Hindawi
Complexity
Volume 2017, Article ID 8718209, 12 pages
https://doi.org/10.1155/2017/8718209

Research Article

An Accurate Approximate-Analytical Technique for Solving Time-Fractional Partial Differential Equations

M. Bishehniasar,[1] S. Salahshour,[2] A. Ahmadian,[3] F. Ismail,[3] and D. Baleanu[4,5]

[1] *Department of Applied Mathematics, Faculty of Mathematical Science, University of Kashan, Kashan, Iran*
[2] *Young Researchers and Elite Club, Mobarakeh Branch, Islamic Azad University, Mobarakeh, Iran*
[3] *Laboratory of Computational Sciences and Mathematical Physics, Institute for Mathematical Research (INSPEM), Universiti Putra Malaysia, 43400 Serdang, Selangor, Malaysia*
[4] *Department of Mathematics, Çankaya University, Balgat, 06530 Ankara, Turkey*
[5] *Institute of Space Sciences, Magurele, Bucharest, Romania*

Correspondence should be addressed to S. Salahshour; soheilsalahshour@yahoo.com

Received 10 August 2017; Accepted 31 October 2017; Published 24 December 2017

Academic Editor: Danilo Comminiello

The demand of many scientific areas for the usage of fractional partial differential equations (FPDEs) to explain their real-world systems has been broadly identified. The solutions may portray dynamical behaviors of various particles such as chemicals and cells. The desire of obtaining approximate solutions to treat these equations aims to overcome the mathematical complexity of modeling the relevant phenomena in nature. This research proposes a promising approximate-analytical scheme that is an accurate technique for solving a variety of noninteger partial differential equations (PDEs). The proposed strategy is based on approximating the derivative of fractional-order and reducing the problem to the corresponding partial differential equation (PDE). Afterwards, the approximating PDE is solved by using a separation-variables technique. The method can be simply applied to nonhomogeneous problems and is proficient to diminish the span of computational cost as well as achieving an approximate-analytical solution that is in excellent concurrence with the exact solution of the original problem. In addition and to demonstrate the efficiency of the method, it compares with two finite difference methods including a nonstandard finite difference (NSFD) method and standard finite difference (SFD) technique, which are popular in the literature for solving engineering problems.

1. Introduction

Many anomalous diffusion processes which existed in some physical and biological areas can be modeled by the time-fractional reaction diffusion wave equation. In the past few decades, high and rapid growing attention related to partial differential equations (PDEs) which contain fractional derivatives and integrals occurred due to their important application in modeling of many anomalous diffusion processes. Fractional partial differential equations (FPDEs) are excellent instrument, bringing into a broader paradigm concepts of science and engineering, such as fluid flow, diffusive transport akin to diffusion, rheology, probability, and electrical networks [1–16]. Consequently, the solution of FPDEs represents nowadays a vigorous research area for scientists and finding approximate and exact solutions to FPDEs is an important task. However, PDEs are commonly hard to tackle, and their fractional-order types are more complicated [1, 2, 17, 18].

In recent years, several analytical and approximate techniques such as the Adomian decomposition [19], homotopy analysis [20], tau method [21], and variational iteration method [22, 23] have been constructed for solving FPDEs. Nevertheless, the study of PDEs with fractional derivative has been impeded because of the nonappearance of low cost and accurate techniques to deal with them. In addition, the derivation of approximate solution of FPDEs remains a hotspot and demands to endeavor some proficient and solid plans are an issue of serious interest.

Based on the depiction above, we undoubtedly need to explore new schemes that propose prompt and obvious typical terms of analytic solutions and additionally numerical approximate solutions without linearization or discretization. Therefore, a sincere attempt has been made in this research to implement relatively new approximate-analytical technique for nonhomogeneous PDEs with a noninteger derivative of order $\alpha \in (0,1]$ as

$$ {}^{c}D_t^{\alpha} u(x,t) - u_{xx}(x,t) = g(t), \quad x \in [0,L], \ t > 0 \tag{1} $$

subject to the initial and boundary conditions:

$$ \begin{aligned} u(x,0) &= f(x), \quad 0 < x < L, \\ u(0,t) &= \theta(t), \\ u(L,t) &= \omega(t), \\ &\qquad t > 0. \end{aligned} \tag{2} $$

The analytical solutions of the FPDEs were investigated in literature by employing of Green's functions or Fourier-Laplace transforms [24–26]. However, explicit analytic solutions of FPDEs are seldom obtainable in the literature due to extra complexity of dealing with fractional derivative. The aim of the current letter is to extend the application of the separation-variables method to solve the approximating PDE corresponding to the FPDE (1). In fact, we combine an algorithm based on the Laplace transform method to convert the FPDE to the relevant PDE [27] and a separation of variables scheme to achieve the analytical solution of the derived PDE which is close to the exact solution of the original FPDE.

The separation of variables method, at least, in its most basic structures, for example, in Cartesian, spherical, or ellipsoidal coordinates, is a key part of the basic mathematical modules. In short, the separation of variables can be portrayed as a tool to reduce a multidimensional problem to series of one-dimensional ones. It behaves similar to most of global numerical techniques for solving complex models arising in real-world systems [28–31]. This urges us to exploit the powerful properties of the separation-variables method for solving FPDEs as well as reducing the difficulty of working with fractional derivatives.

In terms of numerical approach, this strategy provides more efficient tool for computing approximate solutions of FPDEs in comparison with the most common numerical methods such as finite difference techniques. These stepping-type techniques need to manage the matter of stability. In either case, the base stride size for stability can be specified analytically; yet ordinarily the step sizes are not large. Note likewise that stability does not suggest exactness, particularly while approximating noninteger derivatives. Other numerical methods such as finite element methods [32–34] were developed for numerically solving FPDEs but again such a solution requires the discretization of domain into the number of finite domains/points and their computational difficulties increment quickly with the number of testing nodes. Besides that, rounding-off errors solemnly influence the solution precision in the numerical techniques with complicatedness that also raise quickly with the number of testing nodes [35]. To demonstrate our claims which partially motivated us to develop the present scheme for FPDEs, we compare it with a nonstandard finite difference (NSFD) method and standard finite difference (SFD) algorithm [27].

To summarize, solving FPDEs with the proposed scheme offers the solution without coordinate transformations and computational cost does not grow immediately when the quantity of sampling points raises. Moreover, the product solutions constructed by the proposed methodology involve single independent variables regardless of the dimension of the problem and are continuous over all the domain of integration. Our scope in this paper is to give a promising and applicable algorithm for solving FPDEs to achieve an accurate solution which takes conveniences of the characteristics of the separation-variables scheme.

The organization of this letter is as follows. To have the required mathematical background, in Section 2 we describe some necessary definitions and mathematical preliminaries of the fractional calculus theory. In Section 3, we explain how the approximate-analytical method admitted by nonhomogeneous PDEs with fractional derivatives can be implemented. The usefulness of the above method has been illustrated through a number of examples in Section 4. In Section 5, we give a brief outline of our outcomes.

2. Notes on Fractional Calculus

Before embarking into the details of our method to FPDEs, we might want to review some fundamental definitions, results, and characteristics of the fractional calculus operators, utilized as a part of the remaining context of the report. The interested readers are refereed for more details to the following monographs [1, 2, 5, 17].

Definition 1. The Riemann-Liouville fractional integral operator $\mathfrak{J}^{\alpha}$ of order $\alpha \geq 0$ is defined by

$$ \begin{aligned} \left(\mathfrak{J}_t^{\alpha} u\right)(x,t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(x,\tau)\, d\tau, \quad \alpha > 0, \\ \mathfrak{J}^{0} u(t) &= u(t), \end{aligned} \tag{3} $$

in which $\Gamma(\cdot)$ indicates the gamma function.

Definition 2. Let us assume n is the smallest integer that is greater than α; the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as follows:

$$ \begin{aligned} {}^{c}\mathfrak{D}_t^{\alpha} u(x,t) &= \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} \\ &= \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n (u,\tau)}{\partial \tau^n} d\tau, & n-1 < \alpha < n, \\ \dfrac{\partial^n (x,t)}{\partial t^n} & \alpha = n \in \mathbb{N}. \end{cases} \end{aligned} \tag{4} $$

The Caputo-type fractional derivative is also stated as ${}^{c}\mathfrak{D}_t^{\alpha} u(x,t) = \mathfrak{J}_t^{n-\alpha}(\partial^n (x,t)/\partial t^n)$. Furthermore, Caputo's fractional derivatives are the linear operators given as follows:

$$ \begin{aligned} &{}^{c}\mathfrak{D}_t^{\alpha} \left(\eta u(x,t) + \lambda v(x,t)\right) \\ &\quad = \eta\, {}^{c}\mathfrak{D}_t^{\alpha} u(x,t) + \lambda\, {}^{c}\mathfrak{D}_t^{\alpha} v(x,t), \end{aligned} \tag{5} $$

in which η and λ are constants.

