

Distributive Substructural Logics as Coalgebraic Logics over Posets

Marta Bílková, Rostislav Horčík¹

Institute of Computer Science, Academy of Sciences of the Czech Republic

Jiří Velebil¹

Faculty of Electrical Engineering, Czech Technical University in Prague

Abstract

We show how to understand frame semantics of distributive substructural logics coalgebraically, thus opening a possibility to study them as coalgebraic logics. As an application of this approach we prove a general version of Goldblatt-Thomason theorem that characterizes definability of classes of frames for logics extending the distributive Full Lambek logic, as e.g. relevance logics, many-valued logics or intuitionistic logic. The paper is rather conceptual and does not claim to contain significant new results. We consider a category of frames as posets equipped with monotone relations, and show that they can be understood as coalgebras for an endofunctor of the category of posets. In fact, we adopt a more general definition of frames that allows to cover a wider class of distributive modal logics. Goldblatt-Thomason theorem for classes of resulting coalgebras for instance shows that frames for axiomatic extensions of distributive Full Lambek logic are modally definable classes of certain coalgebras, the respective modal algebras being precisely the corresponding subvarieties of distributive residuated lattices.

Keywords: Substructural logics, frame semantics, coalgebras, coalgebraic logic, Goldblatt-Thomason theorem.

1 Introduction

Modal logics are coalgebraic, the relational frames of classical modal logics can be seen as **Set** coalgebras for the powerset functor. Given an endofunctor T on **Set**, a conceptually clear setting of classical coalgebraic logic of T -coalgebras can be based on an adjunction called *logical connection*, linking categories **Set** and **BA** of sets and Boolean algebras [5,6] and capturing syntax and semantics of the propositional part of the language. Such connection can be "lifted" to a connection between categories of T -coalgebras and Boolean algebras with

¹ The authors acknowledge the support of the grant No. P202/11/1632 of the Czech Science Foundation. email: bilkova@cs.cas.cz, horcik@cs.cas.cz, velebil@math.feld.cvut.cz

operators, which is in general "almost" an adjunction, capturing syntax and semantics of the modal part of the language. From certain properties of the lifted connection one automatically obtains soundness, completeness and expressivity of the modal language. One can also explore the connection to obtain the Goldblatt-Thomason definability theorem for classes of T -coalgebras for a reasonable class of **Set** functors [26].

In this paper, lead by a motivation to approach distributive substructural logics in a coalgebraic way, we use an (enriched) logical connection [27,31] between categories **Pos** of posets and **DL** of distributive lattices. We consider a general language of distributive lattices with operators, including the usual language of substructural logics as an instance. We start with requiring no additional axioms the operators should satisfy (not even the residuation laws), obtaining coalgebras for a certain endofunctor T on posets as semantics of this language. As an application of this setting we prove Goldblatt-Thomason definability theorem for classes of T -coalgebras. Classes of T -coalgebras definable by additional axioms of distributive substructural logics then precisely correspond to frames for these logics as surveyed and studied in [32]. Distributive modal logics have been treated coalgebraically before [7,29]. We see the main novelty of this paper in the fact that we use a weaker assumption than a duality of the category of algebras and certain topological spaces, thus resulting in non-topological coalgebras as semantics of distributive modal or substructural logics.

A leading example of a logic, semantics of which we want to cover, is the distributive full Lambek calculus **dFL** [15] in the following language

$$\varphi ::= p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \otimes \varphi \mid \varphi \rightarrow \varphi \mid \varphi \leftarrow \varphi \mid e \quad (1)$$

where p ranges through a given poset of atomic propositions, \wedge and \vee are tied together by a distributive law, and the remaining four connectives \otimes , \leftarrow , \rightarrow , e satisfy additional equational axioms as, for example, the residuation laws. The algebraic semantics of **dFL** are residuated lattices.

We want to take the stance that \wedge and \vee are the only *propositional connectives* of the language, while the remaining four constructions \otimes , \leftarrow , \rightarrow , e are *modalities*. To prove that the study of relational models of the above language falls into the realm of *coalgebraic modal logic* it will be essential to start with a weaker setting, with *no* additional requirements on the modalities, apart from being monotone and preserving \wedge or \vee , i.e. being operators over distributive lattices.

As it turns out, the natural environment for giving models of the above language is the one of *posets* and *monotone* relations. Namely, a relational model will consist of a poset \mathscr{W} and four monotone relations P_\otimes , P_\leftarrow , P_\rightarrow and P_e on \mathscr{W} . For example, P_\otimes will be a monotone relation (i.e., a monotone map $P_\otimes : \mathscr{W}^{op} \times \mathscr{W}^{op} \times \mathscr{W} \rightarrow 2$, where 2 is the two-element chain) that we will denote by $P_\otimes : \mathscr{W} \times \mathscr{W} \dashrightarrow \mathscr{W}$. Hence the "arity" of P_\otimes mirrors the arity of the "modality" \otimes . Analogously, P_e will be a monotone relation of the form $P_e : \mathbb{1} \dashrightarrow \mathscr{W}$ where $\mathbb{1}$ denotes the one-element preorder. Hence P_e will appear

as a “nullary” monotone relation, mirroring the fact that the “modality” e is nullary. We prove that the above quintuple $\mathbb{W} = (\mathscr{W}, P_{\otimes}, P_{\leftarrow}, P_{\rightarrow}, P_e)$ can be seen as a *coalgebra* for an endofunctor T of the category \mathbf{Pos} of posets and monotone maps.

The reasoning does not change much if we incorporate slightly more general languages of the form

$$\varphi ::= p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \heartsuit(\varphi_0, \dots, \varphi_{n-1}) \mid (\varphi_0, \dots, \varphi_{l-1}) \multimap \psi \mid \sim \varphi \quad (2)$$

where p ranges through a poset \mathbf{At} of atomic propositions, the connectives \wedge, \vee are tied together by the distributive law, \heartsuit is an n -ary *fusion-like* connective, \multimap is an l -ary *implication-like* connective, and \sim is a *negation-like* connective. These connectives are required to interact with \wedge and \vee in the sense that the following equalities are valid for each $0 \leq i \leq n$:

$$\begin{aligned} \heartsuit(\dots, \varphi_i \vee \varphi'_i \dots) &= \heartsuit(\dots, \varphi_i, \dots) \vee \heartsuit(\dots, \varphi'_i, \dots) \\ (\dots, \varphi_i \vee \varphi'_i, \dots) \multimap \psi &= ((\dots, \varphi_i, \dots) \multimap \psi) \wedge ((\dots, \varphi'_i, \dots) \multimap \psi) \\ (\varphi_0, \dots, \varphi_{l-1}) \multimap (\psi \wedge \psi') &= ((\varphi_0, \dots, \varphi_{l-1}) \multimap \psi) \wedge ((\varphi_0, \dots, \varphi_{l-1}) \multimap \psi') \\ \sim(\varphi_1 \vee \varphi_2) &= \sim\varphi_1 \wedge \sim\varphi_2 \end{aligned}$$

In slogans: \heartsuit should preserve \vee pointwise, \multimap should pointwise transform \vee in its premises to \wedge , and it should preserve \wedge in its conclusion, \sim should transform \vee into \wedge .²

We will prove that:

- (i) Relational models of the language (2) are precisely the coalgebras for an endofunctor $T : \mathbf{Pos} \rightarrow \mathbf{Pos}$. Moreover, the construction of T copies the syntax of the “modalities” $\heartsuit, \multimap, \sim$ in (2).
- (ii) The algebraic semantics of (2) will be given by a variety $\mathbf{DL}_{\heartsuit, \multimap, \sim}$ of distributive lattices with operators \heartsuit, \multimap and \sim .
- (iii) It is essential to start with *no* requirements on the modalities in order to obtain a coalgebraic description. Any additional equational requirements on the modalities \heartsuit, \multimap and \sim will result in a *modally definable* class of T -coalgebras. We characterize modally definable classes in the spirit of *Goldblatt-Thomason Theorem* known from the classical modal logic.

As an illustration, we explain how various classes of frames for languages of the type (2) can be perceived as modally definable. In particular, we cover all the frames for the distributive substructural logics as studied in [32], namely:

- The class of frames modelling the *distributive full Lambek calculus* is modally definable by the equations for residuated distributive lattices. The modalities are $\otimes, \rightarrow, \leftarrow$ and e .

² The language above, in its greatest generality, allows for finitely many connectives of each kind, all of various arities. In order not to make the notation too heavy, we will assume that there is just one connective of each kind in our signature. The results for the general case are straightforward generalisations of results for our simplification.

- The class of frames modelling the *intuitionistic logic* is modally definable by the equations for Heyting algebras. The modalities are \otimes (coinciding with \wedge) and \rightarrow .
- The class of frames modelling *relevance logic* is modally definable. The modalities are \otimes , \rightarrow , \leftarrow , e and \sim .

Related work: Using relational models on posets for modelling semantics of various nonclassical logics goes back at least to the work of Routley and Meyer [34], and Dunn, see [10,11,12] or [32] for an overview. We see the novelty of our approach in the fact that we can systematically work with such frames as coalgebras, hence one has a canonical notion of a frame morphism as morphism of corresponding coalgebras.³

The original Goldblatt-Thomason theorem for modal logics [23] characterizes modally definable classes of Kripke *frames*. For positive modal logic it was proved in [9]. A definability theorem for classes of models is due to Venema [38]. Our version of the theorem is an analogue of coalgebraic Goldblatt-Thomason theorem for **Set** coalgebras [26, Theorem 3.15(2.)]. Possibilities to generalize the theorem to coalgebras over measurable spaces have been explored in [30]. Coalgebraic Goldblatt-Thomason theorem for classes of *models* can be found in [21] and [26, Theorem 3.15(1.)]. A Goldblatt-Thomason theorem for classes of intuitionistic frames appeared in the thesis [33], a definability theorem for classes of intuitionistic models appeared in [22].

Our approach relates to, but also, due to the coalgebraic formulation, differs from extensive work relating algebraic and frame (or topological) semantics of modal and substructural logics, using dualities and discrete dualities for distributive lattices [17,18,19], distributive lattices with operators [20,35,36,25,29], or posets [13], most of it using canonical extensions: in contrast to this approach we do not use a dual equivalence of distributive lattices and certain topological spaces, a weaker kind of adjunction between DL and posets, called logical connection, is enough. The frames, and thus the coalgebras we consider are not topological as those obtained in [29], [7] or [1], they can however be seen as non-topological analogues of those — the map from algebras to frames can be factored through topological frames.

