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Analyticity, Balance and Non-admissibility of *Cut* in Stoic Logic

Abstract. This paper shows that, for the Hertz–Gentzen Systems of 1933 (without *Thinning*), extended by a classical rule $T1$ (from the Stoics) and using certain axioms (also from the Stoics), all derivations are analytic: every cut formula occurs as a subformula in the cut's conclusion. Since the Stoic cut rules are instances of Gentzen's *Cut* rule of 1933, from this we infer the decidability of the propositional logic of the Stoics. We infer the correctness for this logic of a "relevance criterion" and of two "balance criteria", and hence (in contrast to one of Gentzen's 1933 results) that a particular derivable sequent has no derivation that is "normal" in the sense that the first premiss of each cut is cut-free. We also infer that *Cut* is not admissible in the Stoic system, based on the standard Stoic axioms, the $T1$ rule and the instances of *Cut* with just two antecedent formulae in the first premiss.

Keywords: Sequent, Analyticity, Stoic logic, Proof theory, Decidability, Relevance, Balance, Cut-admissibility.

Introduction and Overview

There is dispute and uncertainty about the exact details of much of the logic of the Stoics. The primary sources are fragmentary and the secondary sources fail to provide a coherent account. We ignore the Stoics' modal logic and consider only their propositional logic, which may be regarded as a fragment of what we now call classical propositional logic. It is a substructural logic and a relevant logic, with that term broadly conceived: the rule of *Thinning* is neither present nor admissible and an atom-sharing property, as in [11], i.e. a "relevance criterion", can be proved, as in Section 8 below.

Stoic deduction is via root-first proof search. It is the Stoic view[1] that an argument's correctness could be established by "analysis" of the argument, i.e. (in modern terms, following [5]) the root-first construction of a proof tree with the argument as root and axioms as leaves. The main rules of inference

[1] We base our claim on two surviving examples of Stoic "analysis" in the work of Sextus Empiricus, for which see [13, pp. 133–5]. See [1] for details.

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are cut rules. Here, choice of a suitable cut formula is required to ensure boundedness of the proof search. This is unproblematic in the simple cases where the first premiss of the cut is an axiom, but in general is unclear. (We are not suggesting that the Stoics used proof trees. In so far as there are completed Stoic proofs in linear style, their transformation into trees is unproblematic.)

We have no evidence for a semantics, despite the arguments of authors such as Milne [8]. Thus, completeness of the proof system w.r.t. a semantics is not the issue. Decidability is however both pertinent and open.

In this paper we show that, thanks to the Stoics' specific choice of axioms, all derivations are analytic: every cut formula is (or is the negation of) a subformula of the conclusion of the cut (equivalently, of the derivation). This limits proof search and we infer decidability. The same meta-theory also provides us with several criteria for constraining proof search even further and allows a proof that *Cut* (even if the premisses and conclusion are restricted to being Stoic sequents) is not admissible in the Stoic system.

1. Background on Stoic Logic

We use modern terminology. Since very little of the original Stoic texts survives, there is no full consensus on details; we refer to [1,2] for presentation of the relevant material, including assessment of various claims about what the Stoics did.[2]

The language is based on Stoic *basic propositions*, such as "Plato is in Athens"; we abbreviate all such propositions by the symbols p, q, r etc. These symbols are not instantiable variables: they are abbreviations. It is implicit that distinct symbols abbreviate different propositions. These are the *atomic formulae* or *atoms* of the language. Thus, wherever we present examples, the symbols p, q, r, etc are abbreviations for distinct basic propositions.

Upper case Roman letters A, B, etc are used as formula meta-variables.

Compound formulae are built up from atoms using negation $\neg$, conjunction $\wedge$, implication $\rightarrow$ and exclusive disjunction $\oplus$. Thus, if A is a formula then $\neg A$ is a formula; and if also B is a formula, then $A \wedge B$, $A \rightarrow B$ and $A \oplus B$ are formulae.

Definition 1.1. The *contradictory* A^* of a formula A is defined by

$$(\neg B)^* =_{def} B$$

[2] [1] discusses all published attempts at reconstructing Stoic propositional logic up to 1995. Of more recent contributions, [8,9] deserve mention.

and otherwise

$$B^* =_{def} \neg B.$$

The sources suggest the possibility of identifying $\neg\neg A$ with A: this simplifies the presentation of our results but is not essential. If done, it has the useful effect that the *contradictory* A^* of a formula A is, in every case, just $\neg A$, since if A is $\neg B$ then $\neg A$ is $\neg\neg B = B$. So we make this identification. But we retain the use of the notion of "contradictory" to match better the sources.

Stoic logic concerns the *correctness* of *arguments*, which we shall call "sequents". A *sequent* has antecedent formulae and a succedent formula, so is of the form $\Gamma \Rightarrow A$, where Γ is a collection of formulae. We assume that such a collection is a set, and that combination of such sets, written using a comma, is set-theoretic union. Our results about analyticity and decidability are unaffected if we assume instead that the Stoics used multisets (and multiset union). A sequent is *Stoic* iff it has at least two distinct antecedent formulae.

Derivations are constructed from *axioms* using *rules of inference* as usual. A sequent is *correct* if it is the end-sequent of a derivation. The axioms (called "indemonstrables" by the Stoics) are the instances of metalogical *axiom schemata*, of which the Stoics had five, and which we express as follows:

$$\frac{}{A \to B, A \Rightarrow B}\ A1 \qquad \frac{}{A \to B, B^* \Rightarrow A^*}\ A2$$

$$\frac{}{\neg(A \wedge B), A \Rightarrow B^*}\ A3 \qquad \frac{}{\neg(A \wedge B), B \Rightarrow A^*}\ A3'$$

$$\frac{}{A \oplus B, A \Rightarrow B^*}\ A4 \qquad \frac{}{A \oplus B, B \Rightarrow A^*}\ A4'$$

$$\frac{}{A \oplus B, B^* \Rightarrow A}\ A5 \qquad \frac{}{A \oplus B, A^* \Rightarrow B}\ A5'$$

In each axiom schema's antecedent there is a "principal" formula: it is that containing the indicated occurrence of a logical connective other than $\neg$. Thus, in the first column, the principal formulae are $A \to B$, $\neg(A \wedge B)$, $A \oplus B$ and $A \oplus B$ respectively.

