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Aggregation of triangular fuzzy neutrosophic set information and its application to multi-attribute decision making

Pranab Biswas[1], Surapati Pramanik[2*], and Bibhas C. Giri[3]

[1]Department of Mathematics, Jadavpur University, Kolkata, 700032, India. E-mail: paldam2010@gmail.com

[2*] Department of Mathematics, Nandalal Ghosh B.T. College, Panpur, P.O.-Narayanpur, District-North 24 Parganas, West Bengal, PIN-743126, India. Email: sura_pati@yahoo.co.in

[3]Department of Mathematics, Jadavpur University, Kolkata, 700032, India. Email:bcgiri.jumath@gmail.com

*Corresponding author's email: sura_pati@yahoo.co.in

Abstract. This paper is devoted to propose triangular fuzzy number neutrosophic sets by combining triangular fuzzy numbers with single valued neutrosophic set and define some of its operational rules. Then, triangular fuzzy number neutrosophic weighted arithmetic averaging operator and triangular fuzzy number neutrosophic weighted geometric averaging operator are defined to aggregate triangular fuzzy number neutrosophic sets. We have also established some of their properties of the proposed operators. The operators have been employed to multi attribute decision making problem to aggregate the triangular fuzzy neutrosophic numbers based rating values of each alternative over the attributes. The collective rating values of each alternative have been ordered with the help of score and accuracy values to find out the best alternative. Finally, an illustrative example has been provided to validate the proposed approach for multi attribute decision making problem.

Keywords: Triangular fuzzy number neutrosophic set, Score and accuracy function, Triangular fuzzy number neutrosophic weighted arithmetic averaging operator, Triangular fuzzy number neutrosophic weighted geometric averaging operator, Multi-attribute decision making problem.

1 Introduction

Zadeh [1] has been credited with having pioneered the development of the concept of fuzzy set in 1965. It is generally agreed that a major breakthrough in the evolution of the modern concept of uncertainty was achieved in defining fuzzy set, even though some ideas presented in the paper were envisioned in 1937 by Black [2]. In order to define fuzzy set, Zadeh [1] introduced the concept of membership function with a range covering the interval [0, 1] operating on the domain of all possible values. It should be noted that the concept of membership in a fuzzy set is not a matter of affirmation or denial, rather a matter of a degree. Zadeh's original ideas blossomed into a comprehensive corpus of methods and tools for dealing with gradual membership and non-probabilistic uncertainty. In essence, the basic concept of fuzzy set is a generalization of classical set or crisp set [3, 4]. The field has experienced an enormous development, and Zadeh's seminal concept of fuzzy set [1] has naturally evolved in different directions. Different sets have been derived in the literature such as L-fuzzy sets [5], flou sets [6], interval-valued fuzzy sets [7-10], intuitionistic fuzzy sets [11-13], two fold fuzzy sets [14], interval valued intuitionistic fuzzy set [15], intuitionistic L-fuzzy sets [16], etc. Interval-valued fuzzy sets are a special case of L-fuzzy sets in the sense of Goguen [5] and a special case of type 2 fuzzy set. Mathematical equivalence of intuitionistic fuzzy set (IFS) with interval-valued fuzzy sets was noticed by Atanassov [17], Atanassov and Gargov [15]. Wang and He [18] proved that the concepts of IFS [11-13] and intuitionistic L-fuzzy sets [5] and the concept of L-fuzzy sets [5] are equivalent. Kerre [19] provided a summary of the links that exist between fuzzy sets [1] and other mathematical models such as flou sets [6], two-fold fuzzy sets [14] and L-fuzzy sets [5]. Deschrijver and Kerre [20] established the relationships between IFS [11], L-fuzzy sets [5], interval-valued fuzzy sets [7], interval-valued IFS [15]. Dubois et al. [21] criticized the term IFSs in the sense of [11-13], and termed it "to be unjustified, misleading, and possibly offensive to people in intuitionistic mathematics and logic" as it clashes with the correct usage of intuitionistic fuzzy set proposed by Takeuti and Titani [22]. Dubois et al. [21] suggested changing the name of IFS as I-fuzzy set. Smarandache incorporated the degree of indeterminacy as independent component in IFS and defined neutrosophic set [23-24] as the generalization of IFSs. Georgiev [25] explored some properties of the neutrosophic logic and defined simplified neutrosophic set. A neutrosophic set is simplified [25] if its elements are comprised of singleton subsets of the real unit interval. Georgiev [25] concluded that the neutrosophic logic is not capable of maintaining modal operators, since there is no normalization rule for the components T, I and F. The author [25] claimed that the IFSs have the chance to become a consistent model of the modal logic, adopting all the necessary properties [26].However certain type of uncertain information such as indeterminate, incomplete and inconsistent information cannot be dealt with fuzzy sets as well as IFSs. Smarandache [27-28] re-established neutrosophic set as the generalization of IFS, which plays a key role to handle uncertain, inconsistent and indeterminacy information existing in real world. In this set [27-28] each element of the universe is characterized by the truth degree, indeterminacy degree and falsity degree lying in the nonstandard unit interval. The neutrosophic set [27-28] emerged as one of the research focus in many branches such as image processing [29-31], artificial intelligence [32], applied physics [33-34], topology [35] and social science [36]. Furthermore, single valued neutrosophic set[37], interval neutrosophic set[38],neutrosophic soft set[39], neutrosophic soft expert set [40], rough neutrosophic set [41], interval neutrosophic rough set, interval valued neutrosophic soft rough set [42], complex neutrosophic set[43], bipolar neutrosophic sets [44] and neutrosophic cube set[45] have been studied in the literature which are connected with neutrosophic set. However, in this study, we have applied single valued neutrosophic set [37] (SVNS), a subclass of NS, in which each element of universe is characterized by truth membership, indeterminacy membership and falsity membership degrees lying in the real unit interval. Recently, SVNS has caught attention to the researcher on various topics such as similarity measure [46-50], medical diagnosis [51] and multi criteria/ attribute decision making [52-58], etc

Aggregation of SVNS information becomes an important research topic for multi attribute decision making in which the rating values of alternatives are expressed in terms of SVNSs. Aggregation operators of SVNSs, usually taking the forms of mathematical functions, are common techniques to fuse all the input individual data that are typically interpreted as the truth, indeterminacy and the falsity membership degree in SVNS into a single one. Ye [59] proposed weighted arithmetic average operator and weighted geometric average operator for simplified neutrosophic sets. Peng et al.[60] developed some aggregation operators to aggregate single valued neutrosophic information, such as simplified neutrosophic number weighted averaging (SNNWA), simplified neutrosophic number weighted geometric (SNNWG), simplified neutrosophic number ordered weighted averaging (SNNOWA), simplified neutrosophic number ordered weighted geometric averaging (SNNOWG), simplified neutrosophic number hybrid ordered weighted averaging operator(SNNHOWA), simplified neutrosophic number hybrid ordered weighted geometric operator (SNNHOWG), generalised simplified neutrosophic number weighted averaging operator(GSNNWA) and generalised simplified neutrosophic number weighted geometric operator(GSNNGA) operators. Peng et al. [60] applied these aggregation operators in multi criteria group decision making problem to get an overall evaluation value for selecting the best alternative. Liu et al. [61] defined some generalized neutrosophic Hamacher aggregation operators and applied them to multi attribute group decision making problem. Liu and Wang [62] proposed a single valued neutrosophic normalized weighted Bonferroni mean operator for multi attribute decision making problem.

Application of SVNS has been extensively studied in multi-attribute decision making problem. However, in uncertain and complex situations, the truth membership, indeterminacy membership, and falsity membership degree of SVNS cannot be represented with exact real numbers or interval numbers. Moreover, triangular fuzzy number can handle effectively fuzzy data rather than interval number. Therefore, combination of triangular fuzzy number with SVNS can be used as an effective tool for handling incomplete, indeterminacy, and uncertain information existing in decision making problems. Recently, Ye [63] defined trapezoidal fuzzy neutrosophic set and developed trapezoidal fuzzy neutrosophic number weighted arithmetic averaging and trapezoidal fuzzy neutrosophic number weighted geometric averaging operators to solve multi attribute decision making problem.

Zhang and Liu [64] presented method for aggregating triangular fuzzy intuitionistic fuzzy information and its application to decision making. However, their approach cannot deal the decision making problems which involve indeterminacy. So new approach is essentially needed which can deal indeterminacy. Literature review reflects that this is the first time that aggregation operator of triangular fuzzy number neutrosophic values has been studied although this number can be used as an effective tool to deal with uncertain information. In this paper, we have first presented triangular fuzzy number neutrosophic sets (TFNNS), score function and accuracy function of TFNNS. Then we have extended the aggregation method of triangular fuzzy intuitionistic fuzzy information [64] to triangular fuzzy number neutrosophic weighted arithmetic averaging (TFNNWA) operator and triangular fuzzy number neutrosophic weighted geometric averaging (TFNNWG) operator to aggregate TFNNSs. The proposed TFNNWA and TFNNWG operators are more flexible and powerful than their fuzzy and intuitionistic fuzzy counterpart as they are capable of dealing with uncertainty and indeterminacy.

The objectives of the study include to:

- propose triangular fuzzy number neutrosophic sets (TFNNS), score function and accuracy function of TFNNS.
- propose two aggregation operators, namely, TFNNWA and TFNNWG.
- prove some properties of the proposed operators namely, TFNNWA and TFNNWG.
- establish a multi attribute decision making (MADM) approach based on TFNNWA and TFNNWG.
- provide an illustrative example of MADM problem.

The rest of the paper has been organized in the following way. In Section 2, a brief overview of IFS, SVNS have been presented. In Section 3, we have defined TFNNS, score function and accuracy function of TFNNS, and some operational rules of TFNNS. Section 4 has been devoted to propose two aggregation operators, namely, TFNNWA and TFNNWG operators to aggregate TFNNSs. In Section 5, applications of two proposed operators have been presented in multi attribute decision making problem. In Section 6, an illustrative example of MADM has been provided. Finally, conclusion and future direction of research have been presented in Section 7.

2 Preliminaries

In this section we recall some basic definitions of intuitionistic fuzzy sets, triangular fuzzy number intuitionistic fuzzy set (TFNIFS), score function and accuracy function of TFNIFS.

2.1 Intuitionistic fuzzy sets

Definition1. (Intuitionistic fuzzy set [13]) An intuitionistic fuzzy set A in finite universe of discourse $X=\{x_1, x_2, ..., x_n\}$ is given by

$$A=\left\{\left\langle x, \mu_A(\mathrm{x}), \nu_A(\mathrm{x})\right\rangle \mid x\in X\right\}, \tag{1}$$

where $\mu_A: X\to[0,1]$ and $\nu_A: X\to[0,1]$ with the condition $0\le\mu_A(x)+\nu_A(x)\le 1$. The numbers $\mu_A(\mathrm{x})$ and $\nu_A(\mathrm{x})$ denote, respectively, the degree of membership degree and degree of non-membership of x in A. In addition $\pi_A(\mathrm{x})=1-\mu_A(\mathrm{x})-\nu_A(\mathrm{x})$ is called a hesitancy degree of $x\in X$ in A. For convenience, $A=(\mu_A(\mathrm{x}), \nu_A(\mathrm{x}))$ is considered as an intuitionistic fuzzy number (IFN).

Definition 2. (Operations rules of IFNs [65-67])

Let $A=\left(\mu_A(\mathrm{x}), \nu_A(\mathrm{x})\right)$ and $B=\left(\mu_B(\mathrm{x}), \nu_B(\mathrm{x})\right)$ be two IFNs, then the basic operations of IFNs are presented as follows:

1. $A\oplus B=(\mu_A(\mathrm{x})+\mu_B(\mathrm{x})-\mu_A(\mathrm{x})\mu_B(\mathrm{x}),\ \nu_A(\mathrm{x})\nu_B(\mathrm{x})),$ (2)
2. $A\otimes B=(\mu_A(\mathrm{x})\mu_B(\mathrm{x}),\ \nu_A(\mathrm{x})\nu_B(\mathrm{x})-\nu_A(\mathrm{x})\nu_B(\mathrm{x})),$ (3)
3. $\lambda A=\left(1-(1-\mu_A(\mathrm{x}))^{\lambda}, (\nu_A(x))^{\lambda}\right)$ for $\lambda>0,$ (4)
4. $A^{\lambda}=\left((\mu_A(x))^{\lambda}, 1-(1-\nu_A(\mathrm{x}))^{\lambda}\right)$ for $\lambda>0.$ (5)

Definition 3. [68] Let X be a finite universe of discourse and $F[0,1]$ be the set of all triangular fuzzy numbers on $[0,1]$. A triangular fuzzy number intuitionistic fuzzy set (TFNIFS) A in X is represented by

$$A=\left\{\left\langle x, \tilde{\mu}_A(x), \tilde{\nu}_A(x)\right\rangle \mid x\in X\right\},$$

where, $\tilde{\mu}_A(x): X\to F[0,1]$ and $\tilde{\nu}_A(x): X\to F[0,1]$.