Organisation of the paper: Section 2 is devoted to fixing the terminology and notation for monotone relations. In Section 3 we briefly recall how the *semantics of the propositional part of coalgebraic logic* is captured by an adjunction of a special kind, called *logical connection*. Relational *frames as coalgebras* are introduced in Section 4. *Complex algebras* and *canonical frames* are studied in Sections 5 and 6. Our main result: the *modal definability theorem* is proved in Section 7. We illustrate this result by examples of distributive *full Lambek calculus*, *relevance logic*, etc. We hint at possible generalizations of our approach in Section 8.

³ The usual notion of morphism for substructural frames is different — it requires equalities $a = f(x), b = f(y)$ in the back condition in 4.6. The same notion of a frame morphism as ours in the special case of frames for fuzzy logics is given in [8].

Remark on the notation we use: We work with posets and monotone relations as with categories enriched over the two-element chain 2 , see Section 2. Therefore our formulas for manipulation monotone relations use the structure of the complete Boolean algebra 2 and are to be computed there. We think that the notation will become convenient in future generalizations to enriched categories, see Section 8.

Due to space limitations we have omitted some of the proofs. §

2 Preliminaries

Recall that a *poset* \mathcal{W} is a set W equipped with a reflexive, transitive and anti-symmetric relation \leq . Instead of writing $x \leq x'$ we will often write $\mathcal{W}(x, x') = 1$ (and writing $\mathcal{W}(x, x') = 0$, if $x \leq x'$ does not hold). This is in compliance with the fact that a poset \mathcal{W} can be seen as a small category *enriched* in the two-element chain 2 . Although we will not use any machinery of enriched category theory explicitly, we find the above notation convenient in the view of further generalizations, see Section 8 below.

An *opposite* \mathcal{W}^{op} of the poset \mathcal{W} has the same set of elements as \mathcal{W} , but we put $\mathcal{W}^{op}(x, x') = \mathcal{W}(x', x)$.

Recall further that a *monotone map* $f : \mathcal{W}_1 \rightarrow \mathcal{W}_2$ consists of an assignment $x \mapsto fx$ such that, for any x and x' , the inequality $\mathcal{W}_1(x, x') \leq \mathcal{W}_2(fx, fx')$ holds in 2 . The poset of all monotone maps from \mathcal{W}_1 to \mathcal{W}_2 , with the order defined pointwise, is denoted by $[\mathcal{W}_1, \mathcal{W}_2]$. A *product* $\mathcal{W}_1 \times \mathcal{W}_2$ of posets $\mathcal{W}_1, \mathcal{W}_2$ is an order on the pairs of elements, defined pointwise. We denote by \mathcal{W}^n the product of n -many copies of \mathcal{W} with itself, writing $\mathcal{W}^0 = \mathbb{1}$ — the one-element poset.

Given posets \mathcal{W}_1 and \mathcal{W}_2 , a *monotone relation from \mathcal{W}_1 to \mathcal{W}_2* , denoted by

$$R : \mathcal{W}_1 \dashrightarrow \mathcal{W}_2$$

is a monotone map of the form $R : \mathcal{W}_1^{op} \times \mathcal{W}_2 \rightarrow 2$. We write $R(x, x') = 1$ to denote that x is related to x' . In what follows we will omit the adjective ‘monotone’ and speak just of relations. A relation of the form

$$R : \mathcal{W}^n \dashrightarrow \mathcal{W}$$

is called an *n -ary relation on \mathcal{W}* , where $n \geq 0$. For $n = 0$ we obtain

$$R : \mathbb{1} \dashrightarrow \mathcal{W}$$

and it is easy to see that such a relation corresponds to an *upper set* of \mathcal{W} , i.e., the set $U = \{x \mid Rx = 1\}$ has the property: if $x \in U$ and $x \leq x'$, then $x' \in U$.

Relations compose in the usual way: the composite of the relations $R : \mathcal{W}_1 \dashrightarrow \mathcal{W}_2$ $S : \mathcal{W}_2 \dashrightarrow \mathcal{W}_3$ is a relation $S \cdot R : \mathcal{W}_1 \dashrightarrow \mathcal{W}_3$ given by

§ The full version of the paper including missing proofs can be found at <http://math.feld.cvut.cz/velebil/research/papers.html>.

the formula

$$S \cdot R(x, z) = \bigvee_y S(y, z) \wedge R(x, y)$$

For every poset \mathcal{W} , the *identity relation* $id_{\mathcal{W}} : \mathcal{W} \dashv\vdash \mathcal{W}$ is defined by putting $id_{\mathcal{W}}(x, x') = 1$ iff $x \leq x'$. Hence it is consistent to write \mathcal{W} instead of $id_{\mathcal{W}}$.

It is easy to see that the above composition is associative and that it has identity relations as units, hence we obtain a category (enriched in posets) of posets and relations. The above definitions are specializations of the theory of *profunctors* (also *distributors*, or, *modules*), known from enriched category theory. See, for example, [37] for more details.

3 The logical connection

The semantics of the propositional part of the language, i.e., of the language

$$\varphi ::= p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \quad (3)$$

where p ranges through a poset \mathbf{At} of atomic propositions and \wedge and \vee are tied by the distributive law, will be given by a *logical connection* of the category \mathbf{Pos} of *posets* and *monotone maps* and the category \mathbf{DL} of *distributive lattices* and *lattice morphisms*. The logical connection

$$\mathit{Stone} \dashv \mathit{Pred} : \mathbf{Pos}^{op} \longrightarrow \mathbf{DL} \quad (4)$$

is given by the two-element chain 2 as a *schizophrenic object*. Recall how the above connection works (we refer to [31] or [27] for more details on logical connections):

- (i) Pred sends a poset \mathcal{W} to the distributive lattice $([\mathcal{W}, 2], \cap, \cup)$ of uppersets on \mathcal{W} . A monotone map f is sent to $[f, 2] : U \mapsto U \cdot f$.
For a poset \mathcal{W} , the distributive lattice $\mathit{Pred}(\mathcal{W})$ is to be considered as the “distributive lattice of truth-distributions on \mathcal{W} ”.
- (ii) For a distributive lattice \mathcal{A} , $\mathit{Stone}(\mathcal{A})$ is the poset $\mathbf{DL}(\mathcal{A}, 2)$ of *prime filters* on \mathcal{A} . The mapping $\mathit{Stone}(h)$ is given by composition: a prime filter F is sent to the prime filter $F \cdot h$.
The poset $\mathit{Stone}(\mathcal{A})$ is the “Stone space” of the distributive lattice \mathcal{A} .
- (iii) The unit $\eta_{\mathcal{A}} : \mathcal{A} \longrightarrow [\mathbf{DL}(\mathcal{A}, 2), 2]$ is the lattice homomorphism sending x in \mathcal{A} to the uperset of all prime filters on \mathcal{A} that contain x .
- (iv) The counit $\varepsilon_{\mathcal{W}} : \mathcal{W} \longrightarrow \mathbf{DL}([\mathcal{W}, 2], 2)$ is the monotone map sending x in \mathcal{W} to the prime filter of those uppersets on \mathcal{W} that contain x .

The semantics of the propositional language (3) is given by the logical connection (4), together with another adjunction

$$F \dashv U : \mathbf{DL} \longrightarrow \mathbf{Pos} \quad (5)$$

where U denotes the obvious forgetful functor and F sends a poset \mathcal{X} to the *free distributive lattice* on \mathcal{X} . More in detail, the semantics is given as follows:

- (i) Fix a poset At of atomic propositions. The distributive lattice $F(\text{At})$ is then the *Lindenbaum-Tarski* algebra of formulas.
- (ii) Observe that $U(\text{Pred}(\mathcal{W})) = [\mathcal{W}, 2]$, for every poset \mathcal{W} . Hence, due to the adjunction $F \dashv U$, monotone maps of the form $\text{val} : \text{At} \rightarrow [\mathcal{W}, 2]$ are in bijective correspondence with lattice morphisms $\|-\|_{\text{val}} : F(\text{At}) \rightarrow \text{Pred}(\mathcal{W})$.

Of course, as the notation suggests, the monotone map val is the *valuation* of atomic propositions, assigning to every p the upperset $\text{val}(p)$ of those x 's in \mathcal{W} , where p is valid. The lattice homomorphism $\|-\|_{\text{val}}$ is then the free extension of the valuation val . It can be described inductively as follows:

$$\|p\|_{\text{val}} = \text{val}(p), \quad \|\varphi_1 \wedge \varphi_2\|_{\text{val}} = \|\varphi_1\|_{\text{val}} \cap \|\varphi_2\|_{\text{val}}, \quad \|\varphi_1 \vee \varphi_2\|_{\text{val}} = \|\varphi_1\|_{\text{val}} \cup \|\varphi_2\|_{\text{val}}$$

We will later add more connectives (fusion-like, implication-like and negation-like) but we are going to consider them as *modal operators* on distributive lattices. In fact, as we will see, such extension of the language will yield extensions of the above two functors *Pred* and *Stone*.

4 Relational frames as coalgebras

We define structures that we call (relational) frames for the language of the type (2). Frames will consist of a poset of states and various relations reflecting the syntax of “modalities” of the language, compare to frames in [32]. We prove that frames are exactly the coalgebras for a certain endofunctor of the category of posets.

Notation 4.1 We will introduce the following “vector” conventions: for a relation $P : \mathcal{W}^n \multimap \mathcal{W}$ we will write $P(\vec{x}; x)$ instead of $P(x_0, \dots, x_{n-1}; x)$. For $P : \mathcal{W} \multimap (\mathcal{W}^{op})^l \times \mathcal{W}$ we will write $P(x; \vec{y}, z)$ instead of $P(x; y_0, \dots, y_{l-1}, z)$. Analogously we will write $\mathcal{W}_2(\vec{a}, f\vec{x})$ instead of $\mathcal{W}_2(a_0, fx_0) \wedge \dots \wedge \mathcal{W}_2(a_{n-1}, fx_{n-1})$, etc.

