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On $\mathcal{F}$-systems

A Graph-Theoretic Model for Paradoxes Involving a Falsity Predicate and its Application to Argumentation Frameworks

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Abstract $\mathcal{F}$-systems are useful digraphs to model sentences that predicate the falsity of other sentences. Paradoxes like the Liar and the one of Yablo can be analyzed with that tool to find graph-theoretic patterns. In this paper we studied this general model consisting of a set of sentences and the binary relation '... affirms the falsity of...' among them. The possible existence of non-referential sentences was also considered. To model the sets of all the sentences that can jointly be valued as true we introduced the notion of *conglomerate*, the existence of which guarantees the absence of paradox. Conglomerates also enabled us to characterize *referential contradictions*, i.e., sentences that can only be false under a classical valuation due to the interactions with other sentences in the model. A Kripke-style fixed-point characterization of groundedness was offered, and complete (meaning that every sentence is deemed either true or false) and consistent (meaning that no sentence is deemed true and false) fixed points were put in correspondence with conglomerates. Furthermore, argumentation frameworks are special cases of $\mathcal{F}$-systems. We showed the relation between *local conglomerates* and admissible sets of arguments and argued about the usefulness of the concept for the argumentation theory.

Keywords The Liar paradox · Yablo's paradox · $\mathcal{F}$-system · Conglomerates · Groundedness · Argumentation frameworks

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1 Introduction

Some semantic paradoxes, like the Liar and the one of Yablo, involve sentences that assert the falsity of other sentences. Liar-like sentences, such as 'This sentence is false,' are clearly self-referential and, hence, circular. On the other hand, S. Yablo [19] introduced a problem, consisting of an infinite series of sentences, each asserting the falsity of all the following ones, to show that a paradox can be obtained without self-reference or circularity.[1] However, is there any common source of paradox behind the Liar and Yablo's paradoxes? What conditions suffice to produce them? A good first step to try to answer these questions is to find a common framework to represent them. At first glance, all we have are sentences that assert the falsity of other sentences (perhaps themselves); hence the approach can begin by considering just a set of sentences and a relation according to which some assert the falsity of others. Following this motivation, Cook [5] introduced the novelty of using graph-theoretic tools for dealing with semantic paradoxes involving a falsity predicate. Rabern, Rabern, and Macauley [15] coined the term '$\mathcal{F}$-systems' to refer to "sentence systems which are restricted in such a way that all the sentences can only say that other sentences in the system are false". The aim of the present article is to show some interesting results about $\mathcal{F}$-systems, mainly in relation to structural characteristics that give rise to the aforementioned semantic paradoxes and to sentences that present an intrinsic truth value in such frameworks. Moreover, we will show connections between Dung's [8] notion of controversial argumentation frameworks (a special case of $\mathcal{F}$-system) and some sufficient conditions for paradox.

The works by Cook and Rabern et al. concentrate on systems where *every* sentence affirms the falsity of some other sentence(s). That is represented by serial or sink-free digraphs (roughly, every node "shoots," at least, one arrow). On the other hand, Beringer and Schindler [2] and Walicki [18] also considered sentences that do not refer to other sentences in their reference graphs. This last approach is more general and, particularly, enables us to take into account the interaction among object language sentences (like 'Snow is white') and metalanguage sentences (like 'The sentence 'Snow is white' is false').[2] The analysis can be done on a general level where the specificities of the underlying languages are (or the hierarchy of languages is) irrelevant to find the graph-theoretic patterns that characterize paradox. The aim is to get the simplest model that allows that. All we need is a set S of nodes, which represent primitive entities we call *sentences* (indeed, they can be understood as names of sentences of a given language) and a binary relation $F \subseteq S \times S$, i.e. a set of directed edges or arrows such that, for any pair of sentences x

[1] A well-known dissenting opinion on the non-circularity of Yablo's paradox is that of Graham Priest [14].

[2] Though we include sinks in $\mathcal{F}$-systems, the systems of Beringer and Schindler [2] and Walicki [18] are still more general to the extent that referential sentences may attribute truth to other sentences as well. This is also the case with Rabern and colleagues's [15] main proposal beyond $\mathcal{F}$-systems.

and y of S, $(x, y) \in F$ is understood as 'x says that y is false.' In this way, for example, we can model the relationship between the (English) sentences 'Snow is white' and 'The sentence 'Snow is white' is false,' through an $\mathcal{F}$-system $\mathcal{F} = \langle S, F \rangle$, where $S = \{a, b\}$ and $F = \{(b, a)\}$, and where a and b represent 'Snow is white' and 'The sentence 'Snow is white' is false,' respectively.

Semantic paradoxes are sets of sentences that cannot all be assigned a classical truth value (true/false) at the same time. We will represent the assignment of truth values through *labellings*[3] on the nodes of the $\mathcal{F}$-systems, in such a way that every node can be labeled with `T` (for `True`), `F` (for `False`) or `U` (for `Undetermined`). Paradoxical $\mathcal{F}$-systems will be such that every labelling can only put the label `U` on some nodes. "Classical" labellings, i.e. those that can put `T` or `F` on every sentence, will be put in correspondence with graph-theoretic patterns that we will call *conglomerates*. The notion of conglomerate extends that of *kernel* used by Cook, which represents a subset of sentences that can be true together. Kernels are suitable for capturing classic assignments of truth values in systems where each sentence refers to other sentences. But if we use this notion in systems that include sinks, kernels will only allow them to be represented as true. Conglomerates, although they will lead to similar formal results regarding paradoxes, will allow a more intuitive representation since object language sentences can be assigned any truth-value.

Another aim of this work is to define a Kripke's style fixed point operator to characterize groundedness in $\mathcal{F}$-systems [12]. Grounded sentences are, roughly, those which truth-value can be tracked through the reference path until a sentence with a definite truth-value (the "ground"). We will define complete and consistent fixed points (meaning that every sentence is deemed either true or false and no sentence is deemed true and false) and show that they correspond exactly to conglomerates.

In $\mathcal{F}$-systems that are free of paradoxes, conglomerates are also useful to characterize *referential contradictions* and *referential tautologies*, i.e. sentences that can only have one of the two classical truth values, due to the interactions with other sentences in the model. These can be related to the intensional concepts of *semi-falsity* and *semi-truth*, respectively [17, 7]. This is another advantage of conglomerates with respect to kernels, which are unable to do that. Furthermore, we will see that referential contradictions and tautologies can be captured in every $\mathcal{F}$-system (not only in non-paradoxical ones) through *local conglomerates*: subsets of sentences that affirm the falsity of all the sentences that affirm the falsity of them.

We find that local conglomerates cover and extend that of *admissibility* in Dung's argumentation frameworks. As shown by Dyrkolbotn [9], argumentation frameworks are special cases of $\mathcal{F}$-systems where arguments play the role of sentences and F is interpreted as an attack relation. Admissibility formalizes the idea of sets of arguments that can be defended each other. Local conglomerates cover that idea and give it a twist: they deem "admissible" also

[3] We borrowed the term 'labelling' from [4]. In [2], the term 'decoration' is used instead.

the sets of arguments that can be defended together against any argument, except those that promote some non-preferred value, which suggests a new semantics for value-based argumentation frameworks [1].

Finally, we show that Dung's contribution on *controversiality* issues in argumentation frameworks covers, at least, two sufficient conditions for paradox in $\mathcal{F}$-systems. One, reported in [5], is transitivity in serial (i.e., sink-free) digraphs, whereas here we identified the other one regarding odd-length cycles in particular conditions.

The paper is organized as follows. In Section 2, we define $\mathcal{F}$-systems, labellings, and the notion of conglomerate. In Section 3, we give a fixed-point characterization of groundedness. Section 4, shows the correspondence among conglomerates and complete and consistent fixed points. In Section 5 we define referential contradictions and tautologies, and show that the transitivity of F in non-paradoxical systems is a sufficient condition for their existence. In Section 6, we define local conglomerates and show their correspondence with maximally consistent fixed points and with labellings that maximize the assignation of classical truth values. In Section 7 we show relationships among local conglomerates and Dung's admissible sets. Moreover, Dung's theory of controversial arguments is shown to cover two sources of paradox in $\mathcal{F}$-systems regarding transitivity and odd-length cycles. We comment on those points in Section 8. Final comments and conclusions are summarized in Section 9.

