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From Pictures to Employments: Later Wittgenstein on 'the Infinite'

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Abstract
With respect to the metaphysics of infinity, the tendency of standard debates is to either endorse or to deny the reality of 'the infinite'. But how should we understand the notion of 'reality' employed in stating these options? Wittgenstein's critical strategy shows that the notion is grounded in a confusion: talk of infinity naturally takes hold of one's imagination due to the sway of verbal pictures and analogies suggested by our words. This is the source of various philosophical pictures that in turn give rise to the standard metaphysical debates: that the mathematics of infinity corresponds to a special realm of infinite objects, that the infinite is profoundly huge or vast, or that the ability to think about infinity reveals mysterious powers in human beings. First, I explain Wittgenstein's general strategy for undermining philosophical pictures of 'the infinite' – as he describes it in *Zettel*; and then show how that critical strategy is applied to Cantor's diagonalization proof in *Remarks on the Foundations of Mathematics* II.

> On mathematics: 'Your idea [*Begriff*] is wrong.—However, I cannot illumine the matter by fighting against your words, but only by trying to turn your attention away from certain expressions, illustrations, images, and *toward the employment* of the words.'
> (Z 463)
>
> Say what you please, so long as it does not prevent you from seeing how things are. (PI 79)

1. Introduction

In this paper, I offer a reading of the later Wittgenstein's most distinctive commentary on 'the infinite'. On my reading, Wittgenstein's discussion is intended to undermine a crucial presupposition of standard debates. With respect to the metaphysics of infinity, the tendency is to either endorse or to deny the reality of 'the infinite'. But how should we understand the notion of 'reality' employed in stating these options? Wittgenstein's critical strategy shows that the notion is grounded in a confusion: talk of infinity naturally takes hold of one's imagination due to the sway of verbal pictures and analogies suggested by our words. This is the source of various philosophical pictures of the infinite: that the mathematics of infinity corresponds to a special realm of infinite objects, that the infinite is profoundly huge or vast, or that the ability to think about infinity reveals mysterious powers in human beings.[1] However, one's imagination in these instances does not

[1] Compare Hilbert's (1926) claim that Cantor's work presents a kind of mathematical 'paradise', Cantor's thought that the mathematics of infinity reveals to us the infinite nature of God (Moore 1990: 127), or Nagel's suggestion that it is mysterious (and possibly inexplicable) how we so much as have an *idea* of the infinite (Nagel 1997: 76). Compare also Leopold Kronecker's early critique of Cantor's work on infinity as a kind of theology rather than genuine mathematics, famously remarking that 'God made the integers, all the rest is the work of man' (see Moore 1990: 120).

determine the meanings of 'infinity' and its surrounding expressions – but rather their meanings are ultimately to be found in their uses and applications, and such uses don't determine what, if anything, to say about the 'reality' of the infinite. To inquire into the reality of the infinite is thus as confused as inquiring into the reality of 'the average person' on the grounds that such talk implies the existence of a special being – distinct from all other human beings.

The reading offered will also allow us to answer two powerful objections raised by A.W. Moore against the later Wittgenstein's critique of 'the infinite'. The objections are intended to reveal an inconsistency between Wittgenstein's meta-philosophical prescriptions and his more local remarks about mathematics.

> **OBJECTION 1.** Wittgenstein claims that he will not in any way interfere with mathematics itself, but 'leave it as it is' (PI 124). And yet Wittgenstein appears to suggest that misleading vocabulary should be *removed* from mathematics. 'To claim that that vocabulary can be peeled off from the underlying calculus *is* to issue a direct challenge to [mathematicians'] work' (Moore 2016: 327).
>
> **OBJECTION 2.** Wittgenstein appears to deny the mathematical results of Cantor's proof due to his own philosophical uneasiness about them. 'Wittgenstein is very uncomfortable with the way in which set theorists claim to have shown that some infinite sets are bigger than others […] [but] that some infinite sets are bigger than others would be accepted by any orthodox set theorist as an unassailable result of set theory' (Moore 2016: 324).[2]

As I will show in my reading of Wittgenstein's *later* commentary on 'the infinite', his remarks are not intended to show that one ought to either remove vocabulary from mathematics or deny any of its established results. Rather, he finds that *when* the vocabulary and the standard descriptions of mathematical results are unwittingly analogized with distinct, especially non-mathematical, uses of said vocabulary, then they are apt to suggest imaginative pictures that are inappropriate to their actual uses or applications. This leaves the uses or applications themselves completely intact (thereby, 'leaving mathematics as it is'), while challenging imaginative extravagances that verbal pictures of those applications might inspire – e.g., of a special realm of profoundly vast entities, requiring mysterious intellectual powers to grasp.[3] To the extent that this critique presents any challenge

Boolos, Burgess, and Jeffrey (2007: 19) recognize the potential of Cantor's proof to play with our imagination, and even (cautiously) indulge in such extravagances (involving the powers of divine beings), albeit for strictly pedagogical reasons.

[2] Compare Dummett (1978: 168): 'Certainly in his discussion of Cantor he displays no timidity about "interfering with the mathematicians".'

[3] See fn. 1. My reading thus bears similarities to Shanker's (1987) discussion of Wittgenstein on (Cantorian) infinity, especially in his emphasis on Wittgenstein's suspicions about the suggestive 'prose' used to describe a mathematical result rather than the mathematical 'proof' itself. However,

to mathematical practices, it *might* encourage one to abandon certain of those practices entirely – e.g., if it disrupts a romantic yet misbegotten picture one initially had of it (cf., LFM, p. 103; RFM II, 62; LC 28). But since one can find many reasons to engage in mathematics without being attracted to such pictures, to that extent Wittgenstein's remarks leave mathematical practice 'as it is'.

The reading I offer highlights a distinctive and compelling critique that is not only consistent with, but indeed cannot be understood independently of Wittgenstein's later conception of philosophical practice as it is expressed in the *Philosophical Investigations*. The critique presented is thus easily overlooked if one assimilates into a singular perspective Wittgenstein's commentary on 'the infinite' throughout different stages of his philosophical thought.[4]

In section 2, I will explain this general critical perspective on 'the infinite' (distinctive to Wittgenstein's later philosophy) and its attending confusions as he describes it in *Zettel*. In section 3, I will show how Wittgenstein's discussion of diagonalization in Manuscript 117 (as presented in *Remarks on the Foundations of Mathematics* II) aims at counteracting the misleading effects it might have on one's imagination via a sober redescription of Cantor's famous proof that 'the set of real numbers is *bigger* than the set of natural numbers'.[5] Again, to counteract such misleading effects and imaginative extravagances – drawing us away from misbegotten 'pictures' and returning us to its genuine 'employments' – will leave the mathematics itself 'as it is', though it might well reduce philosophical (especially metaphysical) excitement about it.

I have avoided that distinction (between 'prose' and 'proof') to emphasize that *later* Wittgenstein's concern (see fn. 4 below) is more specifically about the confused pictures that such 'prose' tends to suggest – in which case even standard 'prose' can be tolerated if it is not misunderstood. For example, via misleading analogies across distinctive uses of the relevant vocabulary. (Compare PI 79, an epigraph to this paper.)

[4] For all that I will say, Moore's objections might be perfectly valid regarding Wittgenstein's *early* or *transitional-period* remarks on 'the infinite'. By Wittgenstein's 'early' remarks, I have in mind *Notebooks 1914-16* and *Tractatus Logico-Philosophicus*. By his 'transitional-period', I have in mind his *Philosophical Remarks*. What about writings such as *Philosophical Grammar* and *The Big Typescript*? It is difficult to draw a sharp line between the true 'later' Wittgenstein and the works of his so-called 'transitional' period. But it is uncontroversial that Wittgenstein saw his *Philosophical Investigations* as a profound shift in his philosophical thought that contrasts significantly with his 'older way of thinking' (PI, Preface). For my purposes, it is sufficient to consider the author of the *Philosophical Investigations* and writings thereafter (such as most of those collected in *Zettel* and Manuscript 117) as the 'later Wittgenstein'. The reading that follows thus differs in general orientation from that offered by Marion (1998: xii) in that it takes seriously (and as its sole focus) Wittgenstein's *later* philosophical methodology – including his explicit aspirations to avoid any 'theory' (PI 109) – and attempts to make sense of particular (later) remarks about infinity by showing how they are expressions of that methodology at work. As with Moore, for all I say about the *later* Wittgenstein, Marion may be perfectly correct in his readings of Wittgenstein's *early* or *transitional-period* remarks on 'the infinite'.

[5] Compare Monk's (2007: 285) reading on which Wittgenstein aims to 'redescribe' mathematics so as to dislodge misbegotten pictures of it.