Definition 4.2 A relational frame for the language (2) is a quadruple $\mathbb{W} = (\mathcal{W}, P_{\heartsuit}, P_{\circ}, P_{\sim})$, consisting of a poset \mathcal{W} , and relations

$$P_{\heartsuit} : \mathcal{W}^n \multimap \mathcal{W}, \quad P_{\circ} : \mathcal{W} \multimap (\mathcal{W}^l)^{op} \times \mathcal{W}, \quad P_{\sim} : \mathcal{W} \multimap \mathcal{W}^{op}$$

A morphism from $\mathbb{W}_1 = (\mathcal{W}_1, P_{\heartsuit}^1, P_{\circ}^1, P_{\sim}^1)$ to $\mathbb{W}_2 = (\mathcal{W}_2, P_{\heartsuit}^2, P_{\circ}^2, P_{\sim}^2)$ is a monotone map $f : \mathcal{W}_1 \rightarrow \mathcal{W}_2$ such that the following three equations hold:

$$P_{\heartsuit}^2(\vec{a}; fy) = \bigvee_{\vec{x}} \mathcal{W}_2(\vec{a}, f\vec{x}) \wedge P_{\heartsuit}^1(\vec{x}; y) \quad (6)$$

$$P_{\circ}^2(fx; \vec{b}, c) = \bigvee_{\vec{y}, z} \mathcal{W}_2(\vec{b}, f\vec{y}) \wedge \mathcal{W}_2(fz, c) \wedge P_{\circ}^1(x; \vec{y}, z) \quad (7)$$

$$P_{\sim}^2(fx; b) = \bigvee_y \mathcal{W}_2(b, fy) \wedge P_{\sim}^1(x; y) \quad (8)$$

We write $f : \mathbb{W}_1 \rightarrow \mathbb{W}_2$ to indicate that f is a morphism of relational frames.

Remark 4.3 We have not defined semantics in a relational frame yet, but the following intuitions about the “meaning” of the individual relations P_{\heartsuit} , P_{\multimap} and P_{\sim} on \mathscr{W} might be useful (see Notation 4.1).

- (i) $P_{\heartsuit}(\vec{x}; y) = 1$ holds, only if $\vec{x} \Vdash \vec{\varphi}$ implies $y \Vdash \heartsuit\vec{\varphi}$.
- (ii) $P_{\multimap}(x; \vec{y}, z) = 1$ holds, only if $x \Vdash \vec{\varphi} \multimap \psi$ and $\vec{y} \Vdash \vec{\varphi}$ imply $z \Vdash \psi$.
- (iii) $P_{\sim}(x; y) = 1$ holds, only if $y \Vdash \varphi$ implies $x \nVdash \sim\varphi$.

See Remark 5.5 below for precisising the above intuitions.

Example 4.4 A relational frame \mathbb{W} for the language (1) consists of a poset \mathscr{W} , together with fusion-like relations $P_{\otimes} : \mathscr{W} \times \mathscr{W} \multimap \mathscr{W}$, $P_e : \mathbb{1} \multimap \mathscr{W}$, and implication-like relations $P_{\rightarrow} : \mathscr{W} \multimap \mathscr{W}^{op} \times \mathscr{W}$ and $P_{\leftarrow} : \mathscr{W} \multimap \mathscr{W}^{op} \times \mathscr{W}$.

Let us stress that the relations P_{\otimes} , P_e , P_{\rightarrow} and P_{\leftarrow} are (as of yet) arbitrary. When one needs special properties as, for example, the frame to be the model of a distributive full Lambek calculus (for such frames see [32]), one needs to invoke modal definability theorem. This is shown in Example 7.7 below.

Example 4.5 Relational frames for the language $\wedge, \vee, \otimes, \rightarrow, e$ and \sim of *relevance logic*, see [12], are posets \mathscr{W} equipped with relations $P_{\otimes} : \mathscr{W} \times \mathscr{W} \multimap \mathscr{W}$, $P_e : \mathbb{1} \multimap \mathscr{W}$, $P_{\rightarrow} : \mathscr{W} \multimap \mathscr{W}^{op} \times \mathscr{W}$ and $P_{\sim} : \mathscr{W} \multimap \mathscr{W}^{op}$. The above relations are as of yet arbitrary. Frames for various classes of relevance logic are modally definable, see Remark 7.8 below.

Remark 4.6 It is very easy to prove that the above equations (6)–(8) can be “split” into six inequalities, giving us the *back & forth* description of morphisms for fusion-like, implication-like and negation-like connectives. More precisely:

- (i) The equation (6) is equivalent to the conjunction of the following two inequalities

$$P_{\heartsuit}^1(\vec{x}; y) \leq P_{\heartsuit}^2(f\vec{x}; fy) \quad (9)$$

$$P_{\heartsuit}^2(\vec{a}; fy) \leq \bigvee_{\vec{x}} \mathscr{W}_2(\vec{a}, f\vec{x}) \wedge P_{\heartsuit}^1(\vec{x}; y) \quad (10)$$

- (ii) The equation (7) is equivalent to the conjunction of the following two inequalities

$$P_{\multimap}^1(x; \vec{y}, z) \leq P_{\multimap}^2(fx; f\vec{y}, fz) \quad (11)$$

$$P_{\multimap}^2(fx; \vec{b}, c) \leq \bigvee_{\vec{y}, z} \mathscr{W}_2(\vec{b}, f\vec{y}) \wedge \mathscr{W}_2(fz, c) \wedge P_{\multimap}^1(x; \vec{y}, z) \quad (12)$$

- (iii) The equation (8) is equivalent to the conjunction of the following two inequalities

$$P_{\sim}^1(x; y) \leq P_{\sim}^2(fx; fy) \quad (13)$$

$$P_{\sim}^2(fx; b) \leq \bigvee_y \mathscr{W}_2(b, fy) \wedge P_{\sim}^1(x; y) \quad (14)$$

We define now three functors

$$T_{\heartsuit} : \text{Pos} \longrightarrow \text{Pos}, \quad T_{\circ} : \text{Pos} \longrightarrow \text{Pos}, \quad T_{\sim} : \text{Pos} \longrightarrow \text{Pos}$$

and prove that their product $T = T_{\heartsuit} \times T_{\circ} \times T_{\sim}$ gives rise to relational frames and their morphisms. Namely: frames are T -coalgebras and frame morphisms are T -coalgebra morphisms.

Definition 4.7

- (i) The functor T_{\heartsuit} sends \mathscr{W} to the poset $[(\mathscr{W}^n)^{op}, 2]$ of lowersets on \mathscr{W}^n . For a monotone map $f : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$, the map $T_{\heartsuit}(f)$ sends $\vec{L} : (\mathscr{W}_1^n)^{op} \longrightarrow 2$ to

$$\vec{b} \mapsto \bigvee_{\vec{x}} \mathscr{W}_2(\vec{b}, f\vec{x}) \wedge \vec{L}\vec{x}$$

- (ii) The functor T_{\circ} sends \mathscr{W} to the poset $[(\mathscr{W}^1)^{op} \times \mathscr{W}, 2]^{op}$. For a monotone map $f : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$, the map $T_{\circ}(f)$ sends $X : (\mathscr{W}_1^1)^{op} \times \mathscr{W}_1 \longrightarrow 2$ to

$$(\vec{b}, c) \mapsto \bigvee_{\vec{y}, z} \mathscr{W}_2(\vec{b}, f\vec{y}) \wedge \mathscr{W}_2(fz, c) \wedge X(\vec{y}, z)$$

- (iii) The functor T_{\sim} sends \mathscr{W} to the poset $[\mathscr{W}^{op}, 2]^{op}$. For a monotone map $f : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$, the map $T_{\sim}(f)$ sends $X : \mathscr{W}_1^{op} \longrightarrow 2$ to

$$b \mapsto \bigvee_y \mathscr{W}_2(b, fy) \wedge Xy$$

Proposition 4.8 Put $T = T_{\heartsuit} \times T_{\circ} \times T_{\sim}$. The category of relational frames and their morphisms is isomorphic to the category Pos_T of T -coalgebras and their morphisms.

Proof.

- (i) To give a monotone map $\gamma : \mathscr{W} \longrightarrow T(\mathscr{W})$ is to give three monotone maps $\gamma_{\heartsuit} : \mathscr{W} \longrightarrow T_{\heartsuit}(\mathscr{W})$, $\gamma_{\circ} : \mathscr{W} \longrightarrow T_{\circ}(\mathscr{W})$ and $\gamma_{\sim} : \mathscr{W} \longrightarrow T_{\sim}(\mathscr{W})$. Each of the three maps, however, can be uncurried to produce monotone maps $P_{\heartsuit} : (\mathscr{W}^n)^{op} \times \mathscr{W} \longrightarrow 2$, $P_{\circ} : \mathscr{W}^{op} \times (\mathscr{W}^1)^{op} \times \mathscr{W} \longrightarrow 2$ and $P_{\sim} : \mathscr{W}^{op} \times \mathscr{W}^{op} \longrightarrow 2$. To conclude: T -coalgebras are exactly the relational frames.

- (ii) To give a monotone map $f : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ such that the square

$$\begin{array}{ccc} \mathscr{W}_1 & \xrightarrow{\gamma_1} & T(\mathscr{W}_1) \\ f \downarrow & & \downarrow T(f) \\ \mathscr{W}_2 & \xrightarrow{\gamma_2} & T(\mathscr{W}_2) \end{array}$$

commutes, is, by Definition 4.7, to give a monotone map f such that equations (6)–(8) hold. To conclude: coalgebra homomorphisms are exactly the morphisms of relational frames. \square

5 Complex algebras

The complex algebra $Pred^\sharp(\mathbb{W})$ of the frame \mathbb{W} will be a distributive lattice $Pred(\mathcal{W})$, equipped with extra operators \heartsuit , \multimap and \sim .

