Small Π^0_1 Classes.

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Abstract

The property of *smallness* for Π^0_1 classes is introduced and is investigated with respect to Medvedev and Muchnik degree. It is shown that the property of containing a small Π^0_1 class depends only on the Muchnik degree of a Π^0_1 class. A comparison is made with the idea of thinness for Π^0_1 classes

1 Introduction

This paper is a continuation of the project to study Π^0_1 classes with reference to their Medvedev and Muchnik degrees. The basic concepts and results in this area have been outlined in [3], [4], [2] and [5]. This paper is an adaptation of Chapter 4 in [3]. We also review the basic ideas below.

It is known that various structural properties of computable binary trees affect the Medvedev and Muchnik degrees of the associated Π^0_1 class. For example, if a Π^0_1 class has positive Lebesgue measure then it is necessarily Muchnik (and therefore Medvedev) incomplete. Similarly if it is thin. Simpson has shown that the property of *productiveness* is equivalent to Medvedev completeness. In this paper we define a new property of trees and show that it guarantees Muchnik

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and Medvedev incompleteness. This work is informed by Post's effort [15] to construct a non-zero c.e. degree strictly below $\mathbf{0}'$. Post's attempt was ultimately unsuccessful and the construction of such a degree needed more sophisticated methods. A discussion of these issues can be found in [20] chapter V, or [17] §9.7.

Perhaps surprisingly, Post's methods are more conducive to solving the corresponding problem in the Medvedev and Muchnik lattices. Here we define two new properties also guaranteeing incompleteness and having properties not shared by thin Π^0_1 classes. Both of these properties relate to some conception of the size of a Π^0_1 class.

1.1 The Medvedev and Muchnik Lattices of Π^0_1 classes

We denote the set of natural numbers by ω. ω^ω is the set of functions from ω to ω and the set of functions from ω to the set $\{0,1\}$ is denoted 2^ω. Subsets of ω are identified with their characteristic functions without further comment. The corresponding sets of finite sequences are denoted $\omega^{<\omega}$ and $2^{<\omega}$ respectively. Each of these sets are given the standard (i.e. product) topology. For other unexplained computability-theoretic notation refer to [20].

Definition 1.1. *A* Π^0_1 class P *is a subclass of* ω^ω *of the following form:*

$$f \in P \Leftrightarrow \forall n \ R(n,f),$$

where $R(\ ,\ ) \subseteq \omega \times \omega^\omega$ *is a computable predicate.*

A second and very useful characterisation of Π^0_1 classes is as follows:

Theorem 1.2. $P \subseteq \omega^\omega$ *is a* Π^0_1 *class if and only if it is the set of infinite paths through some computable tree.*

Definition 1.3. *If* $P \subseteq \omega^\omega$ *is a* Π^0_1 *class, then the* extendible nodes of P*, denoted* $\mathrm{Ext}(P)$*, is the set*

$$\{\sigma \in 2^{<\omega} : \exists f \in P \ f \supset \sigma\}.$$

Definition 1.4. *A* Π^0_1 *class* P *is* computably bounded (c.b.) *if there is a computable function* g *such that, for all* $f \in P$, $g(n) > f(n)$.

Definition 1.5. *A* Π^0_1 *class* P *is* special *if it is non-empty and contains no computable element.*

In this paper we are mainly concerned with special c.b. Π^0_1 classes. We will study these classes from the point of view of their Muchnik and Medvedev degree. These are ideas that can be applied to arbitrary subclasses of ω^ω.

Definition 1.6. *If $\emptyset \neq X, Y \subseteq \omega^\omega$ then we say X is* Medvedev reducible to Y *(denoted $X \leqslant_M Y$) if there is a computable functional $\Phi : Y \to X$.*

Definition 1.7. *If $\emptyset \neq X, Y \subseteq \omega^\omega$, then we say X is* Muchnik reducible to Y *(denoted $X \leqslant_w Y$) if, for every $f \in Y$, there is a $g \in X$ such that $g \leqslant_T f$.*

We say that X is *Muchnik equivalent to* Y ($X \equiv_w Y$) if $X \leqslant_w Y$ and $Y \leqslant_w X$. The *Muchnik degree* of X, $\deg_w(X)$, is the set $\{Y : X \equiv_w Y\}$. $X <_w Y$ means $X \leqslant_w Y$ and $X \not\equiv_w Y$. There is an induced partial ordering on Muchnik degrees, namely $\deg_w(X) \leqslant \deg_w(Y)$ if $X \leqslant_w Y$. Corresponding notation is used for Medvedev degrees.

Both of these partial orders exhibit a distributive lattice structure with the least upper bound and greatest lower bound defined as follows.

Let $X, Y \subseteq \omega^\omega$, and $f \in X, g \in Y$. Define $f \oplus g \in \omega^\omega$ so that for all $n \in \omega$,

$$f \oplus g(n) = \begin{cases} f(n/2) & \text{if } n \text{ is even} \\ g((n-1)/2) & \text{if } n \text{ is odd} \end{cases}$$

We now define two operations:

$$X \vee Y = \{f \oplus g : f \in X,\ g \in Y\}$$

$$X \wedge Y = \{\langle 0 \rangle^\frown f : f \in X\} \cup \{\langle 1 \rangle^\frown g : g \in Y\},$$

where $\langle i \rangle^\frown f$ denotes the element h of ω^ω such that $h(0) = i$ and $h(n+1) = f(n)$ for all n.

These operations induce least upper bounds and greatest lower bounds on the sets of Muchnik and Medvedev degrees. That is:

$$\deg_M(P) \wedge \deg_M(Q) = \deg_M(P \wedge Q)$$

$$\deg_M(P) \vee \deg_M(Q) = \deg_M(P \vee Q)$$

and similarly for the Muchnik degrees.

We now concentrate our attention on Medvedev and Muchnik reducibilities as they relate to the set of non-empty c.b. Π^0_1 classes. We list some basic results. Details can be found in [19].

• If P and Q are non-empty c.b. Π^0_1 classes then so are $P \wedge Q$ and $P \vee Q$. The lattices of the Medvedev (Muchnik) degrees of non-empty c.b. Π^0_1 classes are denoted $\mathcal{P}_M$ and $\mathcal{P}_w$ respectively.

• $\mathcal{P}_M$ and $\mathcal{P}_w$ are distributive lattices, each with a maximum and minimum element. The maximum element in both cases is the degree of PA - the set of completions of Peano arithmetic. Another representative of the maximum degree is the set:

$$\mathrm{DNR}_2 = \{f \in 2^\omega : \forall n\ \{n\}(n) \neq f(n)\},$$

which is readily seen to be a Π^0_1 class. Any Π^0_1 class with a computable element is a representative of the minimum degree. 2^ω will be the conventional representative of this degree.

