qquad\qquad a = \frac{x y_C + (\mu - x)x_D}{x_D y_C}.$$

We will now show, using the bounds on x and μ, that

$$a \in [0,1] \qquad\qquad r \in \left[\frac{y_D}{y_C + y_D}, \frac{x_D}{x_C + x_D}\right]$$

and that

$$\langle x, \mu - x\rangle = \big\langle a(1-r)x_D, a r y_C\big\rangle,$$

so that we are in the second case of (9) and $\langle x, \mu - x\rangle = \Phi\big(r[(aC)(aD)]\big)$ as required. The calculations are quite straightforward and we present here only the proof that $a \leqslant 1$.

$$\begin{aligned} a &= \frac{x y_C + (\mu - x)x_D}{x_D y_C} \\ &= \frac{x(y_C - x_D) + \mu x_D}{x_D y_C} \end{aligned}$$

If $y_C \geqslant x_D$ then, as $x \leqslant \frac{\mu x_D}{d}$,

$$\begin{aligned}
a &\leqslant \frac{\frac{\mu x_D}{d}(y_C - x_D) + \mu x_D}{x_D y_C} \\
&= \frac{\mu(y_C - x_D + d)}{d y_C} \\
&= \mu \cdot \frac{y_C + y_D}{d y_C} \qquad\qquad \text{(as } x_D + y_D = d\text{)} \\
&\leqslant 1.
\end{aligned}$$

If however $y_C \leqslant x_D$, then as $x \geqslant \frac{\mu x_C}{c}$

$$\begin{aligned}
a &\leqslant \frac{\frac{\mu x_C}{c}(y_C - x_D) + \mu x_D}{x_D y_C} \\
&= \mu \cdot \frac{x_C(y_C - x_D) + c x_D}{c x_D y_C} \\
&= \mu \cdot \frac{x_C + x_D}{c x_D} \qquad\qquad \text{(as } x_C + y_C = c\text{)} \\
&\leqslant 1.
\end{aligned}$$

□

Corollary 4.16 (Procrustean Principle)**.** *For any $\langle x, y\rangle \in [0,\infty)^2$ such that $\frac{x_C}{c} \leqslant \frac{x}{x+y} \leqslant \frac{x_D}{d}$, there is an $\alpha \in \mathbb{R}^+$ such that $\langle \alpha x, \alpha y\rangle \in \operatorname{range} \Phi$.*

Proof. Take $\alpha = \frac{\gamma}{x+y}$, where γ is as in Lemma 4.15. Then $\frac{\gamma x}{x+y} \in [\frac{\gamma x_C}{c}, \frac{\gamma x_D}{d}]$ and $\langle \alpha x, \alpha y\rangle = \langle \frac{\gamma x}{x+y}, \gamma - \frac{\gamma x}{x+y}\rangle$ as required to apply the lemma. □

We will use the Procrustean Principle to simplify calculations of distances of elements in $\widehat{\mathcal{H}}_{CD}$. Suppose $X, Y \in \widehat{\mathcal{H}}_{CD}$, and we want to work out $d(X \to Y)$. To do this we take $\Phi(X)$ and $\Phi(Y)$ and construct using their coordinates a third point $Z \in \mathcal{T}$. We want to consider $\Phi^{-1}(Z)$ but have no guarantee that such an element of $\widehat{\mathcal{H}}_{CD}$ exists. So we take using the Procrustean principle a γ such that $\Phi^{-1}(\gamma Z)$ exists and instead compute the distance from γX to γY. The corresponding construction point will be γZ and then we use the fact that $d(X \to Y) = \frac{1}{\gamma} d(\gamma X \to \gamma Y)$. The γ factor will not affect the calculations and will be ommitted, and $\Phi^{-1}(Z)$ will simply be assumed to exist.

