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Title

A quantum probability perspective on borderline vagueness

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running head

Quantifying vagueness

keywords

borderline contradictions, contextualism, fuzzy logic, neuronal network, quantum interference, quantum probability, vagueness.

words (inclusive footnotes and references): 10952; **references**: 1199; **footnotes**: 397.

A quantum probability perspective on borderline vagueness

Reinhard Blutner, Emmanuel M. Pothos, and Peter Bruza

Abstract

The term 'vagueness' describes a property of natural concepts, which normally have fuzzy boundaries, admit borderline cases and are susceptible to Zeno's sorites paradox. We will discuss the psychology of vagueness, especially experiments investigating the judgment of borderline cases and contradictions. In the theoretical part, we will propose a probabilistic model that describes the quantitative characteristics of the experimental finding and extends Alxatib's and Pelletier's (2011) theoretical analysis. The model is based on a Hopfield network for predicting truth values. Powerful as this classical perspective is, we show that it falls short of providing an adequate coverage of the relevant empirical results. In the final part, we will argue that a substantial modification of the analysis put forward by Alxatib and Pelletier and its probabilistic pendant is needed. The proposed modification replaces the standard notion of probabilities by quantum probabilities. The crucial phenomenon of borderline contradictions can be explained then as a quantum interference phenomenon.

1 Introduction

The term ‘vagueness’ describes a property of natural concepts, which normally have fuzzy boundaries, admit borderline cases and are susceptible to Zeno’s sorites paradox (as long as there is a semi-continuous relevant dimension). For example, consider the concept of a ‘tall man’ as applied under usual circumstances. An important characteristic of such concepts is that they apparently lack precise, well-defined extensions. Questions such as ‘what is the smallest size of man called “tall”?’ do not make precise sense, since the boundary between ‘tall’ and ‘not tall’ is not clearly defined. Also, predicates like ‘tall man’ admit *borderline cases*. These are instances where it is unclear whether the predicate applies. The lack of clarity about whether the chosen instance is tall or not cannot be eliminated by further information about the person’s exact height; rather, the underlying issue seems to be associated with the lack of a precise definition for the predicate. Such situations are related to Zeno’s sorites paradox, which aptly illustrates the problem with vagueness. It derives its name from the Greek *soros*, which means *heap*. Obviously, we could formulate a rule stating that if X is a heap of sand, then removing one grain will still result in a heap. However, when we repeat the action of removing grains often enough, one by one, eventually the repeated application of the formulated rule gives the paradoxical result that the last grain left must still count as a heap.

The phenomenon of vagueness has attracted intense interest from philosophers, logicians. Only recently has it become a principal research topic for experimental psychologists. There are three main streams in this experimental work on vagueness: (1) Research pioneered by Hampton and others concerning the structure of vague concepts (Hampton, 1988a, 1988b, 2007; Wallsten et al., 1986; Budescu & Wallsten, 1995); (2) The investigation of Osherson and others regarding compositional theories of conceptual combination (Osherson & Smith, 1981; Osherson & Smith, 1997); (3) the investigation of borderline cases pioneered by Bonini et al. (1999) and Alxatib and Pelletier (2011). Hereby, borderline cases are exemplified by sentences where participants are normally unsure whether the sentence is true or false.

The present paper is devoted to the last research stream. Our main interest concerns the explanation of the acceptance of borderline contradictions, such as ‘X is tall and not tall’, where X refers to a borderline case. We will give an explanation of recent data found by Alxatib and Pelletier (2011) and some related findings by Sauerland (2010). Though we are particularly interested in explaining the data concerning borderline contradictions, we will argue that the present model is more general.

We are looking for a quantitative model of vagueness. Such an endeavor goes beyond purely logic-based approaches, as that by Alxatib and Pelletier (2011). The first model we propose employs classical probability theory, and is based on discrete hidden variables mimicking Alxatib and Pelletier’s (2011) assumption about the underlying logic and pragmatics of super- and sub-valuation. The second model introduces quantum interference effects. It is the superposition of ‘tall’ and ‘not tall’ that introduces additional interference terms, when we calculate the corresponding quantum probabilities. The details of the mathematical treatment are related to earlier work discussing probability judgment errors (Aerts, 2009; Blutner, 2009; Busemeyer, Pothos, Franco & Trueblood, 2011; Conte et al., 2008; Khrennikov, 2006).

In the next section, we introduce psychologically relevant theories of vagueness and outline the findings of Bonini et al. (1999). Section 3 explains the basic data reported by Alxatib and Pelletier (2011) and their theoretical analysis. Based on this theoretical analysis, Section 4 presents a classical probabilistic model quantifying vagueness. We fit this model to the data presented in Alxatib and Pelletier (2011), but conclude that a fully satisfactory fit is elusive. Section 5 develops an alternative probabilistic model, based on interference effects, which can arise in quantum cognitive models. Section 6 concludes the paper with a discussion of our main findings and implications.

2 Psychological theories of vagueness

We introduce four semantic theories of vagueness, which are taken as relevant for the construction of a psychological theory of vagueness (see also Bonini et al., 1999, Alxatib and Pelletier, 2011).

2.1 Gap- and glut theories

According to gap theory (Fine, 1975; Van Fraassen, 1966) a predicate such as ‘bald’ is vague because there is indeterminacy between these various ways of picking out a precise cut-off value separating bald from not bald. The set of these possibilities to make it precise are called ‘precisifications’. In gap theory it is assumed that a sentence such as ‘Peter is bald’ is considered true, without qualifications, if the sentence is true independently of the precise cut-off values, i.e., if the sentence is true for all precisifications. In this case, the term ‘super-true’ is used (and the method used in gap theory is called supervaluation). A sentence such as ‘Peter is bald’ is considered ‘super-false’ if it is false for all precisifications. Sentences which are neither super-true nor super-false for a given system of precisifications are said to be *vague*. They fall into a truth-value gap.

Glut theory (Hyde, 1997; Priest, 1989) is analogous to gap theory. Again, there is a set of precisifications, and sentences and predicates can be multiply precisified. However, instead of ‘super-valuations’, so called ‘sub-valuations’ are considered for determining truth-values. A sentence is called ‘sub-true’ if there is a precisification that makes it true, and a sentence is called ‘sub-false’ if there is a precisification that makes it false. Interestingly, now the logical principle of ‘non-contradiction’ can be violated: there can be sentences which are both sub-true and sub-false. Such sentences are said to fall into a truth-value glut.

Gap and glut theories are logical theories. An assumption about psychological process is needed to connect logical theory with human behavior. Bonini et al. (1999: 379) proposed that: “Speakers typically know what truth-value (if any) results from predicating vague adjectives like *red*, *tall*, and *old* of common objects. They tend to assent to such predications if they consider them true and to dissent from them if they consider them false.” In other words, it is assumed that speakers have access to a lexical base that contains the relevant classification of common objects. A simple example from Alxatib and Pelletier (2011) should illustrate this point and the idea behind gap- and glut theories.

Assume we have five suspects of differing heights (Fig. 1). For some suspects X the sentence ‘X is tall’ is clearly accepted (e.g., for X = 3), for others it is clearly rejected (X=1 and X =4), and for the remaining individuals (X=2 and X=5) we get the typical answer ‘can’t tell’. Note that we somewhat idealize the real situation, since we accept a clear separation between accepting ‘X is tall’, rejecting it and declaring it unclear.

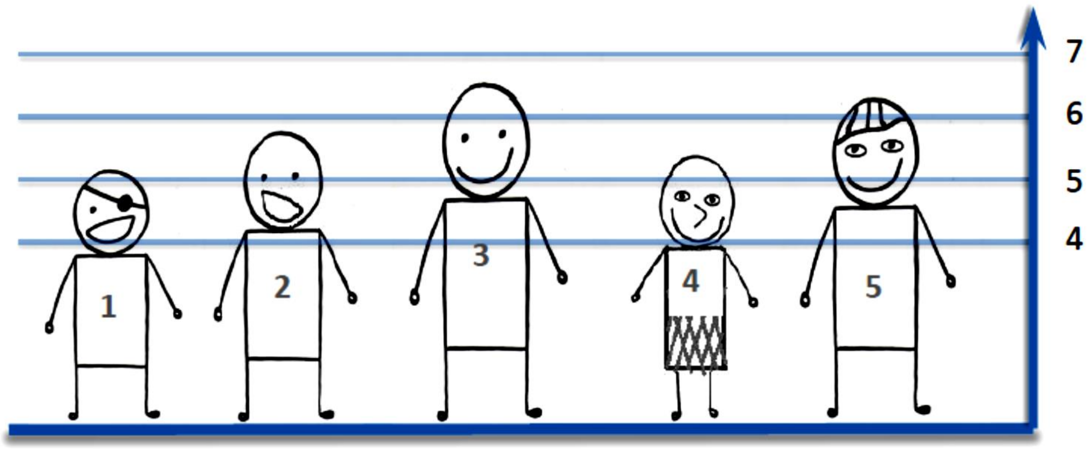

Fig. 1: Suspects of differing heights (adapted from Alxatib & Pelletier, 2011). The scale values shown are heights in feet.

Table 1 (left column) shows the lexical base for the adjective ‘tall’ for the fixed comparison class. It generates the classification F for instance #1 (see Fig. 1), null for instances #2 & #4, and T for instances #3 & #5.

	Lexical base	Precisificat-ions	Super-valuation	Sub-valuation
1	F	FF	F	F
4	null	TF	Gap	T/F (Glut)
2	null	TF	Gap	T/F (Glut)
5	T	TT	T	T
3	T	TT	T	T

Table 1: Example illustrating gaps in super-valuation theory and gluts in sub-valuation theory

Two precisifications are considered in Table 1; one assigns the value T for instance #4 and the value F for instance #2; the other assigns the opposite values to the instances #2 and #4. As a consequence, we get the corresponding super- and sub-valuations, as shown in the last two columns of the table.

On the gap hypothesis, speakers take super-true and -false and vague (null) as the relevant truth values. On the glut hypothesis, speakers take sub-true and -false as the relevant truth values. In the first case the principle of ‘non-contradiction’ is satisfied: there can be no sentences which are both super-true and super-false. Since we can have truth value gaps, the principle of ‘bivalence’ is violated. In the second case sentences can be both sub-true and sub-false: gluts. In the second case, hence, the principle of ‘non-contradiction’ is violated, but the principle of bivalence is satisfied. Paraconsistent logic (Priest, 2002) can handle this case.

It is obvious now how speakers can answer questions such as “who is the smallest person who can be called ‘tall’?” or “who is the biggest person where the property ‘tall’ is rejected?”. In the first case the answer is #5 for gap-theory and #4 for glut theory. In the second case it is #1 for gap-theory and #2 for glut-theory.