The triangular fuzzy numbers $\tilde{\mu}_A(x)=\left(\mu^1_A(\mathrm{x}), \mu^2_A(\mathrm{x}), \mu^3_A(\mathrm{x})\right)$ and $\tilde{\nu}_A(x)=\left(\nu^1_A(\mathrm{x}), \nu^2_A(\mathrm{x}), \nu^3_A(\mathrm{x})\right)$, respectively, denote the membership degree and non-membership degree of x in A and for every $x\in X$:

$$0\le\mu^3_A(\mathrm{x})+\nu^3_A(\mathrm{x})\le 1.$$

For convenience, we consider $A=\langle(\mathrm{a,b,c}),(\mathrm{e,f,g})\rangle$ as the trapezoidal fuzzy number intuitionistic fuzzy values (TFNIFV) where, $\left(\mu^1_A(\mathrm{x}), \mu^2_A(\mathrm{x}), \mu^3_A(\mathrm{x})\right)=(a,b,c)$ and $\left(\nu^1_A(\mathrm{x}), \nu^2_A(\mathrm{x}), \nu^3_A(\mathrm{x})\right)=(e,f,g)$.

Definition 4. [69-70] Let $A_1=\langle(\mathrm{a_1,b_1,c_1}),(\mathrm{e_1,f_1,g_1})\rangle$ and $A_2=\langle(\mathrm{a_2,b_2,c_2}),(\mathrm{e_2,f_2,g_2})\rangle$ be two TFNIFVs, then the following operations are valid:

1. $$A_1 \oplus A_2 = \left\langle \begin{matrix} \left(a_1 + a_2 - a_1a_2, b_1 + b_2 - b_1b_2, c_1 + c_2 - c_1c_2\right), \\ \left(e_1e_2, f_1f_2, g_1g_2\right) \end{matrix} \right\rangle; \quad (6)$$

2. $$A_1 \otimes A_2 = \left\langle \begin{matrix} \left(a_1a_2, b_1b_2, c_1c_2\right), \\ \left(e_1 + e_2 - e_1e_2, f_1 + f_2 - f_1f_2, g_1 + g_2 - g_1g_2\right) \end{matrix} \right\rangle; \quad (7)$$

3. $\lambda A_1 = \left\langle \left(1-(1-a_1)^\lambda, 1-(1-b_1)^\lambda, 1-(1-c_1)^\lambda\right), (e_1^\lambda, f_1^\lambda, g_1^\lambda) \right\rangle$ for $\lambda > 0$, , and (8)

4. $A_1^\lambda = \left\langle \begin{matrix} (a_1^\lambda, b_1^\lambda, b_1^\lambda), \\ \left(1-(1-e_1)^\lambda, 1-(1-f_1)^\lambda, 1-(1-g_1)^\lambda\right) \end{matrix} \right\rangle$ for $\lambda > 0.$ (9)

Definition 5. [69-70]Let $A_1 = \langle (a_1, b_1, c_1), (e_1, f_1, g_1) \rangle$ be a TFNIFV, the score function $S(A_1)$ of A_1 is defined as follows:

$$S(A_1) = \frac{1}{4}\left[\left(a_1 + 2b_1 + c_1\right) - \left(e_1 + 2f_1 + g_1\right)\right],\ S(A_1) \in [-1,1] \quad (10)$$

The score function $S(A^+) = 1$ for the TFNIFV $A^+ = \langle (1,1,1), (0,0,0) \rangle$ and $S(A^-) = -1$ for the TFNIFV $A^- = \langle (0,0,0), (1,1,1) \rangle$.

Definition 6. [69-70] Let $A_1 = \langle (a_1, b_1, c_1), (e_1, f_1, g_1) \rangle$ be a TFNIFV, the accuracy function $H(A_1)$ is of A_1 is defined as follows:

$$H(A_1) = \frac{1}{4}\left[\left(a_1 + 2b_1 + c_1\right) + \left(e_1 + 2f_1 + g_1\right)\right],\ H(A_1) \in [0,1]. \quad (11)$$

2.2 Single valued neutrosophic sets

In this section, some basic definitions of single valued neutrosophic sets are reviewed.

Definition 7. [37] Let X be a space of points (objects) with a generic element in X denoted by x. A single valued neutrosophic set A in X is characterized by a truth membership function $T_A(\mathrm{x})$, an indeterminacy membership function $I_A(\mathrm{x})$, and a falsity membership function $F_A(\mathrm{x})$ and is denoted by

$$\tilde{A} = \left\{x, \left\langle T_{\tilde{A}}(\mathrm{x}), I_{\tilde{A}}(\mathrm{x}), F_{\tilde{A}}(\mathrm{x}) \right\rangle \mid x \in X \right\}.$$

Here $T_{\tilde{A}}(\mathrm{x})$, $I_{\tilde{A}}(\mathrm{x})$ and $F_{\tilde{A}}(\mathrm{x})$ are real subsets of [0, 1] that is $T_{\tilde{A}}(\mathrm{x}): \mathrm{X} \to [0,1]$, $I_{\tilde{A}}(\mathrm{x}): \mathrm{X} \to [0,1]$ and $F_{\tilde{A}}(\mathrm{x}): \mathrm{X} \to [0,1]$. The sum of $T_{\tilde{A}}(\mathrm{x})$, $I_{\tilde{A}}(\mathrm{x})$ and $F_{\tilde{A}}(\mathrm{x})$ lies in [0, 3] that is $0 \le \sup T_{\tilde{A}}(\mathrm{x}) + \sup I_{\tilde{A}}(\mathrm{x}) + \sup F_{\tilde{A}}(\mathrm{x}) \le 3.$

For convenience, SVNS $\tilde{A}$ can be denoted by $\tilde{A} = \left\langle T_{\tilde{A}}(\mathrm{x}), I_{\tilde{A}}(\mathrm{x}), F_{\tilde{A}}(\mathrm{x}) \right\rangle$ for all x in X .

Definition 8. [37] Assume that $\tilde{A} = \left\langle T_{\tilde{A}}(\mathrm{x}), I_{\tilde{A}}(\mathrm{x}), F_{\tilde{A}}(\mathrm{x}) \right\rangle$ and $\tilde{B} = \left\langle T_{\tilde{B}}(\mathrm{x}), I_{\tilde{B}}(\mathrm{x}), F_{\tilde{B}}(\mathrm{x}) \right\rangle$ be two SVNSs in a universe of discourse X . Then the following operations are defined as follows:

1. $$\tilde{A} \oplus \tilde{B} = \left\langle \begin{matrix} T_{\tilde{A}}(\mathrm{x}) + T_{\tilde{B}}(\mathrm{x}) - T_{\tilde{A}}(\mathrm{x})T_{\tilde{B}}(\mathrm{x}), \\ I_{\tilde{A}}(\mathrm{x})I_{\tilde{B}}(\mathrm{x}), F_{\tilde{A}}(\mathrm{x})F_{\tilde{B}}(\mathrm{x}) \end{matrix} \right\rangle; \quad (12)$$

2. $$\tilde{A} \otimes \tilde{B} = \left\langle \begin{matrix} T_{\tilde{A}}(\mathrm{x})T_{\tilde{B}}(\mathrm{x}), I_{\tilde{A}}(\mathrm{x}) + I_{\tilde{B}}(\mathrm{x}) - I_{\tilde{A}}(\mathrm{x})I_{\tilde{B}}(\mathrm{x}), \\ F_{\tilde{A}}(\mathrm{x}) + F_{\tilde{B}}(\mathrm{x}) - F_{\tilde{A}}(\mathrm{x})F_{\tilde{B}}(\mathrm{x}) \end{matrix} \right\rangle; \quad (13)$$

3. $\lambda \tilde{A} = \left\langle 1 - \left(1 - T_{\tilde{A}}(\mathrm{x})\right)^\lambda, \left(I_{\tilde{A}}(\mathrm{x})\right)^\lambda, \left(F_{\tilde{A}}(\mathrm{x})\right)^\lambda \right\rangle$ for $\lambda > 0$, and (14)

4. $(\tilde{A})^\lambda = \left\langle \left(T_{\tilde{A}}(\mathrm{x})\right)^\lambda, 1 - \left(1 - I_{\tilde{A}}(\mathrm{x})\right)^\lambda, 1 - \left(1 - F_{\tilde{A}}(\mathrm{x})\right)^\lambda \right\rangle$ for $\lambda > 0.$ (15)

3 Triangular fuzzy number neutrosophic set

SVNS can represent imprecise, incomplete and inconsistent type information existing in the real world problem. However, decision maker often expresses uncertain information with truth, indeterminacy and falsity membership functions that are represented with uncertain numeric values instead of exact real number values. These uncertain numeric values of truth, indeterminacy and falsity membership functions of SVNSs can be represented in terms of triangular fuzzy numbers.

In this section, we combine triangular fuzzy numbers (TFNs) with SVNSs to develop triangular fuzzy number neutrosophic set (TFNNS) in which, the truth, indeterminacy and falsity membership functions are expressed with triangular fuzzy numbers.

Definition 9. Assume that X be the finite universe of discourse and F [0, 1] be the set of all triangular fuzzy numbers on $[0,1]$. A triangular fuzzy number neutrosophic set (TFNNS) $\tilde{A}$ in X is represented by

$\tilde{A} = \left\{ \left\langle x, \tilde{T}_{\tilde{A}}(x), \tilde{I}_{\tilde{A}}(x), \tilde{F}_{\tilde{A}}(x) \right\rangle | \, x \in X \right\}$,

where, $\tilde{T}_A(x) : X \to F[0,1]$, $\tilde{I}_A(x) : X \to F[0,1]$, and $\tilde{F}_A(x) : X \to F[0,1]$.

The triangular fuzzy numbers $\tilde{T}_{\tilde{A}}(x) = \left(T^1_{\tilde{A}}(\mathrm{x}), T^2_{\tilde{A}}(\mathrm{x}), T^3_{\tilde{A}}(\mathrm{x})\right)$, $\tilde{I}_{\tilde{A}}(x) = \left(I^1_{\tilde{A}}(\mathrm{x}), I^2_{\tilde{A}}(\mathrm{x}), I^3_{\tilde{A}}(\mathrm{x})\right)$, and $\tilde{F}_{\tilde{A}}(x) = \left(F^1_{\tilde{A}}(x), F^2_{\tilde{A}}(x), F^3_{\tilde{A}}(\mathrm{x})\right)$, respectively, denote the truth membership degree, indeterminacy degree, and falsity membership degree of x in $\tilde{A}$ and for every $x \in X$:

$$0 \le T^3_A(\mathrm{x}) + I^3_A(\mathrm{x}) + F^3_A(\mathrm{x}) \le 3 . \tag{16}$$

For notational convenience, we consider $\tilde{A} = \langle (a,b,c), (e,f,g), (r,s,t) \rangle$ as a trapezoidal fuzzy number neutrosophic values (TFNNV) where,
$\left(T^1_{\tilde{A}}(\mathrm{x}), T^2_{\tilde{A}}(\mathrm{x}), T^3_{\tilde{A}}(\mathrm{x})\right) = (a,b,c)$,
$\left(I^1_{\tilde{A}}(\mathrm{x}), I^2_{\tilde{A}}(\mathrm{x}), I^3_{\tilde{A}}(\mathrm{x})\right) = (e,f,g)$,
and $\left(F^1_{\tilde{A}}(\mathrm{x}), F^2_{\tilde{A}}(\mathrm{x}), F^3_{\tilde{A}}(\mathrm{x})\right) = (r,s,t)$.

Definition 10. Let $\tilde{A}_1 = \langle (a_1,b_1,c_1), (e_1, f_1, g_1), (r_1, s_1, t_1) \rangle$ and $\tilde{A}_2 = \langle (a_2,b_2,c_2), (e_2, f_2, g_2), (r_2, s_2, t_2) \rangle$ be two TFNNVs in the set of real numbers. Then the following operations are defined as follows:

1. $$\tilde{A}_1 \oplus \tilde{A}_2 = \left\langle \begin{matrix} \left(a_1 + a_2 - a_1 a_2, b_1 + b_2 - b_1 b_2, c_1 + c_2 - c_1 c_2\right), \\ \left(e_1 e_2, f_1 f_2, g_1 g_2\right), \left(r_1 r_2, s_1 s_2, t_1 t_2\right) \end{matrix} \right\rangle; \tag{17}$$

2. $$\tilde{A}_1 \otimes \tilde{A}_2 = \left\langle \begin{matrix} \left(a_1 a_2, b_1 b_2, c_1 c_2\right), \\ \left(e_1 + e_2 - e_1 e_2, f_1 + f_2 - f_1 f_2, g_1 + g_2 - g_1 g_2\right), \\ \left(r_1 + r_2 - r_1 r_2, s_1 + s_2 - s_1 s_2, t_1 + t_2 - t_1 t_2\right) \end{matrix} \right\rangle; \tag{18}$$

3. $$\lambda \tilde{A} = \left\langle \begin{matrix} \left(1 - (1 - \mathrm{a}_1)^\lambda, 1 - (1 - b_1)^\lambda, 1 - (1 - c_1)^\lambda\right), \\ \left(e_1^{\lambda}, f_1^{\lambda}, g_1^{\lambda}\right), \left(r_1^{\lambda}, s_1^{\lambda}, t_1^{\lambda}\right) \end{matrix} \right\rangle \text{ for } \lambda > 0 \text{ and} \tag{19}$$

4. $$\tilde{A}^\lambda = \left\langle \begin{matrix} \left(a_1^{\lambda}, b_1^{\lambda}, c_1^{\lambda}\right), \\ \left(1 - (1 - e_1)^\lambda, 1 - (1 - f_1)^\lambda, 1 - (1 - g_1)^\lambda\right), \\ \left(1 - (1 - r_1)^\lambda, 1 - (1 - s_1)^\lambda, 1 - (1 - t_1)^\lambda\right) \end{matrix} \right\rangle \text{ for } \lambda > 0 . \tag{20}$$

The operations defined in Definition 10 satisfy the following properties:

1. $\tilde{A}_1 \oplus \tilde{A}_2 = \tilde{A}_2 \oplus \tilde{A}_1$, $\tilde{A}_1 \otimes \tilde{A}_2 = \tilde{A}_2 \otimes \tilde{A}_1$;
2. $\lambda\left(\tilde{A}_1 \oplus \tilde{A}_2\right) = \lambda \tilde{A}_1 \oplus \lambda \tilde{A}_2$, $\left(\tilde{A}_1 \otimes \tilde{A}_2\right)^\lambda = \tilde{A}_1^{\lambda} \otimes \tilde{A}_2^{\lambda}$ for $\lambda > 0$, and
3. $\lambda_1 \tilde{A}_1 \oplus \lambda_2 \tilde{A}_1 = \left(\lambda_1 + \lambda_2\right) \tilde{A}_1$, $\tilde{A}_1^{\lambda_1} \oplus \tilde{A}_1^{\lambda_2} = \tilde{A}_1^{(\lambda_1 + \lambda_2)}$ for $\lambda_1, \lambda_2 > 0$.