• Any non-empty c.b. Π^0_1 class is computably homeomorphic to a Π^0_1 subclass of 2^ω. We can henceforth restrict our attention to non-empty Π^0_1 subclasses of 2^ω.

We now introduce some notation that will be useful for the purposes of this paper.

Notation:

$\|X\| =$ the cardinality of X.

If $f \in \omega^\omega$, $f[n] = \langle f(0), f(1), \ldots f(n-1)\rangle$.

If $P \subseteq \omega^\omega$, $P[n] = \{f[n] : f \in P\}$.

If $X \subseteq \omega^{<\omega}$, $X[n] = \{\sigma \in X : |\sigma| = n\}$.

If $P \subseteq \omega^\omega$, $P[< n] = \{f[m] : m < n,\ f \in P\}$, and similarly for $P[\leqslant n]$, $X[< n]$ and $X[\leqslant n]$.

$\{e\}^\tau[n]$ is the partial sequence γ from $\{0, 1, \ldots n-1\}$ to ω such that $\gamma(m) = \{e\}^\tau(m)$ if this is defined and is undefined otherwise. In particular, if $T \subseteq 2^{<\omega}$ then $\{e\}^\tau[n] \in T$ implies $\{e\}^\tau(m)\downarrow$ for all $m < n$.

$|\{e\}^\tau| = \max\{k : \forall m < k,\ \{e\}^\tau(m)\downarrow\}$.

2 Small Π^0_1 classes

Definition 2.1. *$P \subseteq \omega^\omega$ is* small *if it is non-empty, closed and if there is no computable function, g, such that for all n, $\|P[g(n)]\| \geqslant n$.*

Notice that any finite subclass of ω^ω is small. In fact, one can think of a closed subclass of ω^ω as being small exactly when there is no computable function witnessing its infinitude (in the above sense of witnessing). It will be shown that the property of smallness is invariant under computable homeomorphisms. Rather than arbitrary small subclasses of ω^ω, we will primarily be concerned with small computably bounded Π^0_1 subclasses of ω^ω. In fact, as Corollary 2.13 will show, we can concentrate on small Π^0_1 subclasses of 2^ω without losing generality.

It will be useful here to make the following observations which follow easily from the definitions of $\vee$ and $\wedge$ as operations on Π^0_1 classes.

Observation 2.2. *Let P and Q be c.b. Π^0_1 subclasses of ω^ω. Then for all n,*

$$\|P \vee Q[2n]\| = \|P[n]\| \cdot \|Q[n]\|$$

and

$$\|P \wedge Q[n+1]\| = \|P[n]\| + \|Q[n]\|.$$

Theorem 2.3. *All Medvedev (and therefore Muchnik) degrees of c.b. Π^0_1 classes have a representative that is not small.*

Proof. For any c.b. Π^0_1 class $P \subseteq \omega^\omega$, $P \vee 2^\omega$ is never small because for all n,

$$\|P \vee 2^\omega[2n]\| = \|P[n]\| \cdot \|2^\omega[n]\| \geqslant 2^n \geqslant n.$$

□

Theorem 2.4. *DNR_2 is not small.*

Proof. Let $\langle e_i \rangle_{i \geqslant 0}$ be a computable sequence of indices for the empty function. Then for all $\sigma \in \mathrm{DNR}_2[e_i]$, $\sigma^\frown\langle 0\rangle$ and $\sigma^\frown\langle 1\rangle$ are in $\mathrm{DNR}_2[e_i+1]$. Arguing by induction, and using the fact that $\mathrm{DNR}_2[n]$ is increasing in n we have $\|\mathrm{DNR}_2[e_i+1]\| \geqslant 2^{i+1}$ for all i. If $h(m) =$ the least k such that $2^{k+1} \geqslant m$, then

$$\|\mathrm{DNR}_2[e_{h(m)}+1]\| \geqslant 2^{h(m)+1} \geqslant m$$

for all $m \in \omega$. $m \mapsto e_{h(m)} + 1$ is clearly a computable function, so DNR_2 is not small. ☐

Definition 2.5. *If $A, B \subseteq \omega$, then the* separating class of A and B, *denoted $\mathcal{S}(A, B)$, is the set*

$$\{X \subseteq \omega : X \supseteq A \text{ and } X \cap B = \emptyset\}.$$

If $A \cap B = \emptyset$ then $\mathcal{S}(A, B)$ is non-empty and if A and B are c.e. sets then $\mathcal{S}(A, B)$ is a Π^0_1 class.

Theorem 2.6. *A small Π^0_1 class with no computable path exists.*

Proof. Recall that $A \subseteq \omega$ is *hypersimple* (*h-simple*) if it is c.e. and the principal function (i.e. the function that lists a set in increasing order) of its complement is not dominated by any computable function. If A is h-simple and A^0 and A^1 are disjoint c.e. sets such that $A^0 \cup A^1 = A$, then we claim that $S = \mathcal{S}(A^0, A^1)$ is small. Suppose S were not small, witnessed by the computable function, g. S branches at level n (that is, $S[n+1] > S[n]$) precisely when $n \in \overline{A}$. For such an n, $\|S[n+1]\| = 2\|S[n]\|$. So if p is the principal function of $\overline{A}$, then p has the property that $\|S[p(n)]\| = 2^n$. But $\|S[g(2^n+1)]\| \geqslant 2^n + 1$. So the function $n \mapsto g(2^n + 1)$ is a computable function dominating p, contradicting the fact that A is h-simple.

A^0 and A^1 can be constructed to ensure $\mathcal{S}(A^0, A^1)$ has no computable element (see Theorem 1 [14]).

☐

The next theorems show that the idea of smallness works well with the Muchnik and Medvedev operations.

Theorem 2.7. *P and Q are small Π^0_1 classes if and only if $P \wedge Q$ is a small Π^0_1 class.*

Proof. We make repeated use of Observation 2.2. Suppose $P \wedge Q$ were small and either P or Q were not small. Without losing generality, let it be P. Let f be computable such that $\|P[f(n)]\| \geqslant n$ for all n. Then $\|P \wedge Q[f(n+1)]\| \geqslant \|P[f(n)]\| \geqslant n$ and $P \wedge Q$ is not small. So if $P \wedge Q$ is small, so are both P and Q.