In Bonini et al.’s (1999) experiment, two groups of subjects were asked to estimate (a) the smallest number x of years of age such that the sentence ‘a person is old’ is true (truth-judgers), (b) the largest number x of years of age such that the sentence ‘a person is old’ is false (falsity-judgers). Other conditions involved other properties. In all cases, there was a substantial positive difference between truth-judgers and falsity-judgers. Superficially, these data support the predictions of gap theories of vagueness (super-valuation) and they contradict glut theories (sub-valuations). Interestingly, Bonini et al. (1999) rejected this approach in favor of another account we will introduce in the following subsection: epistemic theories. The authors give the following main reason for rejecting gap-theories: these theories have problems describing higher order vagueness.

2.3 Epistemic theories

Epistemic theories of vagueness (‘vagueness as ignorance’) do not treat vagueness as an ontological problem, but rather attribute vagueness to lack of knowledge (Sorensen, 1991; Williamson, 1994). Epistemic theories assume the existence of an exact definition in each case, but one which is unknown to us. On this account, the indeterminacy connected with a vague expression stems from our inability to determine its exact definition.

Bonini et al. (1999) argue that their data are best explained by an epistemic theory. How, then can one explain the difference between truth judgers and falsity judgers? Why does this gap appear? Bonini et al. argue that gaps appear because speakers are more willing to commit errors of omission, than errors of commission. In other words, speakers have a tendency to prefer type II error over type I. Bonini et al (1999: 387) cite evidence that “people perceive errors of commission as graver than those of omission (Ritov & Baron, 1990; Spranca, Minsk, & Baron, 1991)”.

If the epistemic approach is correct, then there is no need for a lexical semantics that assigns borderline status to some objects. Rather, their ‘borderline status’ becomes an implicit consequence of the mechanism that directs errors of omission and commission. We have already noted that Bonini et al. (1999) do not present any empirical argument against gap theories. Rather they give two theoretical arguments for the departure from gap theory: the existence of higher order vagueness and the alleged identity of metalinguistic and normal judgments.

Not all logicians find epistemic theories attractive, especially in relation to concrete examples. For instance, assume the vagueness of ‘bald’ has an epistemic origin. Therefore, a critical cut-off value of the number of hairs does exist, which separates the bald from the non-bald, but it is unknown to us. Well before this view originated, Russell (1923) criticized this ignorance stance, pointing out that vagueness is determined by the end points only.

2.4 Contextualism

Contextualism (Åkerman & Greenough, 2009; Bosch, 1983; Kamp, 1981) sees vagueness as a result of a particular kind of context-sensitivity. Vagueness is context-dependence with respect to so-called v-standards (certain standards of application, which are distinguished from ordinary contextual elements, such as indexicals etc.).

“The expression ‘here’ is vague, but its vagueness need have nothing to do with the fact that its reference can shift depending on the place of use. Equally, the application of the predicate ‘is tall’ can vary as a function of the operative comparison class and/or what is taken to be typically tall. But such shiftiness in the extension of ‘is tall’ need have nothing as such to do with vagueness. … we will use the term v-standards as a neutral placeholder for whatever contextual parameters are taken to be responsible for the shifts (Åkerman & Greenough, 2009, p.9)

Taking up an idea of Hyde (1997), Odrowaz-Sypniewska (2010) suggested that contextualists should choose sub-valuation rather than super-valuation as their logic. This agrees with an earlier proposal of Bosch (1983), who sees every precisification as provided by a particular definition in a particular context. Looking at Table 1 makes it clear why we get a sub-valuation theory in this way.

2.5 Fuzzy set theory

According to gap and glut theory, the concept of a vague sentence is itself a sharp concept. However, many authors argue that there is no sharp boundary between vague and sharp sentences. Similarly, the notion of a borderline case is itself vague. For that reason, proponents of fuzzy set theory argue that it is natural to allow for a continuum of intermediate truth-values, with a special logic, as proposed by Zadeh (1965). This idea fits nicely with treating vagueness as an ontological phenomenon, as is similarly done within gap- and glut-theories (for an application, see Wallsten et al., 1986; Budescu & Wallsten, 1995).

3 Alxatib and Pelletier (2011)

Alxatib and Pelletier (2011) reported an experiment in which participants were presented with a picture of five suspects of differing heights in a police lineup (similar to Fig. 1). The suspects in the lineup were identified by the numbers #1 (5’4”), #2 (5’11”), #3 (6’6”), #4 (5’7”), and #5 (6’2”) and they were shown in the picture not sorted by height, but with an ordering based on names. Participants also received a form with 20 questions and had to mark one of three check boxes corresponding to three possibilities (true, false, can’t tell). The 20 questions consisted of four questions for each suspect, as demonstrated below for suspect #4. The ordering of the four questions for each suspect was randomized.

(1)	#4 is tall	True □	False □	Can’t tell □
	#4 is not tall	True □	False □	Can’t tell □
	#4 is tall and not tall	True □	False □	Can’t tell □
	#4 is neither tall nor not tall	True □	False □	Can’t tell □

The results, arranged by increasing height of the suspects, are shown in Fig. 2.

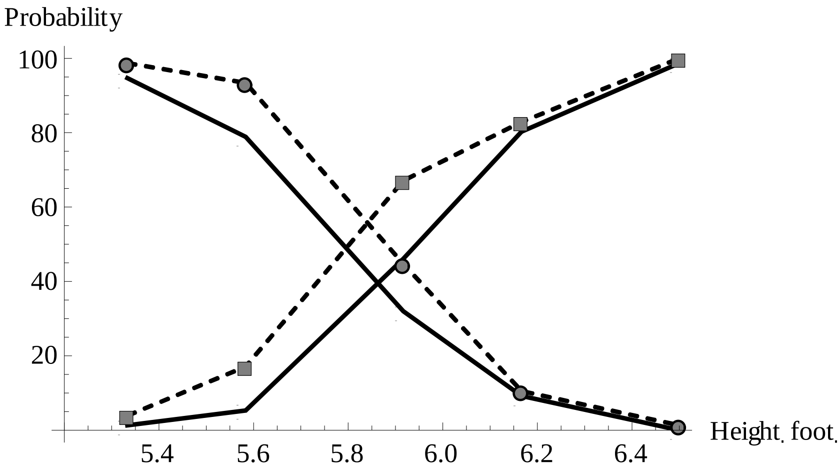

Fig. 2: Alxatib and Pelletier’s (2011) data. —■— stands for accepting a proposition, ---■--- for rejecting a negation, —●— for accepting a negation, and ---●--- for rejecting a proposition

One can see that there is a consistent preference for denying a proposition (●) over accepting its negation (●). Further, there is a substantial preference for rejecting a negation (■) over accepting a proposition (■).

Another important finding is that there are cases (about 30%) where ‘X is tall’ and ‘X is not tall’ are both considered false but ‘X is tall and not tall’ is considered true (Fig. 3). The same applies for ‘X is neither tall nor not tall’. In addition, Fig. 3 illustrates that accepting ‘X is neither tall nor not tall’ is preferred over accepting ‘X is tall and not tall’. This seems to be plausible. However, it is difficult to find a theoretical argument for it. In fact, Alxatib and Pelletier (2011) could not explain the difference.

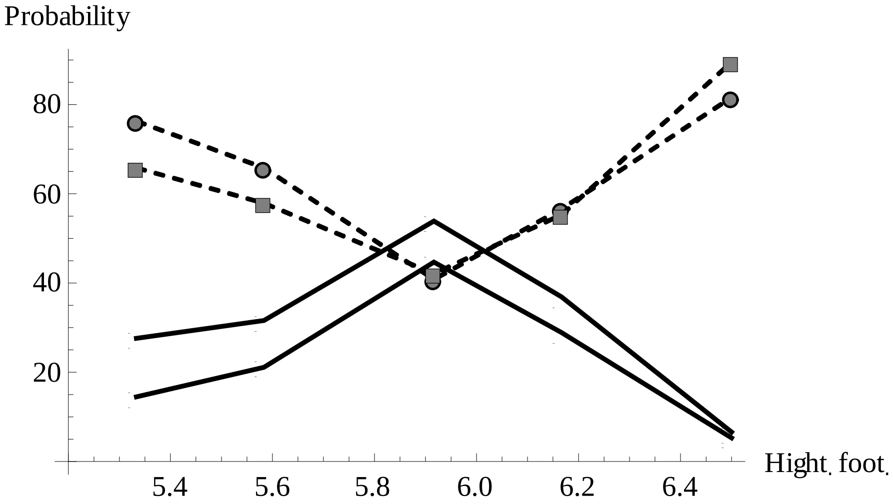

Fig. 3: Alxatib and Pelletier’s (2011) data. —■— stands for accepting ‘and’, ---■--- for rejecting ‘and’, —●— for accepting ‘neither’, and ---●--- for rejecting ‘neither’

Moreover, there is no clear preference for either rejecting ‘neither’ or rejecting ‘and’, at least not for borderline cases.

In order to suggest an explanation for their data, Alxatib and Pelletier (2011) proposed the following assumptions:

1. Each sentence is ambiguous between a super-, and a sub-interpretation.
2. A Gricean mechanism applies in order to select the appropriate interpretation in the given context.
3. The Gricean solution conforms to the ‘strongest meaning hypothesis’ of Dalrymple et al. (1998). As presently relevant, the logically strongest hypothesis is selected.
4. a. For ‘X is tall’ and ‘X is not tall’ the result will be a super-interpretation: ‘X is tall’ is true if and only if ‘X is tall’ is super-true; ‘X is not tall’ is true if and only if ‘X is tall’ is super-false. Note that the resulting super-interpretation eliminates vagueness completely from the truth-conditions. If a sentence is not true, it is assumed to be false (bivalence).

b. For ‘X is tall and not tall’ the super-interpretation is semantically empty. This conflicts with the maxim of quality. Therefore, the super-interpretation cannot apply, but the sub-interpretation does and conforms to the borderline cases, where X is neither tall nor not tall.

Table 2 summarizes the consequences of these assumptions and illustrates it for a simple example. The first row shows the result of the Gricean mechanism in case of ‘X is tall’. The underdetermined value ‘null’ assigned by the lexical base (top level) is now replaced by the value F. The remaining three rows show the corresponding results of the Gricean mechanism for ‘not tall’, ‘tall and not tall’, and ‘neither tall nor not tall’.