The Laplace transform of Caputo fractional derivative of order $\alpha > 0$ is (as presented in [26, 27])

$$L\left\{{}^{c}\mathfrak{D}_{t}^{\alpha} f(t)\right\} = s^{\alpha} f(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \qquad n-1 \leq \alpha \leq n, \; n \in \mathbb{N}. \tag{6}$$

It is worthy to mention that

$$L\left\{{}^{c}D_{t}^{\alpha} f(t)\right\} = s^{\alpha} f(s) - s^{\alpha-1} f(0), \quad 0 < \alpha \leq 1, \tag{7}$$

in which

$$L\{f(t)\} = f(s) = \int_{0}^{\infty} e^{-st} f(t)\, dt. \tag{8}$$

3. Formulation of the Solution Method

In this section we illustrate the strategy that is proposed to approximate the solution of the FPDEs with Caputo-type derivative. In this regard, the FPDE is converted to a PDE by approximating the fractional derivative using the approach proposed in [27]; then, a variable-separation technique is developed to achieve an approximate-analytical solution for the nonhomogeneous FPDE. In fact, the model reduction fulfilled at the first step can enhance the computational productivity.

Let us consider the following Caputo-type FPDE as follows:

$$\begin{gathered} {}^{c}\mathfrak{D}_{t}^{\alpha} u(x,t) = u_{xx}(x,t) + g(x,t), \\ 0 < x < L, \; t > 0, \; 0 < \alpha < 1, \end{gathered} \tag{9}$$

$$\begin{aligned} u(x,0) &= f(x), \quad 0 < x < L, \\ u(0,t) &= \theta(t), \\ u(L,t) &= \omega(t), \\ & \qquad t > 0. \end{aligned} \tag{10}$$

The Caputo fractional term ${}^{c}\mathfrak{D}_{t}^{\alpha} u(x,t)$ is approximated by employing Laplace transform method and a linearization method proposed by Ren et al. [22] which was developed for interval arithmetic in [33] as follows:

$$L\left\{{}^{c}\mathfrak{D}_{t}^{\alpha} u(x,t)\right\} = s^{\alpha} u(x,s) - s^{\alpha-1} u(x,0), \tag{11}$$

in which $u(x,s)$ is the Laplace transform of $u(x,t)$. In this study we assume that $0 < \alpha < 1$, so the term s^{α} is linearized as

$$s^{\alpha} \approx \alpha s^{1} + (1-\alpha) s^{0} = \alpha s + (1-\alpha). \tag{12}$$

By replacing (12) into (11) and using the inverse Laplace transform we have

$${}^{c}\mathfrak{D}_{t}^{\alpha} u(x,t) \approx \alpha u_{t} + (1-\alpha)\left[u(x,t) - u(x,0)\right]. \tag{13}$$

By taking into account (13), the original FPDE (9) is simplified to the following PDE:

$$\begin{aligned} \alpha u_{t}(x,t) &+ (1-\alpha) u(x,t) \\ &= u_{xx}(x,t) + (1-\alpha) f(x) + g(x,t), \\ & \qquad 0 < x < L, \; t > 0, \; 0 < \alpha < 1, \end{aligned} \tag{14}$$

$$\begin{aligned} u(x,0) &= f(x), \quad 0 < x < L, \\ u(0,t) &= \theta(t), \\ u(L,t) &= \omega(t), \\ & \qquad t > 0. \end{aligned} \tag{15}$$

In this step the problem is converted to a PDE and the fractional derivative is removed from the problem which can extremely reduce the computation cost. Now, we present our scheme to get the analytical solution of the PDE (14) which leads to getting an approximate-analytical solution of the FPDE (9).

3.1. Separation of Variables Method for Some PDEs. Let us consider the following PDE:

$$\begin{aligned} u_{t}(x,t) &= c^{2} u_{xx}(x,t), \quad 0 < x < L, \; t > 0, \\ u(0,t) &= u(L,t) = 0, \\ u(x,0) &= f(x). \end{aligned} \tag{16}$$

In order to find nontrivial solutions $u(x,t)$ for the PDE (16), we should assume that $u(x,t)$ is a product of two functions depending on two parameters t and x such that $u(x,t) = \mathcal{F}(x)\mathcal{G}(t)$. Hence, we have $u_t = \mathcal{F}(x)\dot{\mathcal{G}}(t)$ and $u_{xx} = \mathcal{F}''(x)\mathcal{G}(t)$. By substituting these products in (16) we have

$$\mathcal{F}\dot{\mathcal{G}} = c^{2}\mathcal{F}''\mathcal{G} \tag{17}$$

that is implied as follows:

$$\frac{\mathcal{F}''}{\mathcal{F}} = \frac{\dot{\mathcal{G}}}{c^{2}\mathcal{G}}. \tag{18}$$

Since the left-hand side of (18) is a function of x and the right side is a function of t, so it can be concluded that (18) holds such that

$$\frac{\mathcal{F}''}{\mathcal{F}} = \frac{\dot{\mathcal{G}}}{c^{2}\mathcal{G}} = k, \tag{19}$$

in which k is constant. Therefore, by considering the boundary condition (16) we have

$$\begin{aligned} & 0 = u(0,t) = \mathcal{F}(0)\mathcal{G}(t) \longrightarrow \\ & \mathcal{F}(0) = 0, \\ & 0 = u(L,t) = \mathcal{F}(L)\mathcal{G}(t) \longrightarrow \\ & \mathcal{F}(L) = 0. \end{aligned} \tag{20}$$

Note that if we assume $\mathscr{G}(t) = 0$ in the above equations, it leads to a trivial solution $u(x,t) = 0$. So, we have the next relations for $\mathscr{F}(x)$:

$$\begin{aligned} &\mathscr{F}'' - k\mathscr{F} = 0, \\ &\mathscr{F}(0) = \mathscr{F}(L) = 0. \end{aligned} \tag{21}$$

Now we determine the nontrivial solutions for different values of k. It is easy to find that (21) has a trial solution for $k = 0$ and $k > 0$. We have only nontrivial solution for $k = -\lambda^2 < 0$ such that $\lambda = n\pi/L$ for all $n \in \mathbb{N}$; thus $\mathscr{F}_n(x) = b_n \sin(\lambda x)$ is a solution for (21). By substituting $k = -\lambda^2 = -(n\pi/L)^2$ in $\dot{\mathscr{G}} - kc^2\mathscr{G} = 0$, we obtain the solution $\mathscr{G}_n(t) = a_n \exp(-\lambda^2 c^2 t)$ for all $n \in \mathbb{N}$. Therefore, $u_n(x,t) = \mathscr{F}_n(x)\mathscr{G}_n(t), \forall n \in \mathbb{N}$ is a solution of PDE (16) and the general solution of the problem can be formed as

$$\begin{aligned} u(x,t) &= \sum_{n=1} A_n \mathscr{G}_n(t)\mathscr{F}_n(x) \\ &= \sum_{n=1} A_n \exp\left(-\left(\frac{nc\pi}{L}\right)^2 t\right)\sin\left(\frac{n\pi}{L}x\right). \end{aligned} \tag{22}$$

By applying the initial condition $u(x,0) = f(x)$, we can specify the coefficients A_n as follows:

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx. \tag{23}$$

Now let us consider the following problem:

$$\begin{aligned} &u_t(x,t) + \alpha^2 u(x,t) = \beta^2 u_{xx}(x,t) + g(x,t), \\ &\qquad\qquad 0 < x < L,\ t > 0, \\ &u(0,t) = u(L,t) = 0, \\ &u(x,0) = f(x). \end{aligned} \tag{24}$$