3.1 Score and accuracy function of TFNNV

In the following section, we define score function and accuracy function of TFNNV from Definition 5, Definition 6.

Definition 11. Assume that $\tilde{A}_1 = \langle (\mathrm{a}_1, \mathrm{b}_1, \mathrm{c}_1), (\mathrm{e}_1, \mathrm{f}_1, \mathrm{g}_1), (\mathrm{r}_1, \mathrm{s}_1, \mathrm{t}_1) \rangle$ be a *TFNNVs* in the set of real numbers, the score function $S(\tilde{A}_1)$ of $\tilde{A}_1$ is defined as follows:

$$S\left(\tilde{A}_1\right) = \frac{1}{12}\left[\begin{matrix} 8 + (a_1 + 2b_1 + c_1) - (e_1 + 2 f_1 + g_1) \\ - (r_1 + 2s_1 + t_1) \end{matrix}\right]. \tag{21}$$

The value of score function of TFNNV $A^+ = \langle (1,1,1), (0,0,0), (0,0,0) \rangle$ is $S(\mathrm{A}^+) = 1$ and value of accuracy function of TFNNV $A^- = \langle (0,0,0), (1,1,1), (1,1,1) \rangle$ is $S(\mathrm{A}^-) = -1$.

Definition 12. Assume that $\tilde{A}_1 = \langle (\mathrm{a}_1, \mathrm{b}_1, \mathrm{c}_1), (\mathrm{e}_1, \mathrm{f}_1, \mathrm{g}_1), (\mathrm{r}_1, \mathrm{s}_1, \mathrm{t}_1) \rangle$ be a TFNNV in the set of real numbers, the accuracy function $H\left(\tilde{A}_1\right)$ of $\tilde{A}_1$ is defined as follows:

$$H\left(\tilde{A}_1\right) = \frac{1}{4}\left[(\mathrm{a}_1 + 2\,\mathrm{b}_1 + \mathrm{c}_1) - (\mathrm{r}_1 + 2\,\mathrm{s}_1 + \mathrm{t}_1)\right]. \tag{22}$$

The accuracy function $H(\tilde{\mathrm{A}}_1) \in [-1,1]$ determines the difference between truth and falsity. Larger the difference reflects the more affirmative of the TFNNV. The accuracy function $H(\tilde{\mathrm{A}}^+) = 1$ for $A^+ = \langle (1,1,1), (0,0,0), (0,0,0) \rangle$ and $H(\tilde{\mathrm{A}}^-) = -1$ for the TFNNV $A^- = \langle (0,0,0), (1,1,1), (1,1,1) \rangle$.
Based on Definition 11 and Definition 12, we present the order relations between two TFNNVs.

Definition 13. Assume that $\tilde{A}_1 = \langle (\mathrm{a}_1, \mathrm{b}_1, \mathrm{c}_1), (\mathrm{e}_1, \mathrm{f}_1, \mathrm{g}_1), (\mathrm{r}_1, \mathrm{s}_1, \mathrm{t}_1) \rangle$ and $\tilde{A}_2 = \langle (\mathrm{a}_2, \mathrm{b}_2, \mathrm{c}_2), (\mathrm{e}_2, \mathrm{f}_2, \mathrm{g}_2), (\mathrm{r}_2, \mathrm{s}_2, \mathrm{t}_2) \rangle$ be two TFNNVs in the set of real numbers. Suppose that $S(\tilde{\mathrm{A}}_i)$ and $H(\tilde{\mathrm{A}}_i)$ are the score and accuracy functions of TFNNS $\tilde{A}_i (i = 1, 2)$, then the following order relations are defined as follows:

1. If $S(\tilde{A}_1) > S(\tilde{A}_2)$, then $\tilde{A}_1$ is greater than $\tilde{A}_2$ that is $\tilde{A}_1 \succ \tilde{A}_2$;
2. If $S(\tilde{A}_1) = S(\tilde{A}_2)$ and $H(\tilde{A}_1) \geq H(\tilde{A}_2)$ then $\tilde{A}_1$ is greater than $\tilde{A}_2$,that is, $\tilde{A}_1 \succ \tilde{A}_2$;
3. If $S(\tilde{A}_1) = S(\tilde{A}_2)$, $H(\tilde{A}_1) = H(\tilde{A}_2)$ then $\tilde{A}_1$ is indifferent to $\tilde{A}_2$, i.e. $\tilde{A}_1 \approx \tilde{A}_2$.

Example 1. Consider two TFNNVs in the set of real numbers:

$\tilde{A}_1 = \langle (0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20) \rangle$,

$\tilde{A}_2 = \langle (0.40, 0.45, 0.50), (0.40, 0.45, 0.50), (0.35, 0.40, 0.45) \rangle$.

Then from Eqs.(21) and (22), we obtain the following results:

1. Score value of $S(\tilde{A}_1) = (8 + 3 - 0.8 - 0.6) / 12 = 0.80$, and $S(\tilde{A}_2) = (8 + 1.8 - 1.8 - 1.6) / 12 \approx 0.53$;
2. Accuracy value of $H(\tilde{A}_1) = (3 - 0.6) / 4 = 0.60$, and $H(\tilde{A}_2) = (1.8 - 1.6) / 4 = 0.05$.

   Therefore from Definition 13, we obtain $A_1 \succ A_2$.

Example 2. Consider two TFNNVs in the set of real numbers:

$\tilde{A}_1 = \langle (0.50, 0.55, 0.60), (0.25, 0.30, 0.35), (0.20, 0.25, 0.30) \rangle$

$\tilde{A}_2 = \langle (0.40, 0.45, 0.50), (0.40, 0.45, 0.50), (0.35, 0.40, 0.45) \rangle$.

Using Eqs. (21) and (22), we obtain the following results:

1. Score value of $S(\tilde{A}_1) = (8 + 2.2 - 1.2 - 1.0) / 12 \approx 0.67$, and $S(\tilde{A}_2) = (8 + 1.8 - 1.8 - 1.6) / 12 \approx 0.53$;
2. Accuracy value of $H(\tilde{A}_1) = (2.2 - 1.2) / 4 = 0.25$, and $H(\tilde{A}_2) = (1.8 - 1.6) / 4 = 0.05$.

Therefore from Definition 13, we have $\tilde{A}_1 \succ \tilde{A}_2$.

4 Aggregation of triangular fuzzy number neutrosophic sets

In this section, we first recall some basic definitions of aggregation operators for real numbers.

Definition 14. [72] Assume that $W : (\mathrm{Re})^n \to \mathrm{Re}$, and $a_j\,(j = 1, 2, ..., n)$ be a collection of real numbers. The weighted averaging operator WA_w is defined as

$$WA_w(a_1, a_2, ..., a_n) = \sum_{j=1}^{n} w_j a_j \tag{23}$$

where *Re* is the set of real numbers, $w = (w_1, w_2, ..., w_n)^T$ is the weight vector of $a_j\,(j = 1, 2, ..., n)$ such that $w_j \in [0,1]$ $(j = 1, 2, ..., n)$ and $\sum_{j=1}^{n} w_j = 1$.

Definition 15. [73] Assume that $W : (\mathrm{Re})^n \to \mathrm{Re}$, and $a_j\,(j = 1, 2, ..., n)$ be a collection of real numbers. The weighted geometric operator WG_w is defined as follows:

$$WG_w(a_1, a_2, ..., a_n) = \prod_{j=1}^{n} a_j^{w_j}, \tag{24}$$

where *Re* is the set of real numbers, $w = (w_1, w_2, ..., w_n)^T$ is the weight vector of $a_j\,(j = 1, 2, ..., n)$ with $w_j \in [0,1]$ $(j = 1, 2, ..., n)$ and $\sum_{j=1}^{n} w_j = 1$

Based on Definition 14 and Definition 15, we propose the following two aggregation operators of TFNNSs to be used in decision making.

4.1 Triangular fuzzy number neutrosophic arithmetic averaging operator

Definition 16. Assume that $\tilde{A}_j = \langle (a_j, b_j, c_j), (e_j, f_j, g_j), (r_j, s_j, t_j) \rangle$ $(j = 1, 2, ..., n)$ be a collection TFNNVs in the set of real numbers and let $TFNNWA : \Theta^n \to \Theta$. The triangular fuzzy number neutrosophic weighted averaging (TFNNWA) operator denoted by $TFNNWA(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n)$ is defined as

$$TFNNWA(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n) = w_1 \tilde{A}_1 \oplus w_2 \tilde{A}_2 \oplus \cdots w_n \tilde{A}_n = \bigoplus_{j=1}^{n} (w_j \tilde{A}_j), \tag{25}$$

where $w_j \in [0,1]$ is the weight vector of $A_j\,(j = 1, 2, ..., n)$ such that $\sum_{j=1}^{n} w_j = 1$.

In particular, if $w = (1/n, 1/n, ..., 1/n)^T$ then the $TFNNWA(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n)$ operator reduces to triangular fuzzy number neutrosophic averaging (TFNNA) operator:

$$TFNNA(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n) = \frac{1}{n}\left(\tilde{A}_1 \oplus \tilde{A}_2 \oplus \cdots \oplus \tilde{A}_n\right) \tag{26}$$

We can now establish the following theorem by using the basic operations of TFNNVs defined in Definition 10.

Theorem 1.
Let $\tilde{A}_j = \langle (\mathrm{a}_j, \mathrm{b}_j, \mathrm{c}_j), (\mathrm{e}_j, \mathrm{f}_j, \mathrm{g}_j), (\mathrm{r}_j, \mathrm{s}_j, \mathrm{t}_j) \rangle$ $(\mathrm{j} = 1, 2, ..., \mathrm{n})$ be a collection TFNNVs in the set of real numbers. Then the aggregated value obtained by TFNNWA, is also a TFNNV, and

$$TFNNWA_w(\tilde{\mathrm{A}}_1, \tilde{\mathrm{A}}_2, ..., \tilde{\mathrm{A}}_n)$$

$$= w_1\tilde{A}_1 \oplus w_2\tilde{A}_2 \oplus \cdots \oplus w_n\tilde{A}_n = \bigoplus_{j=1}^{n} (\mathrm{w}_j \tilde{\mathrm{A}}_j)$$

$$= \left\langle \begin{array}{l} \left(1 - \prod_{j=1}^{n} (1-\mathrm{a}_j)^{w_j}, 1 - \prod_{j=1}^{n} (1-b_j)^{w_j}, 1 - \prod_{j=1}^{n} (1-c_j)^{w_j} \right), \\ \left(\prod_{j=1}^{n} e_j^{\ w_j}, \prod_{j=1}^{n} f_j^{\ w_j}, \prod_{j=1}^{n} g_j^{\ w_j} \right), \left(\prod_{j=1}^{n} r_j^{\ w_j}, \prod_{j=1}^{n} s_j^{\ w_j}, \prod_{j=1}^{n} t_j^{\ w_j} \right) \end{array} \right\rangle, \tag{27}$$

where $w_j \in [0,1]$ is the weight vector of TFNNV $A_j \, (j = 1, 2, ..., n)$ such that $\sum_{j=1}^{n} w_j = 1$.

Proof: We prove the theorem by mathematical induction.