We prove that taking a complex algebra defines a functor $Pred^\sharp$ from the (opposite of the) category of relational frames and their morphisms to the category $DL_{\heartsuit, \multimap, \sim}$ of distributive lattices equipped with extra operations. Moreover, this construction extends the predicate functor $Pred : \mathbf{Pos}^{op} \rightarrow \mathbf{DL}$ in the sense that the square

$$\begin{array}{ccc}
 (\mathbf{Pos}_T)^{op} & \xrightarrow{Pred^\sharp} & \mathbf{DL}_{\heartsuit, \multimap, \sim} \\
 (V_T)^{op} \downarrow & & \downarrow U_{\heartsuit, \multimap, \sim} \\
 \mathbf{Pos}^{op} & \xrightarrow{Pred} & \mathbf{DL}
 \end{array} \quad (15)$$

commutes. Above, $V_T : \mathbf{Pos}_T \rightarrow \mathbf{Pos}$ is the forgetful functor sending a coalgebra (\mathcal{W}, γ) to the poset \mathcal{W} .

Definition 5.1 *The category $DL_{\heartsuit, \multimap, \sim}$ is defined as follows:*

- (i) *Objects are distributive lattices $\mathcal{A} = (\mathcal{A}_o, \wedge, \vee)$ (where \mathcal{A}_o denotes the underlying poset), together with monotone maps*

$$\llbracket \heartsuit \rrbracket_{\mathcal{A}} : \mathcal{A}_o^n \rightarrow \mathcal{A}_o, \quad \llbracket \multimap \rrbracket_{\mathcal{A}} : (\mathcal{A}_o^l)^{op} \times \mathcal{A}_o \rightarrow \mathcal{A}_o, \quad \llbracket \sim \rrbracket_{\mathcal{A}} : \mathcal{A}_o^{op} \rightarrow \mathcal{A}_o$$

called the interpretations of \heartsuit , \multimap and \sim . We will usually omit the brackets $\llbracket - \rrbracket_{\mathcal{A}}$ and denote $(\mathcal{A}, \heartsuit, \multimap, \sim)$ by \mathbb{A} .

The operations are required to satisfy the following axioms, for each $0 \leq i \leq n$:

$$\begin{aligned}
 \heartsuit(\dots, x_i \vee x'_i, \dots) &= \heartsuit(\dots, x_i, \dots) \vee \heartsuit(\dots, x'_i, \dots) \\
 (\dots, x_i \vee x'_i, \dots) \multimap y &= ((\dots, x_i, \dots) \multimap y) \wedge ((\dots, x'_i, \dots) \multimap y) \\
 \vec{x} \multimap (y_1 \wedge y_2) &= (\vec{x} \multimap y_1) \wedge (\vec{x} \multimap y_2) \\
 \sim(x_1 \vee x_2) &= \sim x_1 \wedge \sim x_2
 \end{aligned}$$

- (ii) *A morphism from \mathbb{A}_1 to \mathbb{A}_2 is a lattice morphism $h : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ preserving the additional operations \heartsuit , \multimap and \sim on the nose.*

The obvious underlying functor will be denoted by $U_{\heartsuit, \multimap, \sim} : DL_{\heartsuit, \multimap, \sim} \rightarrow \mathbf{DL}$.

Remark 5.2 It is clear that $DL_{\heartsuit, \multimap, \sim}$ is a finitary variety over \mathbf{Pos} in the sense of categorical universal algebra. More precisely: the composite $U \cdot U_{\heartsuit, \multimap, \sim} : DL_{\heartsuit, \multimap, \sim} \rightarrow \mathbf{Pos}$ of the obvious forgetful functors is a monadic functor. In particular, the forgetful functor $U \cdot U_{\heartsuit, \multimap, \sim} : DL_{\heartsuit, \multimap, \sim} \rightarrow \mathbf{Pos}$ has a left adjoint, hence there also exists a left adjoint $F_{\heartsuit, \multimap, \sim} : \mathbf{DL} \rightarrow DL_{\heartsuit, \multimap, \sim}$ to $U_{\heartsuit, \multimap, \sim}$. Thus, given a poset \mathbf{At} , we can form $F_{\heartsuit, \multimap, \sim}(F(\mathbf{At}))$. This is the *Lindenbaum-Tarski algebra* of formulas for the language (2) and we denote it by $\mathcal{L}(\mathbf{At})$.

Definition 5.3 The complex algebra $Pred^\sharp(\mathbb{W}) = ([\mathscr{W}, 2], \cap, \cup, \heartsuit, \dashv, \sim)$ is defined as follows:

- (i) Given a vector \vec{U} of uppersets U_0, \dots, U_{n-1} , the upperset $\heartsuit\vec{U}$ is defined by the formula

$$y \mapsto \bigvee_{\vec{x}} \vec{U}\vec{x} \wedge P_{\heartsuit}(\vec{x}; y)$$

- (ii) Given a vector \vec{U} of uppersets U_0, \dots, U_{l-1} , and an upperset W , the upperset $\vec{U} \dashv W$ is defined by the formula

$$x \mapsto \bigwedge_{\vec{y}, z} \vec{U}\vec{y} \wedge P_{\dashv}(x; \vec{y}, z) \Rightarrow Wz$$

- (iii) Given an upperset U , the upperset $\sim U$ is defined by the formula

$$x \mapsto \bigwedge_y P_{\sim}(x; y) \Rightarrow \neg Uy$$

where the \neg sign is negation in $\mathbb{2}$.

The following result is easy to prove, when one uses the back & forth description of morphism of frames, see Remark 4.6.

Proposition 5.4 The assignment $\mathbb{W} \mapsto Pred^\sharp(\mathbb{W})$ can be extended to a functor from $(Pos_T)^{op}$ to $DL_{\heartsuit, \dashv, \sim}$. Moreover, the square (15) commutes.

Proof. It is easy to verify that, given a frame \mathbb{W} , the algebra $Pred^\sharp(\mathbb{W})$ is an object of $DL_{\heartsuit, \dashv, \sim}$.

For a frame morphism $f : \mathbb{W}_1 \rightarrow \mathbb{W}_2$, put $Pred^\sharp(f)$ to be the mapping $[f, 2] : [\mathscr{W}_2, 2] \rightarrow [\mathscr{W}_1, 2]$. We verify that the three operations are preserved on the nose:

- (i) The commutativity of the square

$$\begin{array}{ccc} [\mathscr{W}_2, 2]^n & \xrightarrow{[f, 2]^n} & [\mathscr{W}_1, 2]^n \\ \heartsuit \downarrow & & \downarrow \heartsuit \\ [\mathscr{W}_2, 2] & \xrightarrow{[f, 2]} & [\mathscr{W}_1, 2] \end{array}$$

is the requirement that the equality

$$\bigvee_{\vec{a}} \vec{U}\vec{a} \wedge P_{\heartsuit}^2(\vec{a}; fy) = \bigvee_{\vec{x}} \vec{U}f\vec{x} \wedge P_{\heartsuit}^1(\vec{x}; y)$$

holds for every y . The inequality \geq is obvious: put $\vec{a} = f\vec{x}$ and use that $P_{\heartsuit}^1(\vec{x}; y) \leq P_{\heartsuit}^2(f\vec{x}; fy)$ holds, see (9). The converse inequality follows from the inequality (10).

(ii) The commutativity of the square

$$\begin{array}{ccc} ([\mathcal{W}_2, 2]^l)^{op} \times [\mathcal{W}_2, 2] & \xrightarrow{([f, 2]^l)^{op} \times [f, 2]} & ([\mathcal{W}_1, 2]^l)^{op} \times [\mathcal{W}_1, 2] \\ \downarrow \neg & & \downarrow \neg \\ [\mathcal{W}_2, 2] & \xrightarrow{[f, 2]} & [\mathcal{W}_1, 2] \end{array}$$

is the requirement that the equality

$$\bigwedge_{\vec{b}, c} \vec{U}\vec{b} \wedge P_{\neg}^2(fx; \vec{b}, c) \Rightarrow Wc = \bigwedge_{\vec{y}, z} \vec{U}f\vec{y} \wedge P_{\neg}^1(x; \vec{y}, z) \Rightarrow Wfz$$

holds for every x . The inequality \leq follows from $P_{\neg}^1(x; \vec{y}, z) \leq P_{\neg}^2(fx; f\vec{y}, fz)$, see (11). For the converse inequality, use inequality (12).

(iii) The commutativity of the square

$$\begin{array}{ccc} [\mathcal{W}_2, 2]^{op} & \xrightarrow{[f, 2]^{op}} & [\mathcal{W}_1, 2]^{op} \\ \sim \downarrow & & \downarrow \sim \\ [\mathcal{W}_2, 2] & \xrightarrow{[f, 2]} & [\mathcal{W}_1, 2] \end{array}$$

is the requirement that the equality

$$\bigwedge_b P_{\sim}^2(fx; b) \Rightarrow \neg Ub = \bigwedge_y P_{\sim}^1(x; y) \Rightarrow \neg Ufy$$

holds for every x . The inequality \leq follows from inequality (13). For the converse inequality, use inequality (14). \square

Remark 5.5 The square (15) allows us to give *semantics* of the language. More precisely, we saw in Section 3 that the adjunction $F \dashv U : \mathbf{DL} \rightarrow \mathbf{Pos}$, together with $\text{Stone} \dashv \text{Pred} : \mathbf{Pos}^{op} \rightarrow \mathbf{DL}$, takes care of the semantics $\|-\|_{\text{val}}$ of the propositional part of the logic.

The adjunction $F_{\heartsuit, \neg, \sim} \dashv U_{\heartsuit, \neg, \sim} : \mathbf{DL}_{\heartsuit, \neg, \sim} \rightarrow \mathbf{DL}$, together with square (15), allow us to define, for every frame \mathbb{W} , a *semantics morphism*

$$\|-\|_{\text{val}} : \mathcal{L}(\text{At}) \rightarrow \text{Pred}^{\#}(\mathbb{W})$$

in $\mathbf{DL}_{\heartsuit, \neg, \sim}$ as the transpose under the composite adjunction

$$\mathbf{DL}_{\heartsuit, \neg, \sim} \xleftarrow{F_{\heartsuit, \neg, \sim}} \mathbf{DL} \xleftarrow{F} \mathbf{Pos} \\ \xrightarrow{U_{\heartsuit, \neg, \sim}} \mathbf{DL} \xrightarrow{U} \mathbf{Pos}$$

of a *valuation* $\text{val} : \text{At} \rightarrow [\mathcal{W}, 2]$.