2 $\mathcal{F}$-systems

The following definition is (beyond notation differences) due to Rabern et al. [15]:

Definition 1 An $\mathcal{F}$-*system* is a pair $\mathcal{F} = \langle S, F \rangle$, where S is a set whose elements are primitive entities called *sentences*, and $F \subseteq S \times S$ is a binary relation among sentences.

For every $x \in S$, we define $\overrightarrow{F}(x) = \{y \in S : (x, y) \in F\}$ and $\overleftarrow{F}(x) = \{y \in S : (y, x) \in F\}$, and for every subset $A \subseteq S$, $\overrightarrow{F}(A) = \bigcup_{x \in A} \overrightarrow{F}(x)$ and $\overleftarrow{F}(A) = \bigcup_{x \in A} \overleftarrow{F}(x)$. If $\overrightarrow{F}(x) = \emptyset$, x is said to be a *sink*, and we define $sinks(A) = \{x \in A : x \text{ is a sink}\}$[4]. In order to avoid misrepresentations, we assume that non-sink sentences do not assert anything more than what is represented in F (and, naturally, sink sentences do not assert anything about other sentences). To illustrate the kind of issues to be avoided, consider the following example. Let $\mathcal{F} = (\{x, y\}, \{(y, x)\})$. Then, we want to interpret that x is true if y is false and x is false if y is true. Moreover, we want to interpret that if x has an undetermined truth value, then the value of y is undetermined, too. However, if we accept the interpretation that x = 'Snow is red' and y = 'x is false and the snow is blue,' then the above considerations about the truth and falsity of x and y would not be valid, since x and y could both be false. Though it

[4] This notation is taken from [18].

is true that y affirms the falsity of x, the component 'the snow is blue' of y, which is "hidden" in the representation, can yield anomalous interpretations. The general level of the model does not allow to represent such molecular sentences, since there are no elements to express logical connectives. Hence, we leave that kind of interpretations out of the scope of the model. On the other hand, the only molecular sentences that can be represented in the model, preserving the intuitions about the assignment of truth values, are conjunctions of falsity assertions about other sentences like, for instance, 'x says that both y and z are false,' which can be modeled as $\{(x, y), (x, z)\} \subseteq F$.[5]

Since $\mathcal{F}$-systems define digraphs, we can see the assignment of truth values to the sentences as *labels* on the nodes of a digraph. We consider three labels, `T`, `F` and `U`, for *true*, *false* and *undetermined*, respectively. The non-classical value *undetermined* is intended to express either that the actual value is unknown (as in the case of conjectures) or just the impossibility of assigning a classical truth value (as in the case of paradoxes).

Definition 2 Given $\mathcal{F} = \langle S, F\rangle$, a *labelling* on $\mathcal{F}$ is a total function L such that:

1. $L : S \to \{\mathtt{T}, \mathtt{F}, \mathtt{U}\}$, and
2. for all $x \in S \setminus sinks(S)$
   (a) $L(x) = \mathtt{F}$ iff $L(z) = \mathtt{T}$ for some $z \in \overrightarrow{F}(x)$, and
   (b) $L(x) = \mathtt{T}$ iff $L(z) = \mathtt{F}$ for every $z \in \overrightarrow{F}(x)$.[6]

Note that the assignment of values to sink nodes is unrestricted. Moreover, for every $\mathcal{F}$-system there always exists a labelling that assigns `U` to all nodes. If a labelling is such that all the nodes are labeled as either `T` or `F`, we say that the labelling is *classical*.

Definition 3 A labelling L on $\mathcal{F}$ is *classical* iff for every $x \in S$, $L(x) \neq \mathtt{U}$.[7]

Paradoxes in $\mathcal{F}$-systems can be characterized as follows:

Definition 4 An $\mathcal{F}$-system is *paradoxical* iff it has no classical labellings. Moreover, a sentence x is paradoxical iff $L(x) = \mathtt{U}$ for every labelling L.

[5] We can certainly think of other kinds of sentences that could be expressed. For example, if we have $F = \{(x, y), (y, z)\}$, then we can realize that the model is expressing that x affirms that z is true. However, note that this interpretation does not depend strictly on the model, but on the intended semantics (like the one we will see next). Indeed, we can think of some infectious semantics [13,16] under which z is undetermined, y is false, and x is true, meaning that x is true because it affirms that z is *either* true *or* undetermined.

[6] The conditions for the assignment of `T` and `F` to non-sink nodes are comparable to those of Cook's [5] *acceptable assignments*. We add the label `U` following the general lines of Caminada's [4] labelling semantics, which has the spirit of the strong Kleene three-valued logic. On request of a reviewer, we should say that the weak Kleene logic –i.e., that in which all the connectives receive the undetermined value if any component is undetermined– is not useful here to the aim of characterizing paradoxes according to the intuitions expressed in the following definitions.

[7] Classical labellings play the role here of *acceptable colorings* on serial digraphs, as defined by Cook [6]. `T` and `F` correspond to colors turquoise and fuchsia, respectively.

Example 1 Let $\mathcal{F} = \langle \{a_k\}_{k\in\mathbb{N}}, \{(a_k, a_m)\}_{k<m}\rangle$ (representing Yablo's paradox). Then, $\mathcal{F}$ does not have any classical labelling.

Classical labellings determine partitions of the set of sentences in which there are two subsets, representing the true and the false sentences, respectively. If the system is paradoxical, then we cannot have any such partition. Another way of characterizing the subsets of true and false sentences is by defining suitable properties. We introduce here the notion of *conglomerate*.

Definition 5 Given $\mathcal{F} = \langle S, F\rangle$, a *conglomerate* is a subset $A \subseteq S$ that satisfies:

1. $\overleftarrow{F}(A) \subseteq S \setminus A$, and
2. $(S \setminus A) \setminus sinks(S) \subseteq \overleftarrow{F}(A)$

The idea is that a conglomerate coalesces all the sentences that can share the true value, leaving outside all and only the sentences that can share the false value. Therefore, conglomerates can only exist in systems whose sentences can be "polarized" into true and false. Conglomerates can also be understood in a Kripkean way as defining the extension of the truth predicate of the underlying language, while the set of all the remaining sentences define the anti-extension (we will return to this point in Section 3). If a conglomerate exists, then we can say that the truth predicate is completely defined in the system. Since a conglomerate A is supposed to comprise all true sentences, condition 1 says that it cannot contain two sentences such that one asserts the falsity of the other (i.e., A is *independent*). And $S \setminus A$ is supposed to comprise all false sentences, so condition 2 says that every non-sink sentence must assert the falsity of, at least, one true sentence (i.e., A *absorbs* every external non-sink node). This is different from kernels, which absorb every outer node.[8] This implies that kernels comprise all the sinks, so these can only be interpreted as true sentences in such a model. In this sense, conglomerates seem to be more suitable than kernels to represent the Kripkean view: sinks may or may not belong to the conglomerates, representing object language sentences that may or may not be true. Furthermore, the notion of conglomerate clearly also encompasses that of kernel.

Example 2 Let $\mathcal{F} = \langle \{a, b\}, \{(b, a)\}\rangle$. Assume that $\mathcal{F}$ represents the relation between a: 'I am wearing a hat' and b: 'The sentence 'I am wearing a hat' is false'. $\mathcal{F}$ has only one kernel, $\{a\}$, but it has two conglomerates, $\{a\}$, deeming 'I am wearing a hat' as true and 'The sentence 'I am wearing a hat' is false' as false, and $\{b\}$, deeming 'I am wearing a hat' as false and 'The sentence 'I am wearing a hat' is false' as true. Every kernel is a conglomerate, but not vice versa.