2. 'Infinity' and Imagination in *Zettel*

In a series of passages from *Zettel*, Wittgenstein discusses the temptation to think that the meaning of 'endlessness' or 'infinity' is determined by what takes place in one's imagination. In Z 272, Wittgenstein's interlocutor thinks it makes sense to speak of 'an endless row of trees' because they 'can surely imagine a row of trees going on without end'. Wittgenstein considers the following rationale: since we can say that the row of trees comes to an end, we can surely also say its negation, i.e., that it does not come to an end (cf., PI 344-5). He also considers the 'asseveration', allegedly from Frank Ramsey, that our ability to think such a thing is so obvious that it needs no justification: 'But it *just is* possible to think such a thing' (Z 272). Both reactions are deemed suspect.

> —Well here one has to find out *what* you are thinking. (Your asseveration that this phrase can be *thought*—what can I do with that? For that's not the point. Its purpose is not that of causing a fog to rise in your mind.) (*ibid*)

The interlocutor's appeal to their imagination and Ramsey's view that it *just is* possible to think 'the row of trees is endless' give us little indication of *what* either person is thinking. Neither the insistence that something can be imagined nor the 'asseveration' that it can be thought is the purpose of such a sentence.

Although not its purpose, the fact that 'endlessness' fires one's imagination is a major source of these ideas. Namely, that the meaning of 'endless' – independently of its use – is either determined by our imagination or by brute powers of thought. What is needed is an antidote to these tempting yet misbegotten ideas. Wittgenstein offers one immediately: 'We must patiently examine how this sentence is supposed to be applied. What things look like *round about it*' (*ibid*). This suggestion is not at all surprising coming from the philosopher who famously suggested that if we want to understand the meaning of an expression, we should ('for a *large* class of cases') look at its use (PI 43; cf., LFM, p. 252). Wittgenstein's general reaction to talk of 'endlessness' or 'infinity' is thus to draw us away from tempting pictures of their meaning (manifested by the 'fog' they cause to rise in our minds) and back to their uses or applications. As in the *Investigations*, Wittgenstein warns here that a 'picture [holds] us captive' (PI 115), but that we can find liberation by '[bringing] words back from their metaphysical to their everyday use' (PI 116).

In the next passage of *Zettel*, Wittgenstein shows more explicitly that he is dealing with mystification about 'the infinite' in mathematics and vividly describes his intended methods for demystification.

> Hardy: 'That "the finite cannot understand the infinite" should surely be a theological and not a mathematical war-cry.' True, the expression is clumsy [*ungeshickt*]. But what people are using it to try and say is: 'We mustn't have any juggling! How comes this leap

> from the finite to the infinite?' Nor is the expression all that nonsensical—only the 'finite' that can't conceive the infinite is not 'man' or 'our understanding', but the calculus. And *how* this conceives the infinite is well worth an investigation. (Z 273)

Wittgenstein grants that the phrase Hardy considers is clumsy, but he also thinks that the expression calls out for an investigation into how a mathematical calculus 'conceives' what we call 'the infinite'. Wittgenstein grants that *this* 'is well worth an investigation', especially when that tends to alleviate the mystification that gave rise to the clumsy expression. The expression is not 'all that nonsensical' (however mistaken it might be) when it is understood as saying 'our calculus cannot conceive the infinite'. We can address this concern by examining the calculus and seeing just how 'the infinite' is conceived there. Such an examination will show that the concern is based on a superficial understanding of how this occurs.

> The aim is a synoptic comparative account [*eine übersichtliche, vergleighende Darstellung*] of all the applications, illustrations, conceptions of the calculus. The complete survey [*vollkommene Ubersicht*] of everything that may produce unclarity. And this survey [*diese Übersicht*] must extend over a wide domain, for the roots of our ideas reach a long way.—'The finite cannot understand the infinite' means here: It cannot work *in the way* you, with characteristic superficiality, are presenting it. (*ibid*)

Again, we find the recommended methodology of the *Investigations*, in which Wittgenstein suggests a general source of, as well as an effective antidote to, philosophical misunderstanding: 'A main source of our failure to understand is that we don't have *an overview* [*nicht übersehen*] of the use of our words'; a failure that can be addressed by providing 'A surveyable representation [*Die übersichtliche Darstellung*] [that] produces precisely [the] kind of understanding which consists in "seeing connections"' (PI 122).

Unlike the interlocutor's initial feeling (in Z 272) that the meaning of 'endlessness' manifested itself directly in their imagination or Ramsey's insistence that 'endlessness' *just can* be thought (end of story!), Wittgenstein suggests that a survey of 'the infinite' needs to extend over 'a wide domain, for the roots of our ideas reach a long way' (cf., 'the very nature of the investigation […] compels us to travel crisscross in every direction over a wide field of thought' (PI, Preface)). As we will see later, it requires a comparison with a variety of related terms ('big', 'series', 'denumerable', etc.) as they occur both within and outside the mathematical calculus under investigation (cf., RFM II, 42) – a wide domain indeed.

Wittgenstein ends Z 273 by indicating what has led to the various mistakes thus far: that the meaning of 'infinity' is grounded in our imagination, that it *simply can* be thought, and that 'the finite cannot conceive the infinite'. In each case, we imagine that thought can 'as it were *fly*' beyond what takes place in the mere applications of a word – 'it doesn't have to walk' as do our uses or applications (*ibid*). And so we can *think about* 'the infinite' even if our mere 'finite' uses are incapable of 'conceiving' such a thing. But this idea, that our thought can so thoroughly outstrip the applications of words, arises because 'you do not understand your own transactions, that is to say you do not have a synoptic view of them' (*ibid*). The misunderstanding then leads to a mistaken projection: 'you as it were project your lack of understanding into the idea of a medium [i.e., the mind, our thoughts] in which the most astounding things are possible' (*ibid*). We then compare our 'thoughts' of the infinite to what we find in the calculus, which seems not to contain the vast thing that comes directly to our minds (cf., LFM, pp. 254-5).

Now the major source of misunderstanding is clear: namely, that our thoughts can outstrip the uses that we make of 'the infinite' in our humdrum mathematical applications; one then looks back at the calculus and sees it as inadequate, as unable to fully comprehend the vast reality that is the object of our thoughts (as if the calculus were *too small or modest* to comprehend something *so vast and profound*). But the object of our thoughts (say, 'the fog that rises in one's mind') has no meaning without the calculus and its applications. It is thus an illusion to think that we grasp something with our minds that cannot be comprehended by the calculus.

The succeeding passages state the foregoing diagnosis and intended cure very directly.

> The 'actual infinite' is a 'mere word'. It would be better to say: for the moment this expression merely produces a picture—which still hangs in the air: you owe us an account of its application.
>
> (Z 274; cf. Z 275)

The pictures that come to mind when we think about various kinds of infinity do not tell us on their own how such illustrations are applied – however tempting it might be to treat them as the true arbiters of meaning. At best such pictures provide a 'blueprint' which leaves us with the question 'how to work from it' (Z 275; cf., PI 139). One still owes us 'an account of its application.'

Notice that – against the temptation to think that, say, 'actual infinity' is a 'mere word' – Wittgenstein is granting (against a common brand of Finitism[6]) that 'infinity' is perfectly meaningful in the context of its application – e.g., in our practices of counting through an 'endless' series, in our rule-governed calculations in the context of transfinite arithmetic, or in the construction of mathematical proofs regarding the 'sizes' of various sets (such as Cantor's famous proof which we will examine in detail in section 3). So, by critiquing the hold that *pictures* of 'the infinite' might have on our imagination, Wittgenstein is not thereby denying that there is a meaningful concept of 'infinity' – the instances of which form a family

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[6] See especially Frascolla (1994: 142) for a useful explanation of 'Finitism' as it is typically understood and how it contrasts with Wittgenstein's later philosophy of mathematics.

(PI 67)[7] – but insisting that, if there is such a thing, it is to be found in its various uses and applications.

The foregoing considerations link up neatly with Wittgenstein's general thought 'On mathematics': 'I cannot illumine the matter by fighting against your words, but only by trying to turn your attention away from certain expressions, illustrations, images, and *towards* the *employment* of the words' (Z 463). The hope is that, rather than engaging in a direct theoretical confrontation with someone who endorses (cf., Realism) or denies (cf., Finitism) some evocative picture of 'infinity', the force of such pictures over our imaginations will be weakened by examining the employments of 'infinity' and related words where they are at home. The ideas that such pictures give rise to (e.g., that mathematical infinity corresponds to a special realm of infinities, that the infinite is profoundly huge or vast, or that the ability to think about infinity requires mysterious powers in human beings) can then be seen as mistaken. This common mistake at the root of both Realism and Finitism (that imaginative pictures rather than the uses of words dictate their meanings) is given clear expression in Wittgenstein's lectures:

> Hence we want to see the absurdities both of what the finitists say and of what their opponents say—just as we want in philosophy to see the absurdities both of what the behaviorists say and of what their opponents say. Finitism and behaviourism are as alike as two eggs. The same absurdities, and the same kind of answers. *Both sides of such disputes are based on a particular kind of misunderstanding—which arises from gazing at a form of words and forgetting to ask yourself what's done with it, or from gazing into your own soul to see if two expressions have the same meaning,* and such things. (LFM, p. 111, emphasis added)

Thus, to the extent that Realism and Finitism both rely on such pictures – the former to affirm, the latter to explain what they deny – Wittgenstein is best seen as neither a Realist nor a Finitist on the topic of 'infinity'.[8] Rather, he is undermining a presupposition of their debate: a conception of 'infinity' rooted in misbegotten pictures, affirmed by the former and denied by the latter (cf., RFM II, 61; LFM, p. 141).[9]

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[7] As noted by Moore (2011: 108) and Wheeler (2022: 324).