F	null	T	← lexical base for ‘X is tall’
F	F	T	X is tall
T	F	F	X is not tall
F	T	F	X is tall and not tall
F	T	F	X is neither tall nor not tall

Table 2: Truth-conditional pragmatics for different expressions

Hence, Alxatib and Pelletier’s (2011) theory implies that sentences such as ‘X is tall’ are either accepted or rejected. Their Gricean mechanism has eliminated the gap. The relatively low percentage of cases where subjects say ‘can’t tell’ – about 10% in their experiment – is ignored in the theory.

Consider now the case of negation as in ‘X is not tall’. Table 2 shows that the Gricean mechanism leads to truth conditions that conform to the application of intuitionistic negation to the baseline ‘X is tall’. This contrasts with two other possibilities of inner and outer negation (as defined in Table 3). For the last two expressions in Table 2, no standard analysis is available. Interestingly, both ‘tall and not tall’ and ‘neither tall nor not tall’ result in the same truth conditions, using this sketched mechanism of truth-conditional pragmatics.

F	null	T	p	← semantics of p
T	null	F	$\sim p$	inner negation
T	T	F	$\neg p$	outer negation
T	F	F	$-p$	intuitionistic negation

Table 3: Truth table for inner, outer, and intuitionistic negation

Alxatib and Pelletier’s (2011) analysis has shortcomings, but it illustrates why there is a preference for denying a proposition over accepting its negation: A simple proposition can be denied both in the negative and the neutral region, whereas a negation can be accepted in the negative region only. Similarly, there is an analogous explanation for the preference for rejecting a negation of a proposition over accepting the proposition. Further, the theory makes it clear why considering ‘X is tall’ and ‘X is not tall’ to be both false is associated with accepting ‘X is tall and not tall’ as true. As demonstrated in Table 2, this pattern is realized for borderline cases where the truth-conditional pragmatics of Alxatib and Pelletier’s (2011) theory results in rejections for simple and negated propositions, but predicts acceptance for conjunctions of both expressions.

However, there are shortcomings of Alxatib and Pelletier’s (2011) scheme. The analysis predicts that all instances where ‘X is tall and not tall’ is true are instances where ‘X is tall’ is false and ‘X is not tall’ is false. Empirically, this applies to only about 30% of the borderline cases (Alxatib & Pelletier, 2011: 315). Further, the analysis predicts that all instances where ‘X is tall’ and ‘X is not tall’ are both false are instances where ‘X is tall and not tall’ is true. Empirically, this applies to only about half of the borderline cases (p. 316).

4 Quantifying vagueness: A classical probabilistic model

Alxatib and Pelletier (2011) specified their theory in purely logical terms, and this led to implications, which are hard to reconcile with empirical observation (Section 3). We wonder whether these difficulties can be overcome by recasting their model in probabilistic terms.

The basic assumption is very simple. We assume that the lexical semantics, say for an adjective such as ‘tall’, does not produce a fixed, static characterization of the involved instances. Rather, the assignment of the three truth-values (T, null, F) is stochastic. Obviously, the distribution of the three truth values depends on the height *x* of X and the mean height *a* of individuals in the relevant population (comparison class).

Instead of working with *one* trinary random variable ***Truth***, with the values T, null, and F we will work with two binary random variables, with values 0 and 1, called ***T*** and ***F***. The correspondence between the two systems of random variables is:

(2) ***Truth*** = T iff ***T*** = 1 and ***F*** = 0 ($\boldsymbol{T}\overline{\boldsymbol{F}}$ for short)

Truth = F iff ***T*** = 0 and ***F*** = 1 ($\overline{\boldsymbol{T}}\boldsymbol{F}$ for short)

Truth = null iff ***T*** = 1 and ***F*** = 1 ($\boldsymbol{TF}$ for short)

In this way, the distribution of the values for the random variables ***T*** and ***F*** give the values of the random variable ***Truth***, assuming the combination ***T***=0 and ***F***=0 ($\overline{\boldsymbol{T}}\,\overline{\boldsymbol{F}}$ for short) is excluded. This can be interpreted as the exclusion of gaps. Equally, the correspondence between ‘null’ and ***TF*** is consistent with the underlying glut-ness: A truth-value glut is realized for some proposition if it is true (***T***=1) and false (***F***=1) at the same time.

In order to assign probabilities for the Boolean combinations of the random variables ***T*** and ***F*****,** depending on the height of an observed individual X and the mean height *a* of the individuals in the comparison class, we consider a simple Hopfield network with two output nodes for the random variables ***T*** and ***F***.

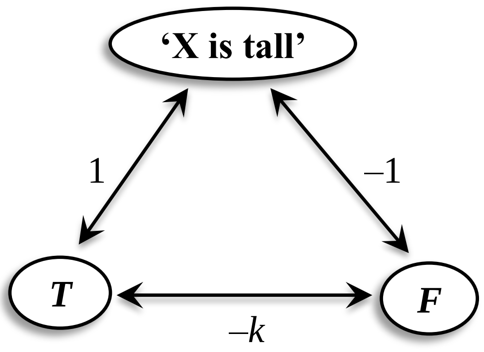

Fig. 4: Simple neural network with one input node (representing ‘X is tall’) and two output nodes (***T*** and ***F***).

Fig. 4 shows a Hopfield network with three neurons (nodes). The input node stands for the activation of the target sentence, e.g. ‘X is tall’. The input node is connected with the two output nodes ***T*** and ***F*** with opposite weights, 1 and –1. We assume that the top node is activated as a function of the value x–*a*, i.e. the difference between the height of individual X and the mean height *a* of individuals in the relevant population. If $x=a$, then the ***T***- and the ***F***-node get the same (zero) net input. The two output nodes are in inhibitory connection with each other. This means that for high values of k there is a strict tendency that they balance each other in opposite directions: if ***T*** gets activation 1 then ***F*** gets activation 0 and vice versa. For lower values of *k* this contrast effect can disappear. This means that it is possible to get the activations ***T***=1 and ***F***=1 expressing the proposition is true and false at the same time (i.e., a glut). Likewise, the output activation ***T***=0 and ***F***=0 is possible, expressing a ‘gap’. Later in this section, we will exclude this last possibility.

Taking the standard formalism for Hopfield networks (Hopfield, 1982), we can calculate the energy function of the activation vectors *s* of our network, $E(s) = -\sum_{i>j} w_{ij}\, s_i s_j$.[1]

The energy describes how stable a certain activation pattern s is, assuming the activation of the input nodes of s is clamped. The lower the energy the more stable is the pattern of activation. Unstable patterns normally decay into patterns of lower energy. A common way to describe the probability for the distributions of the states of a system is the Boltzmann distribution, which was discovered in the context of classical statistical mechanics. According to the Boltzmann distribution, the probability of a state is indirectly proportional to the exponential of the energy of the system: $\mathrm{P}(s) \sim e^{-E(s)/\tau}$. Two characteristics of the Boltzmann make it suitable here, first, the 0/1 endpoints, which can be associated with falsity/ truth, and, second, the parameter τ (called 'temperature'), which allows us to capture graded distinctions between concepts allowing for less and greater degrees of vagueness. Given the activation *x–a* of the top node, we can summarize the corresponding results as in Table 4.

Activation of node *T*	**Activation of node *F***		**Energy E**	**Probability P**
1	0	$\boldsymbol{T\overline{F}}$	$-(x–a)$	$1/Ce^{-(x-a)/\tau}$
0	1	$\boldsymbol{\overline{T}F}$	$x–a$	$1/Ce^{(x-a)/\tau}$
1	1	$\boldsymbol{TF}$	k	$1/Ce^{k/\tau}$
0	0	$\boldsymbol{\overline{TF}}$	0	$1/C$

Table 4: Energies and probabilities for particular activation patterns in the Hopfield network of Fig. 4

In order to get numerical predictions for the probabilities, we have to determine the normalization factor *C*. This factor arises from the requirement that the probabilities of the

[1] In Figure 4 we can see s_1 as the top node and s_2, s_3 as the left and right bottom nodes. In this case, the (symmetric) connection weights are $w_{12} = 1, w_{13} = -1,$ and $w_{23} = -k$.

available alternatives sum up to 1. For now, let us consider only the two activation patterns $\boldsymbol{T}\overline{\boldsymbol{F}}$ (***T***=1, ***F***=0) and $\overline{\boldsymbol{T}}\boldsymbol{F}$ (***T***=0, ***F***=1). This is the case of a binary logic without gluts. Using this case to guide normalization, we get the value $C = e^{-(x-a)/\tau} + e^{(x-a)/\tau}$. We can now calculate the following probability for the activation pattern $\boldsymbol{T}\overline{\boldsymbol{F}}$, that is the probability for accepting the truth of the proposition 'X is tall':

$$(3)\quad \mathrm{P}(\boldsymbol{T}\overline{\boldsymbol{F}}) = \frac{e^{(x-a)/\tau}}{e^{(x-a)/\tau}+e^{-(x-a)/\tau}} = \frac{1}{1+e^{-2(x-a)/\tau}} = \sigma(\frac{2(x-a)}{\tau})$$

The result is a logistic function (also called sigmoid function; Fig. 5). [2]

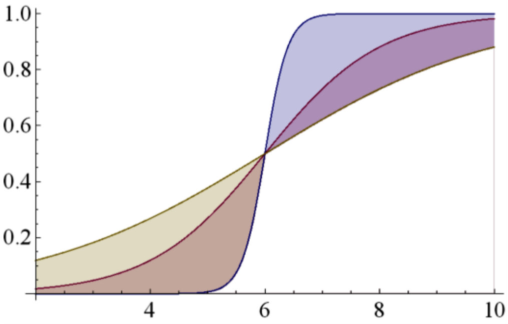

Fig. 5: Sigmoid-function $\sigma(\frac{x-a}{\tau})$ for a=6 and τ = 0.2, 1.0, and 2.0

One crucial assumption in most of the theoretical analyses of vagueness discussed so far concerns the existence of either gaps or gluts. In the present Hopfield model, gaps can be seen as activation pairs (***T***=0, ***F***=0) and gluts as activation pairs (***T***=1, ***F***=1). It is easy to see that for *k*=0 gaps and gluts result in the same probabilities.[3] Hence, from the present Hopfield model perspective, there seems not to be a substantial difference between glut and gap

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[2] A very similar function is used in the Rasch-model. For a discussion of this model and applications to vagueness see Verheyen, Hampton, and Storms (2010)

[3] This result is due to the choice of the numbers we have employed for true (1) and false (0); if we had taken true (0) and false (1) the converse result would appear. Interestingly, the choice true (+1) and false (–1) makes gaps and gluts totally symmetric.

theories. However, for the purposes of the following discussion, we prefer to tell the story in terms of gluts, since gluts are better grounded philosophically (Odrowaz-Sypniewska, 2010). In the following section, we will find another argument in favor of glut-theories: They and only they allow for interference effects, when we look for a generalization of the present model in terms of quantum probabilities.