[8] Kernels differ from conglomerates only in the *absorption* property, which says that $(S \setminus A) \subseteq \overleftarrow{F}(A)$. Therefore, we have essentially the notion of kernel used in Cook's sink-free system, modulo the fact that sinks can be placed on the outside. We are informally saying here that A *absorbs* x with the meaning that $(x, y) \in F$ for some $y \in A$.

The notion of conglomerate is not well-defined, in the sense that some $\mathcal{F}$-systems have no conglomerates. As expected, those systems are the paradoxical ones.

Example 3 Let $\mathcal{F} = \langle \{a\}, \{(a, a)\} \rangle$ (the Liar paradox). Then, $\mathcal{F}$ does not have any conglomerate.

The correspondence between conglomerates and classical labellings is easy to prove:

Theorem 1 *L is a classical labelling iff $A = \{x : L(x) = \mathtt{T}\}$ is a conglomerate.*

Proof Let $\mathcal{F} = \langle S, F \rangle$.
(If) Let A be a conglomerate of $\mathcal{F}$. Let L be such that $\forall x(x \in A \to L(x) = \mathtt{T})$ and $\forall x(x \in S \setminus A \to L(x) = \mathtt{F})$. Then, L trivially satisfies the conditions of a classical labelling.
(Only if) Let L be a classical labelling of $\mathcal{F}$, and let $A = \{x : L(x) = \mathtt{T}\}$ and $B = \{x : L(x) = \mathtt{F}\}$. (i) By definition, if $L(x) = \mathtt{T}$, then for all z such that $z \in \overrightarrow{F}(x)$, $L(z) = \mathtt{F}$. Hence, by construction, $x \in A$ and $z \in B$. Then, for all $x, z \in A$, $z \notin \overrightarrow{F}(x)$. (ii) By definition, for all $x \in S$, if $L(x) = \mathtt{F}$ and x is not a sink, then there exists some $z \in \overrightarrow{F}(x)$ such that $L(z) = \mathtt{T}$. Hence, by hypothesis, $x \in B$ and $z \in A$. Therefore, given (i) and (ii), we have that A is a conglomerate.

Corollary 1 *$\mathcal{F}$ is paradoxical iff it does not have any conglomerate.*

3 Groundedness

The truth value of sentences asserting the falsity of other sentences depends on the truth value of the referred sentences. If the truth value of a sentence does not depend on that of other sentences, "so that the truth value of the original statement can be ascertained, we call the original sentence *grounded*, otherwise *ungrounded*" (Kripke, 1975: 694). In our framework, sentences at sink nodes (for instance, object language sentences) do not depend on other sentences in that sense, so their truth values are determined by material (contingencies) or formal (tautologies or contradictions) facts that are exogenous to the model. Taking as grounded all the sinks that are either true or false, the groundedness of all the remaining sentences of an $\mathcal{F}$-system will be determined in an iterated process very similar to Kripke. In the base case, all the sinks established as true belong to a set S_0^+, and all those established as false belong to a set S_0^-. That is, the *partial set* (S_0^+, S_0^-) models the interpretation of the sink sentences. The systems considered by Cook and Rabern et al. are sink-free, hence, no sentence is grounded in the above sense in those systems. Beringer and Schindler, on the other hand, considered the existence of sinks (representing true arithmetical sentences), and, as in $\mathcal{F}$-systems, grounding can be traced by following dependency to sinks.

Definition 6 Given $\mathcal{F} = \langle S, F \rangle$, a pair (S_0^+, S_0^-) is a *ground base* iff $S_0^+ \cup S_0^- = sinks(S)$ and $S_0^+ \cap S_0^- = \emptyset$.

Then, we can find the other grounded sentences by iterated applications of the following operator:

Definition 7 Given two subsets $S^+, S^- \subseteq S$, we define $\phi((S^+, S^-)) = (S'^+, S'^-)$, where

$S'^+ = sinks(S^+) \cup \{x : \emptyset \neq \overrightarrow{F}(x) \subseteq S^-\}$, and
$S'^- = sinks(S^-) \cup \{x : \emptyset \neq \overrightarrow{F}(x) \cap S^+\}$.

That is, S'^+ includes the sinks that are already known as true plus all the sentences that only affirm the falsity of sentences already known as false; and S'^- includes the sinks that are already known as false plus all the sentences that affirm the falsity of some sentence already known as true. Hence, starting from any ground base (S_0^+, S_0^-), iterated applications of ϕ will lead to a fixed point. A fixed point is any pair $(S^+, S^-) = \phi((S^+, S^-))$. The fixed point $(S^+, S^-) = \phi^\infty((S_0^+, S_0^-))$ reached by the above mentioned iteration procedure is the least one relative to the ground base (S_0^+, S_0^-), in the sense that any other fixed point (S'^+, S'^-), where $S_0^+ \subseteq S'^+$ and $S_0^- \subseteq S'^-$, is such that $S^+ \subseteq S'^+$ and $S^- \subseteq S'^-$. The operator ϕ is monotone. Let $(S^+, S^-) \leq (S'^+, S'^-)$ iff $S^+ \subseteq S'^+$ and $S^- \subseteq S'^-$. Then:

Remark 1 (Monotonicity) If $(S^+, S^-) \leq (S'^+, S'^-)$, then $\phi((S^+, S^-)) \leq \phi((S'^+, S'^-))$.

Proof Assume $(S^+, S^-) \leq (S'^+, S'^-)$. Let $\phi((S^+, S^-)) = (T^+, T^-)$ and $\phi((S'^+, S'^-)) = (T'^+, T'^-)$. Assume now $w \in T^+$ and $z \in T^-$. We have to prove that 1) $w \in T'^+$ and 2) $z \in T'^-$.
1) If $w \in S^+$, the result is obvious. Assume now $w \notin S^+$. Then, $w \in \{x : x \notin S_0$ and $\forall y(y \in \overrightarrow{F}(x) \to y \in S^-)\}$. Then, $w \notin S_0$ and $\forall y(y \in \overrightarrow{F}(w) \to y \in T^-)$. Therefore, $w \in T'^+$.
2) If $z \in S^-$, the result is obvious. Assume now $z \notin S^-$. Then, $z \in \{x : x \notin S_0$ and $\exists y(y \in \overrightarrow{F}(x) \wedge y \in S^+)\}$. Then, $z \notin S_0$ and $\exists y(y \in \overrightarrow{F}(z) \wedge y \in T^+)$. Therefore, $z \in T'^-$.

The existence of the least fixed point is guaranteed by the monotonicity of ϕ and the fact that all the fixed points (relative to the same ground base) form a complete lattice (by Tarski's fixed points theorem. Cf. [10]).

Now, some ground bases may deem all the sentences grounded and others may not. Consider the following example:

Example 4 Let $\mathcal{F} = \langle \{a, b, c\}, \{(b, a), (b, c), (c, b)\} \rangle$. The only sink a determines two possible ground bases: (1) $(\{a\}, \emptyset)$, and (2) $(\emptyset, \{a\})$. For (1), all a, b, and c are grounded (true, false, and true, respectively), and for (2), only a is grounded (false). By way of illustration, the least fixed point in each case is reached as follows:

(1) $\phi((\{a\},\emptyset))$=$(\{a\},\{b\})$
$\phi^2((\{a\},\emptyset))$=$(\{a,c\},\{b\})$
$\vdots$
$\phi^\infty((\{a\},\emptyset))$=$(\{a,c\},\{b\})$

(2) $\phi((\emptyset,\{a\}))$=$(\emptyset,\{a\})$
$\vdots$
$\phi^\infty((\emptyset,\{a\}))$=$(\emptyset,\{a\})$

The following notion of groundedness takes into account those possibilities:

Definition 8 $\mathcal{F} = \langle S, F\rangle$ is (*relatively*) *grounded* iff for every $x \in S$ and for (some) every ground base (S_0^+, S_0^-), $x \in S^+ \cup S^-$, where $(S^+, S^-) = \phi^\infty((S_0^+, S_0^-))$.