Conversely, suppose P and Q are small and $P \wedge Q$ is not small. Let g be a strictly positive computable function such that $\|P \wedge Q[g(n)]\| \geqslant n$ for all n. For all $m > 0$, $\|P \wedge Q[m]\| = \|P[m-1]\| + \|Q[m-1]\|$ so for all n, $\|P[g(n)-1]\| \geqslant n/2$ or $\|Q[g(n)-1]\| \geqslant n/2$. Consider the set

$$X = \{n : \|P[g(n)-1]\| < n/2\}.$$

X is c.e. as P is a Π^0_1 class and it is infinite as P is small (if it were finite then the set $\{n : \|P[g(n)-1]\| \geqslant n/2\}$ would be cofinite, and, modulo finitely many arguments, the function $n \mapsto g(2n)-1$ would prove that P was not small). So X has an infinite computable subset Y. For all $y \in Y$, $\|Q[g(y)-1]\| \geqslant y/2$. If $h(n) =$ the least $y \in Y$ such that $y \geqslant 2n$, then

$$\forall n \ \|Q[g(h(n))-1]\| \geqslant \frac{h(n)}{2} \geqslant n,$$

contradicting the smallness of Q. □

Theorem 2.8. *P and Q are small if and only if $P \vee Q$ is small.*

Proof. The proof follows the proof of Theorem 2.7.

For one direction assume that either P or Q is not small. Let it be P without losing generality. Let f be computable such that $\|P[f(n)]\| \geqslant n$ for all n. Using Observation 2.2 we have for all n,

$$\|P \vee Q[2f(n)]\| = \|P[f(n)]\| \cdot \|Q[f(n)]\| \geqslant \|P[f(n)]\| \geqslant n.$$

The function $n \mapsto 2f(n)$ is computable so $P \vee Q$ is not small.

Conversely, assume that $P \vee Q$ is not small and let g be computable and such that $\|P \vee Q[g(n)]\| \geqslant n$ for all n. The function $n \mapsto \|P[n]\|$ is increasing in n so we also have $\|P \vee Q[2g(n)]\| \geqslant n$ and therefore $\|P[g(n)]\| \cdot \|Q[g(n)]\| \geqslant n$. So for all n, $\|P[g(n)]\| \geqslant \sqrt{n}$ or $\|Q[g(n)]\| \geqslant \sqrt{n}$. As before, the set $\{n : \|P[g(n)]\| < \sqrt{n}\ \}$ is c.e. because P is Π^0_1 and it is infinite because P is small. The proof is then similar to the proof of Theorem 2.7 □

Theorem 2.9. *For every small special Π^0_1 class P there exists a small Π^0_1 class Q such that $P >_M Q >_M 2^\omega$. This is also true with $>_w$ replacing $>_M$.*

Proof. For this we first construct a small Π^0_1 class S using a method similar to the one used in [10]. A uniformly computable sequence of maps, $\psi_s : 2^{<\omega} \to 2^{<\omega}$ is constructed with the properties:

1. for all $s \in \omega$, range(ψ_{s+1})$\subseteq$ range(ψ_s)
2. for all $s \in \omega$ and $\sigma \in 2^{<\omega}$, $\psi_s(\sigma^\frown\langle 0\rangle)$ and $\psi_s(\sigma^\frown\langle 1\rangle)$ are incompatible extensions of $\psi_s(\sigma)$

3. for all $\sigma \in 2^{<\omega}$, $\lim_s \psi_s(\sigma)$ exists.

The range of each ψ_s determines a computable tree T_s after closing under intitial segments and S is then $[\bigcap_s T_s]$. For full details on this method see [10], [3] or [4].

By controlling the construction of the sequence $\langle \psi_s \rangle_s$ we can ensure that S has the property that for all $f \in S$ and $g \in P$, $f \not\geqslant_T g$. This will mean that $S \not\geqslant_w P$. We can also ensure that S has no computable element. The construction of Theorem 4.7 in [10] is sufficient for this. We only need to introduce requirements that ensure S is small. These are as follows (for convenience we begin our enumeration of the partial computable functions at $e = 2$):

$$R_e \equiv \{e\}(e)\downarrow \Rightarrow \|S[\{e\}(e)]\| < e.$$

An exhaustive priority ordering is given to all requirements. R_e will require attention at stage s if $\{e\}_s(e)\downarrow$ and $\|T_s[\{e\}_s(e)]\| \geqslant e$. To ensure that each requirement gets satisfied, we wait for a stage at which R_e is the highest priority requirement requiring attention. To satisfy R_e we take the largest number k such that $2^k < e$ (the reason we require $e \geqslant 2$). Let i be the least number such that for all τ of length $k+i$, $|\psi_s(\tau)| > \{e\}(e)$. If we let 0^i denote the string of i zeroes, we define,

$$\psi_{s+1}(\nu) = \begin{cases} \psi_s(\sigma^\frown 0^{i\frown}\nu') & \text{if } \nu = \sigma^\frown \nu' \text{ and } |\sigma| = k \\ \psi_s(\nu) & \text{if } |\nu| < k \end{cases}$$

This ensures that $\|T_{s+1}[\{e\}(e)]\| < e$ and hence that $\|S[\{e\}(e)]\| < e$. So the function $\{e\}$ cannot be a computable witness to S's not being small. Each requirement will be satisfied for all time after receiving attention so this construction will result in a small Π^0_1 class with the required properties. Finally, $S \wedge P$ will be small by Theorem 2.7 and

$$P >_M S \wedge P >_M 2^\omega$$

because $S \not\geqslant_w P$ and neither S nor P has a computable element. This also establishes that

$$P >_w S \wedge P >_w 2^\omega.$$

□

Theorem 2.10. *Let P and Q be infinite c.b. Π^0_1 subclasses of ω^ω. If P is small, and if $\{e\} : P \to Q$ is a computable surjection, then Q is small.*

Proof. Suppose P, Q and $\{e\}$ are as stated. Let $\langle T_s \rangle_s$ be a computable sequence of computable trees with no end nodes such that $\bigcap_s T_s = \mathrm{Ext}(P)$. Let s and l be computable functions such that for all n

$$\forall \tau \in T_{s(n)}[l(n)],\ |\{e\}^\tau_{s(n)}| \geqslant n.$$

To see that such an l and s exist notice that the compactness of P implies that there is a k such that $\forall \tau \in P[k],\ |\{e\}^\tau| \geqslant n$. Because P is computably bounded a search will eventually find two numbers with the required property for a given n.