In order to solve problem (24), at first we should consider the corresponding homogeneous equation $u_t(x,t) + \alpha^2 u(x,t) = \beta^2 u_{xx}(x,t)$. Similar to the procedure for problem (16), we assume the solution as $u(x,t) = \mathscr{F}(x)\mathscr{G}(t)$. Therefore, by substituting $u(x,t)$ into (16) we have

$$\begin{aligned} &\mathscr{F}\dot{\mathscr{G}} + \alpha^2\mathscr{F}\mathscr{G} = \beta^2\mathscr{F}''\mathscr{G} \longrightarrow \\ &\frac{\dot{\mathscr{G}} + \alpha^2\mathscr{G}}{\beta^2\mathscr{G}} = \frac{\mathscr{F}''}{\mathscr{F}} = k, \end{aligned} \tag{25}$$

where k is a constant. By taking into consideration the boundary condition $\mathscr{F}(0) = \mathscr{F}(L) = 0$, it is proved that $k = -\lambda^2 < 0$. Hence by solving the equation corresponding to $\mathscr{F}(x)$, it is implied that $\lambda = n\pi/L$ for all $n \in \mathbb{N}$ and we acquire $\mathscr{F}_n(x)$ as $\mathscr{F}_n(x) = B_n \sin((n\pi/L)x)$. Now we can get the solution of the nonhomogeneous problem (24) in a form as

$$u(x,t) = \sum_{n=1} \mathscr{G}_n(t)\sin\left(\frac{n\pi}{L}x\right). \tag{26}$$

By replacing (26) into problem (24) we have

$$\begin{aligned} &\sum_{n=1}\left(\dot{\mathscr{G}}_n + \left[\alpha^2 + \left(\frac{\beta n\pi}{L}\right)^2\right]\mathscr{G}_n\right)\sin\left(\frac{n\pi}{L}x\right) \\ &\quad = g(x,t). \end{aligned} \tag{27}$$

Consequently, the coefficients $(\dot{\mathscr{G}}_n + [\alpha^2 + (\beta n\pi/L)^2]\mathscr{G}_n)$ are the coefficients of Fourier sine series of $g(x,t)$ given by

$$\begin{aligned} &\left(\dot{\mathscr{G}}_n + \left[\alpha^2 + \left(\frac{\beta n\pi}{L}\right)^2\right]\mathscr{G}_n\right) \\ &\quad = \frac{2}{L}\int_0^L g(x,t)\sin\left(\frac{n\pi}{L}x\right)dx. \end{aligned} \tag{28}$$

In order to solve the above equation, we employ the initial condition $u(x,0) = f(x)$ as follows:

$$\begin{aligned} &f(x) = u(x,0) = \sum_{n=1}\mathscr{G}_n(0)\sin\left(\frac{n\pi}{L}x\right) \longrightarrow \\ &\mathscr{G}_n(0) = \frac{2}{L}\int_0^L f(x,t)\sin\left(\frac{n\pi}{L}x\right)dx. \end{aligned} \tag{29}$$

Hence, by solving (28) based on the initial condition (29), we can obtain $\mathscr{G}_n(t)$ and finally $u(x,t)$ is achieved, that is, the solution of problem (24).

4. Numerical Experiments

To reveal the usefulness of the proposed plan for the nonhomogeneous PDEs with the Caputo-type derivative, we test a number of FPDEs and compare with the NSFD method and SFD technique [27]. The accuracy of these methods is computed by a maximum norm relative error [27] and error norm L_∞. It is shown that the present technique is a precise and cost efficient tool for the solution of FPDEs. The numerical computations are implemented through Matlab software, version R2010a, and the CPU of system is Intel(R) core(TM) i3-4130 with RAM 4 GB.

Remark 3. To have an accurate comparison with the technique presented in [27], we employ the nonstandard Crank-Nicolson method as a NSFD method and standard Crank-Nicolson method as a SFD technique with the formula proposed in [27].

Example 4. Let us consider the following FPDE [27]:

$$\begin{aligned} &{}^c\mathfrak{D}_t^\alpha u(x,t) = u_{xx}(x,t) \\ &\qquad + \left(\frac{6}{\Gamma(4-\alpha)}t^{3-\alpha} + t^3 + 1\right)\sin x, \\ &u(x,0) = \sin x, \quad 0 < x < \pi,\ 0 < \alpha < 1, \\ &u(0,t) = u(\pi,t) = 0. \end{aligned} \tag{30}$$

The exact solution of the above problem is $u(x,t) = (t^3 + 1)\sin x$. By employing the technique introduced in [27], we approximate the Caputo fractional derivative; hence, the problem (30) is converted to the following system:

$$\begin{aligned}
&\alpha u_t(x,t) + (1-\alpha)u(x,t) \\
&\quad = u_{xx}(x,t) \\
&\qquad + \left(\frac{6}{\Gamma(4-\alpha)}t^{3-\alpha} + t^3 + 2 - \alpha\right)\sin(x), \\
&\qquad\qquad 0 \le x \le \pi,\ t \ge 0, \\
&u(x,0) = \sin(x), \\
&u(0,t) = u(\pi,t) = 0.
\end{aligned} \tag{31}$$

Since the boundary conditions are homogeneous, we can obtain the solution of (31) by using the separation of variables method. So, let us assume that

$$u(x,t) = \sum_{n=1} \mathcal{G}_n(t)\sin(nx), \tag{32}$$

by replacing (32) in (31) we have

$$\begin{aligned}
&\sum_{n=1}\left(\alpha\dot{\mathcal{G}}_n(t) + \left(1-\alpha+n^2\right)\mathcal{G}_n(t)\right)\sin(nx) \\
&\quad = \left(\frac{6}{\Gamma(4-\alpha)}t^{3-\alpha} + t^3 + 2 - \alpha\right)\sin(x),
\end{aligned} \tag{33}$$

in which $\dot{\mathcal{G}}_n(t)$ is the derivative of $\mathcal{G}_n(t)$ with respect to t.

Regarding the equivalency of the right-hand side of (33) with the left-hand side, it can be implied that $\alpha\dot{\mathcal{G}}_n(t) + (1-\alpha+n^2)\mathcal{G}_n(t)$ are the coefficients of the Fourier sin series of the right-hand side function. So, we have for $n = 1$

$$\begin{aligned}
&\alpha\dot{\mathcal{G}}_1(t) + (2-\alpha)\mathcal{G}_1(t) \\
&\quad = \left(\frac{6}{\Gamma(4-\alpha)}t^{3-\alpha} + t^3 + 2 - \alpha\right)
\end{aligned} \tag{34}$$

and for $n \ge 2$

$$\alpha\dot{\mathcal{G}}_n(t) + \left(1-\alpha+n^2\right)\mathcal{G}_n(t) = 0. \tag{35}$$

From the initial condition of problem (31) we have

$$\begin{aligned}
&\sin(x) = u(x,0) = \sum_{n=1}\mathcal{G}_n(0)\sin(nx) \Longrightarrow \\
&\mathcal{G}_1(0) = 1, \\
&\qquad\qquad \forall n \ge 0,\ \mathcal{G}_n(0) = 0.
\end{aligned} \tag{36}$$

By solving the two systems of equations,

$$\begin{aligned}
&\alpha\dot{\mathcal{G}}_1(t) + (2-\alpha)\mathcal{G}_1(t) = \frac{6}{\Gamma(4-\alpha)}t^{3-\alpha} + t^3 + 2 - \alpha, \\
&\mathcal{G}_1(0) = 1, \\
&\alpha\dot{\mathcal{G}}_n(t) + \left(1-\alpha+n^2\right)\mathcal{G}_n(t) = 0, \\
&\mathcal{G}_n(0) = 0,
\end{aligned} \tag{37}$$

Table 1: Maximum norm relative errors of the proposed approximate-analytical solution, NSFD method, and SFD technique for Example 4.

