THREE ESSAYS ON LATER WITTGENSTEIN'S PHILOSOPHY OF MATHEMATICS: REALITY, DETERMINATION, & INFINITY

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ABSTRACT

Philip Bold: Three Essays on Later Wittgenstein's Philosophy of Mathematics:
Reality, Determination, & Infinity
(Under the direction of Alan Nelson)

This dissertation provides a careful reading of the later Wittgenstein's philosophy of mathematics centered around three major themes: reality, determination, and infinity. The reading offered gives pride of place to Wittgenstein's therapeutic conception of philosophy. This conception views questions often taken as fundamental in the philosophy of mathematics with suspicion and attempts to diagnose the confusions which lead to them. In the first essay, I explain Wittgenstein's approach to perennial issues regarding the alleged reality to which mathematical truths or propositions correspond. Wittgenstein diagnoses exotic pictures of mathematical reality as stemming from misleading analogies formed across empirical and mathematical propositions. The second essay explains Wittgenstein's treatment of perplexity regarding the ability of a mathematical rule to determine its applications in advance. This too is found to depend on a misleading analogy, in this case across behavioral and mathematical senses of 'determine'. The third and final essay discusses Wittgenstein's general critical approach to "the infinite". Philosophical perplexity about infinity is shown to depend on the verbal imagery regularly associated with proofs and their power to take hold of one's imagination. Wittgenstein dampens the imaginative excesses often associated with "the infinite" by offering a sober redescription of Cantor's famous method of diagonalization.

To Alan,
without whom this would not be

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Finally, I thank Tamara Fakhoury for her loving support and patience – the therapeutic effects of which go beyond anything I might ever find in Wittgenstein.

TABLE OF CONTENTS

LIST OF FIGURES

ABBREVIATIONS

Wittgenstein's Published Works

BB *The Blue and Brown Books* (Blackwell, Oxford, 1958)

BlB Occasionally used to refer to the Blue Book.

BrB Occasionally used to refer to the Brown Book.

BT *The Big Typescript:* TS 213, tr. and ed. C.G. Luckhardt and M.A.E Aue (Blackwell, Oxford, 2005)

C *On Certainty*, ed. G.E.M. Anscombe and G.H. von Wright, tr. D. Paul and G.E.M. Anscombe (Blackwell, Oxford, 1969)

CV *Culture and Value*, ed. G.H. von Wright in collaboration with H. Nyman, tr. P. Winch (Blackwell, Oxford, 1980)

PI *Philosophical Investigations*, Revised 4th Edition by P.M.S Hacker & Joachim Schulte; English Trans. By G.E.M. Anscombe, P.M.S. Hacker, & Joachim Schulte. (Wiley-Blackwell, 2009)

PR *Philosophical Remarks*, ed. R. Rhees, tr. R. Hargreaves and R. White (Blackwell, Oxford, 1975)

RFM *Remarks on the Foundations of Mathematics,* ed. G.H. von Wright, R. Rhees and G.E.M. Anscombe, rev. edn (Blackwell, Oxford, 1978).

TLP *Tractatus Logico-Philosophicus,* tr. D.F. Pears and B.F. McGuinness (Routledge and Kegan Paul, London, 1961).

Z *Zettel*, ed. G.E.M. Anscombe and G.H. von Wright, tr. G.E.M. Anscombe (Blackwell, Oxford, 1967).

Derivative Primary Sources

AWL *Wittgenstein's Lectures, Cambridge 1932-35, from the Notes of Alice Ambrose and Margaret MacDonald,* ed. Alice Ambrose (Blackwell, Oxford, 1979).

LA *Lectures and Conversations on Aesthetics, Psychology and Religious Beliefs*, ed. C. Barrett (Blackwell, Oxford, 1970).

LFM *Wittgenstein's Lectures on the Foundations of Mathematics, Cambridge 1939,* ed. C. Diamond (Harvester Press, Hassocks, Sussex, 1976).

In general, the validity or invalidity of a prescription – a prescription for baking bread, for example – is demonstrated by whether or not the result it promises is achieved, always presupposing it is carried out correctly.

– Nietzsche, *Daybreak* I.24

Introduction

The Therapeutic Conception of Philosophy

1. Introduction

Each essay in this dissertation will rely on what I refer to as "Wittgenstein's therapeutic conception of philosophy". It is well known that Wittgenstein in his later years thought of philosophy (as he practiced it) as a kind of therapy. But there is little agreement on what exactly is meant by this or how "therapy" as Wittgenstein understood it should be conducted.[1]

The purpose of this introductory chapter is to give a brief and general summary of Wittgenstein's later, therapeutic conception of philosophy as I understand it. My aim won't be to argue for a definitive stance on this matter, but instead to make my assumptions as clear as possible for the purposes of framing the essays that follow. To the extent that there is an argument for my general reading of Wittgenstein on philosophical therapy, it is in the fruitfulness of applying this general understanding to his remarks on the variety of topics that follow.[2] There is much more to say about Wittgenstein's later conception of philosophy than I can include here, so I will restrict myself to the details that are most essential to his engagement with the topics of mathematical reality, determination, and infinity.[3]

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[1] See Stern (2004) and Floyd (2006: 116-18) for surveys of competing interpretations.