1. When $n = 1$, it is a trivial case

When $n = 2$, we have $\bigoplus_{j=1}^{2} (\mathrm{w}_j \tilde{\mathrm{A}}_j) = w_1\tilde{A}_1 \oplus w_2\tilde{A}_2$

---

2. $$= \left(\begin{array}{l} \left\langle \left(1-(1-\mathrm{a}_1)^{w_1}, 1-(1-b_1)^{w_1}, 1-(1-c_1)^{w_1}\right), \left(e_1^{\ w_1}, f_1^{\ w_1}, g_1^{\ w_1}\right), \left(r_1^{\ w_1}, s_1^{\ w_1}, t_1^{\ w_1}\right) \right\rangle \\ \oplus \left\langle \left(1-(1-\mathrm{a}_2)^{w_2}, 1-(1-b_2)^{w_2}, 1-(1-c_2)^{w_2}\right), \left(e_2^{\ w_2}, f_2^{\ w_2}, g_2^{\ w_2}\right), \left(r_2^{\ w_2}, s_2^{\ w_2}, t_2^{\ w_2}\right) \right\rangle \end{array} \right)$$

$$= \left\langle \begin{array}{l} \left(\begin{array}{l} \left(1-(1-\mathrm{a}_1)^{w_1}\right) + \left(1-(1-\mathrm{a}_2)^{w_2}\right) - \left(1-(1-\mathrm{a}_1)^{w_1}\right).\left(1-(1-\mathrm{a}_2)^{w_2}\right), \\ \left(1-(1-b_1)^{w_1}\right) + \left(1-(1-b_2)^{w_2}\right) - \left(1-(1-b_1)^{w_1}\right).\left(1-(1-b_2)^{w_2}\right), \\ \left(1-(1-c_1)^{w_1}\right) + \left(1-(1-c_2)^{w_2}\right) - \left(1-(1-c_1)^{w_1}\right).\left(1-(1-c_2)^{w_2}\right) \end{array} \right), \\ \left(e_1^{\ w_1} e_2^{\ w_2}, f_1^{\ w_1} f_2^{\ w_2}, g_1^{\ w_1} g_2^{\ w_2}\right), \left(r_1^{\ w_1} r_2^{\ w_2}, s_1^{\ w_1} s_2^{\ w_2}, t_1^{\ w_1} t_2^{\ w_2}\right) \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} \left(1 - \prod_{j=1}^{2} (1-\mathrm{a}_j)^{w_j}, 1 - \prod_{j=1}^{2} (1-b_j)^{w_j}, 1 - \prod_{j=1}^{2} (1-b_j)^{w_j} \right), \\ \left(\prod_{j=1}^{2} e_j^{\ w_j}, \prod_{j=1}^{2} f_j^{\ w_j}, \prod_{j=1}^{2} g_j^{\ w_j} \right), \left(\prod_{j=1}^{2} r_j^{\ w_j}, \prod_{j=1}^{2} s_j^{\ w_j}, \prod_{j=1}^{2} t_j^{\ w_j} \right) \end{array} \right\rangle. \tag{28}$$

Thus the theorem is true for n = 2

3. When n = k, we assume that Eq.(27) is also true.

Then, $TFNNWA(\tilde{\mathrm{A}}_1, \tilde{\mathrm{A}}_2, ..., \tilde{\mathrm{A}}_k) = w_1\tilde{A}_1 \oplus w_1\tilde{A}_1 \oplus \cdots \oplus w_n\tilde{A}_n = \bigoplus_{j=1}^{k} (\mathrm{w}_j \tilde{\mathrm{A}}_j)$

$$= \left\langle \begin{array}{l} \left(1 - \prod_{j=1}^{k} (1-\mathrm{a}_j)^{w_j}, 1 - \prod_{j=1}^{k} (1-b_j)^{w_j}, 1 - \prod_{j=1}^{k} (1-b_j)^{w_j} \right), \\ \left(\prod_{j=1}^{k} e_j^{\ w_j}, \prod_{j=1}^{k} f_j^{\ w_j}, \prod_{j=1}^{n} g_j^{\ w_j} \right), \left(\prod_{j=1}^{k} r_j^{\ w_j}, \prod_{j=1}^{k} s_j^{\ w_j}, \prod_{j=1}^{k} t_j^{\ w_j} \right) \end{array} \right\rangle. \tag{29}$$

4. When n = k + 1, we have

$$TNFNWA(\tilde{\mathrm{A}}_1, \tilde{\mathrm{A}}_2, ..., \tilde{\mathrm{A}}_{k+1}) = \bigoplus_{j=1}^{k} (\mathrm{w}_j \tilde{\mathrm{A}}_j) \oplus (\mathrm{w}_{k+1} \tilde{\mathrm{A}}_{k+1})$$

$$=\left\langle \begin{array}{c} \begin{pmatrix} 1-\prod_{j=1}^{k}(1-\mathrm{a}_j)^{w_j}+1-(1-\mathrm{a}_{k+1})^{w_{k+1}}-1-\prod_{j=1}^{k}(1-\mathrm{a}_j)^{w_j}1-(1-\mathrm{a}_{k+1})^{w_{k+1}}, \\ 1-\prod_{j=1}^{k}(1-b_j)^{w_j}+1-(1-b_{k+1})^{w_{k+1}}-1-\prod_{j=1}^{k}(1-b_j)^{w_j}1-(1-b_{k+1})^{w_{k+1}}, \\ 1-\prod_{j=1}^{k}(1-c_j)^{w_j}+1-(1-c_{k+1})^{w_{k+1}}-1-\prod_{j=1}^{k}(1-c_j)^{w_j}1-(1-c_{k+1})^{w_{k+1}} \end{pmatrix}, \\ \left(\prod_{j=1}^{k}e_j^{\ w_j}.\mathrm{e}_{k+1}^{\ w_{k+1}},\prod_{j=1}^{k}f_j^{\ w_j}.f_{k+1}^{\ w_{k+1}},\prod_{j=1}^{n}g_j^{\ w_j}.\mathrm{e}_{k+1}^{\ w_{k+1}}\right),\left(\prod_{j=1}^{k}r_j^{\ w_j}.r_{k+1}^{\ w_{k+1}},\prod_{j=1}^{k}s_j^{\ w_j}.s_{k+1}^{\ w_{k+1}},\prod_{j=1}^{k}t_j^{\ w_j}.t_{k+1}^{\ w_{k+1}}\right) \end{array} \right\rangle$$

$$=\left\langle \begin{array}{l} \left(1-\prod_{j=1}^{k+1}(1-\mathrm{a}_j)^{w_j},1-\prod_{j=1}^{k+1}(1-b_j)^{w_j},1-\prod_{j=1}^{k+1}(1-b_j)^{w_j}\right), \\ \left(\prod_{j=1}^{k+1}e_j^{\ w_j},\prod_{j=1}^{k+1}f_j^{\ w_j},\prod_{j=1}^{k+1}g_j^{\ w_j}\right),\left(\prod_{j=1}^{k+1}r_j^{\ w_j},\prod_{j=1}^{k+1}s_j^{\ w_j},\prod_{j=1}^{k+1}t_j^{\ w_j}\right) \end{array} \right\rangle. \tag{30}$$

We observe that the theorem is true for n = k + 1. Therefore, by mathematical induction, we can say that Eq. (27) holds for all values of *n*. As the components of all three membership functions of $\tilde{A}_j$ belong to [0, 1], the following relations are valid

$$0\le\left(1-\prod_{j=1}^{n}(1-c_j)^{w_j}\right)\le 1,\ \ 0\le\left(\prod_{j=1}^{n}g_j^{\ w_j}\right)\le 1,$$

$$0\le\left(\prod_{j=1}^{n}t_j^{\ w_j}\right)\le 1. \tag{31}$$

It follows that the relation

$0\le\left(1-\prod_{j=1}^{n}(1-c_j)^{w_j}+\prod_{j=1}^{n}g_j^{\ w_j}+\prod_{j=1}^{n}t_j^{\ w_j}\right)\le 3$ is also valid.

This completes the proof of the Theorem 1.

Now, we highlight some necessary properties of TFNNWA operator.

Property 1.(*Idempotency*): If all $\tilde{A}_j\,(\mathrm{j}=1,2,...,\mathrm{n})$ are equal i.e. $\tilde{A}_j=\tilde{A}=\langle(\mathrm{a,b,c}),(\mathrm{e,f,g}),(\mathrm{r,s,t})\rangle$, for all j, then $TFNNWA(\tilde{\mathrm{A}}_1,\tilde{\mathrm{A}}_2,...,\tilde{\mathrm{A}}_k)=\tilde{A}$.

Proof: From Eq.(27), we have

$TFNNWA(\tilde{\mathrm{A}}_1,\tilde{\mathrm{A}}_2,...,\tilde{\mathrm{A}}_n)$

$$=TFNNWA(\tilde{\mathrm{A}},\tilde{\mathrm{A}},...,\tilde{\mathrm{A}})=\bigoplus_{j=1}^{n}(\mathrm{w}_j\tilde{\mathrm{A}})$$

$$=\left\langle \begin{array}{l} \left(1-\prod_{j=1}^{n}(1-\mathrm{a})^{w_j},1-\prod_{j=1}^{n}(1-b)^{w_j},1-\prod_{j=1}^{n}(1-c)^{w_j}\right), \\ \left(\prod_{j=1}^{n}e^{w_j},\prod_{j=1}^{n}f^{\ w_j},\prod_{j=1}^{n}g^{w_j}\right),\left(\prod_{j=1}^{n}r^{w_j},\prod_{j=1}^{n}s^{w_j},\prod_{j=1}^{n}t^{w_j}\right) \end{array} \right\rangle$$

$$=\left\langle \begin{array}{r} \left(1-(1-\mathrm{a})^{\sum_{j=1}^{n}w_j},1-(1-b)^{\sum_{j=1}^{n}w_j},1-(1-c)^{\sum_{j=1}^{n}w_j}\right), \\ \left(e^{\sum_{j=1}^{n}w_j},f^{\sum_{j=1}^{n}w_j},g^{\sum_{j=1}^{n}w_j}\right), \\ \left(e^{\sum_{j=1}^{n}w_j},f^{\sum_{j=1}^{n}w_j},g^{\sum_{j=1}^{n}w_j}\right) \end{array} \right\rangle$$

$$=\langle(\mathrm{a,b,c}),(\mathrm{e,f,g}),(\mathrm{r,s,t})\rangle=\tilde{A}.$$

This completes the proof the Property 1.

Property 2. (*Boundedness*)

Let $\tilde{A}_j=\langle(\mathrm{a}_j,\mathrm{b}_j,\mathrm{c}_j),(\mathrm{e}_j,\mathrm{f}_j,\mathrm{g}_j),(\mathrm{r}_j,\mathrm{s}_j,\mathrm{t}_j)\rangle$ $(\mathrm{j}=1,2,...,\mathrm{n})$ be a collection TFNNVs in the set of real numbers.

Assume $\tilde{A}^{+} = \left\langle \begin{array}{l} \left(\max\limits_j(a_j), \max\limits_j(b_j), \max\limits_j(c_j)\right), \\ \left(\min\limits_j(e_j), \min\limits_j(f_j), \min\limits_j(g_j)\right), \\ \left(\min\limits_j(r_j), \min\limits_j(s_j), \min\limits_j(t_j)\right) \end{array} \right\rangle$ and

$$\tilde{A}^{-} = \left\langle \begin{array}{l} \left(\min\limits_j(a_j), \min\limits_j(b_j), \min\limits_j(c_j)\right), \\ \left(\max\limits_j(e_j), \max\limits_j(f_j), \max\limits_j(g_j)\right), \\ \left(\max\limits_j(r_j), \max\limits_j(s_j), \max\limits_j(t_j)\right) \end{array} \right\rangle$$ for all

$j = 1, 2, ..., n.$

Then $\tilde{A}^{-} \leq TFNNWA(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n) \leq \tilde{A}^{+}$. (32)