α	Our scheme	NSFD method	SFD method [27]
0.01	0.002142897	0.001377269	0.002101727
0.1	0.019659612	0.019617338	0.019617785
0.3	0.045194043	0.045149718	0.045148926
0.5	0.051662902	0.051612374	0.051614489
0.7	0.041504057	0.041454967	0.041453574
0.9	0.017072953	0.017021427	0.017021427
0.99	0.001846904	0.001795412	0.001795165

we can find that $\mathcal{G}_1(t) \ne 0$, $\mathcal{G}_n(t) = 0$ for $n \ge 2$ and

$$u(x,t) = \mathcal{G}_1(t)\sin(x). \tag{38}$$

In [27], the authors employed the following SFD method to approximate the solution of problem (31):

$$\begin{aligned}
&\alpha\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{1-\alpha}{2}\left(u_i^n + u_i^{n+1}\right) \\
&\quad = \frac{1}{2}\left(\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta^2 x} + \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta^2 x}\right) \\
&\qquad + \frac{R_i^n + R_i^{n+1}}{2},
\end{aligned} \tag{39}$$

in which

$$R(x,t) = \left(\frac{6}{\Gamma(4-\alpha)}t^{3-\alpha} + t^3 + 2 - \alpha\right)\sin(x). \tag{40}$$

Now we solve problem (31) by using the NSFD method. Note that the formula of NSFD method is similar to the SFD method (39) with a difference that Δt is replaced with $\phi(\Delta t) = (\exp(\beta\Delta t) - 1)/\beta$, where $\beta = (\alpha - 2)/\alpha$. Hence, an algebraic nonlinear equations system is obtained that can be solved by LU decomposition method to get the approximate solution. Then, the relative error in maximum norm between the numerical solution and the exact solution of (31) is achieved using the following formula:

$$E(\Delta x, \Delta t) = \max_{0\le n\le N}\frac{\max_{1\le j\le M-1}\left|u\left(x_j, t_n\right) - u_j^n\right|}{\max_{1\le j\le M-1}\left|u\left(x_j, t_n\right)\right|}. \tag{41}$$

In Table 1 by using formula (41), the difference between the approximate solutions of the NSFD method, SFD method, and the proposed approximate-analytical solution is compared with the exact solution of the original FPDE (30) for $\Delta x = \pi/100, \Delta t = 1/100$ on $t \in [0,1]$. A comparison of the NSFD and SFD solutions based on the error norm L_∞ for different values of α at $T = 1$ is shown in Table 2. The numerical results illustrated that the present scheme works very well for this problem, specially near $\alpha = 1$. The order of accuracy is approximately the same for all the numerical methods. However, there is no requirement to check the

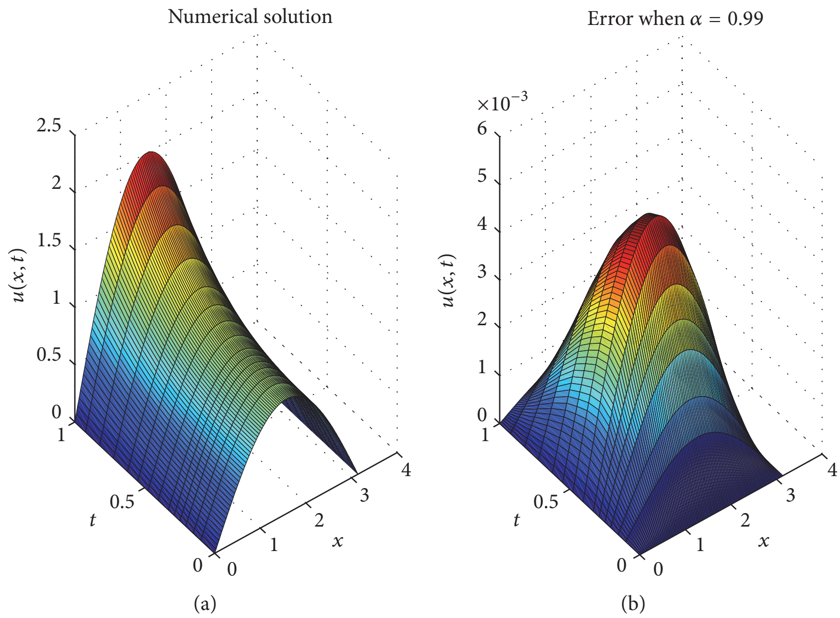

FIGURE 1: The numerical solution (a) with the error function (b) of Example 4 for $\alpha = 0.99$ over $t \in [0, 1]$.

TABLE 2: Error norm L_∞ for Example 4.

α	Our scheme	NSFD method	SFD method [27]
0.01	0.003699	0.001490	0.003617
0.1	0.034897	0.033022	0.034801
0.3	0.084289	0.082783	0.084514
0.5	0.098251	0.098184	0.099550
0.7	0.078375	0.080241	0.081020
0.9	0.029823	0.033207	0.033430
0.99	0.000609	0.003509	0.003529

stability and consistency of the solution for the proposed algorithm while for others it is critical. Moreover, the graphs of the numerical solution (38) with the error function between the exact and numerical solutions over $t \in [0, 1]$ for $\alpha = 0.99$ are plotted in Figure 1.

Example 5. Let us consider the following FPDE [27]:

$$\begin{aligned} {}^{c}\mathfrak{D}_t^{\alpha} u(x,t) &= u_{xx}(x,t) \\ &\quad + \left(\frac{1}{6}\Gamma(4+\alpha)t^3 - t^{3+\alpha}\right)\sin x, \\ u(x,0) &= 1, \quad 0 < x < \pi, \\ u(0,t) &= u(\pi,t) = 1. \end{aligned} \tag{42}$$

The the exact solution is $u(x,t) = t^{3+\alpha}\sin x + 1$. Similar to problem (30), the above problem is converted to the following PDE:

$$\begin{aligned} &\alpha u_t(x,t) + (1-\alpha)u(x,t) \\ &\quad = u_{xx}(x,t) + (1-\alpha) \\ &\qquad + \left(\frac{\Gamma(4+\alpha)}{6}t^3 - t^{3+\alpha}\right)\sin(x), \quad 0 < x < \pi, \\ &u(x,0) = 1, \\ &u(0,t) = u(\pi,t) = 1. \end{aligned} \tag{43}$$

To change the conditions of problem (43) to homogeneous conditions, we replace $u(x,t)$ with $v(x,t) + 1$. By substituting this new variable in (43) we have

$$\begin{aligned} &\alpha v_t(x,t) + (1-\alpha)v(x,t) \\ &\quad = v_{xx}(x,t) + \left(\frac{\Gamma(4+\alpha)}{6}t^3 - t^{3+\alpha}\right)\sin(x), \\ &\qquad\qquad 0 < x < \pi, \\ &v(x,0) = 0, \\ &v(0,t) = v(\pi,t) = 0. \end{aligned} \tag{44}$$

Similar to Example 4, we can get the general solution of problem (44) according the boundary conditions as

$$v(x,t) = \sum_{n=1} \mathcal{G}_n(t)\sin(nx). \tag{45}$$

By replacing (45) into (44) we have

$$\begin{aligned} &\sum_{n=1}\left(\alpha\dot{\mathcal{G}}_n(t) + \left(1-\alpha+n^2\right)\mathcal{G}_n(t)\right)\sin(nx) \\ &\quad = \left(\frac{\Gamma(4+\alpha)}{6}t^3 - t^{3+\alpha}\right)\sin(x). \end{aligned} \tag{46}$$

Hence, $\alpha\dot{\mathcal{G}}_n(t) + (1-\alpha+n^2)\mathcal{G}_n(t)$ are the coefficients of the Fourier sin series of the right-hand side function and we have for $n = 1$

$$\alpha\dot{\mathcal{G}}_1(t) + (2-\alpha)\mathcal{G}_1(t) = \left(\frac{\Gamma(4+\alpha)}{6}t^3 - t^{3+\alpha}\right) \tag{47}$$

and for $n \geq 2$

$$\alpha \dot{\mathcal{G}}_n(t) + \left(1 - \alpha + n^2\right) \mathcal{G}_n(t) = 0. \quad (48)$$

Regarding the initial conditions (43), we have

$$0 = v(x,0) = \sum_{n=1} \mathcal{G}_n(0) \sin(nx) \Longrightarrow \\ \mathcal{G}_n(0) = 0 \quad \forall n. \quad (49)$$

By solving the systems of equations,

$$\begin{aligned} &\alpha \dot{\mathcal{G}}_1(t) + (2-\alpha) \mathcal{G}_1(t) = \frac{\Gamma(4+\alpha)}{6} t^3 - t^{3+\alpha}, \\ &\mathcal{G}_1(0) = 0, \\ &\alpha \dot{\mathcal{G}}_n(t) + \left(1 - \alpha + n^2\right) \mathcal{G}_n(t) = 0, \\ &\mathcal{G}_n(0) = 0, \end{aligned} \quad (50)$$

it is obtained that $\mathcal{G}_1(t) \neq 0$, $\mathcal{G}_n(t) = 0$ for $n \geq 2$ and

$$\begin{aligned} v(x,t) &= \mathcal{G}_1(t) \sin(t), \\ u(x,t) &= \mathcal{G}_1(t) \sin(t) + 1. \end{aligned} \quad (51)$$