The next lemma allows us to eliminate many of the cases when using the definition of Φ in calculations. It also involves introducing a constant γ that scales down the calculations to a point that provides us convenient shortcuts. In this case we show that we can usually assume without losing generality that if $r[(aC)(aD)] \in \widehat{\mathcal{H}}_{CD}$, then $r \in \left[\frac{y_D}{y_C+y_D}, \frac{x_D}{x_C+x_D}\right]$.

Lemma 4.17. *For all $X \simeq_d r[(aC)(aD)] \in \widehat{\mathcal{H}}_{CD}$ and for any $\gamma \leqslant \min\{\frac{y_C}{y_C+y_D}, \frac{x_D}{x_C+x_D}\}$, there is some $r' \in \left[\frac{y_D}{y_C+y_D}, \frac{x_D}{x_C+x_D}\right]$ and some $a' \in [0,1]$, such that $\gamma X \simeq_d r'[(a'C)(a'D)]$.*

Proof. Let X and γ be as given. If $r \in \left[\frac{y_D}{y_C+y_D}, \frac{x_D}{x_C+x_D}\right]$, then

$$\gamma X \simeq_d r[(\gamma aC)(\gamma aD)].$$

If $r < \frac{y_D}{y_C+y_D}$, then let $r' = \frac{y_D}{y_C+y_D}$, and $a' = \frac{\gamma a(1-r)(y_C+y_D)}{y_C}$. Then

$$\begin{aligned}
\Phi(\gamma X) &= \langle \gamma a(1-r)x_D, \gamma a(1-r)y_D\rangle \\
&= \langle \tfrac{y_C}{y_C+y_D} a' x_D, \tfrac{y_C}{y_C+y_D} a' y_D\rangle \\
&= \Phi(r'[(a'C)(a'D)]).
\end{aligned}$$

So $\gamma x \simeq_d r'[(a'C)(a'D)]$ by Lemma 4.13. If $r > \frac{x_D}{x_C+x_D}$, then let $r' = \frac{x_D}{x_C+x_D)}$ and $a' = \frac{\gamma ar(x_C+x_D)}{x_D}$ and argue similarly. In both cases the bound on γ guarantees that $a' \leqslant 1$.

□

The next lemma shows that in order to establish that Φ preserves the directed pseudometric on $\widehat{\mathcal{H}}_{CD}$, it is enough to show that it preserves the distance from the origin.

Lemma 4.18. *If Φ has the property that for all $X \in \widehat{\mathcal{H}}_{CD}$ with $\Phi(X) = \langle x, y\rangle$*

$$\dim(X) = x + y,$$

then for all $X_1, X_2 \in \widehat{\mathcal{H}}_{CD}$ with $\Phi(X_1) = \langle x_1, y_1\rangle$ and $\Phi(X_2) = \langle x_2, y_2\rangle$,

$$d(X_1 \to X_2) = \max\{0, x_2 - x_1\} + \max\{0, y_2 - y_1\}.$$

Proof. Let i range over $\{1, 2\}$ and let Z be such that $\Phi(Z) = \langle \max_i\{x_i\}, \max_i\{y_i\}\rangle$. The Procrustean Principle allows us to assume that such a Z exists (it is straightforward to show that for all $\langle x, y\rangle \in$ range Φ, $\frac{x_C}{c} \leqslant \frac{x}{x+y} \leqslant \frac{x_D}{d}$ and hence that $\frac{x_C}{c} \leqslant \frac{\max_i\{x_i\}}{\max_i\{x_i\}+\max_i\{y_i\}} \leqslant \frac{x_D}{d}$ and that the principle can be applied to Z. The proof is omitted here). Let $X_i \simeq_d r_i[(a_iC)(a_iD)]$. We can use Lemma 4.17 to assume without losing generality that $\frac{y_D}{y_C+y_D} \leqslant r_i \leqslant \frac{x_D}{x_C+x_D}$ and so $\Phi(X_i) = \langle a_i(1-r_i)x_D, a_i r_i y_C\rangle$. Now let