Proof: We have

$$\min_j(c_j) \leq c_j \leq \max_j(c_j),\ \min_j(g_j) \leq g_j \leq \max_j(g_j),$$

$$\min_j(t_j) \leq t_j \leq \max_j(t_j) \text{ for } j = 1,2,...,n. \tag{33}$$

Then

$$1 - \prod_{j=1}^{n}(1 - \min_j(c_j))^{w_j} \leq 1 - \prod_{j=1}^{n}(1 - c_j)^{w_j}$$

$$\leq 1 - \prod_{j=1}^{n}(1 - \max_j(c_j))^{w_j}$$

$$= 1 - \left((1 - \min_j(c_j))\right)^{\sum_{j=1}^{n} w_j} \leq 1 - \prod_{j=1}^{n}(1 - c_j)^{w_j}$$

$$\leq 1 - \left((1 - \max_j(c_j))\right)^{\sum_{j=1}^{n} w_j}$$

$$= \min_j(c_j) \leq 1 - \prod_{j=1}^{n}(1 - c_j)^{w_j} \leq \max_j(c_j).$$

Again from Eq.(33), we have for j = 1, 2, …, n

$$\prod_{j=1}^{n}\left(\min_j(g_j)\right)^{w_j} \leq \prod_{j=1}^{n} g^{w_j} \leq \prod_{j=1}^{n}\left(\max_j(g_j)\right)^{w_j}$$

$$= \left(\min_j(g_j)\right)^{\sum_{j=1}^{n} w_j} \leq \prod_{j=1}^{n} g^{w_j} \leq \left(\max_j(g_j)\right)^{\sum_{j=1}^{n} w_j}$$

$$= \min_j(g_j) \leq \prod_{j=1}^{n} e^{w_j} \leq \max_j(g_j);$$

and $\prod_{j=1}^{n}\left(\min_j(t_j)\right)^{w_j} \leq \prod_{j=1}^{n} t^{w_j} \leq \prod_{j=1}^{n}\left(\max_j(t_j)\right)^{w_j} =$

$$\left(\min_j(t_j)\right)^{\sum_{j=1}^{n} w_j} \leq \prod_{j=1}^{n} t^{w_j} \leq \left(\max_j(t_j)\right)^{\sum_{j=1}^{n} w_j}$$

$$= \min_j(t_j) \leq \prod_{j=1}^{n} t^{w_j} \leq \max_j(t_j).$$

Similarly, we have

$$\min_j(a_j) \leq 1 - \prod_{j=1}^{n}(1 - a_j)^{w_j} \leq \max_j(a_j),$$

$$\min_j(b_j) \leq 1 - \prod_{j=1}^{n}(1 - b_j)^{w_j} \leq \max_j(b_j);$$

$$\min_j(e_j) \leq 1 - \prod_{j=1}^{n}(1 - e_j)^{w_j} \leq \max_j(e_j),$$

$$\min_j(f_j) \leq 1 - \prod_{j=1}^{n}(1 - f_j)^{w_j} \leq \max_j(f_j);$$

$$\min_j(r_j) \leq 1 - \prod_{j=1}^{n}(1 - r_j)^{w_j} \leq \max_j(r_j),$$

$$\min_j(s_j) \leq 1 - \prod_{j=1}^{n}(1 - s_j)^{w_j} \leq \max_j(s_j)$$

for $j = 1,2,...,n.$

Assume that

$TFNNWA_w(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n) = \tilde{A} = \langle(a,b,c),(e,f,g),(r,s,t)\rangle$,

then the score function of $\tilde{A}$

$$S(\tilde{A}) = \frac{1}{12}\left[8 + (a + 2b + c) - (e + 2f + g) - (r + 2s + t)\right]$$

$$\leq \frac{1}{12}\left[\begin{array}{l} 8 + \left(\max\limits_j(a_j) + \max\limits_j(2b_j) + \max\limits_j(c_j)\right) \\ -\left(\min\limits_j(e_j) + 2\min\limits_j(f_j) + \min\limits_j(g_j)\right) \\ -\left(\min\limits_j(r_j) + 2\min\limits_j(s_j) + \min\limits_j(t_j)\right) \end{array}\right], \tag{34}$$

$$= S\left(\tilde{A}^{+}\right).$$

Similarly, the score function of $\tilde{A}$

$$S(\tilde{A}) = \frac{1}{12}\left[8 + (a + 2b + c) - (e + 2f + g) - (r + 2s + t)\right];$$

$$\geq \frac{1}{12}\left[\begin{array}{l} 8 + \left(\min\limits_j(a_j) + \min\limits_j(2b_j) + \min\limits_j(c_j)\right) \\ -\left(\max\limits_j(e_j) + \max\limits_j(2f_j) + \max\limits_j(g_j)\right) \\ -\left(\max\limits_j(r_j) + \max\limits_j(2s_j) + \max\limits_j(t_j)\right) \end{array}\right]$$

$$= S\left(\tilde{A}^{-}\right).$$

Now, we consider the following cases:

1. If $S(\tilde{A}) < S(\tilde{A}^{+})$ and $S(\tilde{A}) > S(\tilde{A}^{-})$ then we have

$$\tilde{A}^{-} < TFNNWA(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n) < \tilde{A}^{+}. \tag{35}$$

2. If $S(\tilde{A}) = S(\tilde{A}^{+})$, then we can take

$$\frac{1}{12}\left[8+(a+2b+c)-(e+2f+g)-(r+2s+t)\right]$$

$$= \frac{1}{12}\left[\begin{array}{l} 8+\left(\max_j(a_j)+2\max_j(b_j)+\max_j(c_j)\right) \\ -\left(\min_j(e_j)+2\min_j(f_j)+\min_j(g_j)\right) \\ -\left(\min_j(r_j)+2\min_j(s_j)+\min_j(t_j)\right) \end{array}\right].$$

3. It follows that

$$(a+2b+c)=\left(\max_j(a_j)+2\max_j(b_j)+\max_j(c_j)\right),$$

$$(e+2f+g)=\left(\min_j(e_j)+2\min_j(f_j)+\min_j(g_j)\right) \text{ and}$$

$$(r+2s+t)=\left(\min_j(r_j)+2\min_j(s_j)+\min_j(t_j)\right).$$

Therefore the accuracy function of $\tilde{A}$

$$H(\tilde{A}) = \frac{1}{4}\left[(a+2b+c)-(r+2s+t)\right]$$

$$= \frac{1}{4}\left[\begin{array}{l} \left(\max_j(a_j)+\max_j(2b_j)+\max_j(c_j)\right) \\ -\left(\min_j(r_j)+\min_j(2s_j)+\min_j(t_j)\right) \end{array}\right].$$

$$= H(\tilde{A}^{+}), \quad (36)$$

From (36),we have $TFNNWA(\tilde{A}_1,\tilde{A}_2,...,\tilde{A}_n)=\tilde{A}^{+}$ (37)

Similarly, for $S(\tilde{A}) = S(\tilde{A}^{-})$, the accuracy function of $\tilde{A}$

$$H(\tilde{A}) = \frac{1}{4}\left[(a+2b+c)-(r+2s+t)\right]$$

$$= \frac{1}{4}\left[\begin{array}{l} \left(\min_j(a_j)+\min_j(2b_j)+\min_j(c_j)\right) \\ -\left(\max_j(r_j)+\max_j(2s_j)+\max_j(t_j)\right) \end{array}\right]$$

$$= H(\tilde{A}^{-}) \quad (38)$$

From (38), we have $TNFNWA(\tilde{A}_1,\tilde{A}_2,...,\tilde{A}_n)=\tilde{A}^{-}$. (39)

Combining Eqs. (35), (37) and (39), we obtain the following result

$$\tilde{A}^{-} \le TFNNWA(\tilde{A}_1,\tilde{A}_2,...,\tilde{A}_n) \le \tilde{A}^{+} \quad (40)$$

This proves the Property 2. □

Property 3. (*Monotonicity*) Suppose that $\tilde{A}_j^1 = \left\langle (a_j^1,b_j^1,c_j^1),(e_j^1,f_j^1,g_j^1),(r_j^1,s_j^1,t_j^1)\right\rangle$ and $\tilde{A}_j^2 = \left\langle (a_j^2,b_j^2,c_j^2),(e_j^2,f_j^2,g_j^2),(r_j^2,s_j^2,t_j^2)\right\rangle$ $(j=1,2,...,n)$ be a collection of two TFNNVs in the set of real numbers.

If $\tilde{A}_j^1 \preccurlyeq \tilde{A}_j^2$ for $j=1,2,...,n$ then

$$TFNNWA(\tilde{A}_1^1,\tilde{A}_2^1,...,\tilde{A}_n^1) \preccurlyeq TFNNWA(\tilde{A}_1^2,\tilde{A}_2^2,...,\tilde{A}_n^2). \quad (41)$$

Proof: We first consider c_j^1, g_j^1, t_j^1 of $\tilde{A}_j^1$ and c_j^2, g_j^2, t_j^2 of $\tilde{A}_j^2$ to prove the property 3.

We can consider $c_j^1 \le c_j^2$, $g_j^1 \ge g_j^2$ and $t_j^1 \ge t_j^2$ for $\tilde{A}_j^1 \preccurlyeq \tilde{A}_j^2$ $(j=1,2,...,n)$.

Then we have

$$\left(1-c_j^1\right)^{w_j} \ge \left(1-c_j^2\right)^{w_j},\ \left(g_j^1\right)^{w_j} \ge \left(g_j^2\right)^{w_j},\ \left(t_j^1\right)^{w_j} \ge \left(t_j^2\right)^{w_j};$$

$$1-\prod_{j=1}^{n}\left(1-c_j^1\right)^{w_j} \le 1-\prod_{j=1}^{n}\left(1-c_j^2\right)^{w_j},\left(g_j^1\right)^{w_j} \ge \left(g_j^2\right)^{w_j} \text{ and}$$

$$\left(t_j^1\right)^{w_j} \ge \left(t_j^2\right)^{w_j}.$$

Therefore,

$$1-\prod_{j=1}^{n}\left(1-c_j^1\right)^{w_j} \le 1-\prod_{j=1}^{n}\left(1-c_j^2\right)^{w_j};\ \prod_{j=1}^{n}\left(g_j^1\right)^{w_j} \ge \prod_{j=1}^{n}\left(g_j^2\right)^{w_j},$$

$$\text{and} \prod_{j=1}^{n}\left(t_j^1\right)^{w_j} \ge \prod_{j=1}^{n}\left(t_j^2\right)^{w_j}. \quad (42)$$

Similarly, we can show

$$1-\prod_{j=1}^{n}\left(1-a_j^1\right)^{w_j} \le 1-\prod_{j=1}^{n}\left(1-a_j^2\right)^{w_j};\ \prod_{j=1}^{n}\left(e_j^1\right)^{w_j} \ge \prod_{j=1}^{n}\left(e_j^2\right)^{w_j},$$

$$\text{and} \prod_{j=1}^{n}\left(r_j^1\right)^{w_j} \ge \prod_{j=1}^{n}\left(r_j^2\right)^{w_j};$$

$$1-\prod_{j=1}^{n}\left(1-b_j^1\right)^{w_j} \le 1-\prod_{j=1}^{n}\left(1-b_j^2\right)^{w_j};\ \prod_{j=1}^{n}\left(f_j^1\right)^{w_j} \ge \prod_{j=1}^{n}\left(f_j^2\right)^{w_j},$$

$$\text{and} \prod_{j=1}^{n}\left(s_j^1\right)^{w_j} \ge \prod_{j=1}^{n}\left(s_j^2\right)^{w_j}.$$

Assume that

$$\tilde{A}^1 = TFNNWA(\tilde{A}_1^1,\tilde{A}_2^1,...,\tilde{A}_n^1)$$

$$= \left\langle (a^1,b^1,c^1),(e^1,f^1,g^1),(r^1,s^1,t^1)\right\rangle \text{ and}$$

$$\tilde{A}^2 = TFNNWA(\tilde{A}_1^2,\tilde{A}_2^2,...,\tilde{A}_n^2)$$

$$= \left\langle (a^2,b^2,c^2),(e^2,f^2,g^2),(r^2,s^2,t^2)\right\rangle, \text{ where}$$

$$a^s = 1-\prod_{j=1}^{n}\left(1-a_j^s\right)^{w_j},\ b^s = 1-\prod_{j=1}^{n}\left(1-b_j^s\right)^{w_j},$$

$$c^s = 1-\prod_{j=1}^{n}\left(1-c_j^s\right)^{w_j};$$

$$e^s = \prod_{j=1}^{n}\left(e_j^s\right)^{w_j},\ f^s = \prod_{j=1}^{n}\left(f_j^s\right)^{w_j},\ g^s = \prod_{j=1}^{n}\left(g_j^s\right)^{w_j} \text{ and}$$

$$r^s = \prod_{j=1}^{n}\left(r_j^s\right)^{w_j},\ s^s = \prod_{j=1}^{n}\left(s_j^s\right)^{w_j},\ t^s = \prod_{j=1}^{n}\left(t_j^s\right)^{w_j} \text{ for } s=1, 2.$$

Now we consider the score function of $\tilde{A}_1$:

$$S\left(\tilde{A}^1\right)=\frac{1}{12}\begin{bmatrix}8+(a^1+2b^1+c^1)-(e^1+2f^1+g^1)\\ -(r^1+2s^1+t^1)\end{bmatrix}$$

$$\le\frac{1}{12}\left[8+(a^2+2b^2+c^2)-(e^2+2f^2+g^2)+(r^2+2s^2+t^2)\right]=S\left(\tilde{A}^2\right)$$

Now we consider the following two cases:

Case 1 . If $S\left(\tilde{A}^1\right)<S\left(\tilde{A}^2\right)$, from Definition-13, we have

$TNFNWA(\tilde{A}_1^1,\tilde{A}_2^1,...,\tilde{A}_n^1)\prec TNFNWA(\tilde{A}_1^2,\tilde{A}_2^2,...,\tilde{A}_n^2)$. (43)

Case 2 . If $S\left(\tilde{A}^1\right)=S\left(\tilde{A}^2\right)$, then by Eq.(21) we can consider

$$\frac{1}{12}\begin{bmatrix}8+(a^1+2b^1+c^1)-(e^1+2f^1+g^1)\\ -(r^1+2s^1+t^1)\end{bmatrix}$$
$$=\frac{1}{12}\begin{bmatrix}8+(a^2+2b^2+c^2)-(e^2+2f^2+g^2)\\ -(r^2+2s^2+t^2)\end{bmatrix}.$$

Thus for $\tilde{A}_j^1\preccurlyeq\tilde{A}_j^2$ $(j=1,2,...,n)$ i.e., for $a_j^1\le a_j^2,\ b_j^1\le b_j^2$ $c_j^1\le c_j^2;\ e_j^1\ge e_j^2\ \ f_j^1\ge f_j^2,\ g_j^1\ge g_j^2$ and $r_j^1\le r_j^2,\ s_j^1\ge s_j^2\ ,t_j^1\ge t_j^2$ we have $a^1=a^2,\ b^1=b^2,\ c^1=c^2,\ e^1=e^2,\ f^1=f^2,\ g^1=g^2,$ $r^1=r^2,\ s^1=s^2$ and $t^1=t^2$.