Let us consider now the case of a binary logic with gluts which can be described by three activation pattern: $\boldsymbol{T\overline{F}}$ (true); $\boldsymbol{\overline{T}F}$ (false); $\boldsymbol{TF}$ (glut). The probability for the first activation pattern can be calculated as follows:

(4) $\mathrm{P}\left(\boldsymbol{T\overline{F}}\right) = \frac{1}{C}\, e^{(x-a)/\tau}$, where $C = e^{(x-a)/\tau} + e^{-(x-a)/\tau} + e^{-k/\tau}$

This is the probability distribution for ‘X is tall’, noting that the normalization is now computed to take into account gluts as well. It is not difficult to see that for $\frac{k}{\tau} \gg 1$, the contribution $1/e^{k/\tau}$ corresponding to the activation ($\boldsymbol{T}$=1, $\boldsymbol{F}$=1) can be ignored, and the resulting distribution is the logistic distribution given by equation (3) (i.e., the probability that a person is tall, ignoring gluts). However, the term $1/e^{k/\tau}$ cannot be ignored if the quotient $\frac{k}{\tau}$ is close to 1 or smaller than 1. In this case, the predictions of equation (4) will deviate from those based on the logistic distribution of equation (3).

What is the probability of rejecting the proposition ‘X is tall’? In their theoretical analysis based on Gricean pragmatics, Alxatib and Pelletier (2011) say that we can ignore the third answer possibility ‘can’t tell’, which accounts for less than 10% of responses. If we accept this idealization, then the probability for rejecting the proposition ‘X is tall’ is given by 1–$P\left(\boldsymbol{T\overline{F}}\right)$.

The probability distribution for accepting ‘X is not tall’ (see Table 2) can be calculated as follows:

(5) $\mathrm{P}(\overline{\boldsymbol{T}}\boldsymbol{F}) = \frac{1}{C}\, e^{-(x-a)/\tau}$, where $C = e^{(x-a)/\tau} + e^{-(x-a)/\tau} + e^{-k/\tau}$

The probability for rejecting the proposition 'X is not tall' is described as $1-P(\overline{\boldsymbol{T}}\boldsymbol{F})$.

Now we are ready to demonstrate how this neural network model can fit Alxatib and Pelletier's (2011) experimental data, with the help of three parameters: a (the mean expectation for x), k (the strength of interdependence between $\boldsymbol{T}$ and $\boldsymbol{F}$), and τ (the temperature). We fit the three parameters (minimizing Pearson chisquare) by using the data shown in Fig. 2. The optimal values for the parameters are: a=5.86, τ=0.24, k=0.29. As can be seen from Fig. 6, the present model fits the shown data fairly well (chisquare(8) =5.34; p =.72). Note that for each of five suspects there are two independent questions (accept tall/ is not tall and reject tall/ is not tall), so that in total we have 4x2=8 degrees of freedom.

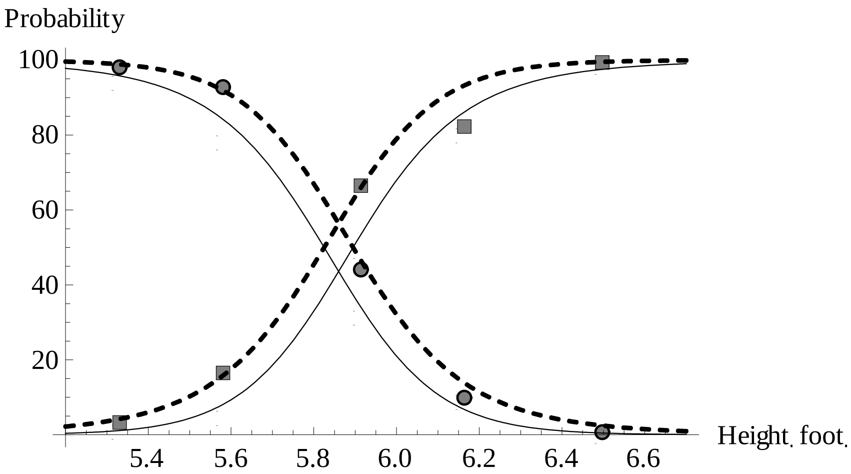

Fig. 6: Fitting the Alxatib and Pelletier (2011) data. ■ stands for accepting a proposition, ■ for rejecting a negation, ● for accepting a negation, and ● for rejecting a proposition.

Given the model fit relative to the frequencies concerning the statements with individual predicates, we can evaluate its ability to capture the data for 'and' and 'neither'. As Table 2 illustrates, in both cases the probability of acceptance of the conjunctive expression with 'and' / 'neither' is due to the 'null' column in the table. In our simple Hopfield network, this is described by the probability for $(\boldsymbol{T} = 1, \boldsymbol{F} = 1)$:

(6) $\mathrm{P}(\boldsymbol{TF}) = \frac{1}{C}\, e^{-k/\tau}$, where $C = e^{(x-a)/\tau} + e^{-(x-a)/\tau} + e^{-k/\tau}$

The probability for rejecting the corresponding expressions is $1 - P(\boldsymbol{TF})$ (again, the 'can't tell' answers are ignored). The distributions for $P(\boldsymbol{TF})$ and 1–$P(\boldsymbol{TF})$ were computed using the parameters identified before and are shown (together with the actual empirical data) in Fig. 7.

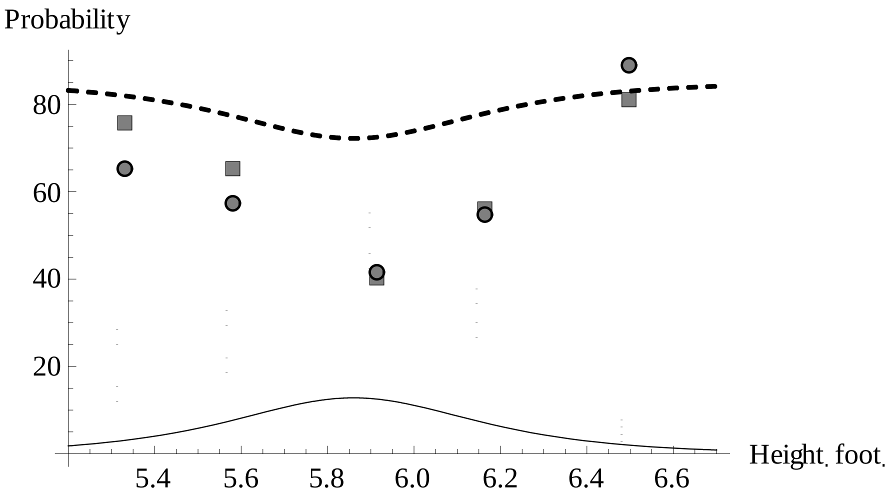

Fig. 7: Fitting the Alxatib and Pelletier (2011) data. ■ stands for accepting 'and', ■ for rejecting 'and', ● for accepting 'neither', and ● for rejecting 'neither'. The lower curve is the prediction for accepting 'and' / 'neither', the upper curve is the prediction for rejecting 'and' / 'neither'

In this case, the calculated predictions are quite different from the empirical finding and the result is not satisfying. The observed and predicted frequencies turned out to be significantly different from each other (chisquare(8)=167, p<.0005).

An alternative fitting approach would be to identify the parameters which minimize predictive error for all the available empirical data concurrently (i.e., the data in both Figures 3, 4). In this case, the optimization procedure led to parameter estimates of a=5.86, τ=0.22, and k=0.01. The predicted frequencies still deviated significantly from the observed ones (chi-square (16)=33.5, p<.005).

The above results indicate that any model based on classical probability theory would fail. Therefore, one could employ a model not based on probability theory at all. For example, in Fuzzy Trace Theory there is a distinction between verbatim and gist information. As the latter can be context/observer dependent, Fuzzy Trace Theory can predict several interesting violations of classical probability theory (Brainerd & Reyna, 2008). Such approaches are clearly valuable. Our interest presently is to explore whether Alxatib and Pelletier's (2011) data can be captured by a formal probabilistic framework. This has the advantage that particular models in different domains all have to obey the same set of basic principles. This both makes individual models more principled and offers the promise of a unified, coherent account for a diverse range of phenomena. We will make use of quantum probabilities and the idea of interference effects, in order to assess whether the shortcoming of the classical model can be overcome. [4]

5 Quantifying vagueness: Quantum probabilities and interference

[4] Another motivation for abandoning the model based on Alxatib's and Pelletier's theoretical analysis is that it does not make any distinction between the 'and' / 'neither' cases. This is a major problem, but one which goes beyond the present paper. The quantum approach shows some promise in terms of addressing this important problem, more so than the classical approach.

One of the main arguments for a quantum approach to cognitive phenomena is the existence of interference effects in higher cognitive processes such as perception, decision making, and reasoning (Aerts, 2009; Blutner, 2009; Bruza, Busemeyer, & Gabora, 2009; Busemeyer et al., 2011; Conte et al., 2008; Franco, 2007; Khrennikov, 2006; Pothos & Busemeyer, 2009; Primas, 2007). In the first subsection, we will introduce some basic concepts of quantum cognition. The second subsection applies the idea of interference to the problem of vagueness and demonstrates that this idea leads to an improved analysis of borderline contradictions. In the third subsection, we discuss the issue of compositionality in connection with recent findings by Sauerland (2010).

5.1 The disjunction puzzle, quantum probabilities and interference

The disjunction fallacy (Tversky & Shafir, 1992) occurs when decision makers prefer option A (versus $\overline{A}$) when knowing that event B occurs and also when knowing that event B does not occur, but they refuse A (or prefer $\overline{A}$) when not knowing whether or not B occurs. The disjunction fallacy is closely connected to violations of the 'sure-thing principle', one of the basic claims made by a (classically) rational theory of decision making. In decision making, this principle is just a psychological version of the law of total probability.

In everyday reasoning, however, human behaviour is not always consistent with the sure thing principle. For example, Tversky and Shafir (1992) reported that more students would purchase a non-refundable Hawaiian vacation if they were to know that they had passed or failed an important exam, compared to a situation where the exam outcome was unknown. Specifically, $\mu(A|B) = 0.54$, $\mu(A|\overline{B}) = 0.57$, and $\mu(A) = 0.32$, whereby A stands for the event of purchasing a Hawaiian vacation, B for the event of passing the exam, $\overline{B}$ for the event of not passing the exam, and μ for the averaged judgements of probability. Disjunction fallacies are fairly common in behaviour.