[8] My reading thus contrasts with those on which Wittgenstein endorses 'Finitism', e.g., Ambrose (1935), Dummett (1997), Kielkopf (1970), Kreisel (1958), Wang (1958), and Wright (1980). My reading agrees with those that emphasize Wittgenstein's explicit disavowal of 'Finitism' as a confused doctrine, e.g., Frascolla (1994: 143), Klenk (1976: 18-24, 92-123), Kripke (1982: 107) and Monk (2007: 276).

[9] Compare especially Frascolla (1994: 143-44): 'The denial of the existence of infinite sets is a mistaken way to draw a grammatical distinction which, though it may be opportune, should be done differently: by showing that the grammar of the word "infinite" cannot in the slightest be clarified by taking into account only the picture of something huge, a picture which usually accompanies the use of the word.'

The reading of *Zettel* given above helps to address Rodych's (2000, 2018) suggestion that 'the overwhelming evidence indicates that the later Wittgenstein still rejects the actual infinite [and thus remains a 'Finitist' in his later philosophy]' (2018). Rodych's claim is especially surprising given that Wittgenstein himself rejected 'Finitism' on various occasions in his later years (see especially LFM, p. 111, quoted above; LFM, p. 141: 'If you say that mathematical propositions are about a mathematical reality—although this is quite vague, it has very definite consequences. And if you deny it, there are also queer consequences—for example, one may be led to finitism. *Both would be quite wrong*' (emphasis added); as well as RFM II, 61). So Rodych's claim immediately requires the uncharitable interpretation that Wittgenstein endorsed 'Finitism' in his later years, despite his own explicit disavowals to the contrary.

But perhaps the evidence demands an exception to interpretive charity. Rodych cites two texts as evidence: Z 274 and RFM V, 21. I provided a reading of Z 274 above (one that is sensitive to its textual context) that shows no commitment to 'Finitism', but that instead helps to explain why it was reasonable for (later) Wittgenstein to distance himself from both 'Finitism' and 'Realism'. This leaves us with RFM V, 21. According to Rodych (2000: 289), this text shows 'a clear rejection of the actual infinite in mathematics'. Let's look at the text:

> Say not: 'the circle has this property because it passes through the two points at infinity…'; but: 'the properties of the circle can be regarded in this (extraordinary) perspective'.
>
> It is essentially a perspective, and a far-fetched one. (Which does not express any reproach.) But it must always be quite clear *how* far-fetched this way of looking at it is. For otherwise its real *significance* is dark. (RFM V, 21)

What is Wittgenstein even saying here? He notes a perspective that seems 'far-fetched' but immediately says that this 'does not express any reproach'. So, he is not rejecting anything (or at least he says he is not). (This already undermines Rodych's claim to 'clear' or 'overwhelming evidence' that Wittgenstein was a Finitist in his later years.) At most, Wittgenstein is drawing attention to a particular way of looking at the matter and noting its extravagance. This is compatible with the reading of *Zettel* provided above as well as the reading I will give of RFM II below. Wittgenstein is not taking issue with whether 'the circle … passes through the two points at infinity' (which would be an objection to mathematics), but rather making us consider a 'far-fetched' 'perspective' we might take on it – i.e., a misleading picture of the mathematical claim – one which distracts us from 'its real *significance*' – i.e., its uses or applications within the broader context of mathematics. So, the reading I offer in this paper (at a minimum) shows that neither of these texts is definitive evidence that Wittgenstein was a 'Finitist' in his later years; indeed, if my reading is correct, these texts are more likely to be expressions of a general critical program that is importantly neither 'Finitist' nor 'Realist'.

3. Cantor's Proof Soberly Redescribed: MS 117

3.1 A Standard Presentation of Cantor's Proof

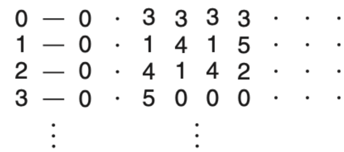
Cantor is widely understood as having shown that not all infinite sets are the same size. However vast the set of natural numbers might have seemed, its vastness can be exceeded, e.g., by the set of real numbers. As A.W. Moore puts it,

> So, *are* all infinite sets the same size? […] Cantor was able to prove that they are not. All infinite sets are big, but some, it transpires, are bigger than others. (Moore, 1990: 118-19)

More specifically, while the natural numbers are enumerable – i.e., "can be arranged in a single list with a first entry, a second entry, and so on, so that every member of the set appears sooner or later on the list" (Boolos, Burgess, & Jeffrey, 2007: 3) – the real numbers are *not* enumerable. The reason typically given is that the set of reals is "too big" to be counted. Boolos, Burgess, & Jeffrey (2007) open their textbook chapter on the subject saying, "Not all sets are enumerable: some are *too big*" (16, emphasis added). These general upshots – about the relative sizes of different infinities – are included in Moore's presentation of Cantor's proof via the famous method of diagonalization. His presentation (which is standard) will serve as a helpful comparison with Wittgenstein's remarks and so I reproduce it here – with emphasis added to phrases that will be particularly relevant to Wittgenstein's critique of diagonalization.

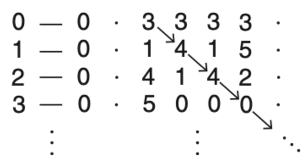
> *Cantor's Diagonal Argument:* Each real number between 0 and 1 can be expressed by means of an infinite decimal expansion, beginning with a 0 and a decimal point; this allows for the case of an expansion that terminates with a never-ending sequence of 0's. Thus
>
> 1/3 = 0.3333 …;
> $\pi - 3$ (that is, the decimal part of π) = 0.1415 …;
> $\sqrt{2} - 1$ (that is, the decimal part of $\sqrt{2}$) = 0.4142 …;
>
> and
>
> ½ = 0.5000 … .
>
> Now consider *any* pairing off of the natural numbers with a selection of these reals. For the sake of argument let us suppose that 0, 1, 2, and 3 have been paired off with the four reals just mentioned, in that

order. Such a pairing off can be represented as an 'infinite square', as shown in [Figure 3.1].



[Figure 3.1]

Suppose next that we start with the first digit in the decimal expansion of the first real in this 'square', then move to the second digit in the decimal expansion of the second real, and so on indefinitely down the 'square diagonal', as shown in [Figure 3.2].



[Figure 3.2]

If, each time we arrive at a new digit, we write down a 3 if that digit is a 4, and a 4 if that digit is anything other than a 4, then we shall find ourselves writing down *what may itself be regarded as the decimal expansion of a real number between 0 and 1* [emphasis added]. In this particular example the expansion will begin with a 4, two 3s, and a 4. That is, the resultant real will be 0.4334 …. Now, has *this* real been paired off with any natural number? That is, is it itself one of the reals listed in the 'square'? No. It has been so defined that it differs from the first of the reals in its first decimal place, from the second in its second, from the third in its third, and so on *ad infinitum.* It cannot itself be any one of them. What this shows is that, *regardless* of what pairing off we start out with, at least one real must inevitably be passed over. It will always be possible to define such a real by means of this kind of 'diagonalization'. The natural numbers cannot after all be paired off with the real numbers between 0 and 1. *There are more of the latter* [emphasis added]. The set of real numbers between 0 and 1, and *a fortiori* the set of all real numbers, *is bigger than the set of natural numbers* [emphasis added]. (Moore, 1990: 119-20)[10]

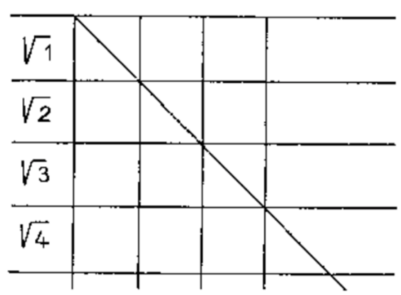
[10] Compare also Papineau (2012): "The example of the reals shows that infinite sets come in *different sizes*. There is the size shared by all the denumerable sets. But the real numbers are *bigger* again" (27, emphasis added); "There is the infinite number that characterizes the denumerable sets, and the distinct and *bigger* infinite number that characterizes all the sets whose members can be paired up with the real numbers" (32, emphasis added).

Before detailing Wittgenstein's critique of diagonalization, it is worth noting that both Moore and Boolos et al explicitly acknowledge the tendency of the proof to fire one's imagination – and in ways that are either mathematically inappropriate or at least diminished with a more technical and sober description of the relevant result.[11] Wittgenstein's critique, as we will see, is in many ways just a thorough and trenchant elaboration of this important reminder – one that is entirely consistent with acceptance of Cantor's *mathematical* result, albeit without many of its usual, and often unnoticed, imaginative interpretations.