The Stoic metalogical formulations of their axiom schemata combine $A3$ and $A3'$, etc., thus yielding their five axiom schemata. Note that each axiom is a Stoic sequent. There are no axioms such as $p \Rightarrow p$.

There are four Stoic inference rules, called *themata*: the rule $T1$ and three "cut" rules: $T2$, $T3$ and $T4$. In each case, the rule has a *premiss* or *premisses* above the line and a *conclusion* below the line.

Since some of the rules are two-premiss rules, it is convenient to follow Hertz and Gentzen in drawing proof trees rather than presenting linear successions of sequents; we do not suggest that this was the Stoic practice.

First, the $T1$ rule:

$$\frac{\Gamma, A \Rightarrow B}{\Gamma, B^* \Rightarrow A^*}\ T1$$

It is implicit in $T1$ that neither A nor B^* is in Γ.

Use of the $T1$ rule would render four of the above eight axiom schemata superfluous: one in each row. But this would upset the terminology of "axiom", so we keep all eight of the schemata.

The actual expression by the Stoics was more verbal, as in (for a restricted version of $T1$) "When from two [propositions] a third follows, then from either of them together with the contradictory of the conclusion [i.e. the succedent] the contradictory of the other follows" [2, p. 111].

Then there are three cut rules: $T2$, $T3$ and $T4$:

$$\frac{A, B \Rightarrow C \quad C, A, [B] \Rightarrow D}{A, B \Rightarrow D}\ T2 \qquad \frac{A, B \Rightarrow C \quad C, \Gamma \Rightarrow D}{A, B, \Gamma \Rightarrow D}\ T3$$

$$\frac{A, B \Rightarrow C \quad C, A, [B], \Gamma \Rightarrow D}{A, B, \Gamma \Rightarrow D}\ T4$$

where $[B]$ in $T2$ and in $T4$ indicates that the occurrence of B is optional. Note that $T2$, $T3$ and $T4$ have as first premiss a restricted form of Stoic sequent, i.e. one with exactly two antecedent formulae.

It is implicit in these cut rules that $A \neq B$ and that neither A nor B is in Γ. It may also be implicit that the *cut formula* C is not in Γ.

It is explicit in the surviving Stoic formulations of $T3$ that Γ is non-empty. Moreover, it must be non-empty, because the second premiss must be a Stoic sequent. It is explicit in the reconstructed formulation of $T4$ that Γ is non-empty. Since—for all we know—the ancients were not familiar with notions like the empty set, we would not expect to find a formulation that covers both $T2$ and $T4$.

There was also in antiquity a single Peripatetic *Cut* rule called the "synthetic theorem", of which Alexander of Aphrodisias (400 years after Chrysippus) says that the Stoics cut it up and made of it the 2nd, 3rd and 4th *themata*. Although this is historically unlikely, since, for all we know, the early Stoics were not acquainted with Aristotle's logic, it is true that $T2$,

$T3$ and $T4$ are instances[3] of this synthetic theorem:

$$\frac{\Gamma \Rightarrow C \quad C, \Gamma' \Rightarrow D}{\Gamma, \Gamma' \Rightarrow D}\ Cut^S_C$$

Again, the actual expression was more verbal: "When from some propositions (Γ) something (C) follows, and from that which follows (C) together with one or more propositions (Γ') something (D) follows, then, too, from those propositions (Γ) from which it (C) follows together with the one or more propositions (Γ') from which together with it (C) something (D) follows, the same thing (D) follows" [1, p. 164].[4]

The suffix C in the rule name is to emphasise the cut formula. We annotate the rule name with a superfix S to emphasise that it is the synthetic theorem rather than in the fuller generality of [5] or [6]. Its sequents are Stoic: thus, Γ has size at least 2, and Γ' has size at least 1. They may overlap.

A more restricted version of Cut^S is Cut^2, where the first premiss is constrained to have exactly two antecedent formulae.

Proposition 1.2. *Every instance of Cut^2 is an instance of one of $T2$, $T3$ and $T4$, and conversely.*

Proof. Let

$$\frac{A, B \Rightarrow C \quad C, \Gamma \Rightarrow D}{A, B, \Gamma \Rightarrow D}\ Cut^2_C$$

be such an instance, with $A \neq B$. This looks exactly like an instance of $T3$. But what if A or B (or both) is in Γ? There are two cases:

1. Neither A nor B is in Γ: then we have an instance of $T3$;
2. One or both are in Γ: then we have two cases:

(a) $\Gamma \backslash \{A, B\}$ is empty: then we have an instance of $T2$;
(b) $\Gamma \backslash \{A, B\}$ is non-empty: then we have an instance of $T4$.

Conversely,

1. $T2$ is the instance with Γ replaced by A, B and Γ' replaced by $A, [B]$;
2. $T3$ is the instance with Γ replaced by A, B and Γ' replaced by Γ;
3. $T4$ is the instance with Γ replaced by A, B and Γ' replaced by $A, [B], \Gamma$.

[3] See Corollary 1.3 below.

[4] The parenthesised symbols are not in the original text; they are added, roughly as in [1], for clarity.

In each case restrictions such as $A \neq B$ may then need to be imposed. ■

So it is certainly the case that $T2$, $T3$ and $T4$ are jointly equivalent to Cut^2. But are they jointly equivalent to Cut^S (i.e. with less restricted first premisses)? We have a partial answer:

COROLLARY 1.3. *$T2$, $T3$ and $T4$ are instances of Cut^S.*

We shall consider this question further in Section 11.

Returning to Alexander's synthetic theorem, with it there are essentially just two inference rules: the rules $T1$ and Cut^S. This (and its extension with unrestricted Cut) is what we will study. The details and restrictions of $T2$, $T3$ and $T4$ can be imposed without changing our main conclusions.