The system in Example 4 is relatively grounded, but not grounded. On the other hand, $\mathcal{F}$-systems representing paradoxes as the Liar or Yablo's ones are neither grounded nor relatively grounded, as expected. Other systems are neither grounded nor relatively grounded, though they are not paradoxical. For instance, $\mathcal{F} = \langle\{a, b\}, \{(a, b), (b, a)\}\rangle$. Since there are no sinks, the only ground base is $(\emptyset, \emptyset)$, and $\phi^\infty((\emptyset, \emptyset)) = (\emptyset, \emptyset)$ is the least fixed point. However, other fixed points like $(\{a\}, \{b\})$ and $(\{b\}, \{a\})$ imply that a and b can be consistently assigned a classical truth value. This makes the difference with paradoxical systems, where no fixed point represents a consistent assignment of truth values to all the sentences.

4 Conglomerates and Fixed Points

Conglomerates of an $\mathcal{F}$-system can be characterized by means of the fixed points of ϕ as follows:

Definition 9 We say that (S^+, S^-) is *complete* iff for every $x \in S$, $x \in S^+ \cup S^-$, and *consistent* iff $S^+ \cap S^- = \emptyset$.

Theorem 2 *A is a conglomerate iff $(A, S \setminus A)$ is a complete and consistent fixed point of ϕ.*

Proof (Only if) Let A be a conglomerate of $\mathcal{F} = \langle S, F\rangle$. Let $(S^+, S^-) = \phi((A, S \setminus A))$. Then, we have:
(i) $S^- = sinks(S \setminus A) \cup \{x : \exists y(y \in \overrightarrow{F}(x) \wedge y \in A)\}$, and
(ii) $S^+ = sinks(A) \cup \{x : x \notin sinks(S)$ and $\forall y(y \in \overrightarrow{F}(x) \to\ y \in S \setminus A)\}$.
To show that $(A, S \setminus A)$ is a fixed point of ϕ, we have to prove that $S^+ = A$ and $S^- = S \setminus A$. From (i), it follows that $S^- = S \setminus A$, since A absorbs every non-sink node and, obviously, every sink of $S \setminus A$ belongs to $S \setminus A$. Let us prove now that $S^+ = A$. (1) $S^+ \subseteq A$: By *reductio*, assume that there exists

$x \notin A$, such that x is not a sink and $\forall y(y \in \overrightarrow{F}(x) \rightarrow\ \ y \in S \setminus A)$. But then, x is not absorbed by A, which contradicts that A is a conglomerate. (2) $A \subseteq S^+$: Of course, $sinks(A) \subseteq S^+$. Let now $x \in A$ be a non-sink node. From the definition of conglomerate, $\overleftarrow{F}(A) \subseteq S \setminus A$. Therefore, it is clear that $\forall y(y \in \overrightarrow{F}(x) \rightarrow\ \ y \in S \setminus A)$. Finally, the fact that $(A, S \setminus A)$ is a complete and consistent fixed point is obvious.
(If) Let (S^+, S^-) be a complete and consistent fixed point of $\mathcal{F} = \langle S, F \rangle$. By way of the absurd, assume $z \in \overrightarrow{F}(x)$ and $x, z \in S^+$. Then, $\phi((S^+, S^-))$ $= (S'^+, S'^-)$ is such that $x \in S'^+$ and $x \in S'^-$. But, this contradicts the consistency property. Therefore, (i) for all $x, z \in S^+$, $z \notin \overrightarrow{F}(x)$. Now, since (S^+, S^-) is complete and consistent, it follows that $S^- = S \setminus S^+$. By this and because (S^+, S^-) is a fixed point of ϕ, we have that $S \setminus S^+ = sinks(S^-) \cup \{x : \exists y(y \in \overrightarrow{F}(x) \wedge y \in S^+)\}$, which in turn implies that (ii) $(S \setminus S^+) \setminus sinks(S)$ $= \{x : \exists y(y \in \overrightarrow{F}(x) \wedge y \in S^+)\} \subseteq \overleftarrow{F}(A)$. Therefore, from (i) and (ii) and by definition, S^+ is a conglomerate.

Example 5 (Continuation of Example 4) There are three conglomerates, $\{a, c\}$, $\{b\}$, and $\{c\}$, that can be put in correspondence with the fixed points ($\{a, c\}$, $\{b\}$), ($\{b\}$, $\{a, c\}$), and ($\{c\}, \{a, b\}$), respectively.

From Theorem 2, it follows immediately that if A is a conglomerate then $(A, S \setminus A)$ is a maximal fixed point (w.r.t. $\leq$). The converse is clearly not true. For instance, $(\{a\}, \{a\})$ is a maximal fixed point in the Liar scenario $\langle \{a\}, \{(a, a)\} \rangle$, but $\{a\}$ is not a conglomerate. The conditions of completeness and consistency are key to relating maximal fixed points to conglomerates.

5 Sentences with Intrinsic Value: Referential Contradictions and Tautologies

We have seen that some $\mathcal{F}$-systems, by their own structural nature, have sentences that can only be labeled as undetermined (`U`): those just characterized as paradoxical. In addition, some systems have sentences that can only be labeled as true (`T`) and others, as false (`F`) by every classical labelling. We will say that those sentences are *referential tautologies/contradictions*, meaning that their truth/falsity is due to structural conditions of the system.

Definition 10 Given a non-paradoxical system $\mathcal{F} = \langle S, F \rangle$, $x \in S$ is a *referential contradiction* (*tautology*) iff $L(x) = \mathtt{F}$ ($L(x) = \mathtt{T}$), for every classical labelling L.[9]

As a consequence, referential contradictions and tautologies are related to conglomerates in the following way:

Proposition 1 *Let $\mathcal{F} = \langle S, F \rangle$ be non-paradoxical. Then, $x \in S$ is a referential contradiction (tautology) iff for every conglomerate A, $x \notin A$ ($x \in A$).*

[9] In Section 6, we will generalize these notions for every $\mathcal{F}$-system (not only the non-paradoxical ones) (Definition 12).

Proof It follows immediately from Theorem 1.

Referential tautologies are just sentences that assert the falsity of a contradiction. However, referential contradictions can exist independently of referential tautologies.

Proposition 2 *Given $\mathcal{F} = \langle S, F \rangle$, if $x \in S$ is a referential tautology, then there exists some $z \in S$, such that z is a referential contradiction (and $z \in \overrightarrow{F}(x)$).*

Proof By definition, if x is a referential tautology, then $L(x) = \mathtt{T}$ for every classical labelling L. By definition of 'labelling', x cannot be a sink (otherwise it could be labeled as $\mathtt{F}$ by some classical labelling). Hence, there exists $z \in \overrightarrow{F}(x)$ and, obviously, $L(z) = \mathtt{F}$ in every classical labelling L.

Let us see now the relation of referential contradictions and tautologies with fixed points. Due to the fact that an $\mathcal{F}$-system can have several ground bases (depending on the number of sinks), each ground base determines a class of fixed points that cannot be related in terms of $\leq$ with those determined by other ground bases. Hence, it is not suitable to talk about *intrinsic* fixed points in the present context in a similar manner as in the context of Kripke's theory. However, we can say that referential contradictions and tautologies have an intrinsic truth value in the sense that that value is the same in every fixed point in which those sentences receive a classical truth value.

Proposition 3 *Let $\mathcal{F} = \langle S, F \rangle$ be non-paradoxical. x is a referential contradiction (resp. tautology) in $\mathcal{F}$ iff for every complete and consistent fixed point (S^+, S^-) of ϕ, $x \in S^-$ (resp. $x \in S^+$).*

Proof x is a referential contradiction iff (by Proposition 1) for every conglomerate A, $x \notin A$ iff for every conglomerate A, $x \in S \setminus A$. Finally, by Theorem 2, for every conglomerate A, $(A, S \setminus A)$ is a complete and consistent fixed point. The case of referential tautologies is obvious.

Example 6 Let $\mathcal{F} = \langle \{a, b, c\}, \{(a, b), (b, c), (a, c)\} \rangle$. There exist two conglomerates, $\{c\}$ and $\{b\}$. Correspondingly, $(\{c\}, \{a, b\})$ and $(\{b\}, \{a, c\})$ are the only two complete and consistent fixed points. Therefore, a is a referential contradiction.