It is possible to give an inductive description of $\|-\|_{\text{val}}$. Namely: the equations

$$\begin{aligned} \|p\|_{\text{val}} &= \text{val}(p), & \|\varphi_1 \wedge \varphi_2\|_{\text{val}} &= \|\varphi_1\|_{\text{val}} \cap \|\varphi_2\|_{\text{val}}, \\ \|\varphi_1 \vee \varphi_2\|_{\text{val}} &= \|\varphi_1\|_{\text{val}} \cup \|\varphi_2\|_{\text{val}}, & \|\heartsuit \vec{\varphi}\|_{\text{val}} &= \heartsuit \|\vec{\varphi}\|_{\text{val}}, \\ \|\vec{\varphi} \multimap \psi\|_{\text{val}} &= \|\vec{\varphi}\|_{\text{val}} \multimap \|\psi\|_{\text{val}}, & \|\sim \varphi\|_{\text{val}} &= \sim \|\varphi\|_{\text{val}} \end{aligned}$$

hold. Above, the symbols \heartsuit , \multimap and \sim on the right-hand sides are to be interpreted as the operations in the complex algebra $\text{Pred}^\sharp(\mathbb{W})$.

Let us call the pair (\mathbb{W}, val) , consisting of a frame and a valuation, a *model*. Then the morphism $\|-\|_{\text{val}}$ defines the notion of *local truth* in the model (\mathbb{W}, val) — we write $x \Vdash_{\mathbb{W}, \text{val}} \varphi$, if x belongs to the upperset $\|\varphi\|_{\text{val}}$, or, equivalently, if $\|\varphi\|_{\text{val}} x = 1$. If rewritten in terms of \Vdash , the above equations give the familiar inductive definition of validity. Namely (omitting the obvious cases of atomic propositions and \wedge and \vee):

- (i) $x \Vdash_{\mathbb{W}, \text{val}} \heartsuit \vec{\varphi}$ holds iff there exists \vec{y} such that both $\vec{y} \Vdash \vec{\varphi}$ and $P_{\heartsuit}(\vec{y}; x)$ hold.
- (ii) $x \Vdash_{\mathbb{W}, \text{val}} \vec{\varphi} \multimap \psi$ holds iff for all \vec{y} and z such that $\vec{y} \Vdash \vec{\varphi}$ and $P_{\multimap}(x; \vec{y}, z)$ hold, $z \Vdash \psi$ holds.
- (iii) $x \Vdash_{\mathbb{W}, \text{val}} \sim \varphi$ iff for all y such that $P_{\sim}(x; y)$ holds, $y \not\Vdash \varphi$ holds.

6 Canonical relational frames

The assignment of the *canonical frame* $\text{Stone}^\sharp(\mathbb{A})$ to an object \mathbb{A} of $\text{DL}_{\heartsuit, \multimap, \sim}$ is, in a way, dual to the formation of complex algebras. We prove below that $\mathbb{A} \mapsto \text{Stone}^\sharp(\mathbb{A})$ is functorial and that the square

$$\begin{array}{ccc} \text{DL}_{\heartsuit, \multimap, \sim} & \xrightarrow{\text{Stone}^\sharp} & (\text{Pos}_T)^{op} \\ U_{\heartsuit, \multimap, \sim} \downarrow & & \downarrow (V_T)^{op} \\ \text{DL} & \xrightarrow{\text{Stone}} & \text{Pos}^{op} \end{array} \quad (16)$$

commutes.

Definition 6.1 Suppose $\mathbb{A} = (\mathcal{A}, \heartsuit, \multimap, \sim)$ is in $\text{DL}_{\heartsuit, \multimap, \sim}$. Define $\text{Stone}^\sharp(\mathbb{A})$ as follows:

- (i) The underlying poset of $\text{Stone}^\sharp(\mathbb{A})$ is the poset $\text{DL}(\mathcal{A}, 2)$ of prime filters on the distributive lattice \mathcal{A} .
- (ii) The relation P_{\heartsuit} is defined as follows:

$$P_{\heartsuit}(\vec{F}; G) = \bigwedge_{\vec{x}} \vec{F} \vec{x} \Rightarrow G(\heartsuit \vec{x})$$

- (iii) The relation P_{\multimap} is defined as follows:

$$P_{\multimap}(F; \vec{G}, H) = \bigwedge_{\vec{x}, y} F(\vec{x} \multimap y) \wedge \vec{G} \vec{x} \Rightarrow Hy$$

(iv) The relation P_{\sim} is defined as follows:

$$P_{\sim}(F; G) = \bigwedge_x Gx \Rightarrow \neg F(\sim x)$$

where the \neg sign is the negation in $\mathcal{2}$.

The above definitions clearly make sense if we work with mere *upper sets* in lieu of prime filters. We will need the following three technical results that slightly generalize the results originating in the work on relevance logic, see Section 6 of [11].

Lemma 6.2 (Squeeze Lemma for \heartsuit) Suppose $P_{\heartsuit}(\vec{F}'; G) = 1$ holds, where \vec{F}' is a vector of filters and G a prime filter. Then there is a vector \vec{F} of prime filters that extends \vec{F}' and $P_{\heartsuit}(\vec{F}; G) = 1$.

Lemma 6.3 (Squeeze Lemma for \multimap) Suppose $P_{\multimap}(F; \vec{G}', \bar{I}') = 1$, where F is a prime filter, \vec{G}' is a vector of filters and \bar{I}' is a complement of an ideal I' . Then there exists a vector \vec{G} of prime filters such that \vec{G} extends \vec{G}' and a prime ideal I that extends I' and $P_{\multimap}(F; \vec{G}, \bar{I}) = 1$, where \bar{I} denotes the complement of I .

Lemma 6.4 (Squeeze Lemma for \sim) Suppose $P_{\sim}(F; G') = 1$, where F is a prime filter and G' is a filter. Then there exists a prime filter G extending G' such that $P_{\sim}(F; G) = 1$.

The above three lemmata allow us to prove that the computation of a canonical frame is a functorial process.

Proposition 6.5 The assignment $\mathbb{A} \mapsto \text{Stone}^{\sharp}(\mathbb{A})$ can be extended to a functor from $\text{DL}_{\heartsuit, \multimap, \sim}$ to $(\text{Pos}_T)^{\text{op}}$. Moreover, the square (16) commutes.

Proof. Given $h : \mathbb{A}_1 \rightarrow \mathbb{A}_2$, we define $\text{Stone}^{\sharp}(h)$ as $\text{DL}(h, \mathcal{2}) : \text{DL}(\mathcal{A}_2, \mathcal{2}) \rightarrow \text{DL}(\mathcal{A}_1, \mathcal{2})$. We only need to prove that equations (6)–(8) are satisfied. For the purposes of better readability we denote $[h, \mathcal{2}]$ by h^{\dagger} in what follows.

(i) The required equality

$$P_{\heartsuit}^1(\vec{K}; h^{\dagger}G) = \bigvee_{\vec{F}} \text{DL}(\mathcal{A}_1, \mathcal{2})(\vec{K}, h^{\dagger}\vec{F}) \wedge P_{\heartsuit}^2(\vec{F}; G)$$

can be rewritten, using the definition of h^{\dagger} , to the equation

$$P_{\heartsuit}^1(\vec{K}; Gh) = \bigvee_{\vec{F}} \text{DL}(\mathcal{A}_1, \mathcal{2})(\vec{K}, \vec{F}h) \wedge P_{\heartsuit}^2(\vec{F}; G)$$

We prove inequalities (9) and (10):

(a) To prove $P_{\heartsuit}^2(\vec{F}; G) \leq P_{\heartsuit}^1(\vec{F}h; Gh)$, suppose $\vec{F}hx = 1$. Then $G(\heartsuit(hx)) = Gh(\heartsuit x) = 1$ and we are done.

(b) We prove $P_{\heartsuit}^1(\vec{K}; Gh) \leq \bigvee_{\vec{F}} \text{DL}(\mathcal{A}_1, 2)(\vec{K}, \vec{F}h) \wedge P_{\heartsuit}^2(\vec{F}; G)$.

Define a vector \vec{K}' of *filters* on \mathcal{A}_2 by putting

$$\vec{K}'\vec{a} = \bigvee_{\vec{x}} \mathcal{A}_2(h\vec{x}, \vec{a}) \wedge \vec{K}\vec{x}$$

We prove $P_{\heartsuit}^2(\vec{K}'; G) = 1$, supposing $P_{\heartsuit}^1(\vec{K}; Gh) = 1$. To that end, suppose $\vec{K}'\vec{a} = 1$ and choose \vec{x} such that $\mathcal{A}_2(h\vec{x}, \vec{a}) \wedge \vec{K}\vec{x} = 1$. Then $Gh(\heartsuit\vec{x}) = G(\heartsuit(h\vec{x})) = 1$, hence $G(\heartsuit\vec{a}) = 1$, since \heartsuit is monotone.

Now use Lemma 6.2 to find a vector \vec{F} of *prime filters* such that \vec{F} extends \vec{K}' and $P_{\heartsuit}^2(\vec{F}; G) = 1$ holds. It remains to prove the equality $\text{DL}(\mathcal{A}_1, 2)(\vec{K}, \vec{F}h) = 1$. This follows immediately from the fact that \vec{F} extends \vec{K}' : if $\vec{K}\vec{x} = 1$, then $\vec{K}'(h\vec{x}) = 1$, hence $\vec{F}h\vec{x} = 1$.

(ii) The required equality

$$P_{\circlearrowleft}^1(h^\dagger F; \vec{L}, M) = \bigvee_{\vec{G}, H} \text{DL}(\mathcal{A}_1, 2)(\vec{L}, h^\dagger \vec{G}) \wedge \text{DL}(\mathcal{A}_1, 2)(h^\dagger H, M) \wedge P_{\circlearrowleft}^2(F; \vec{G}, H)$$

can be rewritten to the equality

$$P_{\circlearrowleft}^1(Fh; \vec{L}, M) = \bigvee_{\vec{G}, H} \text{DL}(\mathcal{A}_1, 2)(\vec{L}, \vec{G}h) \wedge \text{DL}(\mathcal{A}_1, 2)(Hh, M) \wedge P_{\circlearrowleft}^2(F; \vec{G}, H)$$

We prove inequalities (11) and (12):

(a) For proving the inequality $P_{\circlearrowleft}^2(F; \vec{G}, H) \leq P_{\circlearrowleft}^1(Fh; \vec{G}h, Hh)$, assume $P_{\circlearrowleft}^2(F; \vec{G}, H) = 1$. If $Fh(\vec{x} \multimap y) \wedge \vec{G}h\vec{x} = F(h\vec{x} \multimap hy) \wedge \vec{G}h\vec{x} = 1$, then $Hhy = 1$, which was to be proved.