3.2 A Family of Cases

Indeed, in Manuscript 117 (as presented in RFM II), Wittgenstein aims to undermine the various misleading effects Cantor's method of diagonalization might have on one's imagination – especially those that lead to metaphysical wonder and controversy regarding 'the infinite'. He begins by comparing the method of diagonalization as it is typically deployed – viz., as above, to show that some infinite set is 'bigger' than another – with an atypical (and usefully odd) deployment of that method.

> It is of course extremely easy to show that there are numbers that aren't square roots [of natural numbers][12] – but how does *this* method show it?



[11] See Moore (2016: 326): 'We feel a certain heady pleasure when we are told that some infinite sets are bigger than others. We feel considerably less pleasure when we are told that certain one:one correlations yield elements that are not in their ranges'; and Boolos, Burgess, and Jeffrey (2007: 19): '[T]here is no need to refer to [a hypothetically infinite] list, or to a superhuman enumerator: anything we need to say about enumerability can be said in terms of the functions themselves; for example, to say that the set $P*$ is not enumerable is simply to deny the existence of any function of positive integers which has $P*$ as its range. Vivid talk of lists and superhuman enumerators may still aid the imagination, but in such terms the theory of enumerability and diagonalization appears as a chapter in mathematical theology.'

[12] This addition strikes me as the simplest way to make sense of the passage and does nothing to interrupt the major philosophical point.

> Have we a general concept of what it means to show that there is a number that is not included in this infinite set? (RFM II, 1)

Wittgenstein's question at the end shows what the comparison is meant to illustrate. Namely, that it is misleading to say that there is a *general* concept of 'showing that there is a number not included in some infinite set'. On the one hand, we might say, 'Sure, there's a general concept: we can use this general expression for a variety of different cases.' On the other hand, if we consider the variety of examples falling under this 'concept', we will find that they do not have much in common, aside from the trivial feature of 'showing that there's a number not included in some infinite set' (cf., RFM III, 46: 'I should like to say: mathematics is a MOTLEY of techniques of proof'[13]). One would be hard pressed to come up with a singular rule, definition, or essence that somehow governed in every instance the application of this phrase (cf., PI 66-7). Thus, Wittgenstein has us consider several different cases to break one's temptation to think otherwise.[14]

Independently of the diagonal method, it is 'extremely easy to show that there are numbers that aren't square roots [of natural numbers]'. So, we can compare the odd use of diagonalization above with methods that more easily come to mind. Wittgenstein describes more natural methods at the end of RFM II 1 – more natural in that they deploy elementary methods of calculation to get the desired 'result'. The first strategy, involving the $\sqrt[3]{2}$, gives us a specific and familiar number that we can easily see is not included in the list of $\sqrt{n}$'s above. The second likewise gives a specific number that involves a basic subtraction from $\sqrt{2}$ (and then a similar comparison with the list). The diagonal method, especially the bare bones sketch above, gives us a 'number', but it is defined quite differently from numbers that we ordinarily encounter in arithmetic, i.e., $\sqrt[3]{2}$ and $\sqrt{2}$ with the subtraction of 1 from the first decimal. To think that each proof strategy 'shows the very same thing' would be to overlook their important differences, which illustrate different senses of 'showing that there is a number not included in some infinite set' as much as they illustrate different senses of 'number' (cf., 'And likewise the kinds of number, for example, form a family' (PI 67; cf. BB 18-19)).[15]

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[13] Anscombe's translation. But Mülhölzer (2022: 571-2) strongly prefers the more literal translation: 'mathematics is a COLOURFUL *mix* [*ein BUNTES* <u>*Gemisch*</u>] of techniques of proof', which suggests 'the permanent remixing and restructuring of what we do in mathematics' rather than a 'motley' (*ibid*, fn. 2). Both translations, however, suggest a *variety* of techniques ('COLOURFUL' rather than, say, colourless or monochromatic).

[14] This helps to address Moore's (2016) objection that if mathematicians are engaged in the '*modification* of [pre-existing] concepts' (e.g., 'big'), then 'the use of the relevant vocabulary will after all be essential to what mathematicians are doing' (Moore, 2016: 327). The objection is irrelevant if all Wittgenstein hopes to do is to highlight the differences between the uses of certain vocabulary to upset the temptation to think they share a singular rule, definition, or essence.

[15] My reading thus disagrees with Schroeder's (2020: 154): '[Later] Wittgenstein's view is that the diagonal method does *not* yield a new irrational *number*'. The point, as I understand it, is that the diagonal method yields a 'number' *by stipulation* – not by sharing some strict, underlying essence with other 'numbers' (in short: 'number' is a family-resemblance concept; there is no hard-and-fast rule that prevents the concept 'number' from being extended in this manner). However, Schroeder

These differences between the various proof strategies make it reasonable to ask, as Wittgenstein does at the beginning of the passage, '*How* does *this* method show that there is such a number?'. An answer to that question requires a comparison with other ways of doing so. Further, if someone were to say, 'I've shown that there is a number not included in this set!', Wittgenstein's comparison shows why it would make sense to respond, 'I do not know yet *what* you have shown until you show me the proof.' We should have a similar reaction to someone who tells us that they have shown that 'there is a number which is not included in this infinite series of real numbers' – since that could mean various things depending on the method used.

3.3 The Role of Stipulation and Convention in Proof

But there is another sense in which it makes sense to ask '*How* does *this* method show … ?' At first glance, one might have thought that a method of proof is a mere means of showing something independently intelligible, e.g., conveying something about mathematical reality that can be understood independently of its proof.[16] Then there might be a substantive question of how the proof succeeds at representing this independent fact about reality – something that we can intelligibly imagine and ponder all on its own – which might just as easily be shown by other means (cf., Z 273: 'thought can as it were *fly*, it doesn't have to walk'). The next passage, however, indicates that Wittgenstein thinks we should answer the 'How' question quite differently.

> 'Name a number that [dis]agrees[17] with $\sqrt{2}$ at every second decimal place.' What does this task demand? The question is: is it performed by the answer: It is the number got by the rule: develop $\sqrt{2}$ and add 1 or -1 to every second decimal place?
>
> It is the same as the way the task: *Divide an angle into three* can be regarded as carried out by laying 3 equal angles together.
>
> (RFM II, 2)

The comparison with 'dividing an angle into three' provides another atypical example relative to standard mathematics. As it is ordinarily understood, one does *not* 'divide an angle into three' or 'trisect the angle' merely by putting 3 equal angles together. This is because, as a matter of stipulation, 'trisecting the angle' requires doing so using only an unmarked straightedge and a compass. It was

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seems to acknowledge this lesson soon after his initial claim above: 'Note that here Wittgenstein does not object to that usage […] [since] regarding the diagonal decimal expansion as a number would also strike us as a natural use of the word 'number'. However, we should realise that it involves a conceptual extension' (2020: 155).

[16] As Gerrard (1991) notes, Wittgenstein's later philosophy of mathematics centrally aims to undermine the 'Hardyian picture' of an independent and preexisting mathematical reality.

[17] See fn. 13.

famously proved (by Pierre Wantzel in 1837) that this cannot be done. So, to the extent that one can 'divide an angle into three' by laying 3 equal angles together, this depends on a stipulation (or lack thereof) regarding what it means to do such a thing.[18] The connection between *this* 'proof' and its result is thus partly a matter of convention, i.e., regarding how we use the words 'divide an angle into three'. If we asked, '*How* does *this* method show … ?', the answer would be that we defined things in such a way that this is what we should call 'dividing an angle into three'. What is the connection, then, between 'Naming a number that [dis]agrees with $\sqrt{2}$ at every second decimal place' and defining a number by the rule: develop $\sqrt{2}$ and add 1 or -1 to every second decimal place? Likewise, a stipulation about what we call or consider 'naming such-and-such a number' in this context.[19]

However, it is important to contrast Wittgenstein's remarks here with the philosophical position known as 'Conventional-*ism*'. To say that the connection between the 'method' and 'result' of a proof depends on stipulation or convention is *not* to endorse Conventional-ism as this is typically understood. First, to say that something is a matter of convention (in the sense of Conventional-*ism*) typically has the connotation that it is arbitrary or a matter of *mere* convention – that whatever stipulation or convention we happen to choose is just as good as any other. But when Wittgenstein suggests that some aspect of mathematics involves stipulation or convention, he is clear that this need not be arbitrary (RFM I, 74, 116; PI 520). There might be excellent grounds for choosing one stipulation over another. For instance, calling such-and-such a 'number' might prove to be a very useful extension of our concept, it might harmonize well with our natural inclinations, it might open us up to empirical applications that were otherwise impossible, or it might allow for the simplest or most economical expression of what is going on in some new part of mathematics (RFM I, 5, 9, 137; VI, 23, 39; VII, 1; PI 142; PPF 365-6). But we should *not* include – among the reasons why we should call such-and-such a 'number' – either that 'this use corresponds to a pre-existent and independent reality of *numbers*' or that 'this use follows with absolute necessity from the underlying *essence* of number' (cf., RFM I, 8, 119, 156).[20] The use of 'number' in this context is a matter of human practice rather than metaphysical necessity; a practice that is potentially justified by a wide range of

[18] As noted by Floyd (1995: 388-9): 'Archimedes [e.g.] gives us a perfectly good geometrical construction; he does trisect the angle. Only not in what we *now* call "the relevant sense".' Compare Schroeder (2020: 155).