Now suppose Q is not small, witnessed by the computable function g. For all n,

$$\forall \tau \in T_{s(g(n))}[l(g(n))],\ |\{e\}^\tau| \geqslant g(n).$$

As $\{e\}$ is onto,

$$\forall \sigma \in Q[g(n)]\ \exists \tau \in P[l(g(n))]\ \{e\}^\tau \supseteq \sigma.$$

Therefore,

$$\|P[l(g(n))]\| \geqslant \|Q[g(n)]\| \geqslant n.$$

$l(g(n))$ is computable so this contradicts the smallness of P. □

Corollary 2.11. *If $P \geqslant_M Q$ are Π^0_1 subclasses of ω^ω, and if P is c.b. and contains a small Π^0_1 subclass, then Q contains a small Π^0_1 subclass.*

Proof. Suppose $\{e\} : P \to Q$ is computable. The image of any c.b. Π^0_1 class is also Π^0_1 so if $S \subseteq P$ is Π^0_1 and small, then the theorem implies that the image of S under $\{e\}$ is a small Π^0_1 subclass of Q. □

Corollary 2.12. *Smallness is preserved by computable homeomorphisms.*

Proof. All homeomorphisms are surjective. □

Corollary 2.13. *Any small c.b. Π^0_1 subclass of ω^ω is computably homeomorphic to a small Π^0_1 subclass of 2^ω.*

Proof. Any c.b. Π^0_1 subclass of ω^ω is computably homeomorphic to a Π^0_1 subclass of 2^ω and such a homeomorphism preserves smallness. $\square$

Corollary 2.13 allows us to move from small c.b. Π^0_1 subclasses of ω^ω to small Π^0_1 subclasses of 2^ω without losing generality (up to computable homeomorphism).

Corollary 2.14. *No Medvedev complete Π^0_1 subclass of 2^ω has a small Π^0_1 subclass.*

Proof. If some such Medvedev complete Π^0_1 class contained a small Π^0_1 subclass S, then S would also be Medvedev complete. But all Medvedev complete Π^0_1 subclasses of 2^ω are computably homeomorphic [19]. Therefore S would be computably homeomorphic to DNR_2, which would then be small, contradicting Theorem 2.4. $\square$

The following observation by Simpson allows us to transfer a lot of these theorems to the Muchnik lattice. In this respect it is a central lemma in the subject.

Lemma 2.15 (Simpson). *If $P, Q \subseteq 2^\omega$ are Π^0_1, and if $P \geqslant_w Q$, then there exists a Π^0_1 class, $P' \subseteq P$, such that $P' \geqslant_M Q$.*

Proof. Let $f \in P$ be of hyperimmune-free degree. Such an f exists by the hyperimmune-free basis theorem, [10]. Then for some $g \in Q$, $f \geqslant_T g$. The proof of Theorem VI.5.5 [13] (attributed to D.A. Martin) then implies $f \geqslant_{tt} g$. Proposition III.3.2 [13] (Trakhtenbrot, Nerode) then states we can find a *total* computable functional Φ taking f to g. Then $\Phi^{-1}(Q) \cap P$ is a non-empty Π^0_1 subclass of P, and this is a suitable choice for P' because $\Phi(\Phi^{-1}(Q) \cap P) \subseteq Q$. $\square$

Corollary 2.16. *If $P \subseteq 2^\omega$ is a non-empty Π^0_1 class with a small Π^0_1 subclass and $P \geqslant_w Q$, then Q has a small Π^0_1 subclass.*

Proof. Let $S \subseteq P$ be small and let $S' \subseteq S$ be a Π^0_1 class such that $S' \geqslant_M Q$. S' is necessarily small so Q must contain a small Π^0_1 class. $\square$

This means that the property of containing a small Π^0_1 class is a property of Muchnik degree — that is, if P and Q are Π^0_1 classes of the same Muchnik degree then P contains a small Π^0_1 class if and only if Q contains a small Π^0_1 class.

Corollary 2.17. *No Muchnik complete Π^0_1 subclass of 2^ω has a small Π^0_1 subclass.*

Proof. No Π^0_1 subclass of DNR_2 is small as any subclass must be Medvedev complete, and therefore computably homeomorphic to DNR_2. Any Muchnik complete Π^0_1 subclasses of 2^ω is Muchnik equivalent to DNR_2 and therefore cannot contain a small Π^0_1 class. □

Lemma 2.15 also has corollaries for the study of the upper semilattice of c.e. Turing degrees:

Corollary 2.18. *For any h-simple set X and any c.e. partition $X_0 \sqcup X_1 = X$ there exists a separating set of X_0 and X_1 that is not of PA degree.*

Proof. If X is h-simple then $\mathcal{S}(X_0, X_1)$ is small. By Corollary 2.17, it can not be Muchnik complete and so must contain an element not of PA degree. □

The following is a somewhat more general consequence of the proof of Lemma 2.15.

Corollary 2.19. *If $S \subseteq 2^\omega$ is a small Π^0_1 class and $P \subseteq 2^\omega$ is Π^0_1 with no small Π^0_1 subclass, then no hyperimmune-free element of S computes an element of P.*

Many of the previous results can be summed up in the following way.

Theorem 2.20. *The set of Medvedev degrees:*

$$\mathcal{I} = \{deg_M(P) : P \text{ has a small } \Pi^0_1 \text{ subclass}\}$$

forms a (proper, nontrivial) prime ideal in $\mathcal{P}_M$.

Proof. First note that if $P \equiv_M Q$ and P has a small Π^0_1 subclass then so does Q by Corollary 2.11, so in what follows we are free to choose arbitrary representatives of Medvedev degrees.

i. Suppose $\deg_M(P) \in \mathcal{I}$ and $Q \subseteq 2^\omega$ is a Π^0_1 class such that $P \geqslant_M Q$. Corollary 2.11 then implies $\deg_M(Q) \in \mathcal{I}$.

ii. If $\deg_M(P),\ \deg_M(Q) \in \mathcal{I}$ and $S_1 \subseteq P$ and $S_2 \subseteq Q$ are small, then $S_1 \vee S_2 \subseteq P \vee Q$ and by Theorem 2.8, $S_1 \vee S_2$ is small. So $\deg_M(P \vee Q) \in \mathcal{I}$.

iii. No Medvedev complete Π^0_1 class has a small Π^0_1 subclass by Corollary 2.14, so $\mathcal{I}$ is proper.

iv. $\mathcal{I}$ is non-trivial by Theorem 2.6

v. Suppose $P \subseteq 2^\omega$ and $Q \subseteq 2^\omega$ are Π^0_1 and such that $\deg_M(P \wedge Q) \in \mathcal{I}$. If $S \subseteq P \wedge Q$ were small, then either $\{f : \langle 0 \rangle ^\frown f \in S\} \cap P$ or $\{f : \langle 1 \rangle ^\frown f \in S\} \cap Q$ would be non-empty and consequently, small. So either $\deg_M(P)$ or $\deg_M(Q)$ is in $\mathcal{I}$ and $\mathcal{I}$ is prime.