(b) We prove the inequality

$$P_{\circlearrowleft}^1(Fh; \vec{L}, M) \leq \bigvee_{\vec{G}, H} \text{DL}(\mathcal{A}_1, 2)(\vec{L}, \vec{G}h) \wedge \text{DL}(\mathcal{A}_1, 2)(Hh, M) \wedge P_{\circlearrowleft}^2(F; \vec{G}, H)$$

Define

$$\vec{G}'\vec{b} = \bigvee_{\vec{y}} \mathcal{A}_2(h\vec{y}, \vec{b}) \wedge \vec{L}\vec{y}, \quad I'c = \bigvee_z \mathcal{A}_2(c, hz) \wedge \neg Mz$$

and observe that \vec{G}' is a vector of filters and I' is an ideal. Moreover, the complement $\overline{I'}$ of I' is given by the formula

$$\overline{I'}c = \bigwedge_z \mathcal{A}_2(c, hz) \Rightarrow Mz$$

We will prove that $P_{\circlearrowleft}^2(F; \vec{G}', \overline{I'}) = 1$, if we suppose $P_{\circlearrowleft}^1(Fh; \vec{L}, M) = 1$.

To that end, suppose $F(\vec{b} \multimap c) \wedge \vec{G}'\vec{b} = 1$ and suppose z is such that $\mathcal{A}_2(c, hz) = 1$ holds. We need to prove $Mz = 1$.

Pick \vec{y} witnessing $\vec{G}'\vec{b} = 1$. Then $F(h\vec{y} \multimap hz) = Fh(\vec{y} \multimap z) = 1$ and $\vec{L}\vec{y} = 1$. Therefore $Mz = 1$, since we assumed $P_{\multimap}^1(Fh; \vec{L}, M) = 1$.

By Lemma 6.3 there exist \vec{G} and I such that \vec{G} is a vector of prime filters extending \vec{G}' , I is a prime ideal extending I' , and $P_{\multimap}^2(F, \vec{G}, \bar{I}) = 1$ holds. Since a complement of a prime ideal is a prime filter, we can put $H = \bar{I}$.

It remains to show that $\vec{G}h$ extends L and Hh is extended by M . Since $\vec{G}'h$ clearly extends L , so does $\vec{G}h$ (use that \vec{G} extends \vec{G}').

Since $\bar{I}h$ is extended by M , so is Hh . This follows from the fact that I extends I' .

(iii) The required equality

$$P_{\sim}^1(h^\dagger F; L) = \bigvee_G \text{DL}(\mathcal{A}_1, 2)(L, h^\dagger G) \wedge P_{\sim}^2(F; G)$$

can be rewritten to the equality

$$P_{\sim}^1(Fh; L) = \bigvee_G \text{DL}(\mathcal{A}_1, 2)(L, Gh) \wedge P_{\sim}^2(F; G)$$

We prove inequalities in (13) and (14):

- (a) To prove the inequality $P_{\sim}^2(F; G) \leq P_{\sim}^1(Fh; Gh)$, suppose that $P_{\sim}^2(F; G) = 1$ and $Fhx = 1$. Then $\neg G(\sim hx) = \neg Gh(\sim x) = 1$, which had to be proved.
- (b) We prove the inequality $P_{\sim}^1(Fh; L) \leq \bigvee_G \text{DL}(\mathcal{A}_1, 2)(L, Gh) \wedge P_{\sim}^2(F; G)$.

Define the filter G' by the formula

$$G'b = \bigvee_y \mathcal{A}_2(hy, b) \wedge Ly$$

and observe that $P_{\sim}^2(F; G') = 1$ holds, if we assume $P_{\sim}^1(Fh; L) = 1$.

Indeed: suppose $G'b = 1$ and let y witness this equality. We need to prove $\neg F(\sim b) = 1$. But we know $\neg Fh(\sim y) = \neg F(\sim(hy)) = 1$. Therefore $\neg F(\sim b) = 1$, since $\mathcal{A}_2(hy, b) = 1$ and F is an upper set.

By Lemma 6.4 we can find a prime filter G extending G' such that $P_{\sim}^2(F; G) = 1$ holds. Moreover, Gh extends L , since $G'h$ does. \square

7 Modal definability

Our modal definability theorem (Theorem 7.6 below) will identify classes \mathbf{C} of frames such that the image of \mathbf{C} under Pred^\sharp is an ‘‘HSP’’ class in $\text{DL}_{\heartsuit, \multimap, \sim}$, i.e., it is a variety (compare with the version of Goldblatt-Thomason theorem for

modal logics [3, Theorem 5.54] and [26, Theorem 3.15/2.]). Since we work over posets, the notion of HSP-closedness has to take this fact under consideration. Namely, we will use the factorization system $(\mathcal{E}, \mathcal{M})$ on \mathbf{Pos} where \mathcal{E} consists of surjective monotone maps and \mathcal{M} of monotone maps reflecting order, i.e., $f : \mathcal{W}_1 \rightarrow \mathcal{W}_2$ is in \mathcal{M} if $\mathcal{W}_1(x, x') = \mathcal{W}_2(fx, fx')$ holds for every x and x' . That $(\mathcal{E}, \mathcal{M})$ is indeed a factorization system on \mathbf{Pos} is proved in [4]. We will use the HSP Theorem w.r.t. a factorization system, see [28]:

A class \mathbf{A} of algebras in a variety \mathcal{V} over \mathbf{Pos} is definable by equations in \mathcal{V} iff \mathbf{A} satisfies the following conditions ($U : \mathcal{V} \rightarrow \mathbf{Pos}$ denotes the underlying functor):

- (H) If $e : \mathbb{A}_1 \rightarrow \mathbb{A}_2$ is such that $U(e)$ is a split epi in \mathbf{Pos} and \mathbb{A}_1 is in \mathcal{A} , then \mathbb{A}_2 is in \mathbf{A} .
- (S) If $m : \mathbb{A}_1 \rightarrow \mathbb{A}_2$ is such that $U(m)$ is in \mathcal{M} and \mathbb{A}_2 is in \mathcal{A} , then \mathbb{A}_1 is in \mathbf{A} .
- (P) If $\mathbb{A}_i, i \in I$, are in \mathcal{A} , then $\prod_{i \in I} \mathbb{A}_i$ is in \mathbf{A} .

In fact, since the algebraic semantics of our logic takes place in (distributive) lattices, we may as well replace equationally defined classes by inequationally defined. We prefer to introduce the inequational description, since it is often more useful in applications.

Definition 7.1 *Suppose \mathbb{W} is a relational frame. We say that α entails β , and denote this fact by $\alpha \models_{\mathbb{W}} \beta$, provided that $\|\alpha\|_{\text{val}} \leq \|\beta\|_{\text{val}}$ holds, for every valuation $\text{val} : \text{At} \rightarrow [\mathcal{W}, 2]$.*

Given a class Σ of pairs of formulas, we denote by $\text{Mod}(\Sigma)$ the class of frames \mathbb{W} such that $\alpha \models_{\mathbb{W}} \beta$, for all $(\alpha, \beta) \in \Sigma$.

The following result is trivial.

Lemma 7.2 *$\alpha \models_{\mathbb{W}} \beta$ holds iff $\text{Pred}^{\sharp}(\mathbb{W}) \models \alpha \wedge \beta = \alpha$, where the \models sign on the right denotes validity in the sense of universal algebra.*

Although the notation might suggest it, *it is not the case* that the logical connection $\text{Stone} \dashv \text{Pred}$ lifts to an adjunction $\text{Stone}^{\sharp} \dashv \text{Pred}^{\sharp}$. The unit of $\text{Stone} \dashv \text{Pred}$ does lift, however, and we will need it in the proof of Theorem 7.6.

This fact, known in modal logic as the Jónsson-Tarski theorem, is itself interesting since it gives us the modal completeness of the resulting logic w.r.t. T -coalgebras: namely, the T -coalgebra (the canonical frame) corresponding to the Lindenbaum-Tarski $\text{DL}_{\heartsuit, \rightarrow, \sim}$ -algebra provides a counterexample to any unprovable formula.

Lemma 7.3 *The unit η of $\text{Stone} \dashv \text{Pred}$ is a morphism in $\text{DL}_{\heartsuit, \rightarrow, \sim}$, i.e., η lifts along the functor $U_{\heartsuit, \rightarrow, \sim} : \text{DL}_{\heartsuit, \rightarrow, \sim} \rightarrow \text{DL}$ to a natural transformation $\eta^{\sharp} : \text{Id}_{\text{DL}_{\heartsuit, \rightarrow, \sim}} \rightarrow \text{Pred}^{\sharp} \text{Stone}^{\sharp}$.*

Another technical result that we need for Theorem 7.6 is the following one.

Lemma 7.4 *The functor Stone sends maps reflecting order to surjective monotone maps.*

Proof. Suppose $m : \mathcal{A} \rightarrow \mathcal{B}$ is a lattice homomorphism that reflects order. We need to prove that the monotone map $\text{Stone}(m) : \text{Stone}(\mathcal{B}) \rightarrow \text{Stone}(\mathcal{A})$ is surjective. To that end, fix a prime filter F on \mathcal{A} . Define the set

$$\mathbf{E} = \{G \mid G \cdot m = F\}$$

of filters on \mathcal{B} , ordered by inclusion. The set \mathbf{E} is nonempty, since m reflects order: put $Gb = \bigvee_a \mathcal{B}(ma, b) \wedge Fa$ and observe that G is in \mathbf{E} . Furthermore, the union of a nonempty chain of elements of \mathbf{E} is an element of \mathbf{E} . By Zorn's Lemma, \mathbf{E} has a maximal element G_0 . It is easy to prove that it is a prime filter. \square

Finally, before stating Theorem 7.6, we need to introduce the concept of a *prime extension* of a frame.