The exact solution of the original FPDE is $t^{\alpha+3} \sin(x) + 1$. As it is obvious, the difference between the exact solution and the numerical solution $u(x,t)$ is the terms $\mathcal{G}_1(t)$ and $t^{3+\alpha}$. Jiang and Ma [27] approximated the solution using the SFD method (39) such that

$$R(x,t) = (1-\alpha) + \left(\frac{\Gamma(4+\alpha)}{6} t^3 - t^{3+\alpha}\right) \sin(x), \quad (52)$$

and the error of the method is analyzed by using relation (41). Once again we employ NSFD method to have a comparison among our proposed scheme, NSFD method, and SFD technique. Indeed, the formula of NSFD method is similar to the SFD method (39) with a difference that Δt is replaced with $\phi(\Delta t) = (\exp(\beta \Delta t) - 1)/\beta$ in which $\beta = (\alpha - 2)/\alpha$. We also employ (41) to compare the numerical solution $u(x,t)$ with the the SFD method and NSFD techniques by assuming $\Delta x = \pi/100, \Delta t = 1/100$ on $t \in [0,1]$ in Table 3. In addition, error norm L_∞ is used to have a comparison from another point of view that is illustrated in Table 4. From the tables we can see that the order of accuracy for all the present numerical techniques is the same. Besides, error norm L_∞ for the approximate-analytical solution is plotted with $\alpha = 0.9, 0.99$ and 0.999 at $T = 1$ in Figure 2.

Remark 6. It is worth noting here that the results of this problem, by using the SFD method as it is shown in [27], are not correct. Thereby, we modified the results to have a fair comparison for this example.

Example 7. Consider the following FPDE [27]:

$$\begin{aligned} {}^{c}\mathfrak{D}_t^{\alpha} u(x,t) &= u_{xx}(x,t) + \left(\frac{6}{\Gamma(4-\alpha)} t^{3-\alpha} - t^3 - 1\right) e^x, \\ u(x,0) &= e^x, \quad 0 < x < 1, \end{aligned}$$

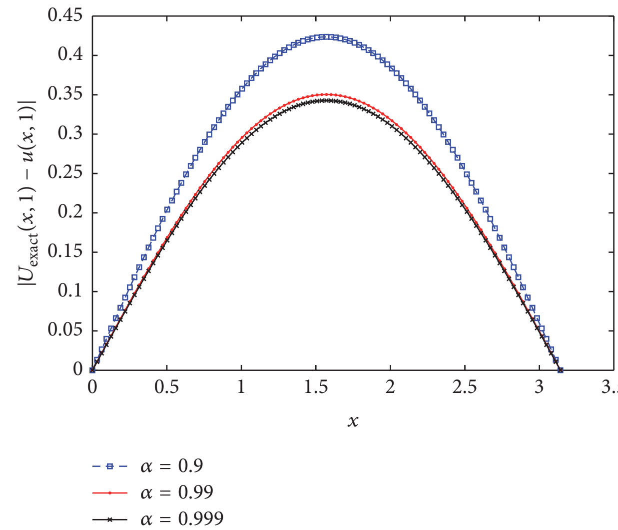

FIGURE 2: Error norm L_∞ for Example 5 ($\alpha = 0.9 - \square$; $\alpha = 0.99 - \circ$, and $\alpha = 0.999 - \times$) at $T = 1$.

$$\begin{aligned} u(0,t) &= t^3 + 1, \\ u(1,t) &= \left(t^3 + 1\right) e, \\ & \qquad t > 0. \end{aligned} \quad (53)$$

The analytical solution of problem (53) is $u(x,t) = (t^3 + 1)e^x$ and $0 < \alpha < 1$. Once again the Caputo-type derivative is approximated by the described technique in the previous section. Therefore, the problem changes to the following PDE:

$$\begin{aligned} &\alpha u_t(x,t) + (1-\alpha) u(x,t) \\ &\quad = u_{xx}(x,t) + (1-\alpha) e^x \\ &\qquad + \left(\frac{6}{\Gamma(4-\alpha)} t^{3-\alpha} - t^3 - 1\right) e^x, \quad 0 < x < 1, \\ &u(x,0) = e^x, \\ &u(0,t) = t^3 + 1, \\ &u(1,t) = \left(1 + t^3\right) e. \end{aligned} \quad (54)$$

In order to change Example (54) to a problem with the homogeneous conditions, we use a change of variables as $u(x,t) = v(x,t) + (t^3 + 1)e^x$. Therefore, we have

$$\begin{aligned} &\alpha v_t(x,t) + (1-\alpha) v(x,t) \\ &\quad = v_{xx}(x,t) \\ &\qquad + \left(\frac{6}{\Gamma(4-\alpha)} t^{3-\alpha} - (1-\alpha) t^3 - 3\alpha t^2\right) e^x, \\ &\qquad\qquad 0 < x < 1, \end{aligned}$$

$$v(x,0)=0,$$
$$v(0,t)=0,$$
$$u(1,t)=0. \tag{55}$$

Analogously to the demonstration of the procedure for the previous examples, we can consider the solution of (55) as $v(x,t) = \sum_{n=1} \mathcal{G}_n(t)\sin(n\pi x)$. By substituting it into (55) we have

$$\sum_{n=1} \left(\alpha \dot{\mathcal{G}}_n(t) + \left(1-\alpha+n^2\pi^2\right)\mathcal{G}_n(t)\right)\sin(n\pi x) = \left(\frac{6}{\Gamma(4-\alpha)}t^{3-\alpha} - (1-\alpha)t^3 - 3\alpha t^2\right)e^x. \tag{56}$$

Therefore, we have

$$\alpha\dot{\mathcal{G}}_n(t) + \left(1-\alpha+n^2\pi^2\right)\mathcal{G}_n(t) = 2\int_0^1 \left(\frac{6}{\Gamma(4-\alpha)}t^{3-\alpha} - (1-\alpha)t^3 - 3\alpha t^2\right) \cdot e^x \sin(n\pi x)\,dx = 2\left(\frac{6}{\Gamma(4-\alpha)}t^{3-\alpha} - (1-\alpha)t^3 - 3\alpha t^2\right)\frac{n\pi\left(1-(-1)^n e\right)}{n^2\pi^2+1}. \tag{57}$$

Now by solving the above differential equation we can get $\mathcal{G}_n(t)$ and finally $v(x,t)$ as

$$v(x,t) = \sum_{n=1}\mathcal{G}_n(t)\sin(n\pi x),$$
$$u(x,t) = \sum_{n=1}\mathcal{G}_n(t)\sin(n\pi x) + \left(t^3+1\right)e^x. \tag{58}$$

From the exact solution of the original FPDE (53), it can be seen that the only difference between the approximate solution, (58), and exact solutions is $v(x,t) = \sum_{n=1}\mathcal{G}_n(t)\sin(n\pi x)$. Table 5 depicts the maximum norm relative error based on formula (41) for the approximate-analytical solution (58), NSFD method, and SFD technique at $\Delta x = 1/100, \Delta t = 1/100$ at $T = 1$. Similar to the previous examples, the authors in [27] solved problem (54) by using the SFD method (39) such that

$$R(x,t) = (1-\alpha)e^x + \left(\frac{6}{\Gamma(4-\alpha)}t^{3-\alpha} - t^3 - 1\right)e^x. \tag{59}$$

Note that the NSFD method is similar to the SFD formula (39) with a difference that Δx is substituted by $\sin(\pi\Delta x)/\pi$ and Δt is replaced with $\phi(\Delta t) = (\exp(\beta\Delta t)-1)/\beta$ in which $\beta = (\alpha - 1 - \pi^2)/\alpha$.

In this experiment, maximum norm relative errors of the numerical methods for different choices of α on $t \in [0,1]$ are displayed in Table 5. Also, the comparison of the numerical results by using the error norm L_∞ at $T = 1$ for different values of α is illustrated in Table 6. We recall that the proposed technique has good accuracy and robustness for different values of α in comparison with others. Also, the error norm L_∞ of the proposed approximate-analytical solution is also plotted for $\alpha = 0.8, 0.9$, and 0.99 at $T = 1$ in Figure 3.

Table 3: Maximum norm relative errors of the proposed approximate-analytical solution, NSFD method, and SFD technique for Example 5.