Assuming a classical (Bayesian) model of probabilities, the law of total probability requires that

(7) $P(A) = P(A|B)\cdot P(B) + P(A|\overline{B})\cdot P(\overline{B})$.

It is helpful to examine how the law of total probability arises. We require three assumptions. First, we assume that the underlying algebra of events is Boolean. That means essentially, that we have distributivity. In the present case, distributivity allows us to derive $A=AB\cup A\overline{B}$. Second, we assume that probability is an additive measure function. In particular, we have $P(A)=P(AB)+P(A\overline{B})$ for the two disjoint conjunctive events AB and $A\overline{B}$. Third, the standard definition of conditional probability allows us to write:

(8) $P(X|Y) = P(XY)/P(Y)$

The above three assumptions readily derive the law of total probability and explain the classical requirement that the predicted disjunction effect must always be zero.

Let us see now how the use of quantum probabilities changes the situation. Instead of using possible worlds as the underlying ontology for constructing propositions, quantum theory makes use of vectors in a Hilbert space $\mathcal{H}$ (i.e., a vector space upon which an inner product is defined and which makes use of complex numbers). Further, one can define linear operators on $\mathcal{H}$. A special kind of linear operator is the so-called projection operator, which projects vectors to certain subspaces of $\mathcal{H}$. The algebraic structure underlying these projection operators is not a Boolean algebra, but an orthoalgebra. An orthoalgebra is similar to a Boolean algebra, but one key principle of Boolean algebra can be violated: the principle of distributivity. Recall, this principle was necessary for deriving equation (7).

In the quantum approach, propositions are modeled by projection operators (or, equivalently, subspaces of $\mathcal{H}$). If $\boldsymbol{A}$ is a projection operator, then the functional composition of it with itself is $\boldsymbol{A}$ again: $\boldsymbol{AA}=\boldsymbol{A}$. In combining projection operators, order can matter. That is, it can be that $\boldsymbol{AB}\neq\boldsymbol{BA}$ for two projection operators $\boldsymbol{A}$ and $\boldsymbol{B}$. Interestingly, if all projection operators relative to a given Hilbert space commute (i.e., $\boldsymbol{AB}=\boldsymbol{BA}$), then we get a Boolean algebra of projectors. The important conclusion is that the algebra of projection operators contains the Boolean algebra as a special case (when projectors obey commutativity).

For what follows, it is essential to appreciate two main differences between the treatment of classical propositions and quantum proposition (projections). First, instead of union $A\cup B$ in the classical case, we consider the sum operation $\boldsymbol{A}+\boldsymbol{B}$ in the quantum case (constructing the smallest subspace that contains the two subspaces corresponding to $\boldsymbol{A}$ and $\boldsymbol{B}$. The second difference refers to complementation. In the quantum case, a negated propositions refers to a subspace orthogonal to the original one. We will write $\overline{\boldsymbol{A}}=\boldsymbol{I}-\boldsymbol{A}$ for the orthogonal projection operator[5] ($\boldsymbol{I}$ is the identity operator mapping any vector to itself). It is easy to see that $\boldsymbol{A}\overline{\boldsymbol{A}} = \overline{\boldsymbol{A}}\boldsymbol{A}=(\boldsymbol{I}-\boldsymbol{A})\boldsymbol{A}=\boldsymbol{IA}-\boldsymbol{AA}=\boldsymbol{A}-\boldsymbol{A}=0$.

Let us see now what happens with equation (7) in the quantum case. Even in the quantum case, a probability function is an additive measure function (now assuming that the two considered parts are orthogonal to each other). It turns out that the direct translation of equation (8) into Hilbert spaces does not work, since $\boldsymbol{XY}$ is not a Hermitian operator, if $\boldsymbol{X}$ and $\boldsymbol{Y}$ do not commute (this means that the operator is not associated with real values, an obvious requirement when considering behavioral models). A definition that does work is given by the following equation (Niestegge, 2008)[6]:

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[5] Sometimes the symbol $\boldsymbol{A}^{\perp}$ is used for indicating the orthocomplement. Since, from the context, it is always clear whether orthocomplementation is meant or the usual set-theoretic complement, we will use the form $\overline{\boldsymbol{A}}$ in both cases.

[6] Niestegge follows the theory of Lüders (1951).

(9) $P(\boldsymbol{X}|\boldsymbol{Y}) = P(\boldsymbol{YXY})/P(\boldsymbol{Y})$

The operator $\boldsymbol{YXY}$ is also called asymmetric conjunction. Note that P(YXY)=P(Y and then X), which is how Busemeyer et al. (2011) modeled conjunction in human decision making (see also Blutner, 2009; Bruza et al., 2011). In order to get the quantum version of equation (7), we can decompose the projector $\boldsymbol{A}$ in the following way:

(10) $\boldsymbol{A}=\boldsymbol{IAI} = (\boldsymbol{B}+\overline{\boldsymbol{B}})\boldsymbol{A}(\boldsymbol{B}+\overline{\boldsymbol{B}}) = \boldsymbol{BAB}+\overline{\boldsymbol{B}}\boldsymbol{A}\overline{\boldsymbol{B}}+\boldsymbol{BA}\overline{\boldsymbol{B}}+\overline{\boldsymbol{B}}\boldsymbol{AB}$

The four parts of this decomposition are orthogonal to each other. Using the definition of equation (9), we get

(11) $P(\boldsymbol{A})=P(\boldsymbol{A}|\boldsymbol{B})\,P(\boldsymbol{B})+P(\boldsymbol{A}|\overline{\boldsymbol{B}})\,P(\overline{\boldsymbol{B}})+\partial(\boldsymbol{B},\boldsymbol{A})$, where $\partial(\boldsymbol{B},\boldsymbol{A})=P(\boldsymbol{BA}\overline{\boldsymbol{B}}+\overline{\boldsymbol{B}}\boldsymbol{AB})$

The term $\partial(\boldsymbol{B},\boldsymbol{A})$ is called the interference term. It is zero if $\boldsymbol{B}$ and $\boldsymbol{A}$ commute, in which case equation (11) reduces to equation (7).

In the general case, by using equation (11), the disjunction effect can be related to the interference term $P(\boldsymbol{BA}\overline{\boldsymbol{B}}+\overline{\boldsymbol{B}}\boldsymbol{AB})$:

(12) $\nabla(\boldsymbol{A},\boldsymbol{B}) = \partial(\boldsymbol{B},\boldsymbol{A}) = P(\boldsymbol{BA}\overline{\boldsymbol{B}}+\overline{\boldsymbol{B}}\boldsymbol{AB})$

The probability of a proposition $\boldsymbol{A}$ is the squared length of the projection of ψ into the subspace generated by the projector $\boldsymbol{A}$:

(13) $P_{\psi}(\boldsymbol{A}) = \langle \boldsymbol{A}\psi | \boldsymbol{A}\psi \rangle = \langle \psi | \boldsymbol{A}\psi \rangle$

With the help of this so-called Born rule, we can calculate the following expression for the interference term, in the case of pure states:

$$\text{(14)}\ \ P_{\psi}(\boldsymbol{BA\overline{B}}+\boldsymbol{\overline{B}AB}) = 2\sqrt{P_{\psi}(\boldsymbol{A}|\boldsymbol{B})P_{\psi}(\boldsymbol{B})}\ \cdot \sqrt{P_{\psi}(\boldsymbol{A}|\boldsymbol{\overline{B}})P_{\psi}(\boldsymbol{\overline{B}})} \cdot \cos(\Delta).$$ [7]

The phase shift relates to the impact of knowing B or $\overline{B}$ for assessing the likelihood of A. This angle is zero if the subspaces corresponding to the events A and then B (or A and then $\overline{B}$) are orthogonal. If they are not orthogonal, the subspaces are incompatible. This means that if a participant decides A, then he/she must necessarily be undecided regarding B. From a psychological perspective, the interference term is the correlation between two decision paths: (1) First considering you won't pass the exam and then considering the trip to Hawaii and (2) first considering you will pass the exam and then considering the trip to Hawaii. A negative correlation corresponds to a negative interference term ($\partial(\boldsymbol{B},\boldsymbol{A})<0$ in (11)), which will negatively impact on the law of total probability (i.e., reduce the probability for the trip, in the unknown case), and conversely for a positive correlation.

5.2 Quantifying vagueness using quantum probabilities

Now we are prepared to apply the quantum formalism and the key idea of interference to the case of vagueness. The first step is to reformulate the approach developed in Section 4 by using the formalism of projection operators. Let us assume that the state of tallness of a suspect X is described by a state vector ψ_X. We have to reconstruct the association between

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[7] Proof of this equation: $P_{\psi}(\boldsymbol{BA\overline{B}} + \boldsymbol{\overline{B}AB}) = \langle\psi|\ \boldsymbol{BA\overline{B}} + \boldsymbol{\overline{B}AB}|\ \psi\rangle = \langle\psi|\ \boldsymbol{\overline{B}AB}|\ \psi\rangle^{*} + \langle\psi|\ \boldsymbol{\overline{B}AB}|\ \psi\rangle =$
$2\ \mathrm{RE}(\langle\psi|\boldsymbol{\overline{B}AB}|\ \psi\rangle) = 2\ \mathrm{RE}(\langle\boldsymbol{A\overline{B}}\psi|\ \boldsymbol{AB}\psi\rangle) = 2\ \|\ \boldsymbol{A\overline{B}}\ \psi\|\ \|\ \boldsymbol{AB}\ \psi\| \cdot \cos(\Delta) =$
$2\sqrt{P_{\psi}(\boldsymbol{A}|\boldsymbol{B})\quad P_{\psi}(\boldsymbol{B})} \cdot \sqrt{P_{\psi}(\boldsymbol{A}|\boldsymbol{\overline{B}})P_{\psi}(\boldsymbol{\overline{B}})} \cdot \cos(\Delta)$. Note that $P_{\psi}(\boldsymbol{BAB}) = \|\ \boldsymbol{AB}\psi\|^2$ and $P_{\psi}(\boldsymbol{\overline{B}A\overline{B}}) = \|\ \boldsymbol{A\overline{B}}\ \psi\|^2$. In other words, as noted in the text, $P_{\psi}(\boldsymbol{BAB})$ can be thought of as $P_{\psi}(B \text{ and then } A)$.