□

Using an argument similar to that used in Corollary 2.17, we can show that Theorem 2.20 is true in $\mathcal{P}_w$ as well.

Theorem 2.21. *The set of Muchnik degrees:*

$$\mathcal{J} = \{deg_w(P) : P \text{ has a small } \Pi^0_1 \text{ subclass}\}$$

forms a (proper, nontrivial) prime ideal in $\mathcal{P}_w$*.*

Proof. ii, iv and v are proved exactly as in Theorem 2.20. iii follows from Corollary 2.17. For i, suppose $\deg_w(P) \in \mathcal{J}$ and $Q \leqslant_w P$. Let $S \subseteq P$ be Π^0_1 and small and let $f \in S$ be hyperimmune-free. As in Corollary 2.17, there is a total computable functional, Φ, such that $\Phi(f) \in Q$. Thus $\Phi[S] \cap Q$ is non-empty and therefore a small subclass of Q. □

So far we have described only one Muchnik (Medvedev) degree that is not in $\mathcal{J}$ ($\mathcal{I}$) - namely the degree of DNR_2. There are in fact infinitely many such degrees:

Theorem 2.22. *The sets* $\mathcal{P}_M \setminus \mathcal{I}$ *and* $\mathcal{P}_w \setminus \mathcal{J}$ *have no minimal elements.*

Proof. Let P be any Π^0_1 class with no small subclass. By Theorem 1 in [4] we can find Π^0_1 classes P_0 and P_1 such that $P_0, P_1 <_M P$ and $P_0 \vee P_1 \equiv_M P$ (similarly for $\leqslant_w$). If both P_0 and P_1 contained small subclasses then so would P by Theorem 2.8. □

We will now consider alternative characterisations of smallness for computably bounded Π^0_1 classes.

Definition 2.23. *If* $P \subseteq 2^\omega$ *is* Π^0_1*, then let* $\mathrm{Br}(P)$*, the branching nodes of* P*, be the set*

$$\{\sigma \in \mathrm{Ext}(P) : \ \sigma ^\frown \langle 0 \rangle \in \mathrm{Ext}(P) \text{ and } \sigma ^\frown \langle 1 \rangle \in \mathrm{Ext}(P)\}.$$

Observation 2.24. $\|\mathrm{Br}(P)[< n]\| + 1 = \|P[n]\|$.

Proof. This is just a matter of counting. Each branching node below a given level of $\mathrm{Ext}(P)$ increases the number of extendible nodes at that level by one. □

We will need the following well-known concepts.

Definition 2.25. *A* disjoint strong array *is a computable sequence of pairwise disjoint canonically indexed finite sets.*

Definition 2.26. $X \subseteq \omega$ *is* hyperimmune (h-immune) *if there is no disjoint strong array* $\langle D_{f(n)} \rangle_n$ *such that for all* n, $D_{f(n)} \cap X \neq \emptyset$.

It is well known (see for example Proposition III.3.8 [13]) that X is h-immune if and only if its principal function is not dominated by any computable function. This means a c.e. set Y is h-simple if and only if $\overline{Y}$ is h-immune.

Theorem 2.27. *For any* Π^0_1 *class,* $P \subseteq 2^\omega$, P *is small if and only if* $\mathrm{Br}(P)$ *is h-immune.*

Proof. $\Rightarrow$) Assume $\mathrm{Br}(P)$ is not h-immune. Let $f(n)$ be a total computable function and let $\langle D_{f(n)} \rangle_{n \geqslant 0}$ be a disjoint strong array such that $D_{f(n)} \cap \mathrm{Br}(P) \neq \emptyset$ for all $n \in \omega$. For all $n \in \omega$, define a total computable function g by:

$$g(n) = \max\{|\sigma| : \sigma \in \bigcup_{i=0}^{n} D_{f(i)}\}.$$

Then for all $n \in \omega$, $\|\mathrm{Br}(P)[\leqslant g(n)]\| \geqslant n+1$. Therefore, by observation 2.24, for all n, $\|P[g(n)+1]\| = \|\mathrm{Br}(P)[\leqslant g(n)]\| + 1 \geqslant n+2 \geqslant n$. So P is not small.

$\Leftarrow$) Assume P is not small and the fact is witnessed by a strictly increasing, computable function, h. We now construct the required strong array as follows: first define the computable function:

$$\hat{h}(n) = \begin{cases} h(0) & \text{if } n = 0 \\ h(2^{\hat{h}(n-1)+1}) & \text{if } n \neq 0. \end{cases}$$

For any Π^0_1 class $Q \subseteq 2^\omega$ and any $m \in \omega$, we have $2^{m+1} > \|Q[m]\|$ and so we get, for all n,

$$\|P[\hat{h}(n+1)]\| \geqslant 2^{\hat{h}(n)+1} > \|P[\hat{h}(n)]\|,$$

and so there must be a $\sigma \in \mathrm{Br}(P)$ such that $\hat{h}(n) \leqslant |\sigma| < \hat{h}(n+1)$. Now define:

$$D_{f(n)} = \{\sigma : \hat{h}(n) \leqslant |\sigma| < \hat{h}(n+1)\}.$$

So $\langle D_{f(n)} \rangle_{n \geqslant 0}$ is a strong array and for each n, $D_{f(n)} \cap \mathrm{Br}(P) \neq \emptyset$. □

Notice that $\mathrm{Br}(P)$ is a co-c.e. set so that P is small if and only if $\overline{\mathrm{Br}(P)}$ is hypersimple.

There is another, closely related characterisation of smallness.

Definition 2.28. *$n \in \omega$ is said to be a* branching level *of P if there exists a $\sigma \in \mathrm{Br}(P)$ such that $|\sigma| = n$. We denote the set of branching levels of P by* $\mathrm{Brl}(P)$

We observe the following which will be used later.

Observation 2.29. *If X is a subset of ω and X_0, X_1 is a partion of X, and $S = \mathcal{S}(X_0, X_1)$, then $\overline{\mathrm{Brl}(S)} = X$.*

Theorem 2.30. *$P \subseteq 2^\omega$ is small if and only if $\overline{\mathrm{Brl}(P)}$ is hypersimple.*

Proof. First observe that $\overline{\mathrm{Brl}(P)}$ is c.e. for any Π^0_1 class P. Assume now that $\overline{\mathrm{Brl}(P)}$ is not hypersimple. Let $\langle D_{f(n)} \rangle_n$ be a disjoint strong array such that for all n, $D_{f(n)} \cap \mathrm{Brl}(P) \neq \emptyset$. Let $D_{g(n)} = \{\sigma \in 2^{<\omega} : |\sigma| \in D_{f(n)}\}$. Then $\langle D_{g(n)} \rangle_n$ forms a disjoint strong array and for all n, $D_{g(n)} \cap \mathrm{Br}(P) \neq \emptyset$.