Then, the accuracy function of $\tilde{A}^1$ yields

$$H(\tilde{A}^1)=\frac{1}{4}\left[(a^1+2b^1+c^1)-(r^1+2s^1+t^1)\right]$$

$$=\frac{1}{4}\left[(a^2+2b^2+c^2)-(r^2+2s^2+t^2)\right]$$
$$=H(\tilde{A}^2)\,. \qquad (44)$$

Thus from Definition-13, we have

$TNFNWA(\tilde{A}_1^1,\tilde{A}_2^1,...,\tilde{A}_n^1)=TNFNWA(\tilde{A}_1^2,\tilde{A}_2^2,...,\tilde{A}_n^2)$. (45)

Finally, from Eqs. (43) and (45), we have the following result

$TFNNWA(\tilde{A}_1^1,\tilde{A}_2^1,...,\tilde{A}_n^1)\preccurlyeq TFNNWA(\tilde{A}_1^2,\tilde{A}_2^2,...,\tilde{A}_n^2)$.

This completes the proof of Property 3. □

Example3. We consider the following four TFNNVs:

$\tilde{A}_1=\langle(0.80,0.85,0.90),\ (0.10,0.15,0.20),$ $(0.05,0.10,0.15)\rangle;\ \tilde{A}_2=\langle(0.70,0.75,0.80),$ $(0.15,0.20,0.25),\ (0.10,0.15,0.20)\rangle;$

$\tilde{A}_3=\langle(0.40,0.45,0.50),\ (0.40,0.45,0.50),$ $(0.35,0.40,0.45)\rangle$ and

$\tilde{A}_4=\langle(0.70,0.75,0.80),\ (0.15,0.20,0.25),$ $(0.10,0.15,0.20)\rangle.$

Using TFNNWA operator defined in Eq.(27), we can aggregate $\tilde{A}_1,\tilde{A}_2,\tilde{A}_3$, and $\tilde{A}_4$ with weight vector $w=(0.30,0.25,0.25,0.20)$ as:

$$\tilde{A}=TFNNWA(\tilde{A}_1,\tilde{A}_2,\tilde{A}_3,\tilde{A}_4)$$
$$=w_1\tilde{A}_1\oplus w_2\tilde{A}_2\oplus w_3\tilde{A}_3\oplus w_4\tilde{A}_4$$

$$=\left\langle\begin{pmatrix}\left(1-(1-0.80)^{0.30}(1-0.70)^{0.25}(1-0.40)^{0.25}(1-0.70)^{0.20}\right),\\ \left(1-(1-0.85)^{0.30}(1-0.75)^{0.25}(1-0.45)^{0.25}(1-0.75)^{0.20}\right),\\ \left(1-(1-0.90)^{0.30}(1-0.80)^{0.25}(1-0.50)^{0.25}(1-0.80)^{0.20}\right)\end{pmatrix},\begin{pmatrix}\left((0.10)^{0.30}(0.15)^{0.25}(0.40)^{0.25}(0.15)^{0.20}\right),\\ \left((0.15)^{0.30}(0.20)^{0.25}(0.45)^{0.25}(0.20)^{0.20}\right),\\ \left((0.20)^{0.30}(0.25)^{0.25}(0.50)^{0.25}(0.25)^{0.20}\right)\end{pmatrix},\right.$$
$$\left.\begin{pmatrix}\left((0.05)^{0.30}(0.10)^{0.25}(0.35)^{0.25}(0.10)^{0.20}\right),\\ \left((0.10)^{0.30}(0.15)^{0.25}(0.40)^{0.25}(0.15)^{0.20}\right),\\ \left((0.15)^{0.30}(0.20)^{0.25}(0.45)^{0.25}(0.20)^{0.20}\right)\end{pmatrix}\right\rangle \qquad (46)$$

$$=\left\langle\begin{matrix}\left(\left(1-0.617\times0.740\times0.880\times0.786\right),\left(1-0.566\times0.707\times0.861\times0.756\right),\left(1-0.501\times0.669\times0.841\times0.725\right)\right),\\ \left(\left(0.501\times0.622\times0.795\times0.684\right),\left(0.566\times0.667\times0.819\times0.725\right),\left(0.617\times0.707\times0.841\times0.887\right)\right),\\ \left(\left(0.407\times0.562\times0.769\times0.631\right),\left(0.501\times0.622\times0.795\times0.684\right),\left(0.566\times0.669\times0.819\times0.725\right)\right)\end{matrix}\right\rangle$$

$= \left\langle (0.6842, 0.7395, 0.7956), (0.1804, 0.2605, 0.3254), (0.1110, 0.1694, 0.2249) \right\rangle$.

4.2 Triangular fuzzy number neutrosophic geometric averaging operator

Definition 17. Suppose that

that $\tilde{A}_j = \left\langle (a_j, b_j, c_j), (e_j, f_j, g_j), (r_j, s_j, t_j) \right\rangle$ $(j = 1, 2, ..., n)$ be a collection TFNNVs in the set of real numbers and $TFNNWG: \Theta^n \to \Theta$. The triangular fuzzy number neutrosophic weighted geometric (TFNNWG) operator denoted by $TFNNWG_w(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n)$ is defined as follows:

$$TFNNWG_w(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n) = \tilde{A}_1^{w_1} \otimes \tilde{A}_2^{w_2} \otimes \cdots \otimes \tilde{A}_n^{w_n} = \bigotimes_{j=1}^{n} (\tilde{A}_j^{w_j}) \quad (47)$$

where $w_j \in [0,1]$ is the exponential weight vector of $\tilde{A}_j$ $(j = 1, 2, ..., n)$ such that $\sum_{j=1}^{n} w_j = 1$. In particular, if $w = (1/n, 1/n, ..., 1/n)^T$ then the $TFNNWG(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n)$ operator reduces to triangular fuzzy neutrosophic geometric(TNFG) operator:

$$TFNNWG_w(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n) = \left(\tilde{A}_1 \otimes \tilde{A}_2 \otimes \cdots \otimes \tilde{A}_n \right)^{\frac{1}{n}}. \quad (48)$$

We now establish the following theorem with the basic operations of TFNNV defined in Definition 10.

Theorem 2. Assume that

$\tilde{A}_j = \left\langle (a_j, b_j, c_j), (e_j, f_j, g_j), (r_j, s_j, t_j) \right\rangle$ $(j = 1, 2, ..., n)$ be a collection TFNNVs in the set of real numbers. Then the aggregated value obtained from TFNNWG, is also a TFNNV, and then we have

$$TFNNWG_w(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n) = \tilde{A}_1^{w_1} \otimes \tilde{A}_2^{w_2} \otimes \cdots \otimes \tilde{A}_n^{w_n} = \bigotimes_{j=1}^{n} (\tilde{A}_j^{w_j})$$

$$= \left\langle \begin{array}{l} \left(\prod_{j=1}^{n} a_j^{w_j}, \prod_{j=1}^{n} b_j^{w_j}, \prod_{j=1}^{n} c_j^{w_j} \right), \\ \left(1 - \prod_{j=1}^{n} (1-e_j)^{w_j}, 1 - \prod_{j=1}^{n} (1-f_j)^{w_j}, 1 - \prod_{j=1}^{n} (1-g_j)^{w_j} \right), \\ \left(1 - \prod_{j=1}^{n} (1-r_j)^{w_j}, 1 - \prod_{j=1}^{n} (1-s_j)^{w_j}, 1 - \prod_{j=1}^{n} (1-t_j)^{w_j} \right) \end{array} \right\rangle \quad (49)$$

where $w_j \in [0,1]$ is the weight vector of TFNNV $\tilde{A}_j$ $(j = 1, 2, ..., n)$ such that $\sum_{j=1}^{n} w_j = 1$.

Similar to arithmetic averaging operator, we can also prove the theorem by mathematical induction.

1. When n = 1, the theorem is true.
2. When n = 2, we have

$$\bigotimes_{j=1}^{2} \left(\tilde{A}_j \right)^{w_j} = \tilde{A}_1^{w_1} \otimes \tilde{A}_2^{w_2}$$

$$= \left(\begin{array}{l} \left\langle \left(a_1^{w_1}, b_1^{w_1}, c_1^{w_1} \right), \left(1-(1-e_1)^{w_1}, 1-(1-f_1)^{w_1}, 1-(1-g_1)^{w_1} \right), \left(1-(1-r_1)^{w_1}, 1-(1-s_1)^{w_1}, 1-(1-t_1)^{w_1} \right) \right\rangle \\ \otimes \left\langle \left(a_2^{w_1}, b_2^{w_1}, c_2^{w_1} \right), \left(1-(1-e_2)^{w_1}, 1-(1-f_2)^{w_1}, 1-(1-g_2)^{w_1} \right), \left(1-(1-r_2)^{w_1}, 1-(1-s_2)^{w_1}, 1-(1-t_2)^{w_1} \right) \right\rangle \end{array} \right)$$

$$=\left\langle \begin{array}{c} \left(a_1^{w_1}a_2^{w_2},b_1^{w_1}b_2^{w_2},c_1^{w_1}c_2^{w_2}\right),\left(\begin{array}{c}\left(1-(1-e_1)^{w_1}\right)+\left(1-(1-e_2)^{w_2}\right)-\left(1-(1-e_1)^{w_1}\right).\left(1-(1-e_2)^{w_2}\right),\\ \left(1-(1-f_1)^{w_1}\right)+\left(1-(1-f_2)^{w_2}\right)-\left(1-(1-f_1)^{w_1}\right).\left(1-(1-f_2)^{w_2}\right),\\ \left(1-(1-c_1)^{w_1}\right)+\left(1-(1-c_2)^{w_2}\right)-\left(1-(1-c_1)^{w_1}\right).\left(1-(1-c_2)^{w_2}\right)\end{array}\right)\\ \left(\begin{array}{c}\left(1-(1-r_1)^{w_1}\right)+\left(1-(1-r_2)^{w_2}\right)-\left(1-(1-r_1)^{w_1}\right).\left(1-(1-r_2)^{w_2}\right),\\ \left(1-(1-s_1)^{w_1}\right)+\left(1-(1-s_2)^{w_2}\right)-\left(1-(1-s_1)^{w_1}\right).\left(1-(1-s_2)^{w_2}\right),\\ \left(1-(1-t_1)^{w_1}\right)+\left(1-(1-t_2)^{w_2}\right)-\left(1-(1-t_1)^{w_1}\right).\left(1-(1-t_2)^{w_2}\right)\end{array}\right)\end{array}\right\rangle$$

$$=\left\langle \begin{array}{c}\left(\prod_{j=1}^{2}a_j^{w_j},\prod_{j=1}^{2}b_j^{w_j},\prod_{j=1}^{2}c_j^{w_j}\right),\left(1-\prod_{j=1}^{2}(1-e_j)^{w_j},1-\prod_{j=1}^{2}(1-f_j)^{w_j},1-\prod_{j=1}^{2}(1-g_j)^{w_j}\right),\\ \left(1-\prod_{j=1}^{2}(1-e_j)^{w_j},1-\prod_{j=1}^{2}(1-f_j)^{w_j},1-\prod_{j=1}^{2}(1-g_j)^{w_j}\right)\end{array}\right\rangle \tag{50}$$

3. When n = k, we assume that Eq.(49) is true then,

$$TNFNWG_w(\tilde{A}_1,\tilde{A}_2,...,\tilde{A}_k)=\tilde{A}_1^{w_1}\otimes\tilde{A}_2^{w_2}\otimes\cdots\otimes\tilde{A}_k^{w_k}$$

$$=\left\langle \begin{array}{c}\left(\prod_{j=1}^{k}a_j^{w_j},\prod_{j=1}^{k}b_j^{w_j},\prod_{j=1}^{n}c_j^{w_j}\right),\left(1-\prod_{j=1}^{k}(1-e_j)^{w_j},1-\prod_{j=1}^{k}(1-f_j)^{w_j},1-\prod_{j=1}^{k}(1-g_j)^{w_j}\right),\\ \left(1-\prod_{j=1}^{k}(1-r_j)^{w_j},1-\prod_{j=1}^{k}(1-s_j)^{w_j},1-\prod_{j=1}^{k}(1-t_j)^{w_j}\right)\end{array}\right\rangle \tag{51}$$