This example leads us to make two observations. First, referential contradictions and tautologies are not related to kernels in the same way as to conglomerates. In the example, the only kernel $\{c\}$ is useless to capture the referential contradiction of a. Second, the same example shows that transitivity is a source of referential contradictions when the antecedent conditions of the transitive property are met in non-paradoxical systems.

Proposition 4 *Let $\mathcal{F} = \langle S, F \rangle$ be non-paradoxical and F be transitive. If $x, y, z \in S$ are such that (x, y), $(y, z) \in F$, then x is a referential contradiction.*

Proof Assume the antecedent of the claim. By the transitivity of F, $(x, z) \in F$. Let L be a classical labelling of $\mathcal{F}$ (which exists due to the non-paradoxicality of $\mathcal{F}$). Then, either (i) $L(z) = \mathtt{T}$ or (ii) $L(z) = \mathtt{F}$. If (i) is the case then $L(x) = \mathtt{F}$. If (ii) is the case, then either (a) $L(y) = \mathtt{T}$ or (b) $L(y) = \mathtt{F}$. If (a) is the case, then $L(x) = \mathtt{F}$, and if (b) is the case, then there exists $w \in S$ such that $(y, w) \in F$ and $L(w) = \mathtt{T}$. But then, by transitivity, $(x, w) \in F$. That implies that $L(x) = \mathtt{F}$. Hence, in any case $L(x) = \mathtt{F}$. Therefore, x is a referential contradiction.

In addition, if transitivity is satisfied by systems where every sentence refers to other sentences, we get paradox, as we will see in Section 8.

The notions of referential contradictions and tautologies can be extended to systems that are partially paradoxical. That is, they can also be identified in systems where paradoxes can be circumscribed, so that other sentences can be assigned a classical truth value. However, there are no conglomerates in such systems as there are no classical labellings. In the next section, we will see a *local* version of conglomerate that will enable us –among other things– to redefine referential contradictions and tautologies with regards to any $\mathcal{F}$-system.

6 Local Conglomerates

Conglomerates capture all the sentences that can be true together in systems that are free of paradoxes. In terms of Dyrkolbotn [9], the absorption condition on conglomerates is *global*, and that inhibits some systems from having conglomerates. However, we can still want to know what sentences can be true together in systems containing paradoxes, even if that class is empty. That can be done by defining a *local* version of the absorption condition. To that aim, in this section, we define the notion of *local conglomerate*.

Definition 11 Given $\mathcal{F} = \langle S, F \rangle$, a *local conglomerate* is a subset $A \subseteq S$ that satisfies:

1. $\overleftarrow{F}(A) \subseteq S \setminus A$, and
2. $\overrightarrow{F}(A) \setminus sinks(S) \subseteq \overleftarrow{F}(A)$.

It is easy to see that conglomerates are also local conglomerates, which allows to establish a hierarchy among the notions, including kernels. Local conglomerates only "absorb" those non-sink nodes to which they point at. As a consequence, the empty set of nodes is always a local conglomerate, which implies that the notion is well-defined: every $\mathcal{F}$-system has, at least, one local conglomerate. Moreover, due to the fact that local conglomerates are local "absorbers," they capture all the sentences that can be assigned the true value in contexts where paradoxes can be isolated.

Example 7 Let $\mathcal{F} = \langle S, F \rangle = \langle \{a, b, c, d, e\}, \{(a, a), (b, c), (c, d), (b, d), (d, e)\} \rangle$. Then, $\mathcal{F}$ has no conglomerates, but has three non-empty local conglomerates, $\{d\}$, $\{e\}$, and $\{c, e\}$ (Figure 1).

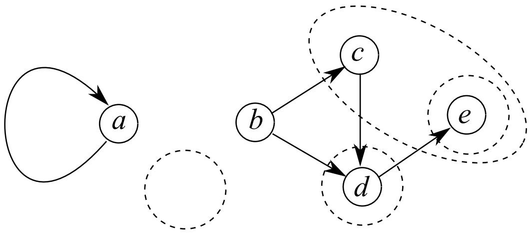

Fig. 1 Local conglomerates in dashed lines. The empty set is always a local conglomerate.

The two following results relate maximal local conglomerates to maximally consistent fixed points and labellings maximizing T.

Theorem 3 *A is a maximal (w.r.t. $\subseteq$) local conglomerate iff $(A, \overrightarrow{F}(A) \cup \overleftarrow{F}(A))$ is a maximally consistent (w.r.t. $\leq$) fixed point of ϕ.*

Proof (If) Let (A, B) be a maximally consistent (w.r.t. $\leq$) fixed point of ϕ. Then,
(i) $\overleftarrow{F}(A) \subseteq S \setminus A$. To prove this, assume $x \in \overleftarrow{F}(A)$. By way of contradiction, assume now that $x \in A$ and let $z \in A$ be such that $z \in \overrightarrow{F}(x)$. Since (A, B) is a fixed point, we have that $B = sinks(B) \cup \{x : \emptyset \neq \overrightarrow{F}(x) \cap A\}$. Then, $z \in B$. Hence, $z \in A \cap B$, which contradicts the consistency of (A, B). Therefore, $x \notin A$, which implies that $x \in S \setminus A$.
(ii) $\overrightarrow{F}(A) \setminus sinks(S) \subseteq \overleftarrow{F}(A)$. To prove this, let $u \in A$ and $w \in \overrightarrow{F}(u) \setminus sinks(S)$ (i.e., $w \in \overrightarrow{F}(A) \setminus sinks(S)$). Since (A, B) is a fixed point, the hypothesis $w \notin \overleftarrow{F}(A)$ implies that $w \notin B$, because $B = sinks(B) \cup \{x : \emptyset \neq \overrightarrow{F}(x) \cap A\}$. But then $u \notin A$, since $A = sinks(A) \cup \{x : \emptyset \neq \overrightarrow{F}(x) \subseteq B\}$. Contradiction. Therefore, $w \in \overleftarrow{F}(A)$.
From (i) and (ii), it follows that A is a local conglomerate. Assume now that A is not maximal, i.e., there exists some local conglomerate A', such that $A \subset A'$. Let $x \in A' \setminus A$. If x is a sink then $x \in sinks(A') \cup \{x : \emptyset \neq \overrightarrow{F}(x) \subseteq B\} = A'$. But this contradicts that (A, B) is a maximal fixed point. Hence, x is not a sink. Let $z \in \overrightarrow{F}(x)$. Then $z \in B'$, for some B' such that $B \subseteq B'$. But then $z \in \overleftarrow{F}(A')$. Then, $(A, B) \leq \phi((A', B'))$ but not $\phi((A', B')) \leq (A, B)$. This also contradicts that (A, B) is a maximal fixed point.
(Only if) Let A be a maximal (w.r.t. $\subseteq$) local conglomerate. We have to prove that $\phi((A, \overrightarrow{F}(A) \cup \overleftarrow{F}(A))) = (A, \overrightarrow{F}(A) \cup \overleftarrow{F}(A))$. This follows from: (i) $A = \{sinks(A) \cup \{x : \emptyset \neq \overrightarrow{F}(x) \subseteq \overrightarrow{F}(A) \cup \overleftarrow{F}(A)\}$. For this, just observe that since A is a maximal local conglomerate, if x is not a sink, then x is such that $\emptyset \neq \overrightarrow{F}(x) \subseteq \overrightarrow{F}(A) \cup \overleftarrow{F}(A)$ iff $x \in A$. (ii) $\overrightarrow{F}(A) \cup \overleftarrow{F}(A) = sinks(\overrightarrow{F}(A) \cup \overleftarrow{F}(A)) \cup \{x : \emptyset \neq \overrightarrow{F}(x) \cap A\}$. To prove this, note that, since $\{x : \emptyset \neq \overrightarrow{F}(x) \cap A\} = \overleftarrow{F}(A)$, it is obvious that $sinks(\overrightarrow{F}(A) \cup \overleftarrow{F}(A)) \cup \{x : \emptyset \neq \overrightarrow{F}(x) \cap A\} \subseteq \overrightarrow{F}(A) \cup \overleftarrow{F}(A)$. Now, since A is a local conglomerate, $\overrightarrow{F}(A) \setminus sinks(\overrightarrow{F}(A)) \subseteq \overleftarrow{F}(A)$, hence, we have that $\overrightarrow{F}(A) \cup \overleftarrow{F}(A) \subseteq sinks(\overrightarrow{F}(A) \cup \overleftarrow{F}(A)) \cup \{x : \emptyset \neq \overrightarrow{F}(x) \cap A\}$.