[19] Compare Schroeder (2020: 144ff). My reading of these passages contrasts with Mülhölzer's (2020: 134), who thinks that the task stated 'Name a number that [dis]agrees with $\sqrt{2}$ at every second decimal place' is obviously *not* satisfied by the suggested answers.

[20] Compare Schroeder (2020: 62): 'It is of course true that elsewhere Wittgenstein calls grammar "arbitrary" […]. But what he means by that is that it is essentially a human artefact that cannot be assessed as true or false to nature. That, however, does not mean that anybody will be allowed to introduce any kind of grammatical rule as the whim takes him.'

considerations (including linguistic, mathematical, psychological, or physical facts) and thus not a *mere* convention.[21]

Second, Wittgenstein's invocation of 'stipulation' or 'convention' is not Conventional-ist in the sense that all our mathematical talk can be reduced to a small list of rules or axioms (cf., Quine 1936, 1963). For Wittgenstein, mathematics is a 'MOTLEY[22] of techniques of proof' (RFM III, 46), which are not fully captured or capturable by a small number of (logical) principles. In short, and more generally, Wittgenstein is not interested in *reducing* (some part of) mathematics to anything at all – whether to human conventions, human psychological facts, or facts of nature. Rather, since mathematical practice depends to some extent on each of these things, reductionism of any sort (including Conventional-ism) for Wittgenstein is mistaken.[23]

The importance of stipulation and convention in connecting 'methods' of proof with their stated 'results' (which, again, is not to be confused with Conventional-*ism*) is further illustrated by Wittgenstein's depiction of an oddly resistant interlocutor.

> If someone says: 'Show me a number different from all these', and is given the rule of the diagonal for answer, why should he not say: 'But I didn't mean it like that!'? What you have given me is a rule for the step-by-step construction of numbers that are successively different from each of these. (RFM II, 3)

To be clear, Wittgenstein himself is *not* the imagined interlocutor, but uses their resistance to illustrate something about the relationship between methods and results in mathematics, as the remainder of this section shows.

> 'But why aren't you willing to call this too a method of calculating a number?'–But what is the method of calculating, and what the result here? You will say that they are *one*, for it makes sense now

[21] This point is made most forcefully by Maddy (2014: Ch. 5-6). See also Floyd (2021: 60).

[22] See fn. 14.

[23] Of course, if by 'Conventional-ism' someone means the watered-down view that '*to some extent* (often overlooked by Realists or Platonists in mathematics) our use of mathematical terms depends on human convention', then Wittgenstein can be called a 'Conventional-ist'. But to avoid confusion, I think it is best to avoid this attribution entirely. Especially seeing as someone could just as easily attribute the following 'Realism' to Wittgenstein: '*to some extent* (often overlooked by Conventional-ists) our use of mathematical terms depends on facts of nature'. Neither is the typical understanding of the relevant 'ism'. That said: I take no issue with Schroeder's (2022: 125) suggestion that Wittgenstein be considered a '*moderate* conventionalist' – to the extent that doing so does not conflict with the considerations above. (Though, again, one should consider how much is gained by this label if we can, via similar conditions and caveats, call him a '*moderate* realist'). See Diamond (1986) and Anscombe (1981) for further discussion. See also Ben-Menahem (1998) on 'convention' more generally in Wittgenstein's later philosophy.

> to say: the number *D* is bigger than . . . and smaller than . . .; it can be squared etc. etc. (*ibid*)

The resistant interlocutor is met with a reaction from someone (perhaps a teacher) who insists on the legitimacy of Cantor's proof ('But why aren't you willing to call this too a method … ?'). What both interlocutors seem to overlook is that there isn't a sharp line between 'method' and 'result' here, or more specifically between the 'method' and the 'number' it is intended to generate.

Wittgenstein suggests by contrast that 'the method of calculating' and 'the result here' are *one*, as this then allows us to perform the sorts of calculations and comparisons that are standard for other 'numbers'.[24] And the fact that they are *one* is a matter of stipulation – it is not a substantive or empirical issue how they are connected, as the resistant interlocutor seems to think. The interlocutor's resistance is confused because they treat 'method' and 'result' as separate and their connection as something other than a matter of convention – misleading them into questioning whether the 'method' is *really* successful in showing the 'result'. As we saw in section 2, it is an illusion to think that a 'result' (stated verbally) is intelligible independently of its application (in this case, its proof) – e.g., via a picture that the interlocutor happens to associate with it.

3.4 The General Philosophical Upshots Regarding Cantor's Proof

The philosophical upshots of these early passages are summed up nicely in a flurry of succeeding remarks. RFM II, 4 expresses the important theme that we cannot intelligibly separate 'method' and 'result' in mathematics – and that if we do we are bound to end up in confusion ('every opportunity of twisting and turning the meanings').[25] RFM II, 5 highlights that there are a variety of cases to consider in this territory that are not easily assimilated to one another as long as we give proper attention to their differences: 'How do I compare infinite series of results? Well, there are very different things that I may call doing that'. RFM II, 6 emphasizes that to avoid these confusions we need to look at the general context in which mathematical 'results' are established: 'The motto here is always: Take a *wider* look round'. RFM II, 7 more specifically encourages us not to be seduced by calculations expressed verbally but to the calculations themselves if we would like to understand the meaning of some mathematical result. Wittgenstein illustrates this lesson with a particularly vivid analogy.

> The verbal expression casts only a dim general glow over the calculation: but the calculation a brilliant light on the verbal expression. (As if you wanted to compare the heights of two

[24] C.f., RFM p. 358: 'How can you say that "…625…" and "… 25 x 25 …" say the same thing?—Only through our arithmetic do they *become one*.'

[25] This contrasts with readings on which Wittgenstein simply *identifies* the meaning of a mathematical proposition (or its 'truth') with its proof (or with some other specific aspect of its use). See especially Dummett (1978), Moore (2016), and Schroeder (2022: 57-8).

> mountains, not by the technique of measurement of heights, but by their apparent relation when looked at from below.) (RFM II, 7)

And, finally, RFM II, 8 illustrates the theme we saw earlier that the connection between 'method' and 'result' in mathematics is a matter of convention or stipulation (rather than a quasi-empirical connection, as it were, between a 'method of verification' and a self-standing and independently intelligible 'fact').

> 'I want to show you a method by which you can serially *avoid* all these developments.' The diagonal procedure is such a method.—'So it produces a series that is different from all of these.' Is that right?—Yes; if, that is, you want to apply these words to the described case. (RFM II, 8)

These lessons together inform how Wittgenstein would like us to think about Cantor's proof regarding the relative 'sizes' of 'infinities'. First, don't take a mathematical result stated verbally as wearing its meaning on its sleeve (e.g., 'the set of real numbers is *bigger* than the set of natural numbers'). Look at the calculation or the proof, along with the broader context of its use, to see what it means. Compare that calculation with others, especially those that, expressed verbally, might on the face of it be seen as having the same or similar results. Otherwise, we will be misled by the verbal similarities between those results as well as with other expressions in or outside mathematics – a mistake akin to comparing the heights of mountains via their apparent relation from below rather than the actual techniques of measurement. And most importantly, whatever words are used to describe the result of a mathematical proof, recognize that this requires a stipulation or convention about the use of those words, rather than simply following from some antecedent condition on them (say, via their underlying definition or essence). This means, perhaps more surprisingly, that to a large extent the connection between a 'method' and its 'result' (which are not sharply distinguished in the first place) is a matter of convention – rather than a 'method' being a mere means to 'verifying' some self-standing result (say, about an independent mathematical realm).

3.5 A Sober Redescription of Cantor's Proof

This last point – about the role of stipulation and convention in connecting 'method' and 'result' – is expressed very sharply in the next passage.

> It means nothing to say: '*Therefore* the X numbers are not denumerable'. One might say something like this: I call number-concept X non-denumerable *if it has been stipulated that* [emphasis added], whatever numbers falling under this concept you arrange in a series, the diagonal number of this series is also to fall under that concept. (RFM II, 10)

Why does it 'mean nothing to say' this? – The use of '*Therefore*' in this expression makes it appear as if 'the X numbers are not denumerable' is intelligible independently of whatever precedes the use of '*Therefore*' (i.e., the method of proof). Is it independently intelligible? Well, perhaps we somehow form a picture of 'the X numbers being non-denumerable' from examples of things that we (conventionally) call 'denumerable'. 'The cardinal numbers are denumerable. We can count through them. To say the X numbers are *not* denumerable is to say they are *not* like that. Of course this is something that we can bring to mind!'—But as we saw earlier in *Zettel*, it is an illusion to think that we understand a mathematical expression about infinity in this way.