the three activation patterns ($\boldsymbol{T}$=1, $\boldsymbol{F}$=0), ($\boldsymbol{T}$=0, $\boldsymbol{F}$=1), and ($\boldsymbol{T}$=1, $\boldsymbol{F}$=1) and the three truth values, T, F, and null (=glut). We assume two commuting projection operators $\boldsymbol{T}$ and $\boldsymbol{F}$ and consider the three combinations $\boldsymbol{T}\overline{\boldsymbol{F}}$, $\overline{\boldsymbol{T}}\boldsymbol{F}$, and $\boldsymbol{TF}$. As mentioned in Section 4, we exclude gaps ($\overline{\boldsymbol{T}}\,\overline{\boldsymbol{F}}$). Hence, we require the assumption:

(15) $\boldsymbol{T}\overline{\boldsymbol{F}}+\overline{\boldsymbol{T}}\boldsymbol{F}+\boldsymbol{TF}=\boldsymbol{I}$

Let us stipulate now the probabilities of the three combinations in the state ψ_X as follows:

(16) a. $\mathrm{P}_X(\boldsymbol{T}\overline{\boldsymbol{F}}) = \langle\psi_X|\boldsymbol{T}\overline{\boldsymbol{F}}|\psi_X\rangle = \frac{1}{C}e^{(x-a)/\tau}$

b. $\mathrm{P}_X(\overline{\boldsymbol{T}}\boldsymbol{F}) = \langle\psi_X|\overline{\boldsymbol{T}}\boldsymbol{F}|\psi_X\rangle = \frac{1}{C}e^{(a-x)/\tau}$

c. $\mathrm{P}_X(\boldsymbol{TF}) = \langle\psi_X|\boldsymbol{TF}|\psi_X\rangle = \frac{1}{C}e^{-k/\tau}$

(17) $C = e^{(x-a)/\tau} + e^{(a-x)/\tau} + e^{-k/\tau}$

It is obvious that equation (16a) exactly corresponds to equation (4) if the normalization constant *C* is set as in equation (17). Similarly, (16b) corresponds to (5) and (16c) to (6). As expected, we can reconstruct these classical probabilities by using the quantum formalism with *commuting* projections.

Let us look now for a quantum solution, assuming that the projection operators $\boldsymbol{T}$ and $\boldsymbol{F}$ no longer commute: $\boldsymbol{TF}\neq\boldsymbol{FT}$. Since we assume a glut theory, the operators $\boldsymbol{T}$ and $\boldsymbol{F}$ are not orthogonal to each other, which implies that $\boldsymbol{TF}\neq 0$. This gives the possibility of interference between $\boldsymbol{T}$ and $\boldsymbol{F}$. The crucial mathematical point is that equation (15) is no longer valid in the case of non-commuting operators. It has to be replaced by the following equations:[8]

(18) a. $\boldsymbol{T\overline{F}T} + \boldsymbol{\overline{T}F\overline{T}} + \boldsymbol{TFT} + \boldsymbol{TF\overline{T}} + \boldsymbol{\overline{T}FT} = \boldsymbol{I}$

b. $\boldsymbol{\overline{F}T\overline{F}} + \boldsymbol{F\overline{T}F} + \boldsymbol{FTF} + \boldsymbol{FT\overline{F}} + \boldsymbol{\overline{F}TF} = \boldsymbol{I}$

The last two terms of the sums in (18a) and (b) are the interference terms (similar to the last two terms in (10)). They vanish if $\boldsymbol{T}$ and $\boldsymbol{F}$ commute and each sum reduces to equation (15).

Introduce the following abbreviations for arbitrary projectors $\boldsymbol{X}$ and $\boldsymbol{Y}$:

(19) a. $\boldsymbol{X.Y} =_{\text{def}} ½\,(\boldsymbol{XYX}+\boldsymbol{YXY})$ (symmetric conjunction)

b. $\partial(\boldsymbol{X}, \boldsymbol{Y}) = \boldsymbol{XY\overline{X}}+\boldsymbol{\overline{X}YX}$ (interference term)

Summing up (18a) and (18b), we get the following decomposition:

(20) $\boldsymbol{T}.\boldsymbol{\overline{F}} + \boldsymbol{F}.\boldsymbol{\overline{T}} + \boldsymbol{T}.\boldsymbol{F} + ½\,\partial(\boldsymbol{T},\boldsymbol{F}) + ½\,\partial(\boldsymbol{F},\boldsymbol{T}) = \boldsymbol{I}$

Again, if $\boldsymbol{T}$ and $\boldsymbol{F}$ commute the interference terms vanish and we obtain the noted correspondence with equation (15). This fact and a comparison with the stipulations in equations (16)(a-c) justifies the following assumption:

[8] This can be proved by noting that $\boldsymbol{F} + \boldsymbol{\overline{F}} = \boldsymbol{I}$ and then $(\boldsymbol{T} + \boldsymbol{\overline{T}})(\boldsymbol{F} + \boldsymbol{\overline{F}})(\boldsymbol{T} + \boldsymbol{\overline{T}}) = \boldsymbol{I}$, and finally eliminating all expressions that contain 'gap terms' (such as $\overline{\boldsymbol{TF}}$).

(21) a. $\mathrm{P}_X(\boldsymbol{T}.\overline{\boldsymbol{F}}) = \langle\psi_X|\boldsymbol{T}.\overline{\boldsymbol{F}}|\psi_X\rangle = \frac{1}{C}e^{(x-a)/\tau}$

b. $\mathrm{P}_X(\boldsymbol{F}.\overline{\boldsymbol{T}}) = \langle\psi_X|\boldsymbol{F}.\overline{\boldsymbol{T}}|\psi_X\rangle = \frac{1}{C}e^{(a-x)/\tau}$

c. $\mathrm{P}_X(\boldsymbol{T}.\boldsymbol{F}) = \langle\psi_X|\boldsymbol{T}.\boldsymbol{F}|\psi_X\rangle = \frac{1}{C}e^{-k/\tau}$

However, because of the existence of the two interference terms in the sum (20), the normalization constant *C* is different from the earlier result in (17) and contains two additional terms, $e^{(x-a-k)/2\tau}\cdot\cos(\Delta_1)$ and $e^{(a-x-k)/2\tau}\cdot\cos(\Delta_2)$, as shown in (22). These additional terms express the probability relating to the interference terms, and have been computed using equation (14).

(22) $C = e^{(x-a)/\tau} + e^{(a-x)/\tau} + e^{-k/\tau} + e^{(x-a-k)/2\tau}\cdot\cos(\Delta_1) + e^{(a-x-k)/2\tau}\cdot\cos(\Delta_2)$

The treatment is entirely analogous to that in Section 4, and so the probabilities for accepting the truth of the propositions 'X is tall', 'X is not tall', and 'X is tall and not tall', are given by equations (21a), (21b), and (21c), respectively. The only difference compared to the classical case is that in the quantum case the normalization factor is given by Equation (22). It is this factor that contains the two interference terms. Further, the treatment of *rejecting* the corresponding propositions is again in exact correspondence to the classical case.[9]

As in Section 4, we fitted the relevant data of Fig. 2 and 3, now by using the quantum model. The optimal parameter values we identified were a=5.86, τ=0.24, k=0, and $\cos(\Delta)=-0.35$ (assuming $\Delta_1=\Delta_2=\Delta$). We received a good confirmation of the quantum model: chisquare(16) =5.3; p>.99. Fig. 8 shows the result for the 'and' data. Further, we did not find

[9] For calculating the corresponding probabilities for rejections we have generally assumed a proportion of 15% for the 'can't tell' answers, which is the mean value reported by Alxatib and Pelletier (2011) for the 'and'/'neither' conditions. Thus, the probability for rejecting a proposition plus the probability for accepting it should sum up to .85 (instead of 1).

a significant difference when fitting the parameters for the data of Fig. 2 separately and applying the found values for describing the 'and' data of Fig. 3 (chisquare(8)=2.98, p>0.9).

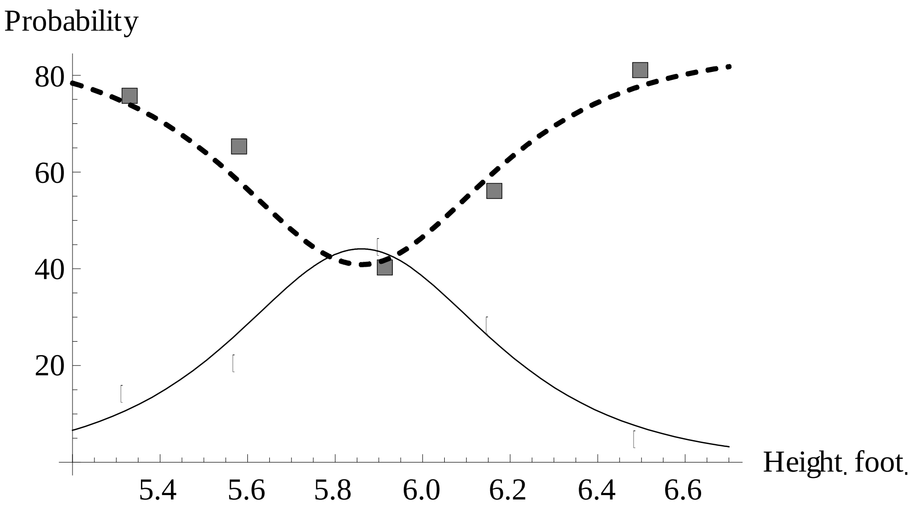

Fig. 8: Fitting the Alxatib and Pelletier (2011) data (both in Fig. 2 and 3). ■ stands for accepting 'and', ■ stands for rejecting 'and'. The curves show the corresponding predictions of the probabilistic model using interference.

In Fig. 9 we show the sum of the probabilities for accepting 'X is tall', accepting 'X is not tall' and accepting 'X is tall and not tall'. The classical model predicts that these three probabilities should sum to 1. Further, the classical model predicts that the probability for rejecting 'X is tall', plus the probability for rejecting 'X is not tall', minus the probability for accepting 'and' should give 1. Fig. 9 shows that empirically this is not the case (chisquare(4)=95.5; p<.005, testing against the null hypotheses that the proportions for all cases add up to 100%).