$$a := \max_i\{a_i r_i\} + \max_i\{a_i(1-r_i)\}$$

and

$$r := \frac{\max_i\{a_i r_i\}}{\max_i\{a_i r_i\} + \max_i\{a_i(1-r_i)\}}.$$

Then $\frac{y_D}{y_C+y_D} \leqslant r \leqslant \frac{x_D}{x_C+x_D}$ (we omit the straightforward proof that, in fact, $\frac{y_D}{y_C+y_D} \leqslant \min\{r_i\} \leqslant r \leqslant \max_i\{r_i\} \leqslant \frac{x_D}{x_C+x_D}$) and so

$$\begin{aligned}
\Phi\big(r[a(C)(aD)]\big) &= \langle a(1-r)x_D, ary_C\rangle \\
&= \langle \max_i\{a_i(1-r_i)\}x_D, \max_i\{a_i r_i\}y_C\rangle \\
&= \langle \max_i\{x_i\}, \max_i\{y_i\}\rangle \\
&= \Phi(Z).
\end{aligned}$$

Thus $Z \simeq_d a[(rC)(rD)]$ and

$$\dim\big(Z\big) = C^*\big(\max_i\{a_i r_i\}C, \max_i\{a_i(1-r_i)\}D\big).$$

Finally

$$\begin{aligned}
d\big(X_1 \to X_2\big) &= C^*\big(X_1, X_2\big) - \dim\big(X_1\big) \\
&= C^*\big(\max_i\{a_i r_i\}C, \max_i\{a_i(1-r_i)\}D\big) - \dim\big(X_1\big) \\
&= \dim\big(Z\big) - \dim\big(X_1\big) \\
&= \max_i\{x_i\} + \max_i\{y_i\} - x_1 - y_1 \qquad \text{(by our original assumption)} \\
&= \max\{0, x_2 - x_1\} + \max\{0, y_2 - y_1\}.
\end{aligned}$$

□

Lemma 4.19. *If $X_i \simeq_d r_i[(a_iC)(a_iD)]$ for $i \in \{1,2\}$, then if $x_1 = x_2$ or $y_1 = y_2$,*

$$d(X_1 \to X_2) = \max\{0, \dim(X_2) - \dim(X_1)\}.$$

Proof. We can assume without losing generality that $x_1 = x_2$ and $y_1 \geqslant y_2$. It is enough then to prove that $d\big(X_1 \to X_2\big) = 0$ as then $d\big(X_2 \to X_1\big) = d\big(X_1 \to X_2\big) + \dim X_1 - \dim X_2 = \dim X_1 - \dim X_2$.

We can use Lemma 4.17 to assume that $r_1, r_2 \in [\frac{y_D}{y_C+y_D}, \frac{x_D}{x_C+x_D}]$, and therefore that $\Phi\big(X_i\big) = \langle a_i(1-r_i)x_D, a_i r_i y_C\rangle$. Now

$$a_1(1-r_1) = a_2(1-r_2) \text{ and } a_1 r_1 \geqslant a_2 r_2.$$

Thus

$$\begin{aligned}
d(X_1 \to X_2) &= C^*(a_1r_1C, a_1(1-r_1)D, a_2r_2C, a_2(1-r_2)D) - C^*(a_1r_1C, a_1(1-r_1)D) \\
&= C^*(a_1r_1C, a_1(1-r_1)D) - C^*(a_1r_1C, a_1(1-r_1)D) \\
&= 0
\end{aligned}$$

□

Lemma 4.20. *For any $\alpha \leqslant 1$, the function $x \mapsto \dim \Phi^{-1}\langle x, \alpha - x\rangle$ is continuous at any $\langle x, \alpha - x\rangle \in \operatorname{range} \Phi$.*

Proof. It is sufficient to prove this for $\alpha = \gamma$ as defined in Lemma 4.15, and $x \in [\frac{\gamma x_C}{c}, \frac{\gamma x_D}{d}]$. Let $i \in \{1, 2\}$ and $\Phi(X_i) = \langle x_i, \gamma - x_i\rangle$. We show that $\dim X_1 \to \dim X_2$ as $x_1 \to x_2$.