4. When n = k+1, we can consider the following expression:

$$TNFNWG_w(\tilde{A}_1,\tilde{A}_2,...,\tilde{A}_{k+1})=\underset{j=1}{\overset{k}{\otimes}}\left(\tilde{A}_j\right)^{w_j}\otimes\left(\tilde{A}_{k+1}\right)^{w_{k+1}}$$

$$=\left\langle\left(\prod_{j=1}^{k}a_j^{w_j}.a_{k+1}^{w_{k+1}},\prod_{j=1}^{k}b_j^{w_j}.b_{k+1}^{w_{k+1}},\prod_{j=1}^{n}c_j^{w_j}.c_{k+1}^{w_{k+1}}\right),\right.$$

$$\left(\begin{array}{c}\left(1-\prod_{j=1}^{k}(1-e_j)^{w_j}\right)+\left(1-(1-e_{k+1})^{w_{k+1}}\right)-\left(1-\prod_{j=1}^{k}(1-e_j)^{w_j}\right).\left(1-(1-e_{k+1})^{w_{k+1}}\right),\\ \left(1-\prod_{j=1}^{k}(1-f_j)^{w_j}\right)+\left(1-(1-f_{k+1})^{w_{k+1}}\right)-\left(1-\prod_{j=1}^{k}(1-f_j)^{w_j}\right).\left(1-(1-f_{k+1})^{w_{k+1}}\right),\\ \left(1-\prod_{j=1}^{k}(1-g_j)^{w_j}\right)+\left(1-(1-g_{k+1})^{w_{k+1}}\right)-\left(1-\prod_{j=1}^{k}(1-g_j)^{w_j}\right).\left(1-(1-g_{k+1})^{w_{k+1}}\right)\end{array}\right),$$

$$\left\langle\begin{pmatrix}\left(1-\prod_{j=1}^{k}(1-r_j)^{w_j}\right)+\left(1-(1-r_{k+1})^{w_{k+1}}\right)-\left(1-\prod_{j=1}^{k}(1-r_j)^{w_j}\right).\left(1-(1-r_{k+1})^{w_{k+1}}\right),\\ \left(1-\prod_{j=1}^{k}(1-s_j)^{w_j}\right)+\left(1-(1-s_{k+1})^{w_{k+1}}\right)-\left(1-\prod_{j=1}^{k}(1-s_j)^{w_j}\right).\left(1-(1-s_{k+1})^{w_{k+1}}\right),\\ \left(1-\prod_{j=1}^{k}(1-t_j)^{w_j}\right)+\left(1-(1-t_{k+1})^{w_{k+1}}\right)-\left(1-\prod_{j=1}^{k}(1-t_j)^{w_j}\right).\left(1-(1-t_{k+1})^{w_{k+1}}\right)\end{pmatrix}\right\rangle \tag{52}$$

$$=\left\langle\begin{matrix}\left(\prod_{j=1}^{k+1}a_j^{w_j},\prod_{j=1}^{k+1}b_j^{w_j},\prod_{j=1}^{k+1}c_j^{w_j}\right),\left(1-\prod_{j=1}^{k+1}(1-e_j)^{w_j},1-\prod_{j=1}^{k+1}(1-f_j)^{w_j},1-\prod_{j=1}^{k+1}(1-g_j)^{w_j}\right),\\ \left(1-\prod_{j=1}^{k+1}(1-r_j)^{w_j},1-\prod_{j=1}^{k+1}(1-s_j)^{w_j},1-\prod_{j=1}^{k+1}(1-t_j)^{w_j}\right)\end{matrix}\right\rangle \tag{53}$$

We observe that the theorem is also true for n = k+1.

Therefore, by mathematical induction, Eq. (49) holds for all values of *n*.
Since the components of all three membership functions of $\tilde{A}_j\,(j=1,2,...,n)$ belong to [0, 1] the following relations are valid

$0\le\left(\prod_{j=1}^{n}c_j^{w_j}\right)\le 1$, $0\le\left(1-\prod_{j=1}^{n}(1-g_j)^{w_j}\right)\le 1$, ,

and $0\le\left(1-\prod_{j=1}^{n}(1-t_j)^{w_j}\right)\le 1$. (54)

It follows that

$$0\le\left(\prod_{j=1}^{n}c_j^{w_j}+1-\prod_{j=1}^{n}(1-g_j)^{w_j}+1-\prod_{j=1}^{n}(1-t_j)^{w_j}\right)\le 3.$$

This completes the proof of Theorem 2.
Now, we discuss some essential properties of TFNNWG operator for TFNNs.

Property 4.(*Idempotency*): If all $\tilde{A}_j\,(\mathrm{j}=1,2,...,\mathrm{n})$ are equal that is $\tilde{A}_j=\langle(\mathrm{a,b,c}),(\mathrm{e,f,g}),(\mathrm{r,s,t})\rangle$, for all *j*, then $TFNNWG_w(\tilde{\mathrm{A}}_1,\tilde{\mathrm{A}}_2,...,\tilde{\mathrm{A}}_k)=\tilde{A}$.
Proof: From Eq.(49), we have
$TFNNWG_w(\tilde{\mathrm{A}}_1,\tilde{\mathrm{A}}_2,...,\tilde{\mathrm{A}}_n)$

$$=TFNNWG_g(\tilde{\mathrm{A}},\tilde{\mathrm{A}},...,\tilde{\mathrm{A}})=\bigotimes_{j=1}^{n}\left(\tilde{\mathrm{A}}_j\right)^{w_j}$$

$$=\left\langle\begin{matrix}\left(\prod_{j=1}^{n}a^{w_j},\prod_{j=1}^{n}b^{w_j},\prod_{j=1}^{n}c^{w_j}\right),\\ \left(1-\prod_{j=1}^{n}(1-e)^{w_j},1-\prod_{j=1}^{n}(1-f)^{w_j},1-\prod_{j=1}^{n}(1-g)^{w_j}\right),\\ \left(1-\prod_{j=1}^{n}(1-r)^{w_j},1-\prod_{j=1}^{n}(1-s)^{w_j},1-\prod_{j=1}^{n}(1-t)^{w_j}\right)\end{matrix}\right\rangle$$

$$=\left\langle\begin{matrix}\left(a^{\sum_{j=1}^{n}w_j},b^{\sum_{j=1}^{n}w_j},c^{\sum_{j=1}^{n}w_j}\right),\\ \left(1-(1-e)^{\sum_{j=1}^{n}w_j},1-(1-f)^{\sum_{j=1}^{n}w_j},1-(1-g)^{\sum_{j=1}^{n}w_j}\right),\\ \left(1-(1-e)^{\sum_{j=1}^{n}w_j},1-(1-f)^{\sum_{j=1}^{n}w_j},1-(1-g)^{\sum_{j=1}^{n}w_j}\right)\end{matrix}\right\rangle$$

$=\langle(a,b,c),(e,f,g),(r,s,t)\rangle=\tilde{A}.$
This completes the Property 4. □

Property 5. (*Boundedness*).
Let $\tilde{A}_j=\langle(\mathrm{a}_j,\mathrm{b}_j,\mathrm{c}_j),(\mathrm{e}_j,\mathrm{f}_j,\mathrm{g}_j),(\mathrm{r}_j,\mathrm{s}_j,\mathrm{t}_j)\rangle$ $(\mathrm{j}=1,2,...,\mathrm{n})$ be a collection TFNNs in the set of real numbers. Assume

$$\tilde{A}^{+}=\left\langle\begin{matrix}\left(\max_j a_j,\max_j b_j,\max_j c_j\right),\left(\min_j e_j,\min_j f_j,\min_j g_j\right),\\ \left(\min_j r_j,\min_j s_j,\min_j t_j\right)\end{matrix}\right\rangle$$

and

$$\tilde{A}^- = \left\langle \left(\min_j a_j, \min_j b_j, \min_j c_j\right), \left(\max_j e_j, \max_j f_j, \max_j g_j\right), \left(\max_j r_j, \max_j s_j, \max_j t_j\right) \right\rangle$$

for all $j = 1, 2, ..., n.$. Then

$$\tilde{A}^- \le TNFNWG_w(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n) \le \tilde{A}^+ . \quad (55)$$

Proof: The proof of the Property 5 is similar to Property 2.

Property 6. (*Monotonicity*).

Let $\tilde{A}_j^1 = \langle (a_j^1, b_j^1, c_j^1), (e_j^1, f_j^1, g_j^1), (r_j^1, s_j^1, t_j^1) \rangle$ and $\tilde{A}_j^2 = \langle (a_j^2, b_j^2, c_j^2), (e_j^2, f_j^2, g_j^2), (r_j^2, s_j^2, t_j^2) \rangle$ $(j = 1, 2, ..., n)$ be a collection of two TFNNVs in the set of real numbers. If $\tilde{A}_j^1 \preccurlyeq \tilde{A}_j^2$ for $j = 1, 2, ..., n$ then

$$TFNNWG_w(\tilde{A}_1^1, \tilde{A}_2^1, ..., \tilde{A}_n^1) \preccurlyeq TFNNWG_w(\tilde{A}_1^2, \tilde{A}_2^2, ..., \tilde{A}_n^2) . \quad (56)$$

Proof: Property 6 can be proved by a similar argument of Property 3. Therefore, we do not discuss again to avoid repetition. □

Example 4. Assume that $\tilde{A}_1 = \langle (0.80, 0.85, 0.90), (0.10, 0.15, 0.20), (0.05, 0.10, 0.15) \rangle$; $\tilde{A}_2 = \langle (0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20) \rangle$; $\tilde{A}_3 = \langle (0.40, 0.45, 0.50), (0.40, 0.45, 0.50), (0.35, 0.40, 0.45) \rangle$ and $\tilde{A}_4 = \langle (0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20) \rangle$ are four TFNNVs. Then using TFNNWG operator defined in Eq.(49), we can aggregate $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3$, and $\tilde{A}_4$ with the considered weight vector $w = (0.30, 0.25, 0.25, 0.20)$ as:

$$\tilde{A} = TFNNWG_w(\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{A}_4)$$
$$= w_1 \tilde{A}_1 \otimes w_2 \tilde{A}_2 \otimes w_3 \tilde{A}_3 \otimes w_4 \tilde{A}_4$$

$$= \left\langle \begin{pmatrix} \left((0.80)^{0.30}(0.70)^{0.25}(0.40)^{0.25}(0.70)^{0.20}\right), \\ \left((0.85)^{0.30}(0.75)^{0.25}(0.45)^{0.25}(0.75)^{0.20}\right), \\ \left((0.90)^{0.30}(0.80)^{0.25}(0.50)^{0.25}(0.80)^{0.20}\right) \end{pmatrix}, \begin{pmatrix} \left(1-(1-0.10)^{0.30}(1-0.15)^{0.25}(1-0.40)^{0.25}(1-0.15)^{0.20}\right), \\ \left(1-(1-0.15)^{0.30}(1-0.20)^{0.25}(1-0.45)^{0.25}(1-0.20)^{0.20}\right), \\ \left(1-(1-0.20)^{0.30}(1-0.25)^{0.25}(1-0.50)^{0.25}(1-0.25)^{0.20}\right) \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} \left(1-(1-0.05)^{0.30}(1-0.10)^{0.25}(1-0.35)^{0.25}(1-0.10)^{0.20}\right), \\ \left(1-(1-0.10)^{0.30}(1-0.15)^{0.25}(1-0.40)^{0.25}(1-0.15)^{0.20}\right), \\ \left(1-(1-0.15)^{0.30}(1-0.20)^{0.25}(1-0.45)^{0.25}(1-0.20)^{0.20}\right) \end{pmatrix} \right\rangle$$

$$= \left\langle \begin{array}{l} \left((0.935\times0.915\times0.795\times0.931), (0.952\times0.930\times0.819\times0.944), (0.969\times0.946\times0.841\times0.956)\right), \\ \left((1-0.969\times0.960\times0.880\times0.968), (1-0.952\times0.946\times0.861\times0.956), (1-0.935\times0.930\times0.841\times0.944)\right), \\ \left((1-0.985\times0.974\times0.898\times0.979), (1-0.969\times0.960\times0.880\times0.968), (1-0.952\times0.946\times0.861\times0.956)\right) \end{array} \right\rangle$$

$$= \left\langle (0.6332, 0.6845, 0.7370), (0.2076, 0.2587, 0.3097), (0.1565, 0.2075, 0.2587) \right\rangle .$$

---

5 Application of TFNNWA and TFNNWG operators to multi attribute decision making

Consider a multi attribute decision making problem in which $Y = \{Y_1, Y_2, ..., Y_m\}$ be the set of n feasible alternatives and $C = \{C_1, C_2, ..., C_n\}$ be the set of attributes. Assume that $w = (w_1, w_2, ..., w_n)^T$ be the weight vector of the attributes, where w_j denotes the importance degree of the attribute C_j such that $w_j \ge 0$ and $\sum_{j=1}^{n} w_j = 1$ for $j = 1, 2, ..., n$.

The ratings of all alternatives $Y_i (i = 1, 2, ..., m)$ with respect to the attributes $C_j (j = 1, 2, ..., n)$ have been presented in a TFNNV based decision matrix $U = (u_{ij})_{m\times n}$ (see the Table 1). Furthermore, in the decision matrix $U = (u_{ij})_{m\times n}$, the rating $u_{ij} = \langle (a_{ij}, b_{ij}, c_{ij}), (e_{ij}, f_{ij}, g_{ij}), (r_{ij}, s_{ij}, t_{ij}) \rangle$ represents a

TFNNV, where the fuzzy number (a_{ij}, b_{ij}, c_{ij}) represents the degree that the alternative $Y_i (i = 1, 2, ..., m)$ satisfies the attribute $C_j (j = 1, 2, ..., n)$, the fuzzy number (e_{ij}, f_{ij}, g_{ij}) represents the degree that the alternative Y_i is uncertain about the attribute C_j and fuzzy number (r_{ij}, s_{ij}, t_{ij}) indicates the degree that the alternative Y_i does not satisfy the attribute C_j such that

$0 \le c_{ij} + g_{ij} + t_{ij} \le 3$, for *i = 1, 2,..., m* and *j = 1, 2, ..., n.*

Based on the TFNNWA and TFNNWG operators, we develop a practical approach for solving MADM problems, in which the ratings of the alternatives over the attributes are expressed with TFNNVs. The schematic diagram of the proposed approach for MADM is depicted in the Figure-1.