Theorem 4 *L is a labelling that maximizes* $\mathtt{T}$ *iff* $A = \{x : L(x) = \mathtt{T}\}$ *is a maximal local conglomerate.*

Proof (If) Let $A = \{x : L(x) = \mathtt{T}\}$ be a maximal local conglomerate. Let us assume, by way of the absurd, that there exists $z \notin A$, such that $L(z) = \mathtt{T}$. Then, for all $w \in \overrightarrow{F}(z)$, $L(w) = \mathtt{F}$. Then, since $\overrightarrow{F}(A) \setminus sinks(S) \subseteq \overleftarrow{F}(A)$ and $L(x) = \mathtt{T}$ for every $x \in A$, it follows that $z \notin \overrightarrow{F}(A)$. Moreover, $z \notin \overleftarrow{F}(A)$ either, otherwise we would have $L(z) = \mathtt{F}$. Then, $A \cup \{z\}$ is a local conglomerate greater than A, contradicting the hypothesis. Therefore, for every $z \notin A$, $L(z) \neq \mathtt{T}$, which implies that L is a labelling that maximizes $\mathtt{T}$.
(Only if) Let L be a labelling that maximizes $\mathtt{T}$. First, it is easy to see that $A = \{x : L(x) = \mathtt{T}\}$ is a local conglomerate. Now assume, by way of the absurd, that A is not maximal. Then, there exists some local conglomerate A' such that $A \subset A'$. Let $x \in A' \setminus A$. Since L maximizes $\mathtt{T}$, then $L(x) \neq \mathtt{T}$. If $L(x) = \mathtt{F}$, then there exists some $z \in A$ such that $z \in \overrightarrow{F}(x)$. This contradicts that A' is a local conglomerate. And if $L(x) = \mathtt{U}$, then x is paradoxical, which also contradicts that A' is a local conglomerate. Therefore, A is a maximal local conglomerate.

Thus, from the two previous theorems, we get the following corollary:

Corollary 2 *For every set of sentences $A \subseteq S$, the following three statements are equivalent:*

1. *A is a maximal local conglomerate.*
2. *There exists a labelling L that maximizes* $\mathtt{T}$ *and* $A = \{x : L(x) = \mathtt{T}\}$.
3. $(A, \overrightarrow{F}(A) \cup \overleftarrow{F}(A))$ *is a consistent maximal (w.r.t. $\leq$) fixed point of ϕ.*

Local conglomerates enable us to characterize the notions of referential contradictions and tautologies with regards to any $\mathcal{F}$-system (not only non-paradoxical ones). First, we redefine the notions as follows.

Definition 12 Let $\mathcal{F} = \langle S, F\rangle$, $x \in S$ is a *referential contradiction* (*tautology*) iff $L(x) = \mathtt{F}$ ($L(x) = \mathtt{T}$), for every labelling L that maximizes $\mathtt{F}$ ($\mathtt{T}$).

From [4], it is known that a labelling maximizes $\mathtt{T}$ if and only if it maximizes $\mathtt{F}$. Then, from Definition 12 and Theorems 3 and 4, we immediately have that x is a referential tautology (resp. contradiction) iff $x \in A$ (resp. $x \in \overrightarrow{F}(A) \cup \overleftarrow{F}(A)$) for every $A \subseteq S$ that is a maximal local conglomerate or, equivalently, for every $A \subseteq S$ such that $(A, \overrightarrow{F}(A) \cup \overleftarrow{F}(A))$ is a maximally consistent (w.r.t. $\leq$) fixed point of ϕ. In Example 8, b can only be assigned $\mathtt{F}$ by labellings that maximize the labels $\mathtt{T}$ and $\mathtt{F}$, hence, it is a referential contradiction. The maximal local conglomerates $\{d\}$ and $\{c, e\}$ are matched with the fixed points $(\{d\}, \{c, e, b\})$ and $(\{c, e\}, \{d, b\})$, respectively.

An interesting question[10] is whether referential tautologies and contradictions can be related to the intensional concepts of *semi-truth* and *semi-falsity*,

[10] Posed by an anonymous reviewer.

respectively [17,7][11]. The idea is that a sentence is semi-true (semi-false) if and only if it is true (false) in some fixed points and false (true) in none. Consider the following example:

Example 8 Let $\mathcal{F} = \langle S, F\rangle = \langle\{a,b,c,d\},\ \{(a,b),\ (b,c),\ (a,c),\ (c,d),\ (d,c)\}\rangle$. Then, a has no classical truth value in the minimal fixed point $(\emptyset,\emptyset)$, but it is false in every consistent fixed point extending the minimal one.

As we know, referential tautologies (contradictions) are true (false) in *every* maximally consistent fixed point and, for each fixed point that consistently assigns truth values, there exists a maximally consistent fixed points extending it. Hence, referential tautologies and contradictions can be considered (the only) semi-true and semi-false sentences, respectively, in the context of $\mathcal{F}$-systems. Moreover, the intentional notions of *semi-classicality* (either true or false in every maximally consistent fixed point) and *paradox* (neither true nor false in any fixed point) can also be understood in a similar way.

7 Local Conglomerates in Dung's Argumentation Frameworks

P.M. Dung [8] has defined *argumentation frameworks*, which are just special cases of $\mathcal{F}$-systems, where S is interpreted as a set of *arguments* and F, as an *attack* relation. The notion of *admissibility* captures the idea of sets of arguments that can be defended jointly: $A \subseteq S$ is *admissible* iff 1) $\overrightarrow{F}(A) \subseteq S\backslash A$ (i.e., the arguments that A attacks are outside A), and 2) $\overleftarrow{F}(A) \subseteq \overrightarrow{F}(A)$ (i.e., A attacks all its attackers). Replacing F with F^{-1}, we obtain a notion that is known in graph theory with the name of *semikernel* [11]; the notion of local conglomerate, in turn, leads to a weaker version of admissibility.

In Dung's theory, the class of all the admissible sets of an argumentation framework forms a complete partial order with respect to the set inclusion (i.e., reflexive, transitive, antisymmetric, and every increasing sequence has a least upper bound). The "fundamental lemma," from which the result follows immediately in [8], can be paraphrased in our framework with the help of the operator ϕ.

Lemma 1 *Let $\phi((A,\ \overrightarrow{F}(A) \cup \overleftarrow{F}(A))) = ((S^+, S^-))$ and A be a local conglomerate. Then, for every $x, z \in S^+$, the set $A' = A \cup \{x\}$ is such that*

1. *A' is a local conglomerate.*
2. *Let $\phi((A',\ \overrightarrow{F}(A') \cup \overleftarrow{F}(A'))) = (S'^+,\ S'^-)$. Then, $z \in S'^+$.*

Proof 1. We only have to prove that $\overleftarrow{F}(A') \subseteq S \setminus A'$. Assume the contrary. Since $\overleftarrow{F}(A) \subseteq S \setminus A$, there exists some $w \in A$, such that either $x \in \overrightarrow{F}(w)$ or $w \in \overrightarrow{F}(x)$. Assume $x \in \overrightarrow{F}(w)$. Since $x \in S^+$, by definition of ϕ, $\emptyset \neq \overrightarrow{F}(x) \subseteq \overrightarrow{F}(A) \cup \overleftarrow{F}(A)$. Hence, $\overrightarrow{F}(x) \cap A = \emptyset$, which implies that A does not absorb w. This contradicts that A is a local conglomerate. Assume now that $w \in \overrightarrow{F}(x)$. Then, $x \notin S^+$. Contradiction.