Conversely, suppose $\langle D_{f(n)} \rangle_n$ is a disjoint strong array such that for all n, $D_{f(n)} \cap \mathrm{Br}(P) \neq \emptyset$. Let $D_{g(n)} = \{|\sigma| : \sigma \in D_{f(n)}\}$. $\langle D_{g(n)} \rangle_n$ is not a disjoint array but it can easily be made so. Let $\varphi(n)$ be defined recursively as follows: $\varphi(0) = 0$ and

$$\varphi(n+1) = \text{ the least } k \text{ such that } D_{g(k)} \cap \bigcup_{i \leqslant n} D_{g(\varphi(i))} = \emptyset.$$

Then $\langle D_{g(\varphi(n))} \rangle_n$ is the required disjoint strong array. □

3 Very Small Π^0_1 classes

The definition of smallness can be strengthened to define a proper subset of the set of small Π^0_1 classes. This new property will have much in common with smallness.

Definition 3.1. $P \subseteq \omega^\omega$ *is* very small *if it is non-empty, closed and the function*

$$n \mapsto \text{the least } k \text{ such that } \|P[k]\| \geqslant n$$

dominates every computable function.

The similarity to smallness can be made more explicit by the observation that P is small if and only if the function $n \mapsto$ the least k such that $\|P[k]\| \geqslant n$ is not dominated by any computable function. This also proves that every very small class is small.

Now theorems analogous to Theorems 2.6 - 2.30 can be established.

Theorem 3.2. *A very small Π^0_1 class with no computable path exists.*

Proof. Recall that a c.e. set, X, is *dense simple* if the principal function of its complement dominates every computable function. Now, if A is dense simple and A^0 and A^1 are disjoint c.e. sets such that $A^0 \cup A^1 = A$, then $\mathcal{S}(A^0, A^1)$ is very small by an argument similar to 2.6. A^0 and A^1 can be constructed to ensure $\mathcal{S}(A^0, A^1)$ has no computable element. □

Theorem 3.3. *P and Q are very small Π^0_1 subclasses of ω^ω if and only if $P \wedge Q$ is a very small Π^0_1 subclass of ω^ω.*

Proof. $\|P \wedge Q[n+1]\| \geqslant \|P[n]\|, \|Q[n]\|$ so if either P or Q were not very small then neither would $P \wedge Q$ be.

Conversely, suppose that $P \wedge Q$ is not very small. Let g be a computable function such that $\|P \wedge Q[g(n)]\| \geqslant n$ for infinitely many n. Then for infinitely many n, either $\|P[g(n)-1]\| \geqslant n/2$ or $\|Q[g(n)-1]\| \geqslant n/2$. Therefore either $\{n : \|P[g(n)-1]\| \geqslant n/2\}$ or $\{n : \|Q[g(n)-1]\| \geqslant n/2\}$ is infinite. Without losing any generality we can assume that $\{n : \|Q[g(n)-1]\| \geqslant n/2\}$ is infinite. Then either $\{2n : n \in \omega\} \cap \{n : \|Q[g(n)-1]\| \geqslant n/2\}$ is infinite, or $\{2n+1 : n \in \omega\} \cap \{n : \|Q[g(n)-1]\| \geqslant n/2\}$ is infinite. If the first case holds, then for infinitely many n $\|Q[g(2n)-1]\| \geqslant n$. If the second case holds, then for infinitely many n $\|Q[g(2n+1)-1]\| \geqslant n+1/2 \geqslant n$. In either case Q is not very small. □

Theorem 3.4. *P and Q are very small Π^0_1 subclasses of ω^ω if and only if $P \vee Q$ is a small Π^0_1 ubclass of ω^ω.*

Proof. The proof imitates Theorem 3.3.

$\|P \vee Q[2n]\| \geqslant \|Q[n]\|, \|P[n]\|$ so if either P or Q were not very small, then neither would $P \vee Q$ be.

For the other direction let g be such that $\|P \vee Q[g(n)]\| \geqslant n$ for infinitely many n. The function $n \mapsto \|P[n]\|$ is increasing in n so we also have $\|P \vee Q[2g(n)]\| \geqslant n$. Using the definition of $\vee$, for infinitely many n either $\|Q[g(n)]\| \geqslant \sqrt{n}$ or $\|P[g(n)]\| \geqslant \sqrt{n}$. Assume as before that $X = \{n : \|Q[g(n)]\| \geqslant \sqrt{n}\}$ is infinite. Let $\{n_0, n_1, n_2 \dots\}$ be an infinite subset of ω such that for all i there exists a $k \in X$ such that $n_i^2 \leqslant k < (n_i+1)^2$. Then for all i

$$\begin{aligned} \|Q[g((n_i+1)^2)]\| &\geqslant \|Q[g(k)]\| \quad \text{for some } k \in X \\ &\geqslant \sqrt{k} \\ &\geqslant n_i. \end{aligned}$$

So there are infinitely many n such that $\|Q[g((n+1)^2)]\| \geqslant n$ and Q is not very small. □

Theorem 3.5. *For every very small special $P \subseteq 2^\omega$ there exists a very small Q such that $2^\omega <_M Q <_M P$. This is also true with $<_w$ replacing $<_M$.*

Proof. We will use the same kind of construction as in Theorem 2.9. We will construct a Π^0_1 class $V \subseteq 2^\omega$ and require that it has no computable path and that no element of V computes an element of P. We then combine these requirements with the following to ensure that it is very small. This time the requirements will be indexed by n and $e \leqslant n$:

$$R_{\langle n,e\rangle} \equiv \{e\}(n)\downarrow \Rightarrow \|V[\{e\}(n)]\| < n$$

$R_{\langle n,e\rangle}$ *requires attention* at stage s if $\{e\}_s(n)\downarrow$ and $\|T_s[\{e\}_s(n)]\| \geqslant n$. Suppose $R_{\langle n,e\rangle}$ is the highest priority requirement requiring attention at stage s. Take k to be the greatest natural number such that $2^k < n$. Let i be the least natural number such that,

$$|\psi_s(\tau)| > \{e\}(n)$$

for all τ of length $k+i$. Now define,

$$\psi_{s+1}(\nu) = \begin{cases} \psi_s(\sigma^\frown 0^{i\frown}\nu') & \text{if } \nu = \sigma^\frown\nu' \text{ and } |\sigma| = k, \\ \psi_s(\nu) & \text{if } |\nu| < k \end{cases}$$

Each requirement is satisfied for all time after receiving attention once. $\lim_s(\sigma)$ exists for all σ as, for each n, there are only finitely many associated values of e and, as n becomes larger, so does the value of k. Therefore, for any σ, k will eventually become larger than $|\sigma|$ forcing $\psi_{s+1}(\sigma)$ to be equal to $\psi_s(\sigma)$ from that stage on. □

Theorem 3.6. *Let P and Q be Π^0_1 subclasses of ω^ω. If P is c.b. and very small, and if $\{e\} : P \to Q$ is a computable surjection, then Q is very small.*

Proof. The proof is virtually identical to Theorem 2.10.