α	Our scheme	NSFD method	SFD method [27]
0.01	0.4968555	0.0741259	0.4968554
0.1	0.4683680	0.3992456	0.4683675
0.3	0.4063310	0.3861797	0.4063114
0.5	0.3461901	0.3362421	0.3461852
0.7	0.2833040	0.2776980	0.2832938
0.9	0.2117990	0.2085503	0.2117820
0.99	0.1741744	0.1726555	0.1751539

Table 4: Error norm L_∞ for Example 5.

α	Our scheme	NSFD method	SFD method [27]
0.01	0.993711	0.080130	0.993710
0.1	0.936737	0.698034	0.936735
0.3	0.812629	0.738043	0.812623
0.5	0.692474	0.655328	0.692337
0.7	0.566690	0.546031	0.566587
0.9	0.423630	0.412194	0.423564
0.99	0.350385	0.341835	0.350307

Table 5: Maximum norm relative errors of the proposed approximate-analytical solution, NSFD method, and SFD technique for Example 7.

α	Our scheme	NSFD method	SFD method [27]
0.01	$3.15e-4$	$2.38e-5$	$3.02e-4$
0.1	$2.84e-3$	$2.60e-3$	$2.82e-3$
0.3	$6.97e-3$	$6.89e-3$	$6.91e-3$
0.5	$8.72e-3$	$8.62e-3$	$8.65e-3$
0.7	$7.69e-3$	$7.60e-3$	$7.62e-3$
0.9	$3.45e-3$	$3.40e-3$	$3.42e-3$
0.99	$3.88e-4$	$3.57e-4$	$3.81e-4$

Table 6: Error norm L_∞ for Example 7.

α	Our scheme	NSFD method	SFD method [27]
0.01	$1.416e-3$	$2.385e-5$	$1.413e-3$
0.1	$1.310e-2$	$8.690e-3$	$1.309e-2$
0.3	$3.169e-2$	$2.919e-2$	$3.168e-2$
0.5	$3.897e-2$	$3.770e-2$	$3.896e-2$
0.7	$3.367e-2$	$3.316e-2$	$3.365e-2$
0.9	$1.475e-2$	$1.460e-2$	$1.474e-2$
0.99	$1.630e-3$	$1.512e-3$	$1.620e-3$

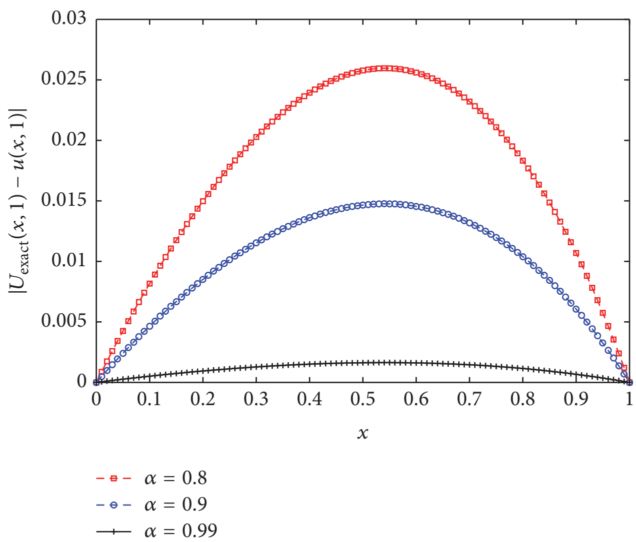

FIGURE 3: Error norm L_∞ for Example 7 ($\alpha = 0.8$ $-\square$; $\alpha = 0.9$ $-\circ$, and $\alpha = 0.99$ $-+$) at $T = 1$.

Example 8. We consider the time-fractional PDE as follows [27]:

$$\begin{aligned} {}^{c}\mathfrak{D}_t^\alpha u(x,t) &= u_{xx}(x,t) \\ &\quad + \left(\frac{6}{\Gamma(4-\alpha)} t^{3-\alpha} + t^3\right)\cos x - e^x, \\ u(x,0) &= e^x, \quad 0 < x < 1, \\ u(0,t) &= t^3 + 1, \\ u(1,t) &= t^3 \cos 1 + e, \end{aligned} \tag{60}$$

and the exact solution of the above problem is $u(x,t) = t^3 \cos(x) + e^x$.

In a similar fashion, we approximate the fractional derivative to covert the time-fractional PDE (60) to the following PDE:

$$\begin{aligned} &\alpha u_t(x,t) + (1-\alpha)u(x,t) \\ &\quad = u_{xx}(x,t) + \left(\frac{6}{\Gamma(4-\alpha)} t^{3-\alpha} + t^3\right)\cos x - \alpha e^x, \\ &\qquad\qquad 0 < x < 1, \ t > 0, \\ &u(x,0) = e^x, \\ &u(0,t) = t^3 + 1, \\ &u(1,t) = t^3 \cos 1 + e. \end{aligned} \tag{61}$$

We define the change of variable $u(x,t) = v(x,t) + (t^3 \cos x + e^x)$ and replace it in the above system. Hence, we need to solve the resulting system as follows:

$$\begin{aligned} &\alpha v_t(x,t) + (1-\alpha)v(x,t) \\ &\quad = v_{xx}(x,t) \\ &\qquad + \left(\frac{6}{\Gamma(4-\alpha)} t^{3-\alpha} - 3\alpha t^2 + (\alpha-1)t^2\right)\cos x, \\ &\qquad\qquad 0 < x < 1, \ t > 0, \\ &v(x,0) = 0, \\ &v(0,t) = v(1,t) = 0. \end{aligned} \tag{62}$$

According to the above system, we can assume the solution of the problem in the form as $v(x,t) = \sum_{n=1} \mathcal{G}_n(t)\sin(n\pi x)$. By substituting this product solution in system (62), we have

$$\begin{aligned} &\sum_{n=1} \left(\alpha \dot{\mathcal{G}}_n(t) + \left(1 - \alpha + n^2\pi^2\right)\mathcal{G}_n(t)\right)\sin(n\pi x) \\ &\quad = \left(\frac{6}{\Gamma(4-\alpha)} t^{3-\alpha} - 3\alpha t^2 + (\alpha-1)t^2\right)\cos x. \end{aligned} \tag{63}$$

Therefore, we have

$$\begin{aligned} &\alpha \dot{\mathcal{G}}_n(t) + \left(1 - \alpha + n^2\pi^2\right)\mathcal{G}_n(t) \\ &\quad = 2\int_0^1 \left(\frac{6}{\Gamma(4-\alpha)} t^{3-\alpha} - 3\alpha t^2 + (\alpha-1)t^2\right)\cos(x) \\ &\quad \cdot \sin(n\pi x)\,dx = \left(\frac{6}{\Gamma(4-\alpha)} t^{3-\alpha} - 3\alpha t^2\right. \\ &\quad \left. + (\alpha-1)t^2\right)\frac{2n\pi}{n^2\pi^2 - 1}\left(1 - (-1)^n \cos 1\right). \end{aligned} \tag{64}$$

Regarding the condition $0 = v(x,0) = \sum_{n=1} \mathcal{G}_n(t)\sin(n\pi x)$, we have $\mathcal{G}_n(0) = 0$. To obtain $\mathcal{G}_n$ for $n \geq 1$, we need to solve the following system:

$$\begin{aligned} &\alpha \dot{\mathcal{G}}_n(t) + \left(1 - \alpha + n^2\pi^2\right)\mathcal{G}_n(t) \\ &\quad = \left(\frac{6}{\Gamma(4-\alpha)} t^{3-\alpha} - 3\alpha t^2 + (\alpha-1)t^2\right) \\ &\quad \cdot \frac{2n\pi}{n^2\pi^2 - 1}\left(1 - (-1)^n \cos 1\right), \\ &\mathcal{G}_n(0) = 0. \end{aligned} \tag{65}$$

Note that the NSFD method here is the same as the formula for Example 7. The accuracy of the proposed method is tested by using the relative error formula (41) over $t \in [0,1]$ and error norm L_∞ for different values of α at $T = 1$ in Tables 7 and 8, respectively. Undoubtedly these results also exhibit the accuracy and proficiency of the present method for this problem. Also, the error norm L_∞ was obtained between the approximate-analytical solution and the exact solution for various values of for α at $T = 1$ in Figure 4. We see that we can accomplish a suitable approximation with the analytical solution by applying the proposed method, taking into account the computational costs of the discretization of domain into finite points in the existing numerical schemes.