We present an example (to which we shall return) of a derivation, satisfying all the restrictions just mentioned:

EXAMPLE 1.4.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p\to q, p \Rightarrow q} \quad \dfrac{\dfrac{}{\neg(p\wedge q), p \Rightarrow \neg q}}{q, p \Rightarrow p \wedge q}\,T1}{p\to q, p \Rightarrow p\wedge q}\,T2}{p\to q, \neg(p\wedge q) \Rightarrow \neg p}\,T1 \qquad \dfrac{\dfrac{}{\neg p, \neg p\to(s\to r) \Rightarrow s\to r} \quad \dfrac{\dfrac{}{\neg p, \neg p\to s \Rightarrow s} \quad \dfrac{}{s, s\to r \Rightarrow r}}{s\to r, \neg p, \neg p\to s \Rightarrow r}\,T3}{\neg p, \neg p\to s, \neg p\to(s\to r) \Rightarrow r}\,T4}{p\to q,\ \neg p\to s,\ \neg p\to(s\to r),\ \neg(p\wedge q) \Rightarrow r}\,T3$$

2. Background on Hertz–Gentzen Systems

Gentzen [5], following earlier work of Hertz, introduced what we call a *Hertz–Gentzen* system: *sequents* consist of a non-empty set Γ of formulae and a formula A, and are written as $\Gamma \Rightarrow A$. Certain sequents are declared to be *initial*, i.e. to be *axioms*. There is a rule of *Thinning*, which we regard as optional; and a *Cut* rule, of the form

$$\frac{\Gamma \Rightarrow C \quad C, \Gamma' \Rightarrow A}{\Gamma, \Gamma' \Rightarrow A}\ Cut_C$$

with the constraint (but only if *Thinning* is present) that $C \notin \Gamma'$. The notion of *derivation* of a sequent from the initial sequents is defined as usual, with the proviso that sequents of the form $A \Rightarrow A$ are also regarded as derivable (but useless).

A derivation can contain only subformulae of the initial sequents. So, if the set $\mathcal{S}_0$ of initial sequents is finite, derivability is decidable, although a loop-checker may be required. But, in general, if the set $\mathcal{S}_0$ is infinite (even just the infinitely many instances of a finite set of schemata), decidability of derivability is not at all obvious. In fact, derivability is in general undecidable [7], by reduction of an undecidable subclass of Horn logic.

Gentzen [5, Theorem III] showed that every derivation can be transformed to a “Goclenian” normal form[5]: all thinnings, if any, are at the root, and the first premiss of every cut is an axiom and thus cut-free. Still, although this can help speed up proof search, it doesn’t show decidability. Gentzen’s proof is semantic and appears to use the finiteness of the set of axioms; but he hints at a syntactic proof, which can be done without such a restriction. As noted by Schroeder-Heister [12], Gentzen’s result is an early statement of the completeness of the “input strategy” for propositional resolution where the (finitely many) axioms and the goal sequent are Horn formulae. See also [10]. From now on we allow no use of *Thinning*.

3. Limiting Proof Search in Stoic Logic

As stated above, a fundamental question about Stoic logic is its decidability. Since the Stoics used root-first proof search, this includes whether there is a method for limiting the root-first search for proofs. Note that there are infinitely many axioms, based on only a finite set of axiom schemata. One candidate method for this limitation might be a normal form theorem, where a derivation is *normal* (i.e. *in normal form*) iff the derivation of the first premiss of every cut is cut-free.

An even stronger limitation for normality would be (as in Gentzen [5]) that every derivation of a cut’s first premiss is just an axiom. But both these approaches fail: the example given above is not normal, and we will see, in Section 10 below, that there is no normal derivation of its end-sequent. However, just as for Gentzen (with infinitely many axioms), even a normal form theorem would not be enough: we need (for example) analyticity.

There are several criteria that limit proof search in Stoic logic. First, every Stoic axiom is classically valid and the rules of inference preserve classical truth (and hence validity). So, use of a classical decision procedure may be helpful: if a sequent is not classically valid, then it isn’t derivable by the Stoic methods. (We call this the “classical validity criterion”.)

Second, as we will see in Sections 8 and 9, there are several other criteria, involving a “variable-sharing property”, i.e. relevance (in the sense of [11]), and what we call “balance”, for constraining proof search.

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[5] Despite its use by Schroeder, Hertz and (in a footnote) Gentzen, the adjective “Goclenian” seems not to have caught on for describing this kind of normal form, distinguished from the “Aristotelian” normal form in which the axioms are, rather, at the other side of the cut.

Although these criteria are helpful in constraining proof search, they are not strong enough to ensure decidability. So we consider the problem of showing analyticity.

4. Subformulae and Occurrences

We recall the standard inductive definition that A *occurs positively* (i.e. with positive *polarity*) in A and in $A \wedge B$ and in $B \wedge A$ and in $B \rightarrow A$ and *negatively* (i.e. with negative *polarity*) in $\neg A$ and in $A \rightarrow B$. We define it to occur *both positively and negatively* in $A \oplus B$ and in $B \oplus A$, because of the interpretation of $A \oplus B$ as "exclusive or". This definition is extended by recursion, with two negatives making a positive, to sub-subformulae, etc.

Since $A = \neg\neg A$, we have that $\neg A$ occurs (negatively) as a subformula of A. (If this assumption isn't allowed, nothing much changes below, except that the presentation of the ideas becomes more complicated.)

Thus, a formula occurs positively, but not negatively, as a subformula of itself; likewise, $\neg A$ occurs negatively, but not positively, as a subformula of A.

If A occurs positively (resp. negatively) as a subformula of E or negatively (resp. positively) as a subformula of a member of Γ, then it *occurs positively* (resp. *negatively*) in the sequent $\Gamma \Rightarrow E$.

We distinguish between A occurring as a *member* of a sequent $\Gamma \Rightarrow E$ and as a *subformula* of the sequent. For example, p occurs as a negative subformula of the sequent $p \wedge q, q \Rightarrow r$ but not as a member of the sequent or even of its antecedent. Of course, if A occurs as a member of a sequent then it also occurs as a subformula.

For brevity in certain parts of the exposition, we define $\mathcal{SF}(A)$ to be the set of subformulae of A; $\mathcal{SF}^+(A)$ to be the set of positively occurring subformulae of A; $\mathcal{SF}^-(A)$ to be the set of negatively occurring subformulae of A; and similarly for sets and for sequents. So, $p \in \mathcal{SF}^-(p \wedge q, q \Rightarrow r)$.

5. Character, Super-Analyticity and Analyticity

DEFINITION 5.1. The *character* $\chi(A)$ of a formula A is defined as follows: $\chi(P)$ (for atomic P) is both positive and negative, $\chi(\neg A)$ is the opposite of $\chi(A)$ (i.e. if the latter is positive (resp. negative) then $\chi(\neg A)$ is negative (resp. positive)), $\chi(A \wedge B)$ is positive, $\chi(A \rightarrow B)$ is negative and $\chi(A \oplus B)$ is negative. (The character is determined thus by the form of the axioms: for example, $A \rightarrow B$ is principal in the antecedent of some axioms but never in the succedent.)