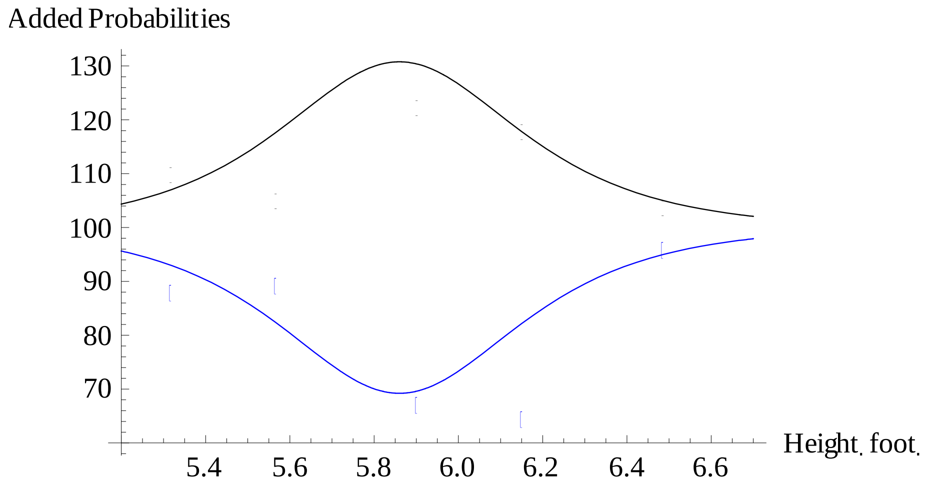

Fig. 9: Sums of probabilities. ▲ stands for the sum of the measured probabilities for accepting 'X is tall', accepting 'X is not tall' and accepting 'X is tall and not tall'; ■ stands for the probability for rejecting 'X is tall' plus the probability for rejecting 'X is not tall' minus the probability for accepting 'and'. The curves show the corresponding predictions of the quantum interference model, with the same parameters as those employed in Fig. 8.

In contrast to the classical model (which has to predict a uniform 100% probability for all cases in Fig. 9), the quantum model produced a fairly satisfying prediction: chi-square(4)=5.47; p=0.24.

Note that the present version of the quantum model does not explain the difference between 'X is tall and not tall' and 'X is neither tall nor not tall'. Both reduce to the logical expression 'X is tall and X is not tall' vs. 'X is not tall and X is tall' (assuming the law of double negation).

Finally, note that the quantum model appears to have one more parameter, than the classical one. In practice this was not the case. When fitting the quantum model, it proved difficult to identify the optimal solution, without fixing either k or Δ, as the two parameters appeared to strongly interact.[10] Thus, the quantum model fits were obtained after having set k=0 and so the quantum and the classical model both had the same number of parameters (note that equally good fits can be obtained if we fix Δ and optimize k instead). Interestingly, the choice of k=0 corresponds to the case where the three probabilities defined in equation (21) are equal for borderline cases (x=a). Hence, the decision k=0 corresponds to a kind of entropy maximization.

5.3 Compositionality and quantum probabilities

In Section 2.5, fuzzy logic was mentioned as a formalism that allows for a continuum of intermediate truth-values, between totally true (=1) and totally false (=0): $1 \geq \mu_X(A) \geq 0$. The symbol A stands for a particular concept (such as 'chair') and X stands for a particular instance. A special logic was proposed for combining concepts A and B. For example, in the original literature (Zadeh 1965) the min-function was proposed for the conjunction of two concepts: $\mu_X(A\&B) = \min(\mu_X(A), \mu_X(B))$.

A common characteristic of all the different approaches within fuzzy logic concerns compositionality. In the present case, we can express the requirement from compositionality by demanding the existence of a two-place function f such that $\mu_X(A\&B) = f[(\mu_X(A),\mu_X(B)]$. This function takes the values $\mu_X(A)$ and $\mu_X(B)$ to form the value for $\mu_X(A \,\&\, B)$.

Recently, Sauerland (2010) has discussed the potential of fuzzy logic for modeling borderline contradictions. He suggested that several arguments against fuzzy logic – including the arguments put forward by Kamp (1975), Fine (1975), Kamp and Partee (1997) – are based on the claim that a sentence of the form 'A and not A' is always logically false, even if A does not have a definite truth value. It goes without saying that philosophers and logicians have not sought an empirical verification of this claim, but have taken this as self-evident. Sauerland

[10] This strong interactions do not mean that the parameters are not mathematically independent. Though we do not have a mathematical proof that the parameters are fully identifiable, simulations suggest that the parameters k and Δ are not functions of each other.

(2010) concluded that the results by Alxatib and Pelletier (2010) and others "show that the argument against fuzzy logic for linguistic semantics by Kamp (1975) and Fine (1975) is less clear-cut than it was previously made out to be" (p. 9).

However, this does not mean that we can adopt fuzzy logic for understanding vagueness and the interpretation of borderline characterizations. Sauerland (2010) gives an argument in support of this view based on violations of compositionality. Consider two properties *A* (say, 'being tall') and *B* (say, 'being rich') and a particular instance *X* (say 'a 5' 10''guy who has $100,000'). Let us assume that the instance *X* is a borderline case for both the property *A* and the property *B*. Let us further assume that $\mu_X(A) \approx \mu_X(B) \approx 50\%$. Assuming compositionality, we are led to predict that $\mu_X(A\&\neg A) \approx \mu_X(A\&\neg B)$, i.e., that the membership values for borderline contradictions and non-contradictory conjunctions should be approximately the same. This hypothesis was falsified empirically. Sauerland (2010) found on average 47.3% agreement to the contradictions $\mu_X(A\&\neg A)$ and only 34.4% agreement to the non-contradictory conjunctions $\mu_X(A\&\neg B)$, which is a substantial difference. This example makes it clear that fuzzy logic cannot provide a complete account of human judgments, concerning such conjunctions.

Violations of compositionality demand alternative models. Models based on quantum probabilities have the potential to solve the raised problems. They are clearly non-compositional, and they are able to solve the conjunction/disjunction puzzle and other puzzles of human behavior (cf. Busemeyer & Bruza, 2012). One approach taken is to adapt general probabilistic methods developed in quantum physics to determine whether a system is compositional, or not (e.g., Aerts & Sozzo, 2011; Bruza et al., forthcoming). These analytic methods build on formal results that imply non-compositionality equates with not being able to model the system in a single probability space. If such results carry across to cognitive science, then non-compositional conjunctions cannot be modeled in a single probability space, a surprising result with potentially significant modeling consequences, as modeling usually proceeds under that assumption that a single, appropriately defined probability space is sufficient.

6 General discussion and conclusions

In a recent paper, Aerts (2009) points out that “there is a well-established corpus of literature in theoretical physics describing methods to prove the presence of quantum structures by ‘only looking at experimental data’...” (p. 315). There are some fields of research within cognitive science where the situation is similar to theoretical physics. First, there is a long and established empirical literature showing deviations from set theoretic rules in conceptual combination (Hampton, 1988a, 1988b; Storms, De Boeck, Van Mechelen, & Ruts, 1996). The relevant empirical results include violations of the conjunction and disjunction rules, the famous ‘guppy effect’, and cases of ‘dominance’, ‘over- and underextension’, which were all successfully described on the basis of quantum principles (Aerts, 2009; Aerts & Gabora, 2005). Second, in one of the most impactful empirical traditions in psychology, Tversky, Kahneman and their colleagues demonstrated how naïve observers often produce judgments at odds with many of the key axioms of classical probability theory, such as the law of total probability or the requirement that a conjunction can never be more probable than individual constituents (Tversky & Shafir, 1992). Such and related results also have natural and intuitive explanations with quantum schemes (e.g., Busemeyer & Bruza, 2012; Busemeyer et al., 2011; Pothos & Busemeyer, 2009). Finally, recent analyses have revealed many non-classical effects in areas as diverse as memory (Bruza, Kitto, Nelson & McEvoy, 2009), perception of bistable figures (Atmanspacher, Filk & Römer, 2004), and concepts (Blutner, 2009, 2012; Busemeyer & Bruza, 2012).

The present paper considers the linguistic phenomenon of vagueness, especially in relation to borderline contradictions. In this domain, quantitative models are rare. Of course, there is fuzzy set theory and supervaluation theory, and some authors (including Kamp &

Partee, 1997) have tried to model quantitative judgments of vagueness by elaborating on these models. However, such attempts have not been very successful. A real breakthrough, both empirically and theoretically, was made by Alxatib and Pelletier (2011), as we discussed in Section 3. Empirically, their paper reports data showing that the same participants, who consider the sentences 'X is tall' and 'X is not tall' as false, consider the apparently contradictory sentence 'X is tall and not tall' as acceptable.

Unfortunately, the analysis of Alxatib and Pelletier is not sufficient for a quantitative model of the data. This shortcoming was overcome in Section 4, where we proposed a probabilistic model for vagueness, based on Alxatib and Pelletier's (2011) ideas. We have shown that this classical model can lead to a satisfying quantitative description for the data of the distributions for accepting and rejecting the clauses 'X is tall' and 'X is not tall'. However, the classical model was not able to fit the additional data for borderline contradictions, such as 'X is tall and not tall'.

A reformulation of the classical model in terms of commuting projection operators was given at the beginning of Section 5. This reformulation gives the term 'classical model' a formal expression: classical models are models based on commuting operators. Consequently, the notion of 'classical' we are using and the notion of 'classical', as used in theoretical physics, are one and the same. The proposal to look for quantum effects then is the proposal to look for models based on non-commuting operators. A consequence of such an approach is the possibility of interference effects analogous to those arising in quantum models in physics. In the present case of vagueness, these interference terms introduce the right corrections to the probability estimates for different statements, which are necessary to account for borderline contradictions.

Intuitively, it is the superposition of 'tall' and 'not tall' which can lead to interference effects. An interference effect can appear, since the superposed terms are not 'orthogonal', i.e. there is some overlap between 'tall' and 'not tall', as required by the assumed glut theory.

This is an interesting point, since a formulation in terms of gap theory would have *excluded* the potential of interference effects (without some overlap, there can be no interference). While in the classical case (Section 4) and the framework of Hopfield networks, gap-theories and glut theories can be seen as 'notational variants', this is not true if a quantum approach is adopted. Hence, the quantum approach can help to resolve the old philosophical issue of how to decide between gap and glut theories (cf. Odrowaz-Sypniewska, 2010).

The present model of quantifying vagueness is restricted in several respects. First, it only considers unmarked gradable adjectives like 'tall', which involve an ordering along a dimension of linear extent and which have relative (context-dependent) standards (Kennedy, 2007; Toledo & Sassoon, 2011). We did not consider adjectives like 'full', 'open', 'closed', 'wet', 'dark' etc., which have absolute (maximum/minimum) standards, but still allow for graduation. Further, we did not consider the distributional details of the comparison class (Solt, 2011). Second, we did not consider the observed differences between the acceptance/rejection data for 'X is tall and not tall' and 'X is neither tall nor not tall'. From a quantum modeling perspective, such discrepancies suggest that the subspace corresponding to the concept "tall and not tall" is not orthogonal to the subspace corresponding to the concept "neither tall nor not tall". The quantum model presented in this paper could, in principle, be generalized to model the interference term generated by the incompatibility of these two subspaces.