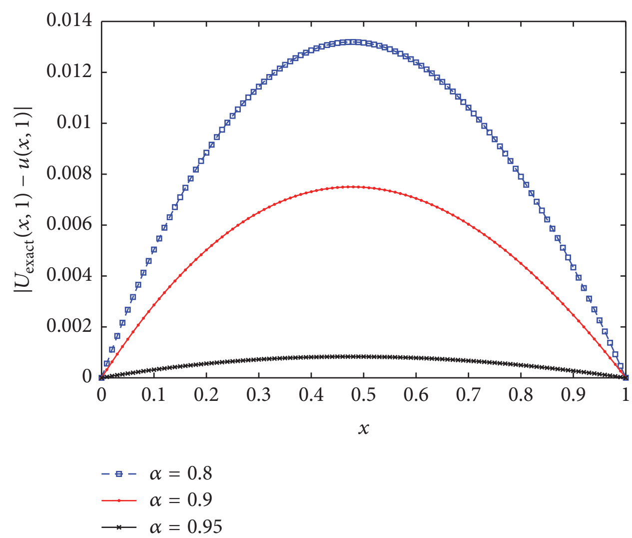

Figure 4: Error norm L_∞ for Example 8 ($\alpha = 0.8$ $-\square$; $\alpha = 0.9$ $-\circ$; and $\alpha = 0.95 - \times$) at $T = 1$.

Table 7: Maximum norm relative errors of the proposed approximate-analytical solution, NSFD method, and SFD technique for Example 8.

α	Our scheme	NSFD method	SFD method [27]
0.01	$2.29e-4$	$2.38e-5$	$2.21e-4$
0.1	$2.13e-3$	$1.54e-3$	$2.05e-3$
0.3	$5.17e-3$	$4.87e-3$	$4.98e-3$
0.5	$6.40e-3$	$6.15e-3$	$6.17e-3$
0.7	$5.57e-3$	$5.35e-3$	$5.37e-3$
0.9	$2.47e-3$	$2.36e-3$	$2.37e-3$
0.99	$2.66e-4$	$2.46e-4$	$2.63e-4$

Table 8: Error norm L_∞ for Example 8.

α	Our scheme	NSFD method	SFD method [27]
0.01	$7.19e-4$	$4.45e-5$	$7.17e-4$
0.1	$6.65e-3$	$4.39e-3$	$6.65e-3$
0.3	$1.61e-2$	$1.48e-2$	$1.60e-2$
0.5	$1.98e-2$	$1.91e-2$	$1.97e-2$
0.7	$1.71e-2$	$1.68e-2$	$1.70e-2$
0.9	$7.49e-3$	$7.43e-3$	$7.49e-3$
0.99	$8.31e-4$	$7.79e-4$	$8.23e-4$

Example 9. Let us consider the FPDE as follows:

$$\begin{aligned} {}^{c}\mathfrak{D}_t^{1/2} u(x,t) &= u_{xx}(x,t) + x(1-x)\frac{6t^{2.5}}{\Gamma(3.5)} \\ &\quad + 2\left(t^3+1\right), \quad t>0 \\ u(x,0) &= x(x-1), \quad 0<x<1, \\ u(0,t) &= u(1,t) = 0, \end{aligned} \tag{66}$$

and the exact solution of the above problem is $u(x,t) = x(x-1)(t^3+1)$.

Once again we approximate the fractional derivative using the plan explained in Section 3. Therefore, the FPDE (66) is converted to the following PDE:

$$\begin{aligned} u_t(x,t) + u(x,t) &= 2u_{xx}(x,t) + x(1-x) \\ &\quad + 2x(1-x)\frac{6t^{2.5}}{\Gamma(3.5)} \\ &\quad + 4\left(t^3+1\right), \quad 0<x<1,\ t>0, \\ u(x,0) &= x(x-1), \\ u(0,t) &= u(1,t) = 0. \end{aligned} \tag{67}$$

Since the boundary conditions are homogeneous, we can hypothesize the solution as $u(x,t) = \sum_{n=1}\mathcal{G}_n(t)\sin(n\pi x)$ and replace it in the above system. So, we have

$$\begin{aligned} &\sum_{n=1}\left(\dot{\mathcal{G}}_n(t) + \left(1+2n^2\pi^2\right)\mathcal{G}_n(t)\right)\sin(n\pi x) \\ &\quad = x(1-x) + 2x(1-x)\frac{6t^{2.5}}{\Gamma(3.5)} + 4\left(t^3+1\right). \end{aligned} \tag{68}$$

Hence, we have

$$\begin{aligned} \dot{\mathcal{G}}_n(t) + \left(1+2n^2\pi^2\right)\mathcal{G}_n(t) &= 2\int_0^1\left(x(1-x)\right. \\ &\quad \left. + 2x(1-x)\frac{6t^{2.5}}{\Gamma(3.5)} + 4\left(t^3+1\right)\right)\sin(n\pi x)\,dx \\ &= \frac{4\left(1-(-1)^n\right)}{n^3\pi^3}\left(1+12\frac{t^{2.5}}{\Gamma(3.5)}\right) \\ &\quad + \frac{8\left(1-(-1)^n\right)}{n\pi}\left(t^3+1\right). \end{aligned} \tag{69}$$

By considering the assumption $x(1-x) = u(x,0) = \sum_{n=1}\mathcal{G}_n(0)\sin(n\pi x)$, we have

$$\mathcal{G}_n(0) = 2\int_0^1 x(1-x)\sin(n\pi x)\,dx = \frac{4\left(1-(-1)^n\right)}{n^3\pi^3}, \tag{70}$$

and to find $\mathcal{G}_n(t)$, the following system should be solved:

$$\begin{aligned} \dot{\mathcal{G}}_n(t) &+ \left(1+2n^2\pi^2\right)\mathcal{G}_n(t) \\ &= \frac{4\left(1-(-1)^n\right)}{n^3\pi^3}\left(1+12\frac{t^{2.5}}{\Gamma(3.5)}\right) \\ &\quad + \frac{8\left(1-(-1)^n\right)}{n\pi}\left(t^3+1\right), \\ \mathcal{G}_n(0) &= \frac{4\left(1-(-1)^n\right)}{n^3\pi^3}. \end{aligned} \tag{71}$$

It is easy to see that $\mathcal{G}_n(t) = 0$ for n is even.

It is worth noting that the demonstration of NSFD method is analogous to the SFD formula (39) with a difference that Δt is replaced with $\phi(\Delta t) = (\exp(\beta\Delta t)-1)/\beta$ in which $\beta = (-1/2-n^2)/(1/2)$.

Table 9: Maximum norm relative errors and error norm L_∞ of the proposed approximate-analytical solution, NSFD method, and SFD technique for Example 9.

	Our scheme	NSFD method	SFD method [27]
Relative error	0.01178	0.01158	0.01158
L_∞	0.004797	0.004656	0.004795

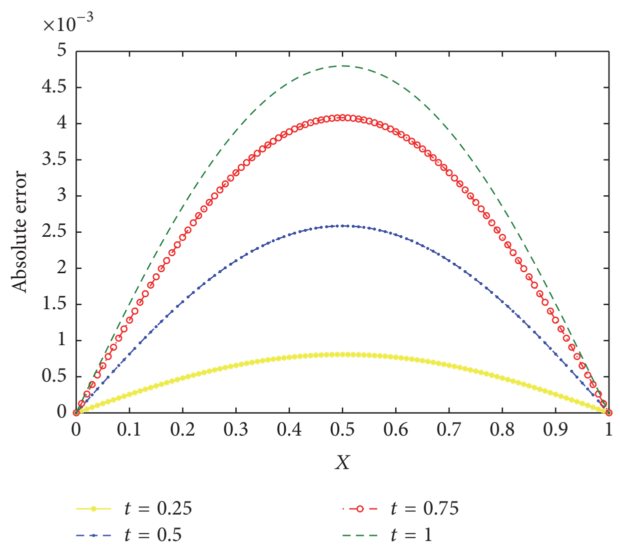

Figure 5: Absolute errors for different values of t, Example 9.

In Table 9, we compute the maximum norm relative errors and error norm L_∞ for approximate-analytical scheme, NSFD method, and SFD technique. The numerical results demonstrate the appropriateness of the present technique for this case and good accuracy in comparison with the others. Besides that, to observe the behavior of the approximate solution over the time interval, we plot the absolute error of the method for $t = 0.25, 0.5, 0.75$, and 1 in Figure 5. From the observation, the method has a smooth behavior over the time domain and obtains a satisfactory result.