Thus, if $\chi(D)$ is positive, D must be either an atom, the negation of an atom, a conjunction $A \wedge B$, a negated implication $\neg(A \rightarrow B)$ or a negated disjunction $\neg(A \oplus B)$.

Definition 5.2.

1. A cut is *super-analytic* iff the cut formula C occurs with polarity $\chi(C)$ as a subformula of the cut's conclusion. In particular, if C is either an atom or the negation of an atom, it must occur both positively and negatively as a subformula of the cut's conclusion.
2. A derivation is *super-analytic* iff every cut therein is super-analytic.

Definition 5.3.

1. A cut is *analytic* iff the cut formula C occurs as a subformula of the cut's conclusion.
2. A derivation is *analytic* iff every cut therein is analytic.

Super-analytic implies analytic. The point of analyticity is that, in root-first search for a derivation, a major source of non-determinism is the choice of cut formula. If the derivation is to be analytic, one may restrict attention to subformulae of the conclusion of the cut. This is in general more restrictive than the subformulae of the conclusion of the derivation: some formulae may, as we move leafwards, have disappeared. Nevertheless:

Proposition 5.4. *In an analytic derivation, if a formula A occurs as a subformula of a sequent in the derivation, then it occurs as a subformula of the conclusion of the derivation.*

Proof. By induction on the derivation structure and case analysis.

1 If the derivation is an axiom, the only sequent is already the conclusion.

2 The $T1$ rule leaves subformula occurrences unchanged.

3 The Cut rule provides two subcases:

(a) The occurrence is not as a subformula of the cut formula; this is routine.
(b) The occurrence is as a subformula of one of the two main occurrences of the cut formula C; by analyticity, there is an occurrence of C as a subformula of the conclusion. This gives us the desired occurrence of A as a subformula of the conclusion.

■

6. Examples

We present some examples using different forms of cut formula with various characters to illustrate the ideas about polarity and character:

Example 6.1.

$$\dfrac{\dfrac{\dfrac{}{\neg r, (p \wedge q) \to r \Rightarrow \neg(p \wedge q)}\,A2 \qquad \dfrac{}{\neg(p \wedge q), q \Rightarrow \neg p}\,A3'}{\neg r, q, (p \wedge q) \to r \Rightarrow \neg p}\,Cut_{\neg(p\wedge q)}}{p, q, (p \wedge q) \to r \Rightarrow r}\,T1$$

the cut formula is $\neg(p \wedge q)$ (with negative character); this is a negative subformula of $p \wedge q$, a negative subformula of $(p \wedge q) \to r$, itself (since it is in the antecedent) a negative member of the sequent $\neg r, q, (p \wedge q) \to r \Rightarrow \neg p$. Three negatives makes a negative.

Example 6.2.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(p \wedge q), p \Rightarrow \neg q}\,A3}{p, q \Rightarrow p \wedge q}\,T1 \qquad \dfrac{}{p \wedge q, p \wedge q \to r \Rightarrow r}\,A1}{(p \wedge q) \to r, q, p \Rightarrow r}\,Cut_{p\wedge q}}{(p \wedge q) \to r, \neg r, p \Rightarrow \neg q}\,T1$$

the cut formula is $p \wedge q$ (with positive character); it is a negative subformula of $(p \wedge q) \to r$, a negative member of the sequent $(p \wedge q) \to r, q, p \Rightarrow r$. Two negatives make a positive.

Example 6.3.

$$\dfrac{\dfrac{}{p, p \to (q \to r) \Rightarrow q \to r}\,A1 \qquad \dfrac{}{q \to r, q \Rightarrow r}\,A1}{p, p \to (q \to r), q \Rightarrow r}\,Cut_{q\to r}$$

the cut formula is $q \to r$ (with negative character); it is a positive subformula of $p \to (q \to r)$, a negative member of the sequent $p, p \to (q \to r), q \Rightarrow r$. One positive combined with a negative makes a negative.

Example 6.4.

$$\dfrac{\dfrac{\dfrac{}{p \to p, p \Rightarrow p}\,A1}{p, \neg p \Rightarrow \neg(p \to p)}\,T1 \qquad \dfrac{}{\neg(p \to p), \neg(p \to p) \to q \Rightarrow q}\,A1}{p, \neg p, \neg(p \to p) \to q \Rightarrow q}\,Cut_{\neg(p\to p)}$$

the cut formula is $\neg(p \to p)$ (with positive character); it is a negative subformula of $\neg(p \to p) \to q$, a negative member of the sequent $p, \neg p, \neg(p \to p) \to q \Rightarrow q$. Two negatives make a positive.

7. Main Results

We recall the five kinds of Stoic axiom from Section 1, the notions of positive and negative subformula occurrences from Section 4 and the notion of character of a formula from Section 5.

LEMMA 7.1. *Let $\mathcal{S}$ be an axiom. Let $\chi(D)$ be positive. Suppose that $D \in \mathcal{SF}^-(\mathcal{S})$. Then also $D \in \mathcal{SF}^+(\mathcal{S})$.*

PROOF. We consider the various forms of axiom; in each subcase (other than those ruled out) we obtain $D \in \mathcal{SF}^+(\mathcal{S})$:

1. If the axiom is $A \to B, A \Rightarrow B$, then there are four options:

(a) D is the formula $A \to B$; but this is ruled out by $\chi(D)$ being positive while $\chi(A \to B)$ is negative (and neither is an atom or the negation of an atom).

(b) D occurs positively in $A \to B$ but is distinct from it: there are two subcases: D occurs negatively in A or positively in B.

(c) D occurs positively in A: since A occurs negatively in $A \to B$, the formula D occurs negatively in $A \to B$.

(d) D occurs negatively in B: since B occurs positively in $A \to B$, the formula D occurs negatively in $A \to B$.

2. The case of the axiom $A \to B, \neg B \Rightarrow \neg A$ is similar.

3. If the axiom is $\neg(A \wedge B), A \Rightarrow \neg B$, then there are four options for the negative occurrences of D as a subformula of $\mathcal{S}$:

(a) D is the formula $\neg(A \wedge B)$: but this is ruled out since $\chi(D)$ is positive while $\chi(\neg(A \wedge B))$ is negative (and neither is an atom or the negation of an atom).