A third limitation is that our way of modeling borderline contradictions can potentially be applied to the conceptual combination data of Hampton (1987, 1988a, 1988b; see also Storms et al., 1996), but we have not pursued this direction in this paper. We think that an adaptation of the present model may well be able to account for these data. This is an interesting task, which could possibly be contrasted with Aerts' (2009) thesis of the need of introducing ideas from quantum field theory, in order to deal with the Hampton data.

It is somewhat remarkable that in 1982, just as John Hopfield proposed his model of a recurrent neural network with content-addressable memory (Hopfield, 1982), Richard Feynman published his first paper on quantum computation (Feynman, 1982). In Section 4, we made an attempt of quantifying vagueness by using a simple Hopfield network. It would be an interesting task to look for a quantum version of the original Hopfield model, following the line of quantum-inspired neural architectures (e.g. Menneer & Narayanan, 1995; Ventura & Martinez, 1998). Possibly, a powerful generalization of the present approach can be developed in this way.

References

Aerts, D. (2009). Quantum structure in cognition. *Journal of Mathematical Psychology, 53*, 314-348.

Aerts, D., & Gabora, L. (2005). A state-context-property model of concepts and their combinations I: The structure of the sets of contexts and properties. *Kybernetes*, *34* (1), 151-175.

Aerts, D. & Sozzo, S. (2011). Quantum structure in cognition: why and how concepts are entangled. In *Proceedings of the Quantum Interaction Conference*. Berlin: Springer.

Åkerman, J., & Greenough, P. (2009). Vagueness and non-indexical contextualism. In S. Sawyer (Ed.), *New Waves in Philosophy of Language* (pp. 8-23). Basingstoke: Palgrave Macmillan.

Alxatib, S., & Pelletier, J. (2011). On the psychology of truth-gaps. In R. Nouwen, R. van Rooij, U. Sauerland, & H.-C. Schmitz (Eds.), *Vagueness in Communication* (pp. 13-36). Berlin: Springer.

Atmanspacher, H., Filk, T., & Römer, H. (2004). Quantum Zeno features of bistable perception. *Biological Cybernetics, 90*, 33-40.

Blutner, R. (2009). Concepts and bounded rationality: An application of Niestegge's approach to conditional quantum probabilities. In L. Accardi, G. Adenier, C. Fuchs, G. Jaeger, A. Y. Khrennikov, J.-Å. Larsson, & S. Stenholm (Eds.), *Foundations of Probability and Physics-5* (pp. 302-310). New-York: American Institute of Physics Conference Proceedings.

Blutner, R. (2012). Questions and answers in an orthoalgebraic approach. *JoLLI*, *21*, 237-277.

Bonini, N., Osherson, D., Viale, R., & Williamson, T. (1999). On the psychology of vague predicates. *Mind & Language, 14*, 377-393.

Bosch, P. (1983). 'Vagueness' is context-dependence. A solution to the Sorites paradox. In T. Ballmer, & M. Pinkal (Eds.), *Approaching Vagueness* (pp. 189-210). Amsterdam: North Holland.

Brainerd, C. J. & Reyna, V. F. (2008). Episodic over-distribution: a signature effect of recollection without familiarity. *Journal of Memory and Language, 58*, 765-786.

Bruza, P., Busemeyer, J. R., & Gabora, L. (2009). Special issue on quantum cognition. *Journal of Mathematical Psychology, 53* (5).

Bruza, P., Kitto, K., Nelson, D., & McEvoy, C. L. (2009). Is there something quantum-like about the human mental lexicon? *Journal of Mathematical Psychology, 53*, 362-377.

Bruza, P. D., Kitto, K., Ramm, B., & Sitbon, L. (forthcoming). The non-compositionality of concept combinations.

Budescu, D., & Wallsten, T. (1995). Processing Linguistic Probabilities: General principles and Empirical Evidence. In J. R. Busemeyer, R. Hastie, & D. L. Medin (Eds.), *Decision making from a cognitve perspective* (Vol. Psychology of Learning and Motivation, Volume 32, pp. 275-318). San Diego, CA: Academic Press.

Busemeyer, J. R., & Bruza, P. D. (2012). *Quantum cognition and decision*. Cambridge, UK Cambridge University Press.

Busemeyer, J. R., Pothos, E. M., Franco, R., & Trueblood, J. S. (2011). A quantum theoretical explanation for probability judgment errors. *Psychological Review, 118* (2), 193-218.

Conte, E., Khrennikov, A. Y., Todarello, O., Robertis, R. D., Federici, A., & Zbilut, J. P. (2008). A Preliminary experimental verification on the possibility of Bell inequality violation in mental states. *NeuroQuantology, 6* (3), 214-221.

Dalrymple, M., Kanazawa, M., Kim, Y., Mchombo, S., & Peters, S. (1998). Reciprocal expressions and the concept of reciprocity. *Linguistics and Philosophy, 21*, 159-210.

Feynman, R. (1982). Simulating physics with computers. *International Journal of Theoretical Physics, 21*, 467-488.

Fine, K. (1975). Vagueness, truth and logic. *Synthese, 30* (3), 265-300.

Franco, R. (2007). The conjunction fallacy and interference effects. *Arxiv preprint arXiv:0708.3948*.

Hampton, J.A.(1987). Inheritance of attributes in natural concept conjunctions. *Memory and Cognition, 15*, 55-71.

Hampton, J.A. (1988a). Disjunction of natural concepts. *Memory and Cognition, 16*, 579-591.

Hampton, J.A. (1988b). Overextension of conjunctive concepts: Evidence for a unitary model of concept typicality and class inclusion. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 14* (1), 12-32.

Hampton, J.A. (2007). Typicality, graded membership, and vagueness. *Cognitive Science, 31*, 355-384.

Hopfield, J. J. (1982). Neural networks and physical systems with emergent collective computational abilities. *Proceedings of the National Academy of Sciences, 79*, 2554-2558.

Hyde, D. (1997). From heaps and gaps to heaps of gluts. *Mind, 106*, 641-660.

Kamp, H. (1975). Two theories about adjectives. In E. Keenan (Ed.), *Formal semantics of natural language* (pp. 123-155). Cambridge: Cambridge University Press.

Kamp, H. (1981). The Paradox of the Heap. In U. Mönnich (Ed.), *Aspects of philosophical logic* (pp. 225-277). Dordrecht: Reidel.

Kamp, H., & Partee, B. (1997). Prototype theory and compositionality. *Cognition, 64* (2), 189-206.

Kennedy, C. (2007). Vagueness and grammar: The semantics of relative and absolute gradable adjectives. *Linguistics and Philosophy, 30*, 1-45.

Khrennikov, A. Y. (2006). Quantum-like brain: "Interference of minds". *BioSystems, 84* (3), 225-241.

Lüders, G. (1951). Über die Zustandsänderung durch den Meßprozeß. *Annalen der Physik, 8,* 322-328.

Menneer, T., & Narayanan, A. (1995). *Quantum-inspired neural networks* (Technical report R329). Exeter, UK.: Department of Computer Science.

Niestegge, G. (2008). An Approach to quantum mechanics via conditional probabilities. *Foundations of Physics, 38*, 241-256.

Odrowaz-Sypniewska, J. (2010). Vagueness and contextualism. In P. Stalmaszczyk (Ed.), *Philosophy of Language and Linguistics. Volume 1: The formal turn* (pp. 169-182). Heusenstamm: Ontos Verlag.

Osherson, D. N., & Smith, E. E. (1981). On the adequacy of prototype theory as a theory of concepts. *Cognition, 9*, 35-58.

Osherson, D. N., & Smith, E. E. (1997). On typicality and vagueness. *Cognition, 64,* 189-206.

Pothos, E. M., & Busemeyer, J. R. (2009). A quantum probability explanation for violations of 'rational' decision theory. *Proceedings of the Royal Society B: Biological Sciences, 276*.

Priest, G. (1989). Reasoning about truth. *Artificial Intelligence, 39* (2), 231-244.

Priest, G. (2002). Paraconsistent logic. In D. M. Gabbay & F. Guenthner (Eds.), *Handbook of philosophical logic* (pp. 287-393). Dordrecht: Kluwer Academic Publishers.

Primas, H. (2007). Non-Boolean descriptions for mind-matter problems. *Mind & Matter, 5* (1), 7-44.

Ritov, I., & Baron, J. (1990). Reluctance to vaccinate: omission bias and ambiguity. *Journal of Behavioral Decision Making, 3,* 263-277.

Russell, B. (1923). Vagueness. *Australasian Journal of Philosophy and Psychology, 1*, 84-92.

Sauerland, U. (2010). *Vagueness in language: The case against fuzzy logic Revisited*. Unpublished manuscript, Berlin, ZAS.

Solt, S. (2011). Notes on the comparison class. In R. Nouwe, R. v. Rooij, U. Sauerland & H.-C. Schmitz (Eds.), *Vagueness in communication* (pp. 189-206). Berlin: Springer.

Sorensen, R. (1991). Vagueness within the language of thought. *The Philosophical Quarterly, 41*, 389-413.

Spranca, M., Minsk, E., & Baron, J. (1991). Omission and commission in judgment and choice. *Journal of Experimental Social Psychology, 27*, 76-105.

Storms, G., De Boeck, P., Van Mechelen, I., & Ruts, W. (1996). The dominance effect in concept conjunctions: generality and interaction aspects. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 22*, 1-15.

Toledo, A., & Sassoon, G. W. (2011). Absolute vs. relative adjectives-variance within vs. between individuals. *Proceedings of SALT, 21*, 135-154.

Tversky, A., & Shafir, E. (1992). The disjunction effect in choice under uncertainty. *Psychological Science, 3* (5), 305-309.

Van Fraassen, B. C. (1966). Singular terms, truth-value gaps, and free logic. *The journal of Philosophy, 63* (17), 481-495.

Ventura, D., & Martinez, T. (1998). *Quantum associative memory with exponential capacity.* Paper presented at the IEEE World Congress on Computational Intelligence. The 1998 IEEE International Joint Conference.

Verheyen, S., Hampton, J.A., & Storms, G. (2010). A probabilistic threshold model: Analyzing semantic categorization data with the Rasch model. *Acta Psychologica*, *135*, 216-225.

Wallsten, T. S., Budescu, D. V., Rapoport, A., Zwick, R., & Forsyth, B. H. (1986). Measuring the vague meanings of probability terms. *Journal of Experimental Psychology: General*, *115*, 348-365.

Williamson, T. (1994). *Vagueness*. London: Routledge.

Zadeh, L. A. (1965). Fuzzy sets. *Information and control*, *8* (3), 338-353.