Whatever one 'brings to mind' when they consider 'the X numbers are not denumerable', the meaning of *that* will have to be clarified by looking to its use. Hence Wittgenstein's question in RFM II, 12 (my emphasis): 'What can the concept "non-denumerable" *be used for*?'. The use is most straightforwardly found in what comes before: i.e., the method. So the 'method' (in large part) reveals the meaning of the 'result' ('they are *one*'), rather than being a mere means to an independently intelligible proposition. A more adequate description of what is going on here, which removes the potentially misleading use of '*Therefore*', is thus that '*I call* number-concept X non-denumerable *if it has been stipulated that*, whatever numbers falling under this concept you arrange in a series, the diagonal number of this series is also to fall under that concept' (my emphasis).

Perhaps one might object that, even if we cannot quite articulate it, we do have an independently intelligible conception of 'non-denumerability', given that we have an independent grasp of 'denumerability' via specific examples (say, the 'denumerability' of the cardinal numbers). But according to Wittgenstein, this is exactly where 'the mistake begins' (RFM II, 16).

> For what concept has one of this ordering? One has of course a concept of an infinite series, but here that gives us at most a vague idea, a guiding light for the formation of a concept. For the concept itself is *abstracted* from this and from other series; or: the expression stands for a certain analogy between cases, and it can e.g. be used to define provisionally a domain that one wants to talk about. (*ibid*)

The mistake alluded to in this paragraph is that of thinking we have a concept of 'ordering' or 'infinite series' clear enough to determine precisely what we mean by the expressions 'the real numbers can/cannot be ordered'. But the analogy one forms from the cardinal numbers and other infinite series (the even numbers, the odd numbers, the negative numbers, etc.) at most gives us a vague idea as to what that might mean. Further stipulations need to be made to bridge the gap from the examples to the application of 'orderability' to the real numbers. Hence, Wittgenstein thinks that an analogy from these examples does not give a clear sense to the question whether *the real numbers* can be 'ordered'.

> Asked: 'Can the real numbers be ordered in a series?' the conscientious answer might be: 'For the time being I can't form any

> precise idea of that'.—'But you can order the roots and the algebraic numbers for example in a series; so you surely understand the expression!' [cf., Ramsey's 'asseveration' in Z 272]—To put it better, I *have got* certain analogous formations, which I call by the common name 'series'. But so far I haven't any certain bridge from these cases to that of 'all real numbers'. Nor have I any general method of trying whether such-and-such a set 'can be ordered in a series'. (*ibid*)

Compare our discussion in section 3.2 of the question 'Have we a general concept of "showing that there is a number not included in this infinite set"?'. The same considerations are applied here to the question 'Have we general concepts of "ordered", "series", "infinite series", etc.?' Well, we have a 'common name' applied to certain analogous formations. Perhaps that is enough to say that we have a 'general concept'. But one should not mistake this with thinking that, from these analogous formations, we have a general and well-defined method of showing whether *any given set* 'can be ordered in a series' – such as the set of real numbers. Such a method would have to be invented, certain conventions and stipulations established to make such an application of those words intelligible. As Wittgenstein puts it in RFM II, 38, 'Such an employment is not: yet to be discovered, but: still to be *invented*' and in RFM I, 168, 'The mathematician is an inventor, not a discoverer.'

It is also tempting to think that we have a general concept of '(non)denumerability' via the concept of a 'one-to-one correspondence'. That is (according to this tempting thought), to say that the real numbers are non-denumerable is just to say that they cannot be brought into a one-to-one correspondence with the natural numbers. This seems to make the result of Cantor's proof intelligible independently of the proof itself. But the reasoning above applies equally to our concept of 'one-to-one correspondence'.[26] As Wittgenstein puts it in RFM V, 40:

> A general formulation of a procedure. The effect is similar to that of introducing the word 'correlation' with a view to the general definition of functions. A general way of talking is introduced, which is very useful for the characterization of a mathematical procedure […]. But the danger is that one will think one is in possession of the complete explanation of the individual cases when one has this general way of talking.

[26] This point about 'correlation' or 'correspondence' is made frequently in the LFM. See especially LFM VII, XVI, XVII, XXVII, and XXX. The point is essentially that 'one-to-one correspondence' (as much as 'number', 'denumerability', etc.) is a *family resemblance concept*: the instances of which do not adhere to a strict, unifying essence, but are connected by evolving and overlapping similarities (PI 66-7).

We have various examples of 'one-to-one correspondence' or 'correlation'. For instance, a one-to-one correspondence between plates and napkins, a one-to-one correspondence between the planets in our solar system and mugs in my cabinet, a one-to-one correspondence between some finite set of numbers and another finite set of numbers, a one-one correlation between the natural numbers and the even numbers, and so on. 'One-to-one correspondence' is a 'common name' applied to these analogous examples – and perhaps to that extent we have a 'general concept'. But the methods used to establish any one of these correlations is quite different from the others. The examples themselves do not decide in any clear way how *those* methods (or *which* of those methods) should be extended to 'a one-to-one correspondence between the natural and the real numbers'. (It is worth noting that, despite all their disagreements throughout Wittgenstein's lectures, Alan Turing agreed with Wittgenstein on this very point.[27]) Cantor had to invent such a method for this concept ('one-to-one correspondence', or lack thereof) to be applied to the natural and real numbers.

RFM II, 16 concludes with the major punchline for the relationship between the 'method' of diagonalization and the 'result' that the real numbers are non-denumerable.

> Now I am shown the diagonal procedure and told: 'Now here you have the proof that this ordering can't be done here'. But I can reply: 'I don't know—to repeat—what it is that *can't be done here*'. (*ibid*)

One can reply in this way because, merely given the analogous formations of 'series' or 'orderings' or 'one-to-one correspondences', one does not yet know what it would even mean to say that 'the real numbers cannot be ordered' or that 'they do not form an infinite series', or that 'they cannot be one-to-one correlated with the natural numbers'. The resistance here is to the idea that the statement of this result is intelligible *all on its own*, i.e., independently of the method of diagonalization. This is why Wittgenstein emphasizes that what the method really reveals is a difference between the stipulated uses of words such as 'root', 'algebraic number', 'real number', and the like, rather than a self-standing and independently intelligible result (RFM II, 16).

> Such a difference as e.g. this: roots are *called* [my emphasis] 'real numbers', *and so too* is the diagonal number formed from the roots. And similarly for all series of real numbers. For this reason it makes no sense to talk about a 'series of all real numbers', just because the

[27] "[*Wittgenstein:*] Suppose you had correlated cardinal numbers, and someone said, 'Now correlate *all* the cardinals to all the squares.' Would you know what to do? Has it already been decided what we must call a one-one correlation of the cardinal numbers to another class? Or is it a matter of saying, 'This technique we might call correlating the cardinals to the even numbers'?

Turing: The order points in a certain direction, but leaves you a certain margin.

Wittgenstein: Yes, but is it a mathematical margin or a psychological and practical margin? That is, would one say, 'Oh no, no one would call this a one-one correlation'?

Turing: The latter.

Wittgenstein: Yes.—It is not a mathematical margin" (LFM, p. 168).

> diagonal number for each series is also *called* [my emphasis] a 'real number'. (*ibid*)

So now – given the stipulated connection between the diagonal method and the result that 'the real numbers cannot be ordered' – it makes no sense to talk of 'the series of real numbers'. This illustrates a difference between how we (will) use the words 'cardinal number' and 'real number', rather than a deep fact about 'real numbers' that was simply awaiting our discovery. As he puts it in two later remarks,

> The dangerous, deceptive thing about the idea: 'The real numbers cannot be arranged in a series', or again 'The set … is not denumerable' is that it makes the determination of a concept—concept formation—look like a fact of nature. (RFM II, 19)

> The following sentence sounds sober: 'If something is *called* a series of real numbers, then the expansion given by the diagonal procedure is also *called* a "real number", and is moreover *said to be* different from all members of the series'. (RFM II, 20, my emphasis)

And in a separate manuscript discussing related themes,

> One would like to say of it e.g.: it introduces us to the mysteries of the mathematical world. *This* is the aspect against which I want to give warning. (RFM II, 40)

3.6 Downstream Applications of Diagonalization

To understand an expression such as 'the real numbers are non-denumerable', one might look at the method which yields it, but one might also look at what, as it were, comes *after* the statement of this result.[28] Wittgenstein describes some such downstream applications of the diagonal method to give a sobering picture of its general significance.