(b) D occurs positively in $\neg(A \wedge B)$ but is distinct from it: so there are two subcases: D occurs negatively in A or negatively in B (and hence positively in $\neg B$).

(c) D occurs positively in A: since A occurs negatively in $\neg(A \wedge B)$, the formula D occurs negatively in $\neg(A \wedge B)$.

(d) D occurs negatively in $\neg B$: since $\neg B$ occurs positively in $\neg(A \wedge B)$, the formula D occurs negatively in $\neg(A \wedge B)$.

4. The case of the axiom $\neg(A \wedge B), B \Rightarrow \neg A$ is similar.

5. If the axiom is $A \oplus B, A \Rightarrow \neg B$, then we remark that both A and B occur both positively and negatively in $A \oplus B$. So there are the following options:

(a) D is the formula $A \oplus B$: but this is ruled out since $\chi(D)$ is positive while $\chi(A \oplus B)$ is negative (and neither is an axiom or the negation of an axiom).
(b) D occurs positively in $A \oplus B$ but is distinct from it: so D occurs in A or in B; by the remark just made, D occurs with both polarities (and thus with negative polarity) in $A \oplus B$.
(c) D occurs positively in A: by the same remark, D occurs with negative polarity in $A \oplus B$.
(d) D occurs negatively in $\neg B$: similar.

6. The case of the axiom $A \oplus B, A \Rightarrow \neg B$ is similar.
7. The cases of the axioms $A \oplus B, \neg A \Rightarrow B$ and $A \oplus B, \neg B \Rightarrow A$ are similar.

■

Remark 7.2. The result of the lemma does not hold of the formula $D = A \wedge B$ (with $\chi(D)$ positive) in the non-derivable schema

$$A \wedge B,\ A \rightarrow (B \rightarrow C) \Rightarrow C.$$

This illustrates that even minor tinkering with the Stoic axiom schemata, such as adding this as an axiom schema, would upset the theory here being developed.

Corollary 7.3. *Let $\mathcal{S}$ be an axiom. Let $\chi(D)$ be negative. Suppose that $D \in \mathcal{SF}^+(\mathcal{S})$. Then also $D \in \mathcal{SF}^-(\mathcal{S})$.*

Proof. Apply the Lemma to $\neg D$. ■

We now need to discuss derivations, for which $\mathcal{D}$ is a good notation; so we abandon use of D as a formula meta-variable in favour of F.

Lemma 7.4. *Let $\mathcal{D}$ be an analytic derivation of $\Delta \Rightarrow E$, ending in*

$$\dfrac{\begin{matrix}\mathcal{D}' \\ \Delta' \Rightarrow C\end{matrix} \quad \begin{matrix}\mathcal{D}'' \\ C, \Delta'' \Rightarrow E\end{matrix}}{\Delta', \Delta'' \Rightarrow E}\ Cut$$

where $\Delta = \Delta', \Delta''$. Suppose that F occurs both negatively and positively in C. Then F also occurs positively in E or negatively in a member of Δ.

Proof. By analyticity, C occurs as a subformula in the conclusion.

Whether it occurs positively or negatively, this implies that F occurs positively in the conclusion, as required. ■

Lemma 7.5. *Let $\mathcal{D}$ be a super-analytic derivation of $\mathcal{S}$. Let $\chi(F)$ be positive. Suppose that $F \in \mathcal{SF}^-(\mathcal{S})$. Then also $F \in \mathcal{SF}^+(\mathcal{S})$.*

PROOF. By induction on the derivation structure and case analysis:

1. The last step is an axiom: we use Lemma 7.1.
2. The last step is by $T1$: we use the induction hypothesis.
3. If the last step is by Cut, as follows:

$$\dfrac{\begin{matrix}\mathcal{D}' \\ \Delta' \Rightarrow C\end{matrix} \quad \begin{matrix}\mathcal{D}'' \\ C, \Delta'' \Rightarrow E\end{matrix}}{\Delta', \Delta'' \Rightarrow E} \; Cut$$

   then there are two cases:

   (a) F occurs negatively in E or positively in Δ''. By the IH on $\mathcal{D}''$, we get either a positive occurrence in E (and we are done) or a negative one in Δ'' (and we are done) or a negative one in C. In the last subcase, by the IH on $\mathcal{D}'$, we get a negative one in Δ' (and we are done) or a positive one in C. So (unless we are done) we have both a negative and a positive occurrence of F in C. By Lemma 7.4, we get a positive occurrence of F in $\Delta', \Delta'' \Rightarrow E$.

   (b) F occurs positively in Δ'. By the IH on $\mathcal{D}'$, we get either a negative occurrence of F in Δ' (and we are done) or a positive occurrence of F in C. In the last subcase, by the IH on $\mathcal{D}''$, we get a negative one in Δ'' (and we are done) or a positive one in E (and we are done) or a negative one in C. So (unless we are done) we have both a positive and a negative occurrence of F in C. By Lemma 7.4, we get a positive occurrence of F in $\Delta', \Delta'' \Rightarrow E$.

■

COROLLARY 7.6. *Let $\mathcal{D}$ be a super-analytic derivation of $\mathcal{S}$. Let $\chi(F)$ be negative. Suppose that $F \in \mathcal{SF}^+(\mathcal{S})$. Then also $F \in \mathcal{SF}^-(\mathcal{S})$.*

PROOF. Apply the Lemma to $\neg F$. ■

THEOREM 7.7. *Every derivation is super-analytic.*

PROOF. By induction on the derivation structure and case analysis. Let $\mathcal{D}$ be a derivation of $\Delta \Rightarrow E$. There are three cases:

1. $\Delta \Rightarrow E$ is an axiom. $\mathcal{D}$ has no cut-formulae, so is certainly super-analytic.
2. $\mathcal{D}$ follows by $T1$ from $\mathcal{D}'$; by the IH, the latter is super-analytic and following it by $T1$ doesn't change this.