Suppose a_i and $r_i \in \left[\frac{y_D}{y_C + y_D}, \frac{x_D}{x_C + x_D}\right]$ are such that $X_i \simeq_d r_i[(a_i C)(a_i D)]$ for $i \in \{1, 2\}$. As $r_i \in \left[\frac{y_D}{y_C + y_D}, \frac{x_D}{x_C + x_D}\right]$ and the sum of the coordinates of $\Phi(X_i)$ is γ,

$$a_1 r_1 \geqslant a_2 r_2 \iff a_1(1 - r_1) \leqslant a_2(1 - r_2).$$

We assume without losing generality that $a_1 r_1 \geqslant a_2 r_2$.

Now

$$\begin{aligned}
\dim X_1 - \dim X_2 &\leqslant d(X_2 \to X_1) \\
&= C^*(a_1 r_1 C, a_2(1 - r_2)D) - C^*(a_2 r_2 C, a_2(1 - r_2)D) \\
&= d(a_2(1 - r_2)D \to a_1 r_1 C) - d(a_2(1 - r_2)D \to a_2 r_2 C) \\
&\leqslant d(a_2 r_2 C \to a_1 r_1 C) \\
&= (a_1 r_1 - a_2 r_2)c.
\end{aligned}$$

But $a_1 r_1 - a_2 r_2 \to 0$ as $x_1 \to x_2$.

□

Now we are able to provide the final proof for Theorem 4.11

Theorem 4.21. *If C, D satisfy Equation (6), then for all $X_1, X_2 \in \widehat{\mathcal{H}}_{CD}$ with $\Phi(X_i) = \langle x_i, y_i\rangle$*

$$d(X_1 \to X_2) = \max\{0, x_2 - x_1\} + \max\{0, y_2 - y_1\}.$$

Proof. By Lemma 4.18 it is enough to show that the dimension of any $X \in \widehat{\mathcal{H}}_{CD}$ is the sum of the coordinates of $\Phi(X)$. By the Procrustean Principle it is enough to show that this is true for any real $X = \Phi^{-1}(\langle x, \gamma - x\rangle)$ where γ is sufficiently small (but positive) and $x \in [\frac{\gamma x_C}{c}, \frac{\gamma x_D}{d}]$. How small we need to choose γ will be determined during the proof. We begin by assuming $\gamma \leqslant \min\{\frac{c x_D}{x_C + x_D}, \frac{d y_D}{y_C + y_D}\}$ as in Lemma 4.15.

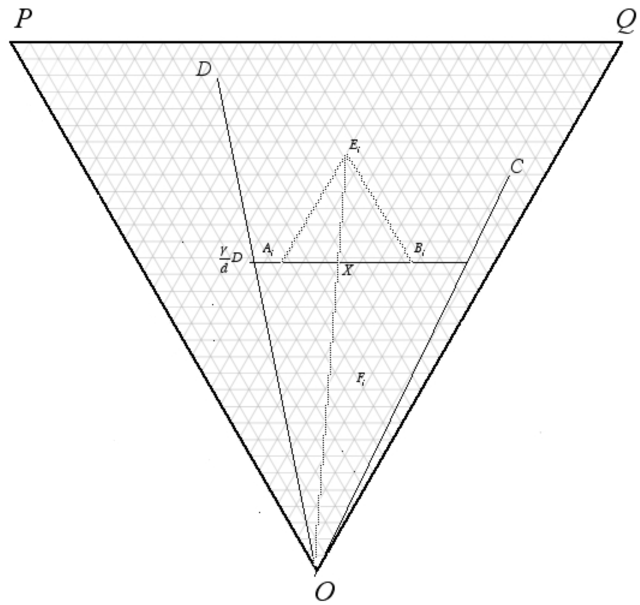
For convenience let $a = \frac{\gamma x_C}{c}$ and $b = \frac{\gamma x_D}{d}$ so that $x \in [a, b]$. We need to show that $\dim \Phi^{-1}(\langle x, \gamma - x\rangle) = \gamma$ for all $x \in [a, b]$. Now consider the sets

$$S = \{x \in [a, b] : \dim \left(\Phi^{-1}(\langle x, \gamma - x\rangle\right) < \gamma\}$$

$$S^c = \{x \in [a, b] : \dim \left(\Phi^{-1}(\langle x, \gamma - x\rangle\right) \geqslant \gamma\}.$$