Definition 7.5 *The frame $\mathbb{W}^* = \text{Stone}^\sharp \text{Pred}^\sharp(\mathbb{W})$ is called the prime extension of \mathbb{W} .*

Theorem 7.6 *Suppose \mathbf{C} is a class of relational frames that is closed under prime extensions (if \mathbb{W} is in \mathbf{C} , then \mathbb{W}^* is in \mathbf{C}). Then the following are equivalent:*

- (i) *There is Σ such that $\mathbf{C} = \text{Mod}(\Sigma)$.*
- (ii) *\mathbf{C} satisfies the following four conditions:*
 - (a) *\mathbf{C} is closed under “surjective coalgebraic quotients”, i.e., if $e : \mathbb{W}_1 \rightarrow \mathbb{W}_2$ is surjective and \mathbb{W}_1 is in \mathbf{C} , so is \mathbb{W}_2 .*
 - (b) *\mathbf{C} is closed under “subcoalgebras”, i.e., if $m : \mathbb{W}_1 \rightarrow \mathbb{W}_2$ reflects order and \mathbb{W}_2 is in \mathbf{C} , so is \mathbb{W}_1 .*
 - (c) *\mathbf{C} is closed under coproducts.*
 - (d) *\mathbf{C} reflects prime extensions: if \mathbb{W}^* is in \mathbf{C} , so is \mathbb{W} .*

Proof. 1 implies 2. Suppose $\mathbf{C} = \text{Mod}(\Sigma)$. We will verify the four conditions for \mathbf{C} .

- (a) Suppose $e : \mathbb{W}_1 \rightarrow \mathbb{W}_2$ is a surjective coalgebra morphism. We prove that if $\alpha \models_{\mathbb{W}_1} \beta$, then $\alpha \models_{\mathbb{W}_2} \beta$.

Consider $y \in \mathcal{W}_2$ and a valuation $\text{val} : \text{At} \rightarrow [\mathcal{W}_2, 2]$. We can define a new valuation $\text{val}' : \text{At} \rightarrow [\mathcal{W}_1, 2]$ by the composition

$$\text{At} \xrightarrow{\text{val}} [\mathcal{W}_2, 2] \xrightarrow{[e, 2]} [\mathcal{W}_1, 2]$$

Then the diagram

$$\begin{array}{ccc} \mathcal{L}(\text{At}) & \xrightarrow{\models_{\text{val}}} & \text{Pred}^\sharp(\mathbb{W}_2) \xrightarrow{\text{Pred}^\sharp(e)} \text{Pred}^\sharp(\mathbb{W}_1) \\ & \underbrace{\hspace{10em}}_{\models_{\text{val}'}} & \end{array}$$

commutes in $\text{DL}_{\heartsuit, \rightarrow, \sim}$.

Let x be such that $ex = y$. Then, by assumption, $x \models_{\text{val}'} \alpha \leq \beta$, hence

$$\begin{aligned} \|\alpha \wedge \beta\|_{\text{val}} ex &= [e, 2](\|\alpha \wedge \beta\|_{\text{val}})x = \|\alpha \wedge \beta\|_{\text{val}'} x = \|\alpha\|_{\text{val}'} x = [e, 2](\|\alpha\|_{\text{val}})x \\ &= \|\alpha\|_{\text{val}} ex \end{aligned}$$

Therefore $ex \Vdash_{\text{val}} \alpha \leq \beta$, i.e., $y \Vdash_{\text{val}} \alpha \leq \beta$.

- (b) Suppose $m : \mathbb{W}_1 \rightarrow \mathbb{W}_2$ is a coalgebra morphism with m reflecting order. We prove that if $\alpha \models_{\mathbb{W}_2} \beta$, then $\alpha \models_{\mathbb{W}_1} \beta$.

Observe that $[m, 2] : [\mathscr{W}_2, 2] \rightarrow [\mathscr{W}_1, 2]$ is a split epimorphism in Pos . Indeed: there exists a monotone map $z : [\mathscr{W}_1, 2] \rightarrow [\mathscr{W}_2, 2]$ such that $[m, 2] \cdot z = id$. Given $u : \mathscr{W}_1 \rightarrow 2$, define $v : \mathscr{W}_2 \rightarrow 2$ by the formula

$$vy = \bigvee_x \mathscr{W}_2(mx, y) \wedge ux$$

Then $z : u \mapsto v$ is monotone and the equalities

$$vmx' = \bigvee_x \mathscr{W}_2(mx, mx') \wedge ux = \bigvee_x \mathscr{W}_1(x, x') \wedge ux = ux'$$

prove $[m, 2] \cdot z = id$ (above, we have used that m reflects order).

Suppose $\text{val} : \text{At} \rightarrow [\mathscr{W}_1, 2]$ is given. To prove $x \in \|\alpha\|_{\text{val}}$, consider

$$\text{val}' \equiv \text{At} \xrightarrow{\text{val}} [\mathscr{W}_1, 2] \xrightarrow{z} [\mathscr{W}_2, 2]$$

By assumption, $\|\alpha \wedge \beta\|_{\text{val}'} mx = \|\alpha\|_{\text{val}'} mx$. But the diagram

$$\begin{array}{ccc} \mathcal{L}(\text{At}) & \xrightarrow{\|\cdot\|_{\text{val}'}} & \text{Pred}^\#(\mathbb{W}_2) \xrightarrow{\text{Pred}^\#(m)} \text{Pred}^\#(\mathbb{W}_1) \\ & \searrow \text{\scriptsize } \|\cdot\|_{\text{val}} \nearrow & \\ & & \end{array}$$

commutes in $\text{DL}_{\heartsuit, \rightarrow, \sim}$ due to $[m, 2] \cdot z = id$. Hence $\|\alpha \wedge \beta\|_{\text{val}} x = \|\alpha\|_{\text{val}} x$.

- (c) Suppose $\alpha \models_{\mathbb{W}_i} \beta$, for all $i \in I$. We prove that $\alpha \models_{\prod_{i \in I} \mathbb{W}_i} \beta$.

The functor $\text{Pred}^\#$ preserves products (in fact, it preserves all limits). Products in $(\text{Pos}_T)^{op}$ are, of course, coproducts in Pos_T .

Consider x in $\prod_{i \in I} \mathscr{W}_i$. Since coproducts of frames are formed on the level of posets, there is $i \in I$ such that x is in \mathscr{W}_i . Let $\text{val} : \text{At} \rightarrow \prod_{i \in I} [\mathscr{W}_i, 2]$ be any valuation. Then, by assumption, $x \Vdash_{\text{val}_i} \alpha \wedge \beta = \alpha$, where

$$\text{val}_i \equiv \text{At} \xrightarrow{\text{val}} \prod_{i \in I} [\mathscr{W}_i, 2] \xrightarrow{p_i} [\mathscr{W}_i, 2]$$

and where p_i denotes the i -th projection.

This proves $\|\alpha \wedge \beta\|_{\text{val}} x = \|\alpha\|_{\text{val}} x$.

- (d) Suppose $\alpha \models_{\mathbb{W}^*} \beta$. We prove that $\alpha \models_{\mathbb{W}} \beta$.

Take x in \mathscr{W} and $\text{val} : \text{At} \rightarrow [\mathscr{W}, 2]$. Recall that, by Lemma 7.3, η lifts to $\eta^\#$, hence we can consider the valuation

$$\text{val}' \equiv \text{At} \xrightarrow{\text{val}} [\mathscr{W}, 2] \xrightarrow{UU_{\heartsuit, \rightarrow, \sim}(\eta^\#_{\text{Pred}^\#(\mathbb{W})})} [\text{StonePred}(\mathscr{W}), 2]$$

and therefore the diagram

$$\mathcal{L}(\text{At}) \xrightarrow{\|\cdot\|_{\text{val}}} \text{Pred}^{\sharp}(\mathbb{W}) \xrightarrow{\eta^{\sharp}_{\text{Pred}^{\sharp}(\mathbb{W})}} \text{Pred}^{\sharp} \text{Stone}^{\sharp} \text{Pred}^{\sharp}(\mathbb{W}) \quad (17)$$

$\underbrace{\hspace{15em}}_{\|\cdot\|_{\text{val}'}}$

commutes in $\text{DL}_{\heartsuit, \rightarrow, \sim}$. Thus, we obtain a commutative diagram

$$U_{\heartsuit, \rightarrow, \sim} \mathcal{L}(\text{At}) \xrightarrow{U_{\heartsuit, \rightarrow, \sim}(\|\cdot\|_{\text{val}})} \text{Pred}(\mathcal{W}) \xrightarrow{\eta_{\text{Pred}(\mathcal{W})}} \text{Pred} \text{Stone} \text{Pred}(\mathcal{W})$$

$\underbrace{\hspace{15em}}_{U_{\heartsuit, \rightarrow, \sim}(\|\cdot\|_{\text{val}'})}$

in DL (apply $U_{\heartsuit, \rightarrow, \sim}$ to diagram (17) and use that $U_{\heartsuit, \rightarrow, \sim}(\eta^{\sharp}_{\text{Pred}^{\sharp}(\mathbb{W})}) = \eta_{\text{Pred}(\mathcal{W})}$).

Hence also the diagram

$$\begin{array}{ccc} U_{\heartsuit, \rightarrow, \sim} \mathcal{L}(\text{At}) & \xrightarrow{U_{\heartsuit, \rightarrow, \sim}(\|\cdot\|_{\text{val}})} & \text{Pred}(\mathcal{W}) \\ & \searrow^{U_{\heartsuit, \rightarrow, \sim}(\|\cdot\|_{\text{val}'})} & \downarrow \eta_{\text{Pred}(\mathcal{W})} \\ & & \text{Pred} \text{Stone} \text{Pred}(\mathcal{W}) \quad (*) \\ & \searrow_{U_{\heartsuit, \rightarrow, \sim}(\|\cdot\|_{\text{val}})} & \downarrow \text{Pred}(\varepsilon_{\mathcal{W}}) \\ & & \text{Pred}(\mathcal{W}) \end{array} \quad \text{id} \quad (18)$$

commutes in DL. In fact, the area (*) in the above diagram is just one of the triangle equalities for $\text{Stone} \dashv \text{Pred}$.