□

Corollary 3.7. *If $P \geqslant_M Q$ are Π^0_1 subclasses of ω^ω, and if P is c.b. and contains a very small Π^0_1 subclass, then Q contains a very small Π^0_1 subclass.*

Proof. See the proof of Corollary 2.11 □

Corollary 3.8. *Very smallness is preserved by computable homeomorphisms.*

Corollary 3.9. *Any very small c.b. Π^0_1 subclass of ω^ω is computably homeomorphic to a very small Π^0_1 subclass of 2^ω.*

Theorem 3.10. *The set of Medvedev degrees:*

$$\mathcal{K} = \{deg_M(P) : P \text{ has a very small } \Pi^0_1 \text{ subclass}\}$$

forms a (proper, nontrivial) prime ideal in $\mathcal{P}_M$.

Proof. The proof of this is essentially the same as Theorem 2.20.

□

Theorem 3.11. *The set of Muchnik degrees:*

$$\mathcal{L} = \{deg_w(P) : P \text{ has a very small } \Pi^0_1 \text{ subclass}\}$$

forms a (proper, nontrivial) prime ideal in $\mathcal{P}_w$.

Proof. See the proof of Theorem 2.21. □

Theorem 3.12. *For any Π^0_1 class, $P \subseteq 2^\omega$, P is very small if and only if $\overline{\mathrm{Br}(P)}$ is dense simple.*

Proof. It is convenient here to provide an alternative characterisation of dense simplicity.

Lemma 3.13. *A c.e. set is dense simple if and only if for all strong arrays* $\langle D_{f(n)}\rangle_n$

$$\{n : \|\overline{X} \cap \bigcup_{i=0}^{n} D_{f(i)}\| \geqslant n\} \text{ is finite.}$$

Proof. Suppose that for some computable function f there are infinitely many n such that $\|\overline{X} \cap \bigcup_{i=0}^{n} D_{f(i)}\| \geqslant n$. If we let $m(n) = \max(\bigcup_{i=0}^{n} D_{f(i)})$, then for infinitely many n,

$$\|\{x : x \in \overline{X} \text{ and } x \leqslant m(n)\}\| \geqslant n.$$

Therefore, if $p_{\overline{X}}$ is the principal function of $\overline{X}$, $p_{\overline{X}}(n) \leqslant m(n)$ for infinitely many n. But m is computable so X is not dense simple.

Conversely, suppose there is a computable function ϕ such that $p_{\overline{X}} \leqslant \phi(n)$ for infinitely many n. Let

$$D_{f(n)} = \begin{cases} \big[0, \phi(0)\big] & \text{if } n = 0 \\ \big(\phi(n-1), \phi(n)\big] & \text{otherwise} \end{cases}$$

where the notation $(a, b]$ represents the appropriate interval in ω. Then whenever $p_{\overline{X}}(n) \leqslant \phi(n)$ we have $\|\overline{X} \cap \bigcup_{i=0}^{n} D_{f(i)}\| \geqslant n$.

□

Now we complete the proof of the theorem. Suppose P is not very small and let g be computable such that for infinitely many n $\|P[g(n)]\| \geqslant n$. Let $g'(n) = g(n+1)$ so that for infinitely many n $\|P[g'(n)]\| \geqslant n+1$. By Observation 2.24 it follows that for infinitely many n $\|\mathrm{Br}(P)[< g'(n)]\| \geqslant n$. Let

$$D_{f(n)} = \begin{cases} \{\sigma \in 2^{<\omega} : g'(n-1) \leqslant |\sigma| < g'(n)\} & \text{if } n \neq 0 \\ \{\sigma \in 2^{<\omega} : |\sigma| < g'(0)\} & \text{otherwise.} \end{cases}$$

Then for infinitely many n

$$\begin{aligned} \|\mathrm{Br}(P) \cap \textstyle\bigcup_{i=0}^{n} D_{f(i)}\| &= \|\mathrm{Br}(P)[< g'(n)]\| \\ &\geqslant n \end{aligned}$$

and $\overline{\mathrm{Br}(P)}$ is not dense simple.

Conversely, suppose $\langle D_{f(n)} \rangle$ is such that $\|\mathrm{Br}(P) \cap \bigcup_{i=0}^{n} D_{f(i)}\| \geqslant n$ for infinitely many n. Let $m(n) = \max(\bigcup_{i=0}^{n} D_{f(i)})$. Then for infinitely many n $\|\mathrm{Br}(P)[\leqslant m(n)]\| \geqslant n$, which implies, using Observation 2.24, that $\|P[m(n)+1]\| \geqslant n$ and so P is not very small. $\square$

Theorem 3.14. *P is very small if and only if $\overline{\mathrm{Brl}(P)}$ is dense simple.*

Proof. Similar to the proof of Theorem 2.30. If $\langle D_{f(n)} \rangle_n$ is a disjoint strong array such that for infinitely many n $\|\mathrm{Brl}(P) \cap \bigcup_{i=0}^{n} D_{f(i)}\| \geqslant n$, then define $D_{g(n)} = \{\sigma : |\sigma| \in D_{f(n)}\}$. This disjoint strong array then witnesses the fact that $\overline{\mathrm{Br}(P)}$ is not dense simple.

In the other direction, let $\langle D_{f(n)} \rangle_n$ be a disjoint strong array witnessing the fact that $\overline{\mathrm{Br}(P)}$ is not dense simple. As before define $D_{g(n)} = \{|\sigma| : \sigma \in D_{f(n)}\}$. Then let

$$D_{h(n)} = \bigcup_{i=0}^{n} D_{g(i)} \setminus \bigcup_{i=0}^{n-1} D_{g(i)}.$$

And $\langle D_{h(n)} \rangle_n$ is a disjoint strong array witnessing the fact that $\overline{\mathrm{Brl}(P)}$ is not dense simple.