Table 1. Triangular fuzzy number neutrosophic value based decision matrix

	C_1	C_2	...	C_n
Y_1	$\left\langle (a_{11}, b_{11}, c_{11}), (e_{11}, f_{11}, g_{11}), (r_{11}, s_{11}, t_{11}) \right\rangle$	$\left\langle (a_{12}, b_{12}, c_{12}), (e_{12}, f_{12}, g_{12}), (r_{12}, s_{12}, t_{12}) \right\rangle$	...	$\left\langle (a_{1n}, b_{1n}, c_{1n}), (e_{1n}, f_{1n}, g_{1n}), (r_{1n}, s_{1n}, t_{1n}) \right\rangle$
Y_2	$\left\langle (a_{21}, b_{21}, c_{21}), (e_{21}, f_{21}, g_{21}), (r_{21}, s_{21}, t_{21}) \right\rangle$	$\left\langle (a_{22}, b_{22}, c_{22}), (e_{22}, f_{22}, g_{22}), (r_{22}, s_{22}, t_{22}) \right\rangle$	...	$\left\langle (a_{2n}, b_{2n}, c_{2n}), (e_{2n}, f_{2n}, g_{2n}), (r_{2n}, s_{2n}, t_{2n}) \right\rangle$
....	....	....	....	...
Y_m	$\left\langle (a_{m1}, b_{m1}, c_{m1}), (e_{m1}, f_{m1}, g_{m1}), (r_{m1}, s_{m1}, t_{m1}) \right\rangle$	$\left\langle (a_{m2}, b_{m2}, c_{m2}), (e_{m2}, f_{m2}, g_{m2}), (r_{m2}, s_{m2}, t_{m2}) \right\rangle$	...	$\left\langle (a_{mn}, b_{mn}, c_{mn}), (e_{mn}, f_{mn}, g_{mn}), (r_{mn}, s_{mn}, t_{mn}) \right\rangle$

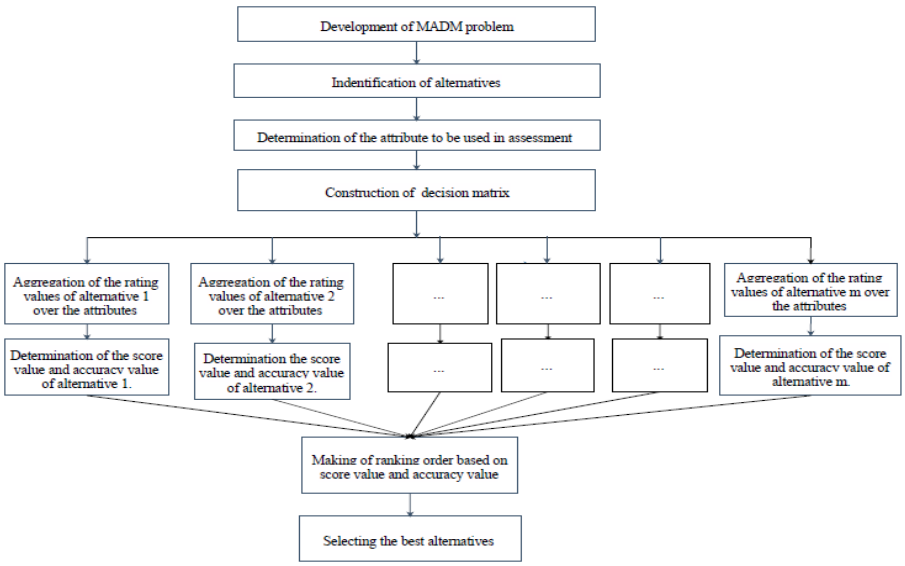

Figure-1. Framework for the proposed MADM method

Therefore, we design the proposed approach in the following steps:

Step 1: First aggregate all rating values p_{ij} $(j = 1, 2, ..., n)$ of the i -th row of the decision matrix $(p_{ij})_{m\times n}$ defined in Table 1.

Step 2: Determine the aggregation value u_i corresponding to the alternative Y_i obtained from TFNNWA operator:

$$u_i = \langle (a_i, b_i, c), (e_i, f_i, g_i), (r_i, s_i, t_i) \rangle = TFNNWA_w(p_{i1}, p_{i2}, ..., p_{in}) \quad (57)$$

or by the TFNNWG operator as

$$u_i = \langle (a_i, b_i, c), (e_i, f_i, g_i), (r_i, s_i, t_i) \rangle = TFNNWG_w(p_{i1}, p_{i2}, ..., p_{in}) \quad (58)$$

Step 3: For each alternative $A_i (i = 1, 2, ..., m)$, calculate the score values $S(u_i)$ and accuracy values $A(u_i)$ of the aggregated rating values obtained by TFNNWA or TFNNWG operators that are in Eqs. (21) and (22).

Step 4: Using Definition 11 to Definition 13, determine the ranking order of aggregated values obtained in Step 3.

Step 5: Select the best alternative in accordance with the ranking order obtained in Step 4.

6 An illustrative example of multi attribute decision making

In this section, we consider an illustrative example of medical representative selection problem to demonstrate and applicability of the proposed approach to multi attribute decision making problem.

Assume that a pharmacy company wants to recruit a medical representative. After initial scrutiny four candidates $Y_i (i = 1, 2, 3, 4)$ have been considered for further evaluation with respect to the five attributes $C_j (j = 1, 2, 3, 4, 5)$ namely,

1. oral communication skill (C_1);
2. past experience (C_2);
3. general aptitude (C_3);
4. willingness (C_4) and
5. self confidence (C_5).

The ratings of the alternatives $Y_i (i = 1, 2, 3, 4)$ with respect to the attributes $C_j (j = 1, 2, 3, 4, 5)$ are expressed with TFNNVs shown in the decision matrix $P = (p_{ij})_{4\times 5}$ (see Table 2.). Assume w = (0.10, 0.25, 0.25, 0.15, 0.25)^T be the relative weight vector of all attributes $C_j (j = 1, 2, 3, 4, 5)$.

Table 2. Triangular fuzzy number neutrosophic value based rating values

	C_1	C_2	C_3	C_4	C_5
Y_1	$\langle (0.80, 0.85, 0.90), (0.10, 0.15, 0.20), (0.05, 0.10, 0.15) \rangle$	$\langle (0.50, 0.55, 0.60), (0.25, 0.30, 0.35), (0.20, 0.25, 0.30) \rangle$	$\langle (0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20) \rangle$	$\langle (0.80, 0.85, 0.90), (0.10, 0.15, 0.20), (0.05, 0.10, 0.15) \rangle$	$\langle (0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20) \rangle$
Y_2	$\langle (0.50, 0.55, 0.60), (0.25, 0.30, 0.35), (0.20, 0.25, 0.30) \rangle$	$\langle (0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20) \rangle$	$\langle (0.80, 0.85, 0.90), (0.10, 0.15, 0.20), (0.05, 0.10, 0.15) \rangle$	$\langle (0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20) \rangle$	$\langle (0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20) \rangle$
Y_3	$\langle (0.40, 0.45, 0.50), (0.40, 0.45, 0.50), (0.35, 0.40, 0.45) \rangle$	$\langle (0.50, 0.55, 0.60), (0.25, 0.30, 0.35), (0.20, 0.25, 0.30) \rangle$	$\langle (0.40, 0.45, 0.50), (0.40, 0.45, 0.50), (0.35, 0.40, 0.45) \rangle$	$\langle (0.40, 0.45, 0.50), (0.40, 0.45, 0.50), (0.35, 0.40, 0.45) \rangle$	$\langle (0.50, 0.55, 0.60), (0.25, 0.30, 0.35), (0.20, 0.25, 0.30) \rangle$
Y_4	$\langle (0.40, 0.45, 0.50), (0.40, 0.45, 0.50), (0.35, 0.40, 0.45) \rangle$	$\langle (0.50, 0.55, 0.60), (0.25, 0.30, 0.35), (0.20, 0.25, 0.30) \rangle$	$\langle (0.40, 0.45, 0.50), (0.40, 0.45, 0.50), (0.35, 0.40, 0.45) \rangle$	$\langle (0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20) \rangle$	$\langle (0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20) \rangle$

Here, we apply two proposed aggregation operators TFNNWA and TFNNWG to solve the medical representative selection problem by using the following steps.

6.1 Utilization of TFNNWA operator:

Step 1: Aggregate the rating values of the alternative Y_i (i= 1, 2, 3, 4) defined in the i -th row of decision matrix $P=(p_{ij})_{4\times5}$ (see Table 2.) with TFNNWA operator.

Step 2: The aggregated rating values u_i corresponding to the alternative Y_i are determined by Eq.(27) and the values are shown in Table 3.

Table 3. Aggregated TFNNV based ratings

	Aggregated ratings
u_1	$\langle(0.6920, 0.7451, 0.8000), (0.1540, 0.2026, 0.2572), (0.1000, 0.1540, 0.2060)\rangle$
u_2	$\langle(0.7147, 0.7667, 0.8197), (0.1426, 0.1938, 0.2445), (0.0901, 0.1426, 0.1938)\rangle$
u_3	$\langle(0.4523, 0.5025, 0.5528), (0.3162, 0.3674, 0.4183), (0.2646, 0.3162, 0.3674)\rangle$
u_4	$\langle(0.5655, 0.6184, 0.6722), (0.2402, 0.2940, 0.3466), (0.1844, 0.2402, 0.2940)\rangle$

Step 3: The score and accuracy values of alternatives Y_i (i= 1, 2, 3, 4) are determined by Eq.(21) and Eq.(22) in Table 4.

Table 4. Score and accuracy values of aggregated rating values

Alternative	Score values $S(u_i)$	Accuracy values $A(u_i)$
Y_1	0.7960	0.5921
Y_2	0.8103	0.6247
Y_3	0.6464	0.1864
Y_4	0.6951	0.3789

Step 4: The order of the alternatives Y_i (i= 1, 2, 3, 4) is determined according to the descending order of the score and accuracy values shown in Table 4. Thus the ranking order of the alternatives is presented as follows:

$$Y_2 \succ Y_1 \succ Y_4 \succ Y_3.$$

Step 5: The ranking order in Step 4 reflects that, Y_2 is the best medical representative.

6.2 Utilization of TFNNWG operator:

Step 1: Using Eq.(49), we aggregate all the rating values of the alternative Y_i (i= 1, 2, 3, 4) for the i- throw of the decision matrix $P=(p_{ij})_{4\times5}$ (see Table 2.).

Step 2: The aggregated rating values u_i corresponding to the alternative Y_i are shown in the Table 5.

Table 5. Aggregated TFNN based rating values

	Aggregated rating values
u_1	$\langle(0.6654, 0.7161, 0.7667), (0.1643, 0.2144, 0.2646), (0.1142, 0.1643, 0.2144)\rangle$
u_2	$\langle(0.6998, 0.7502, 0.8002), (0.1485, 0.1986, 0.2486), (0.0984, 0.1485, 0.1986)\rangle$
u_3	$\langle(0.4472, 0.4975, 0.5477), (0.3292, 0.3795, 0.4299), (0.2789, 0.3292, 0.3795)\rangle$
u_4	$\langle(0.5291, 0.5804, 0.6316), (0.2707, 0.3214, 0.3721), (0.2202, 0.2707, 0.3214)\rangle$

Step 3: The score and accuracy values of alternatives Y_i (i= 1, 2, 3, 4) are determined by Eqs.(21) and (22) and the results are shown in the Table 6.

Table 6. Score and accuracy values of rating values

Alternative	Score values $S(u_i)$	Accuracy values $A(u_i)$
Y_1	0.7791	0.5518
Y_2	0.8010	0.6016
Y_3	0.5962	0.1683
Y_4	0.6627	0.3096

Step 4: The order of alternatives Y_i (i = 1, 2, 3, 4) has been determined according to the descending order of score and accuracy values shown in Table 4.

Thus the ranking order of the alternative is presented as follows:

$Y_2 \succ Y_1 \succ Y_4 \succ Y_3$.

Step 5: The ranking order in Step 4 reflects that Y_2 is the best medical representative.

7 Conclusions

MADM problems generally takes place in a complex environment and usually connected with imprecise data and uncertainty. The triangular neutrosophic fuzzy numbers are an effective tool for dealing with impreciseness and incompleteness of the decision maker's assessments over alternative with respect to attributes. We have first introduced TFNNs and defined some of its operational rules. Then we have proposed two aggregation operators called TFNNWAA and TFNNWGA operators and score function and applied them to solve multi attribute decision making problem under neutrosophic environment. Finally, the effectiveness and applicability of the proposed approach have been illustrated with medical representative selection problem. We hope that the proposed approach can be also applied in other decision making problems such as pattern recognition, personnel selection, medical diagnosis, etc.

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