By assumption, $\varepsilon_{\mathcal{W}}(x) \Vdash_{\text{val}'} \alpha \wedge \beta = \alpha$. From the lower triangle in (18) it follows that $x \Vdash_{\text{val}} \alpha \wedge \beta = \alpha$:

$$\begin{aligned} \|\alpha \wedge \beta\|_{\text{val}} x &= [\varepsilon_{\mathcal{W}}, 2](\|\alpha \wedge \beta\|_{\text{val}'} x) = \|\alpha \wedge \beta\|_{\text{val}'} \varepsilon_{\mathcal{W}}(x) = \|\alpha\|_{\text{val}'} \varepsilon_{\mathcal{W}}(x) \\ &= [\varepsilon_{\mathcal{W}}, 2](\|\alpha\|_{\text{val}'} x) = \|\alpha\|_{\text{val}} x \end{aligned}$$

2 implies 1. Denote by Σ the set of pairs (α, β) such that $\alpha \Vdash_{\mathbb{W}} \beta$, for all \mathbb{W} in \mathbf{C} . Hence $\mathbf{C} \subseteq \text{Mod}(\Sigma)$ by definition.

Suppose \mathbb{W}_0 is in $\text{Mod}(\Sigma)$, we want to prove that \mathbb{W}_0 is in \mathbf{C} .

Define \mathbf{A} to be the closure of $\{\text{Pred}^{\sharp}(\mathbb{W}) \mid \mathbb{W} \in \mathbf{C}\}$ under products, subalgebras along monotone maps reflecting order and images along split epis in Pos . Therefore $\text{Pred}^{\sharp}(\mathbb{W}_0)$ is in \mathbf{A} and there is a diagram

$$\text{Pred}^{\sharp}(\mathbb{W}_0) \xleftarrow{e} \mathbf{A} \xrightarrow{m} \prod_{i \in I} \text{Pred}^{\sharp}(\mathbb{W}_i)$$

in $\text{DL}_{\heartsuit, \rightarrow, \sim}$, where \mathbf{A} is in \mathbf{A} , \mathbb{W}_i are in \mathbf{C} , for all $i \in I$, and m reflects orders, and e is split epi in Pos .

Consider the image of the above diagram

$$\text{Stone}^{\sharp} \text{Pred}^{\sharp}(\mathbb{W}_0) \xleftarrow{\text{Stone}^{\sharp}(e)} \text{Stone}^{\sharp}(\mathbf{A}) \xrightarrow{\text{Stone}^{\sharp}(m)} \text{Stone}^{\sharp}(\prod_{i \in I} \text{Pred}^{\sharp}(\mathbb{W}_i))$$

under $Stone^\sharp : DL_{\sqcap, \multimap, \sim} \longrightarrow (Pos_T)^{op}$.

When reading the above diagram in Pos_T , i.e., when reversing the arrows, we obtain a diagram

$$Stone^\sharp Pred^\sharp(\mathbb{W}_0) \xrightarrow{Stone^\sharp(e)} Stone^\sharp(\mathbb{A}) \xleftarrow{Stone^\sharp(m)} Stone^\sharp(Pred^\sharp(\coprod_{i \in I} \mathbb{W}_i))$$

Then:

- (i) $Stone^\sharp(Pred^\sharp(\coprod_{i \in I} \mathbb{W}_i))$ is in \mathbf{C} , since it is a prime extension of a coproduct of elements of \mathbf{C} .
- (ii) $Stone^\sharp(\mathbb{A})$ is in \mathbf{C} .
 - (a) By Lemma 7.4, $Stone^\sharp(m)$ is a surjective coalgebra homomorphism. Indeed, the underlying map of $Stone^\sharp(m)$ is $Stone(m)$ by (16).
 - (b) Since $Stone^\sharp(Pred^\sharp(\coprod_{i \in I} \mathbb{W}_i))$ is in \mathbf{C} , so is $Stone^\sharp(\mathbb{A})$. Use properties of \mathbf{C} .
- (iii) $Stone^\sharp Pred^\sharp(\mathbb{W}_0)$ is in \mathbf{C} .

This will follow after we prove that $Stone^\sharp(e)$ reflects orders. Its underlying map is restriction along e from the poset of prime filters on \mathbb{A} to the poset of prime filters on $Pred(\mathbb{W}_0)$. Recall that e is a split epimorphism, denote by z the monotone map satisfying $e \cdot z = id$. Consider two prime filters u, u' on \mathbb{A} such that $u \cdot e \leq u' \cdot e$ holds. Then $u = u \cdot e \cdot z \leq u' \cdot e \cdot z = u'$ holds.

Since we proved that $Stone^\sharp Pred^\sharp(\mathbb{W}_0)$ is in \mathbf{C} , we know that \mathbb{W}_0 is in \mathbf{C} , since \mathbf{C} reflects ultrafilter extensions. \square

Example 7.7 The *distributive and associative full Lambek calculus* (denoted by **dFL**) is given by the grammar (1), where \otimes is required to be associative, to have e as a unit and to satisfy the residuation laws $\varphi \otimes \psi \leq \chi$ iff $\psi \leq \varphi \rightarrow \chi$ iff $\varphi \leq \chi \leftarrow \psi$. Thus, the subvariety of $DL_{\otimes, e, \rightarrow, \leftarrow}$ that we want to deal with is exactly that of distributive residuated lattices.

The frames that are definable by the above (in)equations are precisely the quintuples $(\mathscr{W}, P_\otimes, P_\rightarrow, P_\leftarrow, P_e)$ that satisfy the following conditions (for details see [32, Chapter 11]):

- (i) P_\otimes is associative: $\bigvee_z (P_\otimes(x, y; z) \wedge P_\otimes(z, u; v)) = \bigvee_w (P_\otimes(y, u; w) \wedge P_\otimes(x, w; v))$
- (ii) and has P_e as a (left and right) unit:
 $W(x, y) = \bigvee_z (P_e(z) \rightarrow P_\otimes(z, x; y)) = \bigvee_z (P_e(z) \rightarrow P_\otimes(x, z; y))$
- (iii) The equalities $P_\otimes(x_0, x_1; y) = P_\rightarrow(x_1; x_0, y) = P_\leftarrow(x_0; x_1, y)$ hold.

Class \mathbf{C} of frames satisfying the above conditions is easily seen to verify the conditions in Theorem 7.6.

Example 7.8 Many interesting examples can be found among the extensions of (associative) **dFL** with, e.g., the structural rules, or when expanding the language by negation. Instances of the first possibility are: **dFL** extended with

any combination of: exchange, weakening, contraction. See [32] for details on what follows.

- (i) The exchange rule corresponds to the commutativity of P_{\otimes} , i.e. to the equality $P_{\otimes}(x, y; z) = P_{\otimes}(y, x; z)$.
- (ii) Weakening corresponds to: $P_{\otimes}(x_0, x_1; y)$ implies $x_0 \leq y$ and $x_1 \leq y$.
- (iii) Contraction corresponds to the equality $P_{\otimes}(x, x; x) = 1$.

This includes, for example, intuitionistic logic, obtained as an extension of **dFL** with all the three structural rules.⁴ Instances of the second possibility include, e.g., the relevance logic **R**, see [12] or [32]. Here the language $\otimes, \rightarrow, \leftarrow, e$ is extended by a negation connective \sim .

The frames $(\mathscr{W}, P_{\otimes}, P_{\rightarrow}, P_{\leftarrow}, P_e, P_{\sim})$ for the relevance logic **R** are the frames for **dFL** satisfying, in addition, the contraction equality together with the following three axioms ([32]):

- (a) $P_{\sim}(x; y) = P_{\sim}(y; x)$,
- (b) $\bigvee_y P_{\otimes}(x_0, x_1; y) \wedge P_{\sim}(y; u) \leq \bigvee_s P_{\otimes}(u, x_0; s) \wedge P_{\sim}(x_1; s)$,
- (c) $\bigvee_y (P_{\sim}(x; y) \wedge \bigwedge_z (P_{\sim}(y; z) \Rightarrow \mathscr{W}(z, x))) = 1$.

The class **C** of frames satisfying these axioms is easily seen to verify the conditions of Theorem 7.6. It is modally definable by corresponding axioms of **R**.

8 Conclusions and further work

We have shown that frames for various kinds of distributive substructural logic can be perceived naturally as modally definable classes of poset coalgebras. It seems natural to construct first frames for logics that have minimal necessary restrictions on the modalities — these frames are exactly the coalgebras for a certain endofunctor of the category of posets. Such an approach yields the notion of frame morphisms for free: the morphisms of frames are exactly the coalgebra morphisms. Any (in)equational requirement on the modalities results in singling out a subclass of frames that is modally definable in the sense of Goldblatt-Thomason Theorem. Hence any subvariety of modal algebras (= distributive lattices with operators) defines a Goldblatt-Thomason subclass of frames, and vice versa, which has been illustrated by well-known examples of frames for distributive full Lambek calculus, relevance logic, etc.

The limitation of our result lies certainly in the presence of the distributive law for the propositional part of the logic since it leaves out nondistributive substructural logics. We believe that this can be easily overcome by passing to general lattices and using a two-sorted representation of lattices in the sense of [24]. The underlying logical connection will be two-sorted, hence the “state space” will consist of two posets connected with a monotone relation. This

⁴ A usual frame (X, \leq) for intuitionistic logic can be perceived as a relational frame defining $P(x, y; z) = x \leq z \wedge y \leq z$. Then coalgebraic morphisms correspond precisely to bounded morphisms.

is in compliance with various notions of generalized frames, as studied, e.g., in [16] and [14]. Furthermore, this approach will also allow to pass naturally from posets to categories enriched in a general commutative quantale. In the latter framework, we believe to be able to study, e.g., many-valued modal and substructural logics in a rather conceptual way.

A natural further direction would be to prove a more general Goldblatt-Thomason theorem for coalgebras over posets or categories enriched in a general commutative quantale, obtaining an analogue of [26, Theorem 3.15]. Another line of research explores the fact that the coalgebraic functor we obtained is easily seen to satisfy the Beck-Chevalley Condition in the sense of [2]. Hence it will be possible to develop the theory of cover modalities over coalgebras for distributive substructural logics.

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