S is open and S^c closed by Lemma 4.20 and so S is a countable union of pairwise disjoint open intervals (a_i, b_i), and S^c is the union of an isolated set of points with a countable union of pairwise disjoint non-degenerate closed intervals $[c_k, d_k]$.

Figure 6: A representation of the argument in Theorem 4.21. Letters represent their images under Φ in $\mathcal{T}$ and $X = \frac{\gamma}{\gamma+b_i-a_i}\Phi(E_i)$.



Consider an arbitrary $(a_i, b_i) \subseteq S$. Let $A_i = \Phi^{-1}\big(\langle a_i, \gamma - a_i\rangle\big)$, $B_i = \Phi^{-1}\big(\langle b_i, \gamma - b_i\rangle\big)$ and $E_i = \Phi^{-1}\big(\langle b_i, \gamma - a_i\rangle\big)$. We can take γ to be chosen small enough so that all the E_i exist (in fact the sum of the coordinates of $\Phi(E_i)$, $b_i + \gamma - a_i \leqslant \gamma x_D + \gamma y_C$ so we can chose $\gamma \leqslant \frac{1}{x_D+y_C}\min\{\frac{cx_D}{x_C+x_D}, \frac{dy_D}{y_C+y_D}\}$ to ensure this).

Now

$$d\big(A_i \to B_i\big) \leqslant d\big(A_i \to E_i\big) + d\big(E_i \to B_i\big),$$

but by Lemma 4.19 and the fact that $b_i > a_i$, $d\big(A_i \to E_i\big) = \dim E_i - \dim A_i$ and $d\big(E_i \to B_i\big) = 0$. As a_i is the endpoint of an interval in S, Lemma 4.20 implies that $\dim A_i = \gamma$ so we have that

$$d\big(A_i \to B_i\big) \leqslant \dim E_i - \gamma.$$

But now consider the point

$$\left\langle \frac{\gamma b_i}{\gamma + b_i - a_i}, \gamma - \frac{\gamma b_i}{\gamma + b_i - a_i} \right\rangle = \frac{\gamma}{\gamma + b_i - a_i}\Phi\big(E_i\big) = \Phi\left(\frac{\gamma}{\gamma + b_i - a_i}E_i\right),$$

which is the intersection of the line $\overline{O\Phi(E_i)}$ with the line $\overline{\Phi(A_i)\Phi(B_i)}$. See Figure 6.

It is straightforward to show that $\frac{\gamma b_i}{\gamma+b_i-a_i} \in (a_i, b_i)$ and so

$$\dim \Phi^{-1}\left\langle \frac{\gamma b_i}{\gamma + b_i - a_i}, \gamma - \frac{\gamma b_i}{\gamma + b_i - a_i} \right\rangle < \gamma.$$

Hence $\dim \frac{\gamma}{\gamma+b_i-a_i} E_i < \gamma$ and

$$\dim E_i < \gamma + b_i - a_i.$$

So we have finally that $d(A_i \to B_i) < b_i - a_i$.

There is also however a similar argument to show that if $[c_k, d_k]$ is one of the above closed intervals and

$$C_k := \Phi^{-1}\big(\langle c_k, \gamma - c_k\rangle\big) \text{ and } D_k := \Phi^{-1}\big(\langle d_k, \gamma - d_k\rangle\big),$$