3. $\mathcal{D}$ is constructed by *Cut* from super-analytic derivations $\mathcal{D}'$ and $\mathcal{D}''$, with cut formula C, as follows:

$$\dfrac{\begin{array}{c}\mathcal{D}'\\ \Delta' \Rightarrow C\end{array} \quad \begin{array}{c}\mathcal{D}''\\ C, \Delta'' \Rightarrow E\end{array}}{\Delta', \Delta'' \Rightarrow E}\ Cut_C$$

where $\Delta = \Delta', \Delta''$. By hypothesis, $\mathcal{D}'$ and $\mathcal{D}''$ are super-analytic; so we only need to consider the cut on C. We have to show that C occurs with polarity $\chi(C)$ in $\Delta', \Delta'' \Rightarrow E$. Note that, if C is an atom (or an atom's negation), then $\chi(C)$ is both positive and negative, so the two cases that follow are not disjoint (but they are exhaustive). There are the following two cases:

(a) $\chi(C)$ is positive: Now, $C \in \mathcal{SF}^-(C, \Delta'' \Rightarrow E)$. By Lemma 7.5, $C \in \mathcal{SF}^+(C, \Delta'' \Rightarrow E)$ i.e. $C \in \mathcal{SF}^-(C)$ or $C \in SF^-(\Delta'')$ or $C \in SF^+(E)$. But C cannot have a negative subformula occurrence in C; so the occurrence is positive in $\Delta'' \Rightarrow E$ and thus also in $\Delta', \Delta'' \Rightarrow E$.

(b) $\chi(C)$ is negative: So, $\chi(\neg C)$ is positive and $\neg C$ occurs negatively in $\Delta' \Rightarrow C$. By Lemma 7.5, $\neg C \in \mathcal{SF}^+(\Delta' \Rightarrow C)$ i.e. $\neg C \in \mathcal{SF}^+(C)$ or $\neg C \in SF^-(\Delta')$. But $\neg C$ cannot have a positive subformula occurrence in C; so this gives us a negative occurrence of $\neg C$ in Δ' and hence a negative occurrence of C in the conclusion $\Delta', \Delta'' \Rightarrow E$.

■

Corollary 7.8. *Every derivation is analytic.*

Proof. Super-analytic implies analytic. ■

Corollary 7.9. *Stoic derivability is decidable.*

Proof. The sequents in a derivation are constrained by consisting of subformulae of the end-sequent, finite in number. With a loop-checker, this ensures decidability. ■

By way of clarification, we add that this is a result about a system with the Stoic axioms, the Stoic rule $T1$ and the unrestricted *Cut* rule from Gentzen [5] (but allowing the cut formula C to belong to the rest of the antecedent of the cut's second premiss). Restricting the cuts by using either what we have called Cut^S or the yet more restrictive rules $T2$, $T3$ and $T4$ does not affect the argument for analyticity; it just affects the choice of rules used in the root-first search used in the decision procedure.

8. Atom Sharing

Definition 8.1. Consider an occurrence of an atom or negated atom in a sequent. An occurrence thereof in a different formula of the sequent and with opposite polarity (w.r.t. the sequent) is said to be *complementary*. For this purpose, the succedent of a sequent is regarded as a formula different from any of the antecedent formulae.

Lemma 8.2. *If a sequent is derivable and if an atom or a negated atom occurs therein, there is also a complementary occurrence.*

Proof. By induction and case analysis.

1. Trivial for axioms.
2. Trivial for $T1$ steps.
3. If $\Gamma, \Gamma' \Rightarrow B$ follows by Cut from $\Gamma \Rightarrow C$ and $C, \Gamma' \Rightarrow B$, consider a positive occurrence in $\Gamma, \Gamma' \Rightarrow B$. There are various cases:

(a) it is a negative occurrence in Γ. So there is (by the IH on premiss 1) either a positive occurrence in a different formula of Γ (and we are done) or a negative occurrence in C, i.e. an occurrence in the antecedent formula C of $C, \Gamma' \Rightarrow B$ and thus a positive occurrence in that sequent. In this case (by the IH on premiss 2), there is a negative occurrence in $\Gamma' \Rightarrow B$ (and we are done).

(b) it is a negative occurrence in Γ'. So there is (by the IH on premiss 2) either a complementary occurrence in $\Gamma' \Rightarrow B$ (and we are done) or a positive occurrence in C . In this case, (by the IH on premiss 1) there is a complementary occurrence in $\Gamma \Rightarrow$ (and we are done).[6]

(c) It is a positive occurrence in B. So there is (by the IH on premiss 2) a complementary occurrence either in $\Gamma' \Rightarrow$ (and we are done) or in $C \Rightarrow$. If in $C \Rightarrow$ then (by the IH on premiss 1) there is a complementary occurrence in $\Gamma \Rightarrow$ (and we are done).

4. Negative occurrences are dealt with similarly.

The checking that the assertions about polarities in this are correct is routine (but tedious). One case was done in full; the others are left for the reader. ■

Corollary 8.3. *If a sequent is derivable (by Axioms, $T1$ and Cut), and if an atom or negated atom occurs in the succedent, then there is a complementary occurrence in the antecedent.*

[6] Although $\Gamma \Rightarrow$ is not a sequent, its use indicates where the occurrence should lie.

Proof. Trivial. ■

Connections with the "variable-sharing condition" for relevant logic (as described in [11]) may now be apparent. We therefore refer to this as a "relevance criterion"; in practice we find the slightly weaker "balance criterion" (below) as effective and more convenient.

9. Balance

Definition 9.1. A sequent $\mathcal{S}$ is *balanced* iff every atom occurring in $\mathcal{S}$ has both a negative and a positive occurrence.

Root-first proof search for a derivation of a sequent can be restricted to the analysis of balanced sequents:

Proposition 9.2. *Every derivable sequent* $\mathcal{S}$ *is balanced.*

Proof. We present two proofs: a third will follow. First, it is an immediate consequence of Lemma 8.2. Second, by Theorem 7.7, the derivation of $\mathcal{S}$ is super-analytic. Suppose an atom occurs as a subformula of S; then its character is both positive and negative, so both Lemma 7.5 and its Corollary 7.6 apply. ■

But we can do better:

Definition 9.3. A sequent $\mathcal{S}$ is *strongly balanced* iff every subformula D that occurs therein has an occurrence of polarity $\chi(D)$.

Example 9.4. $(p \wedge q) \rightarrow r, p, q \Rightarrow r$ is strongly balanced.