> Surely—if anyone tried day-in day-out 'to put all irrational numbers into a series' we could say: 'Leave it alone; it means nothing; don't you see, if you established a series, I should come along with the diagonal series!' This might get him to abandon his undertaking. Well, that would be useful. (RFM II, 13; cf. RFM II, 18)

When one hears the 'result' of diagonalization, it might appear exciting and surprising – it can, as Wittgenstein puts it, 'fire our imagination most strongly' (RFM II, 11). But these expressions are less apt to excite when we look at their humdrum roles in ordinary life (cf., LFM, pp. 251-2, 253-4). There is nothing particularly exciting about stopping someone from putting the real numbers into a

[28] See fn. 26.

list. Such a result is 'useful', but not enthralling. The same lesson is repeated by describing a child's use of diagonalization in a classroom setting (RFM II 18). These passages make clear that Wittgenstein's concern is not that Cantor's work is uninteresting because it *lacks* any non-mathematical applications.[29] Rather, his concern seems to be that the downstream applications are not particularly *exciting* (metaphysically, psychologically, or otherwise).[30] Hence his famous remark that Cantor's work is not a 'paradise' (LFM, p. 110). And rather than showing that there is anything wrong *per se* with the stipulations linking up the method of diagonalization with its 'results' (after all, 'this calculation is itself useful'), Wittgenstein's remarks seem instead to diminish the philosophical wonder or awe that a presentation of Cantor's proofs might have initially inspired.

3.7 A Change in Aspect

Thus, Wittgenstein's remarks are largely aimed at effecting a certain change in aspect – a different way of looking at Cantor's proofs from what a mere verbal statement of its results might have initially inspired. Cantor's 'discovery' might have seemed to consist in revealing a deep 'fact of nature' (RFM II, 19) about the infinite realms of numbers, or 'the mysteries of the mathematical world' (RFM II, 40). But a simple thought experiment (along with all the foregoing) might help to dislodge such a reaction.

> Here it is very useful to imagine the diagonal procedure for the production of a real number as having been well-known before the invention of set theory, and familiar even to school-children, as indeed might very well have been the case. For this changes the aspect of Cantor's discovery. The discovery might very well have consisted *merely* in the interpretation of this long familiar elementary calculation. (RFM II, 17)

The hypothetical scenario allows us to look at the method of diagonalization and its applications on their own terms – isolated from the usual descriptions of the proof's significance. In the alternative, fictional history Wittgenstein asks us to imagine, diagonalization was originally a simple classroom exercise that did not inspire much wonder (cf., RFM II, 18), but allowed us a simple and clever answer to a certain mathematical puzzle. Such an exercise would then appear quite mundane. This in turn would make it much harder for someone to get excited about any pronouncement by Cantor or his acolytes suggesting that, say, the calculation

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[29] As is suggested by Moore (2011: 119).

[30] Mülhölzer (2020: 162) is right that Wittgenstein's examples do not *exhaust* the applications of diagonalization. The major lesson of the passage still stands: our excitement about Cantor's proof should go no further than our excitement about its applications. (I leave the reader to decide their own level of excitement about its 'central place in set theory […] [and] recursion theory' (Mülhölzer (2020: 162)) – by contrast with the excitement that might be inspired by its alleged ability to reveal secrets of the various and hidden realms of infinity – or of God (see fn. 33)).

reveals a deep fact about the nature of 'the infinite'.[31] But perhaps this also allows us more specifically to separate diagonalization from Cantor himself, who viewed the significance of his mathematical work in explicitly theological and metaphysical terms.[32] In short, Wittgenstein's thought experiment allows us to disassociate, at least for a moment, the method of diagonalization from its usual imaginative extravagances.

3.8 Conclusion: 'a future generation will laugh at this hocus pocus'

Manuscript 117 concludes with a particularly forceful passage, explaining why Wittgenstein hopes that 'a future generation' will laugh at the 'hocus pocus' expressed in the common descriptions of Cantor's method of diagonalization.

> The usual expression creates the fiction of a procedure, a method of ordering which, though applicable here, nevertheless fails to reach its goal *because* [emphasis added] of the number of objects involved, which is greater even than the number of all cardinal numbers. (RFM II 22)

This bears directly on the standard explanation of diagonalization that we find in most textbooks on mathematical logic (see section 3.1). According to the standard explanation, the major upshot is that 'the set of real numbers is *bigger* than the set of natural numbers', i.e., the set of real numbers is susceptible to diagonalization '*because* it is bigger than the set of natural numbers'. This description might encourage one to think that we have an independent grasp of the result (sparking our fascination and wonder all on its own), prior to looking at the method of diagonalization itself and on its own terms (cf., LFM, p. 254). It might encourage one to think that 'the set of real numbers' is a vast reality existing independently outside our talking or thinking about it. Nowhere in the standard textbook explanations is it emphasized that the 'result' of Cantor's proof requires a collection of stipulations regarding the uses of 'size', 'series', 'ordering', 'countable', 'denumerable', 'one-to-one correspondence' and the like – perhaps because this would make it more difficult to attract student interest (cf., LC 28).[33]

This crucial point – that the standard description of Cantor's result relies on a series of verbal stipulations – is overlooked in Moore's (2011: 117; 2016: 324)

[31] C.f., the thought experiment of RFM V 7: 'Imagine set theory's having been invented by a satirist as a kind of parody on mathematics.—Later a reasonable meaning was seen in it and it was incorporated into mathematics. (For if one person can see it as a paradise of mathematicians, why should not another see it as a joke?) […] But isn't it evident that there are concepts formed here—even if we are not clear about their application?'.

[32] See for instance Moore (1990: 127): 'And this Absolute that had revealed itself in his own formal work, in a way that was so reminiscent of more traditional views of the infinite, was embraced by Cantor as a vital part of his conception of God.'

[33] Compare again Boolos, Burgess, and Jeffrey's (2007: 19) pedagogical use of theological imagery to illustrate the proof's significance.

reading of Manuscript 117, leading to his accusation that Wittgenstein is here, contrary to his own philosophical prescriptions (PI 124), *denying* a mathematical claim (e.g., 'that the set of real numbers is *bigger* than the set of natural numbers'). The point is rather (a) that the mathematical claim should not be confused with other uses of 'bigger', etc., and (b) that the use of 'bigger' in this context is a matter of stipulation, rather than an inevitable extension from other uses of 'bigger'. (a) & (b) are of course mutually supporting; a failure to appreciate them can lead to a kind of excitement or metaphysical wonder about Cantor's work that Wittgenstein (for that very reason) finds suspect.[34]

However technical and rigorous the standard textbook presentations of Cantor's proof might otherwise be, Wittgenstein would still think that they – by glossing and framing the major result as a matter of the 'size' of the real numbers – encourage the 'fiction of a procedure' that 'fails to reach its goal *because* of the number of objects involved' (my emphasis). He would insist on a yet more sober description, e.g.: when one set of numbers admits of diagonalization, and another set does not, we will *call* the former set 'bigger' than the other. And now it is difficult to get excited about Cantor's proof on the grounds that it reveals features of the vast and incredible realm of infinity (cf., LC 28). Likewise,

> If it were said: 'Consideration of the diagonal procedure shows you that the *concept* "real number" has much less analogy with the concept "cardinal number" than we, being misled by certain analogies, are inclined to believe', that would have a good and honest sense. But just the *opposite* happens: one pretends to compare the 'set' of real numbers in magnitude with that of cardinal numbers. The difference in kind between the two conceptions is represented, by a skew form of expression, as difference of extension. I believe, and hope, that a future generation will laugh at this hocus pocus. (RFM II, 22)

Wittgenstein thus offers a redescription of Cantor's proof – emphasizing the roles of conceptual innovation, stipulation, and convention – in order to counteract a kind of 'hocus pocus' that it tends to inspire, a feeling that Cantor's work reveals a *fact of nature* regarding the relative 'sizes' of various infinite 'magnitudes'.

Thus, rather than *denying* the existence of 'infinity in extension' or 'the actual infinite', Wittgenstein is insisting that the meaning of such notions does not go beyond the calculus that such phrases are used to describe – and when we look there, we do not find anything especially intriguing, i.e., unless we succumb to tempting analogies across different uses of 'quantity', 'extension', 'size', 'order', and the like (encouraging our thought to '*fly*' rather than 'walk') (cf., LFM, p. 253). The 'hocus pocus' that Wittgenstein warns us against is that of thinking that when

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[34] Compare Fogelin (1987: 220): 'But surely nothing forces us to extend our concepts in these ways, and thus the idea that Cantor has proved the existence of a hierarchy of transfinite cardinals is simply an exaggeration.' On my reading, this is an exaggeration only if it suggests the wrong picture to one's imagination – a picture that is significantly dampened by reminders about the various uses of relevant vocabulary and the role of stipulation in proof.

we are 'comparing the sizes of different infinities' we are doing something perfectly analogous to, say, 'comparing the sizes of two bags of potatoes'. Consider Wittgenstein's motto 'Say what you please, so long as it does not prevent you from seeing how things are' (PI 79). We can continue, if we like, to say 'the set of real numbers is *bigger* than the set of natural numbers', but we should see the crucial differences between this use of 'bigger' with other uses.