[11] Cook [7] gave the canonical example of the tautology-teller ('This sentence is either true or false') as semi-true, but that is clearly not expressible in the present framework.

2. Since $z \in S^+$, we have that $\overrightarrow{F}(z) \subseteq \overrightarrow{F}(A) \cup \overleftarrow{F}(A)$. Then, given that $A \subseteq A'$, $\overrightarrow{F}(z) \subseteq \overrightarrow{F}(A') \cup \overleftarrow{F}(A')$. Therefore, $z \in S'^+$.

Intuitively, the lemma says that, if A is a local conglomerate and ϕ can expand it to incorporate x and z, then $A \cup \{x\}$ is a local conglomerate and ϕ can expand it to incorporate z. As a consequence, the class of all local conglomerates forms a complete partial order with respect to $\subseteq$. Maximally (w.r.t. $\subseteq$) admissible sets of arguments are called *preferred extensions* in Dung's theory. They represent maximal subsets of arguments that a "credulous" agent (i.e., an agent willing to accept arguments beyond groundedness) can accept. Caminada [4] showed that preferred extensions correspond to labellings that maximize $\mathtt{T}$ (or, equivalently, $\mathtt{F}$). In a similar way, we put maximal local conglomerates also in correspondence with such labellings in our framework (Theorem 4). Preferred extension semantics is different from *stable* extension semantics, in correspondence with the difference between semikernels and kernels. Intuitively, the difference is that a stable extension is a set of arguments that not only responds to every external attack, but also attacks *every* external argument. As kernels, stable extensions are not defined for every argumentation framework, and those argumentation frameworks with no stable semantics involve some kind of argumentative paradox or anomaly. Dung argued that it is not necessarily all wrong with those argumentation frameworks. Analogously, it is not necessarily all wrong with $\mathcal{F}$-systems having no conglomerates, in the sense that they may have many meaningful, non-paradoxical parts, and those parts can be captured via local conglomerates.

The notion of local conglomerate clearly extends that of admissible set (again, provided that F is changed to F^{-1} appropriately). Let us argue why local conglomerates not corresponding to admissible sets can make sense in argumentation frameworks. Consider the system $\langle \{a, b\}, \{(a, b)\} \rangle$. Interpreted as an argumentation framework, we have that argument a attacks argument b, and the only maximally admissible set is $\{a\}$ (by the way, $\emptyset$ is also admissible, but not maximally). On the other hand, there are two maximal local conglomerates: $\{a\}$ and $\{b\}$. In what sense can $\{b\}$ be "admissible"? The notion of admissibility was introduced by Dung as a model of the "principle" *"The one who has the last word laughs best."* As such, it is a good model. However, we can also think of situations in which the last word is wrong. For example, assume b is an argument without objections except for a, which is an argument that everybody in the audience would reject. Then that audience would possibly deem the argument b *in* inasmuch as a is *out*, and even if a is the last word. The local conglomerate $\{b\}$ enables us to capture that possibility. A similar intuition is also modeled by value-based argumentation frameworks (VAF's) [1]: if a attacks b, but b promotes a value that is preferred to the value promoted by a, then a does not *defeat* b. For example, consider, on the one hand, an argument that promotes the legalization of abortion as a public health necessity in view of a significant magnitude of women's deaths as a result of clandestine abortion practices and, on the other hand, an argument that promotes the prohibition of abortion as a need for protection of the fundamental human right to the life of an embryo. Suppose the second argument is advanced as an attack on the first. However, if the audience values public health over the right to the life of a human embryo, it will result that the expected defeat will have no effect (the same applies in the opposite case). In this setting, we can capture the opposing opinions with two different (local) conglomerates. In any case, these ideas are only exploratory and require more in-depth analysis that is beyond the scope of the present work.

8 Sufficient Conditions for Paradox in $\mathcal{F}$-systems

In this section, we summarize some conditions found in the literature in relation to different systems, which are sufficient for paradoxes to occur and which apply to general $\mathcal{F}$-systems. We will express those conditions using our notation. Rabern et al. and Beringer and Schindler identified structural properties of the digraphs as necessary conditions for paradox. The conditions are, basically, the existence of directed cycles (as in the case of the Liar) [15] or double paths[12] (as in the case of the Yablo's paradox) [2]. Previously, Cook [5] defined *f-forced* sentences as those x such that there exists an infinite sequence $x, x_1, x_2, \ldots, x_n$ in F, where n is even, and either $x \in \overrightarrow{F}(x_n)$ or $x_n \in \overrightarrow{F}(x)$; then, he showed that (i) if L is a classical labelling and x is f-forced, then $L(x) = \mathtt{F}$ (i.e., x is a referential contradiction), and (ii) if x is f-forced and for every $y \in \overrightarrow{F}(x)$, y is f-forced, then the $\mathcal{F}$-system is paradoxical. Even before Cook, Dung [8] observed, in the context of argumentation frameworks, a similar issue due to what he called *controversial* arguments. x is said to be *controversial w.r.t.* y iff (i) there exists a sequence $x_0, \ldots, x_{2n+1}$, such that $y = x_0$, $x = x_{2n+1}$, and $(x_{i+1}, x_i) \in F$, and (ii) there exists a sequence $x_0, \ldots, x_{2n}$, such that $y = x_0$, $x = x_{2n}$, and $(x_{i+1}, x_i) \in F$. x is said to be *controversial* if it is controversial w.r.t. some y.[13] Then, Dung defined the property of "limited controversiality" for argumentation frameworks which, in terms of our general setting, says that an $\mathcal{F}$-system is *limited controversial* if there exists no infinite sequence $x_0, \ldots, x_n, \ldots$ such that x_i is controversial with respect to x_{i+1}. Note that both Yablo's and the Liar paradoxical systems are particular cases of non-limited controversial systems. We can discern two different ways of not complying with the limited controversiality, and both are sufficient conditions for paradox. One has to do with transitivity in sink-free systems and the other, with odd-length cycles in F. The first one was observed by Cook [5], while the second one, will be precised here.

[12] A double path is a graph consisting of two non-trivial paths, both with common origin and end.

[13] In terms of argumentation, x is controversial w.r.t. y if and only if x indirectly attacks y (odd-length path) and indirectly defends y (even-length path). In terms of sentences, we would be tempted to say that x indirectly affirms the falsity of y and indirectly affirms the truth of y, but this is not necessarily the case, since the possible existence of shortcuts between both paths could give rise to different interpretations.

8.1 Transitivity

F is transitive iff for all $x, y, z \in S$, if $y \in \overrightarrow{F}(x)$ and $z \in \overrightarrow{F}(y)$, then $z \in \overrightarrow{F}(x)$. For instance, in Figure 1, $F|_{\{b,c,e\}}$ (i.e., the restriction of F to $\{b, c, e\}$) is transitive. Note that b is a referential contradiction: it cannot be labelled as `T`, but it can be labeled as `F` whenever d is labeled as `T` or `F`. Moreover, if *every* sentence refers to the falsity of some other sentences, then transitivity will also prevent the assignment of the label `F`. This result was showed by Cook [5].

Definition 13 $\mathcal{F} = \langle S, F\rangle$ is *unlimited transitive* iff (i) F is transitive and (ii) $\mathcal{F}$ is a serial digraph (i.e., no node is a sink).

Proposition 5 *(Cook [5]) If $\mathcal{F}$ is unlimited transitive, then it is paradoxical.*

Proof Assume $\mathcal{F} = \langle S, F\rangle$ is not paradoxical. Then it has a classical labelling L. Then, for every $x \in S$, either $L(x) = \mathtt{T}$ or $L(x) = \mathtt{F}$. Assume $L(x) = \mathtt{T}$. Then, for all $y \in \overrightarrow{F}(x)$, $L(y) = \mathtt{F}$. Let now $y \in \overrightarrow{F}(x)$. Then, for some $z \in \overrightarrow{F}(y)$, $L(z) = \mathtt{T}$. But, by transitivity, $z \in \overrightarrow{F}(x)$, which implies that $L(z) = \mathtt{F}$. Contradiction. Assume now $L(x) = \mathtt{F}$. Then, for some $y \in \overrightarrow{F}(x)$, $L(y) = \mathtt{T}$. Then we can apply on y the same argument as before to get a contradiction. Therefore, $\mathcal{F}$ is paradoxical.