Proof. Its positive subformula occurrences are

$$r, \neg p, \neg q, \neg r, p \wedge q, p, q, \neg((p \wedge q) \rightarrow r)$$

and its negative ones are

$$\neg r, p, q, r, \neg(p \wedge q), \neg p, \neg q, (p \wedge q) \rightarrow r.$$

Apart from atoms (which occur with both polarities, because of their having both characters), and their negations (ditto), the subformulae here of positive character are $p \wedge q$, $\neg((p \wedge q) \rightarrow r)$ and those of negative character are $\neg(p \wedge q)$, $(p \wedge q) \rightarrow r$ and it is routine to check that they do occur with the correct polarity. ■

Proposition 9.5. *Strongly balanced implies balanced.*

Proof. Consider an atom occurring in $\mathcal{S}$. Its character is both positive and negative; it follows that it has both a positive and a negative occurrence. ■

Theorem 9.6. *Every derivable sequent $\mathcal{S}$ is strongly balanced.*

Proof. Similar to the proof of Proposition 9.2. Thus, by Theorem 7.7, the derivation of $\mathcal{S}$ is super-analytic. Suppose a formula D occurs as a subformula of $\mathcal{S}$ and its character $\chi(D)$ is positive. There are two cases:

1. Suppose that $D \in \mathcal{SF}^-(\mathcal{S})$. By Lemma 7.5, $D \in \mathcal{SF}^+(\mathcal{S})$, as required.
2. Otherwise, $D \in \mathcal{SF}^+(\mathcal{S})$, again as required.

The argument if the character is negative is similar, using Corollary 7.6. ■

Corollary 9.7. *Every derivable sequent is balanced.*

Proof. (This is the promised third proof.) By Theorem 9.6 and Proposition 9.5. ■

These are important syntactic constraints that can be used to prune the search, desperately needed when (in root-first search) generating the premisses to a cut. If one just took over all the conclusion's formulae into each premiss, there would be little to check; but in general one just takes some of them.

Example 9.8. $(p \wedge q) \wedge r, \neg p, \neg q \Rightarrow r$ is balanced but not strongly balanced. So it is not derivable.

Proof. (Balanced is easy.) $p \wedge q$ occurs just with negative polarity; but it has positive character. ■

Example 9.9. $p \wedge q, p \rightarrow (q \rightarrow r) \Rightarrow r$ is not derivable.

Proof. The sequent is not strongly balanced: consider $p \wedge q$. ■

Theorem 9.10. *If the conclusion of an analytic cut is strongly balanced, then the cut is super-analytic.*

Proof. Routine. ■

In other words, if the only sequents we generate in root-first proof search are strongly balanced, and a cut formula is chosen from among the subformulae of the proposed cut's conclusion, then the cut is automatically super-analytic.

We remark that our proofs are insensitive to details such as whether we identify $A \wedge B$ with $B \wedge A$, or whether we restrict the first premiss of a cut to having exactly two antecedent formulae. What we cannot do is identify

A with the conjunction $A \wedge A$; the character of $p \oplus q$ is negative but the character of $(p \oplus q) \wedge (p \oplus q)$ is positive. Likewise, we cannot add axioms willy-nilly; the idea from Gentzen [6] that (*mutatis mutandis*, i.e. using rules rather than axioms) an introduction axiom such as $A, B \Rightarrow A \wedge B$ must be complemented by an elimination axiom such as $A \wedge B, A \rightarrow (B \rightarrow C) \Rightarrow C$ is unStoic and has to be suppressed.

10. Absence of a Normal Form Theorem for Stoic Logic

We illustrate the use of super-analyticity and the balance criteria in demonstrating that there is no simple normal form theorem in Stoic logic. Recall that a derivation is *normal* (i.e. in normal form) iff the derivation of the first premiss of every cut is cut-free.

Example 1.4 was of a sequent and a derivation thereof. It is not a normal derivation: we can now show, since all derivations are super-analytic, that the sequent has no normal derivation. It follows that there is no normal form theorem for Stoic logic, unless we can relax the notion of normality somehow.

The sequent is

$$p \rightarrow q,\ \neg p \rightarrow s,\ \neg p \rightarrow (s \rightarrow r),\ \neg(p \wedge q) \Rightarrow r.$$

In addition to both positive and negative occurrences of the atoms p, q, r and s (all of both positive and negative polarity) we have the following negative subformula occurrences: $p \rightarrow q$, $\neg p \rightarrow s$, $\neg p \rightarrow (s \rightarrow r)$, $s \rightarrow r$ and $\neg(p \wedge q)$ (all of negative polarity). We also have their negations (all of positive polarity) as positive subformula occurrences. That is, 18 occurrences in total.

The sequent is not an axiom. Suppose it is the conclusion of a cut. If the cut is to be Stoic, the first premiss must be of the form $A, B \Rightarrow C$, where A and B are in the sequent's antecedent. For the unordered pair A, B there are just six possibilities:

1. $A = p \rightarrow q$, $B = \neg p \rightarrow s$: but then C has to contain both q and s. By super-analyticity, this can't be done.
2. $A = p \rightarrow q$, $B = \neg p \rightarrow (s \rightarrow r)$: but then C has to contain all of q, s and r, which can't be done.
3. $A = p \rightarrow q$, $B = \neg(p \wedge q)$: so C has to contain p negatively and yet be distinct from A and B. The only possibility is $C = \neg p$, and the derivation given earlier illustrates one of the two ways by which this approach can

be continued. But, note that $p \to q, \neg(p \wedge q) \Rightarrow \neg p$, although derivable, is not cut-free derivable. So, this cannot lead to a normal derivation.

4. $A = \neg p \to s$, $B = \neg p \to (s \to r)$: but then C (distinct from B) has to contain r and $\neg p$, which can't be done.
5. $A = \neg p \to s$, $B = \neg(p \wedge q)$: but then C has to contain q and s, which can't be done.
6. $A = \neg p \to (s \to r)$, $B = \neg(p \wedge q)$; a similar argument applies.

If however the sequent is the conclusion of a $T1$ step, similar considerations apply: adding $\neg r$ to the antecedent doesn't help.

This procedure finds four derivations satisfying the Stoic criterion that the first premiss of a cut should have just two antecedent formulae, but there are none where every cut has a cut-free first premiss.