$\square$

Very smallness is a strictly stronger property than smallness as the next theorem shows. First we will need the following lemma.

Lemma 3.15. *(Lachlan [11] and Robinson [16]) There is a hypersimple set that has no dense simple superset.*

Robinson and Lachlan actually proved that there is an r-maximal set with no dense-simple superset, but as all r-maximal sets are hypersimple (see for example [20] chapter X) the lemma follows.

Theorem 3.16. *There exists a small Π_1^0 subclass of 2^ω with no computable path that has no very small subclass.*

Proof. Let X be hypersimple with no dense simple superset and let $X_0 \cup X_1 = X$ be any c.e. partition of X with no computable separating set. The claim is that $S = \mathcal{S}(X_0, X_1)$ is small with no very small Π_1^0 subclass.

We first observe that S is small as X is hypersimple (as in Lemma 2.6). Suppose $V \subseteq S$ is a very small Π_1^0 subclass. Then $\overline{\mathrm{Brl}(V)}$ is dense simple by Theorem 3.12. But $\mathrm{Brl}(V) \subseteq \mathrm{Brl}(S)$, so $\overline{\mathrm{Brl}(V)} \supseteq \overline{\mathrm{Brl}(S)} =$

X by observation 2.29. This contradicts the assumption that X has no dense simple superset.

□

The previous theorem means that small Π^0_1 classes and very small Π^0_1 classes can be distinguished by their Muchnik degree — that is, that there is a Muchnik degree that contains a small Π^0_1 class but no very small Π^0_1 class.

The density of the Muchnik lattice is still an open problem although some partial results have been obtained. The previous theorem gives one such result.

Corollary 3.17. *If P and V are Π^0_1 subclasses of 2^ω such that V is very small, P has no small Π^0_1 subclass, and $P >_w V$, then there exists a Π^0_1 class $Q \subseteq 2^\omega$ such that $V <_w Q <_w P$.*

Proof. Let S be small with no very small Π^0_1 subclass. Then we claim $V \vee (P \wedge S)$ is the required Q. $V \wedge S$ is small and so $V \wedge S \not\geqslant_w P$ (using Lemma 2.15). Therefore $V \vee (P \wedge S) \equiv_w P \wedge (V \vee S) <_w P$. But also $V \not\geqslant_w P \wedge S$ as neither P nor S has a very small Π^0_1 subclass. Therefore $V \vee (P \wedge S) >_w V$. □

4 Small Π^0_1 classes and thinness

In this sections we compare smallness with the well-established propery of thinness. But first we establish the perhaps not surprising fact that all small classes have zero measure.

Let μ be the standard fair-coin measure on subclasses of 2^ω. Observe that if P is a closed subclass of 2^ω, then the function $n \mapsto \|P[n]\|/2^n$ is decreasing and $\mu(P) = \lim_{n\to\infty} \|P[n]\|/2^n$.

Theorem 4.1. *If $P \subseteq 2^\omega$ is closed, and $\mu(P) > 0$, then P is not small.*

Proof. Choose some computable $r \in \mathbb{R}$ such that $0 < r \leqslant \mu(P)$. Then for all n $\|P[n]\| \geqslant r \cdot 2^n$, and if $g(n) =$ the least k such that $k \geqslant \log_2(n/r)$, then $\|P[g(n)]\| \geqslant n$. □

A Π^0_1 class P is *thin* if every Π^0_1 subclass of P is the intersection of P with some clopen set. Equivalently, P is thin if and only if its lattice (under $\cap, \cup$) of Π^0_1 subclasses forms a Boolean algebra. The notion has been studied by Cholak, Coles, Downey, Jockusch, Hermann, Stob and others in [6], [8], [9] and elsewhere. As both small and thin classes are "diminutive" in some sense it is natural to ask at this stage how the notions of thinness and smallness relate to each other.

Theorem 4.2. *There exists a very small (and hence small) Π^0_1 class that is not thin.*

Proof. If V is any very small Π^0_1 class, then by Lemma 3.4 so is $V \vee V$. However $V \vee V$ is never thin as $\{f \oplus f : f \in V\}$ is a Π^0_1 subclass of $V \vee V$ that is not the intersection of $V \vee V$ with any clopen set (it is easy to see its complement in $V \vee V$ is not closed). $\square$

Theorem 4.3. *There is a thin Π^0_1 class that is not very small*

Proof. We first show that for any perfect Π^0_1 class $P \subseteq 2^\omega$, $\mathrm{Ext}(P) \equiv_T \mathrm{Br}(P)$. One direction is clear because $\sigma \in \mathrm{Br}(P) \Leftrightarrow \sigma^\frown\langle 0\rangle, \sigma^\frown\langle 1\rangle \in \mathrm{Ext}(P)$. So $\mathrm{Br}(P) \leqslant_T \mathrm{Ext}(P)$. For the other direction, $\sigma \in \mathrm{Ext}(P) \Leftrightarrow \exists \tau \in \mathrm{Br}(P)\ \tau \supseteq \sigma$. So $\mathrm{Ext}(P)$ is c.e. in $\mathrm{Br}(P)$. But $\mathrm{Ext}(P)$ is a co c.e. set so it is in fact computable in $\mathrm{Br}(P)$. That is, $\mathrm{Ext}(P) \leqslant_T \mathrm{Br}(P)$.

The rest of the proof follows from results in [8] about the Turing degree of the extendible nodes of thin Π^0_1 classes. In [8], Downey, Jockusch and Stob introduce a class of c.e. degrees called the anr degrees (later called anc degrees). They prove that there are thin separating classes whose extendible nodes are of anc degree (viz. the Π^0_1 sets associated with Martin Pour-El theories), and indeed that every anc degree contains $\mathrm{Ext}(T)$ for some thin separating Π^0_1 class,T. They also show in [8] that there are low anc degrees.

Let T be a thin separating Π^0_1 class such that $\mathrm{Ext}(T)$ is of low degree. Suppose T is very small. Then $\overline{\mathrm{Br}(T)}$ would be dense simple, and therefore of high degree (see [12] or Theorem XI.1.3 [20]). As $\overline{\mathrm{Br}(T)} \equiv_T \mathrm{Br}(T) \equiv_T \mathrm{Ext}(T)$, this is a contradiction. $\square$

Theorem 4.4. *There exists a thin, very small Π^0_1 class*

Proof. This is just a matter of combining the requirements from theorem 3.5 with the requirements for thinness (see for example [6]). $\square$

It is unknown as yet if every thin Π^0_1 class is small although we conjecture the answer is no. In a future paper this question and similar ones will be investigated.

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