then

$$d\big(C_k \to D_k\big) \leqslant d_k - c_k.$$

To see this consider the point $F_k = \Phi^{-1}\big(\langle c_k, \gamma - d_k\rangle\big)$. Our choice of γ and Lemma 4.15 guarantees that F_k exists as $c_k + \gamma - d_k < \gamma$. Then $d\big(C_k \to D_k\big) \leqslant d\big(C_k \to F_k\big) + d\big(F_k \to D_k\big)$. But $d\big(C_k \to F_k\big) = 0$ and $d(F_k \to D_k\big) = \dim D_k - \dim F_k = \gamma - \dim F_k$ by the previous argument. Therefore $d\big(C_k \to D_k\big) \leqslant \gamma - \dim F_k$. But now the point

$$\left\langle \frac{\gamma c_k}{c_k + \gamma - d_k}, \gamma - \frac{\gamma c_k}{c_k + \gamma - d_k} \right\rangle = \frac{\gamma}{c_k + \gamma - d_k} \Phi\big(F_k\big) = \Phi\left(\frac{\gamma}{c_k + \gamma - d_k} F_k\right)$$

lies between $\Phi\big(C_k\big)$ and $\Phi\big(D_k\big)$ as $\frac{\gamma c_k}{c_k+\gamma-d_k} \in [c_k, d_k]$. Therefore

$$\dim \frac{\gamma}{c_k + \gamma - d_k} F_k \geqslant \gamma,$$

and so

$$\dim F_k \geqslant c_k + \gamma - d_k.$$

Therefore $d\big(C_k \to D_k\big) \leqslant d_k - c_k$.

By the triangle inequality, the sum of all the distances $d(A_i \to B_i)$ and $d(C_k \to D_k)$ must be at least the distance from $\frac{\gamma}{c}C$ to $\frac{\gamma}{d}D$, and now using the original assumption on the size of $\angle CD$ from Theorem 4.11,

$$\sum_i d(A_i \to B_i) + \sum_k d(C_k \to D_k) \geqslant d\big(\frac{\gamma}{c}C \to \frac{\gamma}{d}D\big) = \gamma \angle CD = \gamma\left(\frac{x_D}{d} - \frac{x_C}{c}\right).$$

However,

$$\sum_i (b_i - a_i) + \sum_k (d_k - c_k) = \gamma\left(\frac{x_D}{d} - \frac{x_C}{c}\right),$$

and thus there can be no i with $d(A_i \to B_i) < b_i - a_i$.

Therefore $S = \emptyset$ and $\dim \Phi^{-1}\big(\langle x, \gamma - x\big) \geqslant \gamma$ for all $x \in [a, b]$.

A completely symmetrical argument can be used to show that $\dim \Phi^{-1}\big(\langle x, \gamma - x\big) \leqslant \gamma$ for all $x \in [a, b]$. We divide the interval $[a, b]$ into the sets

$$T = \{x \in [a, b] : \dim\big(\Phi^{-1}\big(\langle x, \gamma - x\rangle\big) > \gamma\}$$

$$T^c = \{x \in [a,b] : \dim\big(\Phi^{-1}\big(\langle x, \gamma - x\rangle\big) \leqslant \gamma\},$$

and then show $T = \emptyset$.

$\square$

Which completes the proof of Theorem 4.11.

$\square$

References

[1] Rod Downey and Noam Greenberg. Turing degrees of reals of positive effective packing dimension. *Information Processing Letters*, 108:198–203, 2008.

[2] Rod G. Downey and Denis Hirschfeldt. *Algorithmic Randomness and Complexity*. Springer, 2010. 588 pages.

[3] Jack H. Lutz Krishna B. Athreya, John M. Hitchcock and Elvira Mayordomo. Effective strong dimension, algorithmic information, and computational complexity. *SIAM Journal on Computing*, 37(3):671–705, 2007.

[4] André Nies. *Computability and Randomness*. Oxford University Press, USA, 2009. 420 pages.

[5] Jan Reimann. *Computability and Fractal dimension*. PhD thesis, Ruprecht-Karls-Universität, Heidelberg, 2004. 130 pages.

[6] Claude Tricot. Two definitions of fractional dimension. *Mathematical Poceedings of the Cambridge Philosophical Society*, 91:57–74, 1982.