We have illustrated the use of super-analyticity and the balance criteria: use of the classical validity criterion would have pruned the search earlier, but hardly needs illustration. The details just given establish not just that there is no normal derivation but that (in this case at least) the search space for derivations can be constrained by the cuts being Stoic, by classical validity, by super-analyticity and by balance.

11. Non-admissibility of *Cut*

As mentioned above, Alexander implies that the Peripatetic "synthetic theorem" Cut^S is admissible, in the system consisting of axioms, the $T1$ rule and the instances (namely $T2$, $T3$ and $T4$) of Cut in which the first premiss is restricted to having exactly two antecedent formulae.

In this section we present a derivation, which uses Cut^S, of a sequent not derivable in the reduced system: this refutes Alexander's above-mentioned implication. We maintain the restriction that all sequents are Stoic, i.e. have at least two antecedent formulae. We use Cut^n_A to indicate a cut, with n antecedents in the first premiss, on the formula A.

Example 11.1. Consider the cut

$$\frac{p \to q, p \to (q \to r), \neg r \Rightarrow \neg p \qquad \neg p, \neg p \to s, \neg p \to (s \to t) \Rightarrow t}{p \to q, p \to (q \to r), \neg r, \neg p \to s, \neg p \to (s \to t) \Rightarrow t}\ Cut^3_{\neg p}$$

of which the two premisses are easily derivable using Cut^2. There are several other derivations of the conclusion using both Cut^3 and Cut^2. The example uses just negation and implication: by avoiding conjunction we sidestep suggestions about extra rules for dealing with it.

Proposition 11.2. *Every derivation of the conclusion of this example uses* Cut^n *for some* $n > 2$.

Proof. To see this, consider first of all how a derivation might end in Cut^2, using just two of the five antecedent formulae as antecedents in the first premiss. We consider what to take as the cut formula, maintaining the analyticity and the balance criteria, and consider the ten cases, done in detail to illustrate the methods and show avoidance of pitfalls:

1. $p \to q$ and $p \to (q \to r)$: so the cut formula has to contain p negatively and r positively—but, there is no suitable subformula, other than $p \to (q \to r)$ itself. But then, if we avoid looping, the other premiss can only be one of
$$p \to (q \to r), \neg p \to s, \neg p \to (s \to t) \Rightarrow t$$
and variants of this with one of $p \to q$ and $\neg r$ in the antecedent; this is unbalanced, so is not derivable.
2. $p \to q$ and $\neg r$: so the cut formula has to contain p and r negatively and q positively—but, there is no suitable subformula.
3. $p \to q$ and $\neg p \to s$: so the cut formula has to contain q and s positively—but, there is no suitable subformula.
4. $p \to q$ and $\neg p \to (s \to t)$: so the cut formula has to contain s negatively and q and t positively—but, there is no suitable subformula.
5. $p \to (q \to r)$ and $\neg r$: similar to the first case.
6. $p \to (q \to r)$ and $\neg p \to s$: so the cut formula has to contain q negatively and r and s positively—but, there is no suitable subformula.
7. $p \to (q \to r)$ and $\neg p \to (s \to t)$: so the cut formula has to contain q and s negatively and r and t positively—but, there is no suitable subformula.
8. $\neg r$ and $\neg p \to s$: so the cut formula has to contain r negatively and p and s positively—but, there is no suitable subformula.
9. $\neg r$ and $\neg p \to (s \to t)$: so the cut formula has to contain r and s negatively and p and t positively—but, there is no suitable subformula.
10. $\neg p \to s$ and $\neg p \to (s \to t)$: similar to the first case.

Similar arguments apply for each of the five possible premisses of a $T1$ step, such as
$$\neg t, p \to (q \to r), \neg r, \neg p \to s, \neg p \to (s \to t) \Rightarrow \neg(p \to q)$$
i.e. for each of these five possibilities we look, but in vain, for a Cut^2 step having it as conclusion. ■

These results were confirmed by the implementation mentioned in the next section. The same example can, we believe, be extended to show the inadmissibility of Cut^{n+1} in a system allowing use of cuts up to Cut^n.

Hence

THEOREM 11.3. *Cut^S, the "synthetic theorem", is not admissible in the system (axioms and the four* themata*) of the Stoics.*

PROOF. Immediate,[7] since each of the *themata* $T2$, $T3$ and $T4$ requires the antecedent of the first premiss to be of size exactly 2. ■

COROLLARY 11.4. *The Cut rule of Gentzen* [6] *is not admissible in the system (axioms and the four* themata*) of the Stoics.*

PROOF. Immediate: Cut^S is an instance of this rule. ■

12. A Proof Search Algorithm

Given a sequent to be decided: we execute root-first search with, for example, a depth-first strategy, constrained by a loop-checker, the classical validity check, the strong balance criterion and the analyticity criterion; we ensure (if we choose) at each stage that the instances of *Cut* to be used are Stoic, in that the first premiss should have just two antecedent formulae. We can also add use of the relevance criterion. Backtracking is generally required: we don't have the nice features of classical propositional logic. Implementation in Prolog is straightforward (but not yet efficient) [4]. It would be interesting to try, instead, a forward search algorithm, as (for intuitionistic logic) in Gentzen's thesis [6].

13. Further Work

We have provided methods for limiting the search space; but we do not change in any way the space of derivations, as can be done, for example, in intuitionistic logic [3]. One way of doing this would be to permute some instances of $T1$ upwards; but it is not yet clear that this is effective in reducing the derivation space. Put bluntly, there is (so far) nothing significant that counts as "proof theory", in the sense of Gentzen's programme (going back to Hertz) of transforming proofs.

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[7] *Pace* [1, p. 166].

14. Conclusion

We have shown that in Stoic logic (as reconstructed in [1]) all derivations are analytic and thus the logic is decidable. By the criteria of strong analyticity, strong balance, relevance and classical validity, the search space can be limited considerably. There is no normal form theorem in Stoic logic (where a derivation is *normal* if the derivation of the first premiss of every cut is cut-free), and neither the lightly restricted Cut^S rule, i.e. the "synthetic theorem", nor Gentzen's *Cut* rule, as e.g. in [6], is admissible in the Stoic system. We have sketched a proof search algorithm for Stoic logic.

We do not claim that the Stoics knew or suspected any of this metatheory. On the other hand, we do not rule out that they may have developed their own approximations to some of these results.

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