5. Concluding Remarks

This article deals with the numerical solution of nonhomogeneous time-fractional PDEs by an approximate-analytical method based on a separation-variables technique. A comparative study is presented by applying two different finite difference methods and the proposed techniques. The strategy exploits an approximating process to reduce the FPDE to the corresponding PDE. This combination is rational and efficient, because, on one hand, the separation-variables technique is very suitable to the homogeneous PDE and, more importantly, can provide an exact solution and reduce the computational cost by avoiding extra conditions on the stability conditions of the method, and on the other hand the adaptive method can cope with different kinds of FPDEs with less effort. The utilization of approximating technique for the fractional derivative radically decreases the computational expense of this variable-separation algorithm. The results of numerical examples demonstrate that this method is accurate similar to finite difference methods.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] I. Podlubny, *Fractional Differential Equations*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1999.

[2] D. Baleanu, J. A. T. Machado, and A. C. J. Luo, *Fractional Dynamics and Control*, Springer Science & Business Media, New York, NY, USA, 2011.

[3] D. Baleanu, K. Diethelm, E. Scalas, and J. J. Trujillo, *Fractional Calculus Models and Numerical Methods*, vol. 3, World Scientific, 2009.

[4] Y. Zhou, *Basic Theory of Fractional Differential Equations*, vol. 6, World Scientific, Singapore, 2014.

[5] V. E. Tarasov, *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer Science & Business Media, 2010.

[6] S. Chen, X. Jiang, F. Liu, and I. Turner, "High order unconditionally stable difference schemes for the Riesz space-fractional telegraph equation," *Journal of Computational and Applied Mathematics*, vol. 278, pp. 119–129, 2015.

[7] X.-J. Yang, J. A. Tenreiro Machado, and H. . Srivastava, "A new numerical technique for solving the local fractional diffusion equation: two-dimensional extended differential transform approach," *Applied Mathematics and Computation*, vol. 274, pp. 143–151, 2016.

[8] F. A. Rihan, "Computational methods for delay parabolic and time-fractional partial differential equations," *Numerical Methods for Partial Differential Equations*, vol. 26, no. 6, pp. 1556–1572, 2010.

[9] H. Tariq and G. Akram, "Quintic spline technique for time fractional fourth-order partial differential equation," *Numerical Methods for Partial Differential Equations*, vol. 33, no. 2, pp. 445–466, 2017.

[10] W. Fan, F. Liu, X. Jiang, and I. Turner, "A novel unstructured mesh finite element method for solving the time-space fractional wave equation on a two-dimensional irregular convex domain," *Fractional Calculus and Applied Analysis*, vol. 20, no. 2, pp. 352–383, 2017.

[11] B. Yu, X. Jiang, and C. Wang, "Numerical algorithms to estimate relaxation parameters and Caputo fractional derivative for a fractional thermal wave model in spherical composite medium," *Applied Mathematics and Computation*, vol. 274, pp. 106–118, 2016.

[12] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204, Elsevier Science Limited, New York, NY, USA, 2006.

[13] X.-L. Ding and J. J. Nieto, "Analytical solutions for the multi-term time-space fractional reaction-diffusion equations on an infinite domain," *Fractional Calculus and Applied Analysis*, vol. 18, no. 3, pp. 697–716, 2015.

[14] T. Bakkyaraj and R. Sahadevan, "An approximate solution to some classes of fractional nonlinear partial differential-difference equation using Adomian decomposition method,"

Journal of Fractional Calculus and Applications, vol. 5, no. 1, pp. 37–52, 2014.

[15] M. Dehghan, J. Manafian, and A. Saadatmandi, “Solving nonlinear fractional partial differential equations using the homotopy analysis method,” *Numerical Methods for Partial Differential Equations*, vol. 26, no. 2, pp. 448–479, 2010.

[16] S. Karimi Vanani and A. Aminataei, “Tau approximate solution of fractional partial differential equations,” *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1075–1083, 2011.

[17] S. Das, “Analytical solution of a fractional diffusion equation by variational iteration method,” *Computers & Mathematics with Applications*, vol. 57, no. 3, pp. 483–487, 2009.

[18] A. Elsaid, “The variational iteration method for solving Riesz fractional partial differential equations,” *Computers & Mathematics with Applications*, vol. 60, no. 7, pp. 1940–1947, 2010.

[19] P. Zhuang, F. Liu, V. Anh, and I. Turner, “New solution and analytical techniques of the implicit numerical method for the anomalous subdiffusion equation,” *SIAM Journal on Numerical Analysis*, vol. 46, no. 2, pp. 1079–1095, 2008.

[20] F. Liu, C. Yang, and K. Burrage, “Numerical method and analytic technique of the modified anomalous subdiffusion equation with a nonlinear source term,” *Journal of Computational and Applied Mathematics*, vol. 231, no. 1, pp. 160–176, 2009.

[21] S. Salahshour, T. Allahviranloo, and S. Abbasbandy, “Solving fuzzy fractional differential equations by fuzzy Laplace transforms,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 3, pp. 1372–1381, 2012.

[22] J. Ren, Z.-Z. Sun, and W. Dai, “New approximations for solving the Caputo-type fractional partial differential equations,” *Applied Mathematical Modelling*, vol. 40, no. 4, pp. 2625–2636, 2016.

[23] F. E. Harris, *Mathematics for physical science and engineering: symbolic computing applications in Maple and Mathematica*, Academic Press, 2014.

[24] L. Gao, X. Tan, and E. T. Chung, “Application of the generalized multiscale finite element method in parameter-dependent (PDE) simulations with a variable-separation technique,” *Journal of Computational and Applied Mathematics*, vol. 300, pp. 183–191, 2016.

[25] M. Kostoglou, “On the analytical separation of variables solution for a class of partial integro-differential equations,” *Applied Mathematics Letters*, vol. 18, no. 6, pp. 707–712, 2005.

[26] V. R. Subramanian and R. E. White, “New separation of variables method for composite electrodes with galvanostatic boundary conditions,” *Journal of Power Sources*, vol. 96, no. 2, pp. 385–395, 2001.

[27] Y. Jiang and J. Ma, “High-order finite element methods for time-fractional partial differential equations,” *Journal of Computational and Applied Mathematics*, vol. 235, no. 11, pp. 3285–3290, 2011.

[28] H. Zhang, F. Liu, and V. Anh, “Galerkin finite element approximation of symmetric space-fractional partial differential equations,” *Applied Mathematics and Computation*, vol. 217, no. 6, pp. 2534–2545, 2010.

[29] N. J. Ford, J. Xiao, and Y. Yan, “A finite element method for time fractional partial differential equations,” *Fractional Calculus and Applied Analysis*, vol. 14, no. 3, pp. 454–474, 2011.

[30] A. Ahmadian, S. Salahshour, D. Baleanu, H. Amirkhani, and R. Yunus, “Tau method for the numerical solution of a fuzzy fractional kinetic model and its application to the oil palm frond as a promising source of xylose,” *Journal of Computational Physics*, vol. 294, pp. 562–584, 2015.

[31] A. Ahmadian, S. Salahshour, and C. S. Chan, “Fractional differential systems: a fuzzy solution based on operational matrix of shifted chebyshev polynomials and its applications,” *IEEE Transactions on Fuzzy Systems*, vol. 25, no. 1, pp. 218–236, 2017.

[32] A. Ahmadian, F. Ismail, S. Salahshour, D. Baleanu, and F. Ghaemi, “Uncertain viscoelastic models with fractional order: a new spectral tau method to study the numerical simulations of the solution,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 53, pp. 44–64, 2017.

[33] A. Ahmadian, S. Salahshour, M. Ali-Akbari, F. Ismail, and D. Baleanu, “A novel approach to approximate fractional derivative with uncertain conditions,” *Chaos, Solitons & Fractals*, vol. 104, pp. 68–76, 2017.

[34] K. Moaddy, S. Momani, and I. Hashim, “The non-standard finite difference scheme for linear fractional PDEs in fluid mechanics,” *Computers & Mathematics with Applications*, vol. 61, no. 4, pp. 1209–1216, 2011.

[35] K. C. Patidar, “Nonstandard finite difference methods: recent trends and further developments,” *Journal of Difference Equations and Applications*, vol. 22, no. 6, pp. 1–33, 2016.