Both the Liar and Yablo's paradoxes can be modeled as unlimited transitive $\mathcal{F}$-systems. Moreover, as we have seen before (Proposition 4), transitivity is also a source of other untruthful sentences like referential contradictions.

8.2 Odd-Length Cycles

Unlike Yablo's paradox, the Liar paradox contains a referential cycle. Referential cycles imply direct or indirect self-reference, but not every referential cycle leads to paradox. Bolander [3] distinguished between vicious and innocuous self-reference and claimed that the first sort can only occur if it involves negation or something equivalent (like in 'not true' or 'untrue'). That is guaranteed in $\mathcal{F}$-systems. However, the conditions for vicious self-reference can be further narrowed. Next, we define some conditions involving cycles that suffice to yield paradox in $\mathcal{F}$-systems.

Definition 14 Given $\mathcal{F} = \langle S, F\rangle$, a subset $O \subseteq S$ is an *odd core* of $\mathcal{F}$ iff for some $n \geq 0$ there exist sentences $x_1, \ldots, x_{2n+1} \in S$, such that $O = \{x_1, \ldots, x_{2n+1}\}$, $\{(x_i, x_{i+1}) : 1 \leq i \leq 2n\} \cup \{(x_{2n+1}, x_1)\} \subseteq F$, and for all $x \in O$, $|\overrightarrow{F}(x)| = 1$ (i.e., the restriction of F to O is *functional*). Moreover, $\mathcal{F}$ is *odd* iff it has an odd core.

Informally, the definition says that an odd $\mathcal{F}$-system is such that there exists an odd-length cycle in F, the nodes of which can shoot exactly one arrow each (no matter how many arrows point to them).

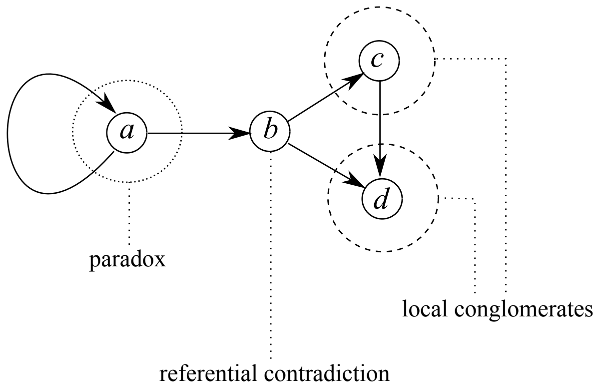

Fig. 2 Odd-length cycles yield paradox whenever all their nodes point only to untrue sentences (either paradoxical or not)

Proposition 6 *If $\mathcal{F}$ is odd, then it is paradoxical.*

Proof Let $\mathcal{F} = \langle S, F\rangle$ be odd and let $O = \{x_1, \ldots, x_{2n+1}\}$ be an odd core of $\mathcal{F}$. By way of contradiction, let us assume that A is a conglomerate of $\mathcal{F}$. Assume, without lost of generality, that $x_1 \in A$. Then, since for all $x \in O$, $|\overrightarrow{F}(x)| = 1$, by the absorption condition we have $\{x_1, x_3, \ldots, x_{2n+1}\} \subseteq A$. However, since $(x_{2n+1}, x_1) \in F$, that contradicts the independence property of A. Hence, we should have $x_1 \in S \setminus A$. This implies that $\{x_2, x_4, \ldots, x_{2n}, x_1\} \subseteq A$. However, since $(x_1, x_2) \in F$, we get again a contradiction with the independence property. Therefore, $\mathcal{F}$ cannot have any conglomerate, which means that it is paradoxical.

This sufficient condition can be relaxed to some extent. We will still have paradox by allowing the nodes involved in odd-length cycles to point to referential contradictions, i.e., nodes that do not belong to any local conglomerate. This is easy to see, since no sentence referred from the cycle can be labeled as true, and at least one referred sentence has undetermined value (see an example in Figure 2).

From the last two propositions we can see that the source of paradox in the Liar paradox is twofold: the $\mathcal{F}$-system is both unlimited transitive and odd. Yablo's paradox, in turn, only suffers from unlimited transitivity. Thus, we have identified more specific cases of directed cycles and double paths that yield paradox in $\mathcal{F}$-systems.

It is worth mentioning that Dyrkolbotn [9] resumed some known sufficient conditions to avoid paradox in presence of odd-length cycles, though they are limited to finite systems. In our framework, the result can be expressed as follows:

Proposition 7 *(Dyrkolbotn) Any finite $\mathcal{F}$-system has a conglomerate if every odd-length cycle has one of the following conditions*

1. *at least two symmetric nodes*
2. *at least two crossing consecutive chords (a* chord *is an arrow on a cycle connecting two non-consecutive nodes)*
3. *at least two chords with consecutive targets.*

The lesson seems to be that paradoxes are related to transitivity in infinite contexts, and to cycles of odd length in finite ones. However, both issues can be connected in some way. An important finding by Cook (maybe in collaboration with Thomas Bolander, as suggested in [5]) is that every finite paradoxical set of sentences is, in some sense, equivalent to an infinite set of sentences with no cycles. Cook introduced *unwinding*, an ingenious paradox-preserving operation that, applied to an arbitrary system, returns an infinite acyclic system (see [5] for details, and [15] for a clear illustration). In that way, the unwinding of the Liar paradox yields the Yablo's paradox. In sum, circularity is not a *necessary* condition for paradox in systems with infinite sentences, while in finite contexts it remains to specify what kinds of circularities constitute *necessary* conditions.

9 Final Comments and Conclusions

The notions of conglomerate and local conglomerate enabled us to essentially capture the same results as that of kernel regarding semantic paradoxes. However, while kernels can only represent situations in which all the sink (object language) sentences in the system are true, conglomerates also allow to represent the false cases. An important result is that our concepts allow for the characterization of referential contradictions and tautologies, i.e., sentences that can be assigned only the false value or only the true value, respectively. Referential contradictions cannot belong to any local conglomerate, while referential tautologies belong to every maximal local conglomerate.

$\mathcal{F}$-systems offer a treatment of paradoxes and other sentence interaction problems at an abstract level with respect to language, similar to how Dung's argumentation frameworks treat the interaction among arguments by abstracting the specific nature and qualities of arguments. The level of abstraction is key to subsuming argumentation frameworks in the most general framework of $\mathcal{F}$-systems. However, the analogy goes further, since predication of falsity among sentences and refutation among arguments generate similar patterns of paradoxical behavior. Cook's acceptable assignments for a propositional language of paradox [5] and Caminada's labelling semantics for Dung's systems [4] provide essentially the same semantic device for analyzing the aforementioned phenomena. Furthermore, Dung showed that Kripke's original ideas about fixed points of an operator to trace dependency and its application to groundedness and intrinsic truth value of sentences are useful for characterizing complete semantics in argumentation. Dyrkolbotn [9] has previously treated paradox, kernel theory, and argumentation frameworks on a common ground. He showed the connection between local kernels and admissible sets. Here, we extended that connection with the notion of local conglomerate, which in turn gives a twist to the notion of argument admissibility. In that respect, we plan to explore in the future the use of local conglomerates as a semantics for value-based argumentation frameworks [1], as suggested in Section 7.

Finally, we have shown two sufficient conditions for paradox, to wit, transitivity in serial digraphs (previously discussed by Cook [5]) and odd-length cycles involving single reference sentences, which are special cases of the controversy problem treated by Dung. As we can see, much of the studies on semantic paradoxes and argumentation theory can contribute to each other within the common ground of $\mathcal{F}$-systems.

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