Thus, Z and MS 117 together reveal a powerful style of critique applied to 'the infinite', one that is distinctive to the later philosophy of Wittgenstein. As we have seen in each of these texts, Wittgenstein thinks that talk of the infinite naturally takes hold of one's imagination due to the sway of verbal pictures and analogies suggested by our words. This is the source of various philosophical pictures of the infinite: that mathematical infinity corresponds to a special realm of infinities, that the infinite is profoundly huge or vast, or that the ability to think about infinity reveals mysterious powers in human beings. However, one's imagination in these instances does not determine the meanings of 'infinity' and its surrounding expressions – but rather their meanings are ultimately to be found in their uses and applications. Thus, when the verbal pictures associated with our expressions take us far beyond their uses or perhaps even conflict with them, then the conditions are ripe for confusion.

By Wittgenstein's lights, to avoid such confusion, we need to weaken the hold of said pictures and analogies on our imagination by reminding ourselves of the applications of these expressions. When we do this, a certain romantic quality that tends to be inspired by talk of 'the infinite' is diminished. More specifically, the picture of 'the infinite' as a self-standing realm admitting of exploration or discovery by mathematicians is shown to be a product of illusion. Instead, *talk* of 'the infinite' is just that – a collection of language-games (or 'employments') governed by (non-arbitrary) human conventions rather than independent facts of nature.[35]

References

Primary Sources

NB Wittgenstein, L. 1979. *Notebooks 1914-16,* ed. G.H. von Wright and G.E.M. Anscombe, tr. G.E.M. Anscombe, 2nd edition, Blackwell, Oxford.

TLP Wittgenstein, L. 1961. *Tractatus Logico-Philosophicus*. Trans. D.F. Pears & B.F. McGuinness. Routledge and Kegan Paul, London.

PR Wittgenstein, L. 1975. *Philosophical Remarks,* Ed. R. Rhees, tr. R. Hargreaves & R. White, Blackwell, Oxford.

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[35] Compare Stern's (2004: 169) characterization of the so-called 'quietist' position: 'Wittgenstein's invocation of forms of life is not the beginning of a positive theory of practice […] but rather is meant to help his readers get over their addiction to theorizing about mind and world, language and reality.'

BT Wittgenstein, L. 2005. *The Big Typescript:* TS 213, Tr. and Ed. C.G. Luckhardt & M.A.E. Aue, Blackwell, Oxford.

BB Wittgenstein, L. 1958. *The Blue and Brown Books.* Blackwell, Oxford.

LFM Wittgenstein, L. 1976. *Lectures on the Foundations of Mathematics,* C. Diamond (ed.), Chicago: The University of Chicago Press. (LFM)

RFM Wittgenstein, L. 1983. *Remarks on the Foundations of Mathematics*, ed. G.H. von Wright, Rush Rhees, and G.E.M. Anscombe, trans., G.E.M. Anscombe. Cambridge: The MIT Press. (RFM)

LC Wittgenstein, L. 1970. *Lectures and Conversations on Aesthetics, Psychology and Religious Beliefs.* Ed. C. Barrett. Blackwell, Oxford.

PI Wittgenstein, L. 2009. *Philosophical Investigations,* trans., G.E.M. Anscombe, P.M.S. Hacker, and Joachim Schulte. West Sussex: Blackwell Publishing, Ltd. (PI)

Z Wittgenstein, L. 1970. *Zettel,* ed. G.E.M. Anscombe & G.H. von Wright, trans. G.E.M. Anscombe. Berkeley & Los Angeles: University of California Press.

Secondary Sources

Ambrose, A. 1935. 'Finitism in Mathematics', *Mind*, 44: 186-203 & 317-340.

Anscombe, E. 1981. 'The Question of Linguistic Idealism', in *From Parmenides to Wittgenstein*, Minneapolis: University of Minnesota Press.

Ben-Menahem, Y. 1998. 'Explanation and Description: Wittgenstein on Convention'. *Synthese* 115: 99-130.

Boolos, Burgess, Jeffrey. 2007. *Computability and Logic*. 5th Edition, Cambridge University Press.

Diamond, C. 1986. 'Realism and the realistic spirit', reprinted in *The Realistic Spirit,* Cambridge, MA: MIT Press, 1991, pp. 39–72.

Dummett, M. 1978. "Wittgenstein's Philosophy of Mathematics". In *Truth and Other Enigmas*, Harvard University Press, Cambridge, MA.

Dummett, M. 1997. "Wittgenstein on Philosophy and Mathematics". *Journal of Philosophy*, 94: 359-374.

Floyd, J. 1995. "On Saying What You Really Want to Say: Wittgenstein, Gödel and the Trisection of the Angle". In J. Hintikka (ed.), *From Dedekind to Gödel*, Synthese Library, Vol. 251, pp. 373-426.

Floyd, J. 2021. *Wittgenstein's Philosophy of Mathematics.* Cambridge University Press.

Fogelin, R. 1987. *Wittgenstein*, 2nd ed., Routledge & Kegan Paul, New York.

Frascolla, P. 1994. *Wittgenstein's Philosophy of Mathematics*. Routledge: London & New York.
Gerrard, S. 1991. 'Wittgenstein's Philosophies of Mathematics', *Synthese*, 87:1, 125–142.
Hilbert, D. 1926. "On the Infinite". Reprinted in *Philosophy of Mathematics: Selected Readings*, 1983, Eds. Benacerraf, P., & Putnam, H.
Kielkopf, C.F. 1970. *Strict Finitism: An Examination of Ludwig Wittgenstein's Remarks on the Foundations of Mathematics.* The Hague: Mouton.
Klenk, V. 1976. *Wittgenstein's Philosophy of Mathematics*. The Hague: Nijhoff.
Kreisel, G. 1958. "Review of Wittgenstein's 'Remarks on the Foundations of Mathematics'", *British Journal for the Philosophy of Science*, 9: 135-158.
Kripke, 1982. *Wittgenstein on Rules and Private Language.* Harvard University Press, Cambridge, MA.
Maddy, P. 2014. *The Logical Must*. Oxford: Oxford University Press.
Marion, M. 1998. *Wittgenstein, Finitism, and the Foundations of Mathematics*. Oxford University Press.
Monk, R. 2007. 'Bourgeois, Bolshevist, or Anarchist?: The Reception of Wittgenstein's Philosophy of Mathematics'. In Guy Kahane, Edward Kanterian, & Oskari Kuusela (eds.), *Wittgenstein and his Interpreters: Essays in Memory of Gordon Baker.* Wiley-Blackwell.
Moore, A.W. 1990. *The Infinite.* Routledge, London & New York.
Moore, A.W. 2011. 'Wittgenstein and Infinity'. In *The Oxford Handbook of Wittgenstein,* Oxford University Press.
Moore, A.W. 2016. "Wittgenstein's Later Philosophy of Mathematics". In *The Blackwell Companion to Wittgenstein*, eds. Glock, H.J., & Hyman, J., Wiley-Blackwell.
Mülhölzer, F. 2020. 'Wittgenstein on Cantor's Diagonal Method'. In Floyd & Mülhölzer, *Wittgenstein's Annotations to Hardy's Course of Pure Mathematics*, pp. 125-191.
Mülhölzer, F. 2022. 'Mathematics as Calculus and as Grammar'. *Grazer Philosophische Studien* 99, 570-597.
Nagel, T. 1997. *The Last Word*. Oxford University Press.
Papineau, D. 2012. *Philosophical Devices: Proofs, Probabilities, Possibilities, and Sets.* Oxford University Press.
Quine, W. V., 1936. 'Truth by Convention', reprinted in *The Ways of Paradox*, 2nd ed., Cambridge: Harvard University Press, 1976.
Quine, W.V. 1963. 'Carnap and Logical Truth', reprinted *The Ways of Paradox*, 2nd ed., Cambridge: Harvard University Press, 1976.
Rodych, V. 2000. 'Wittgenstein's Critique of Set Theory'. *The Southern Journal of Philosophy*, Vol. 38, 281-319.
Rodych, V. "Wittgenstein's Philosophy of Mathematics", *The Stanford Encyclopedia of Philosophy* (Spring 2018 Edition), Edward N. Zalta (ed.), URL = <https://plato.stanford.edu/archives/spr2018/entries/wittgenstein-mathematics/>.
Schroeder, S. 2022. *Wittgenstein on Mathematics*. London: Routledge.

Shanker, S.G. 1987. *Wittgenstein and the Turning-Point in the Philosophy of Mathematics*. State University of New York Press.
Stern, D. 2004. *Wittgenstein's Philosophical Investigations.* Cambridge University Press.
Wang, H. 1958. "Eighty Years of Foundational Studies", *Dialectica*, 12: 466-497.
Wheeler, S.J. 2022. "Defending Wittgenstein's Remarks on Cantor from Putnam". *Philosophical Investigations,* 45: 3.
Wright, C. 1980. *Wittgenstein on the Foundations of Mathematics.* London: Duckworth.