[2] Further, to the extent that my general reading makes Wittgenstein's approaches to these topics persuasive or at least philosophically valuable, perhaps it will go some way toward addressing the generally negative response that Wittgenstein's later philosophy of mathematics has seen. A paradigmatic instance of this comes from Dummett (1978): "Many of the thoughts [in *Remarks on the Foundations of Mathematics*] are expressed in a manner which the author recognized as inaccurate or obscure; some passages contradict others; some are quite inconclusive; […] other passages again […] are of poor quality or contain definite errors" (Dummett, 1978: 166). See Floyd (1991, 2006), Monk (2007), and Moore (2016) for discussion of the reception of Wittgenstein's later remarks on mathematics.

2. Philosophy as "Therapy": Initial Clarifications

What, then, is Wittgenstein's therapeutic conception of philosophy? That is, what is his conception of how philosophy is to be done, by contrast with more traditional ways of doing philosophy (which by and large aim at 'theory' rather than 'therapy')?[4]

Perhaps the most fundamental component of Wittgenstein's later conception of philosophy is the following: "The philosopher [i.e., Wittgenstein] treats a question like an illness" (PI 255). That is to say, according to the therapeutic conception, traditional philosophical questions themselves are an object of suspicion and require investigation; something to be treated as a problem, rather than to be answered directly with a theory, hypothesis, account, or definition (PI 109).[5] More specifically, philosophical questions are submitted to diagnosis ("where did this question come from? why is it being asked?") and therapy ("how might I counteract the hold it has over me?"). Both of these procedures are complicated and subject to variation: "There is not a single philosophical method, though there are indeed methods, different therapies, as it were" (PI 133). Despite my attempt here to schematize his conception of philosophy, Wittgenstein makes clear that his way of dealing with problems is, "demonstrated by examples" (*ibid*), and thus is most adequately displayed by working

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[3] By putting Wittgenstein's later conception of philosophy front and center in my reading of his remarks on mathematics, my approach throughout these essays can be seen as a "left-wing interpretation" in Chihara's (1982) sense: "Left-wing interpretations emphasize Wittgenstein's radical views about the nature of philosophy; they stress the ideas that philosophical problems arise from misconceptions about grammar and meaning and that these problems should be resolved by a kind of therapy in which the therapist puts forward no theses, explanations, or theories of any kind. Right-wing interpretations emphasize Wittgensteinian *doctrines*" (Chihara, 1982: 105). Right-wing interpretations include those offered by Dummett (1978) and Wright (1980). See Monk (2007) for further discussion of so-called "left-" and "right-wing" interpretations.

[4] So, to be clear, by "Wittgenstein's conception of philosophy", I mean his own conception of how philosophy should be done, rather than a description or explanation of the massive variety of things we might call "philosophy". Wittgenstein was quite aware that his conception of philosophy is a radical departure from philosophy as it has standardly been practiced throughout (Western) history – indeed, his conception is largely aimed at undermining that tradition (PI 116, 118). One must be careful to distinguish Wittgenstein's uses of "philosophy" or "philosophers", which sometimes are used to target the activity or people he is challenging (PI 11, 38, 52, 81, 116, 131), while at other times are used to articulate his radical new approach and the people who follow it (PI 108, 109, 119, 121, 123-128, 133). In each case, context makes clear which Wittgenstein has in mind.

[5] Compare Floyd (1991: 144): "The profundity and impressiveness of philosophy retreats from purported answers to the nature and character of the questions themselves."

through a variety of cases. (Not unlike how one might learn the meaning of a word by being shown examples of its application, rather than a strict general definition or rule (PI 71).) Let us bear this in mind as I proceed with a broad outline of his conception of philosophy – in the end it will need to be clarified and substantiated with examples. In other words, this outline is at best a heuristic; the real proof of the pudding is in the eating.

Thus far we have a metaphorical description of Wittgenstein's later, therapeutic conception of philosophy. But what is meant here by "therapy" or "diagnosis"? In what sense is a philosophical question "like an illness"? Indeed, these metaphorical characterizations deserve unpacking. It is worth emphasizing that, on my reading, Wittgenstein's characterization of his philosophy as "therapeutic" is best seen as an *analogy* or *metaphor*, which is not by any means intended to undermine its importance. (C.f., CV 1: "A good simile refreshes the intellect.") According to Wittgenstein, "The philosopher treats a question *like* an illness" (PI 255, emphasis added); "there are indeed methods, different therapies, *as it were*" (PI 133, emphasis added). Given that this is an analogy, one should not haphazardly import features of 'illness' or 'therapy' in, say, the medical sense into Wittgenstein's conception of philosophy. (It's a truism about analogy that when 'A is *like* B', it does not follow that *all features of B are also features of A* – reading is like riding a bike (once you learn, it's easy to pick back up), but reading doesn't thereby require pedaling, wearing a helmet, or shifting gears).

What follows is my own unpacking of what exactly is intended by 'therapy' and 'illness' in the context of the PI and Wittgenstein's later philosophy as a whole. Differences between Wittgenstein's philosophical methods and medical practices should be clear enough throughout the discussion. The most crucial analogy is simply this: just as a doctor aims to study the sources of the pain in your leg in order to relieve it, likewise Wittgenstein studies the sources of philosophical questions in order to relieve them and make them go away. A crucial disanalogy is this: whereas a doctor may require some 'theory' or other in order to properly diagnose the pain in your leg or administer physical therapy, Wittgenstein intends to proceed by example alone – and thus (by his own lights) does not depend on some theory or other (say, a theory of 'meaning' or of 'language').

That said: both a doctor and Wittgenstein might rely on standard heuristics, techniques, and a general sense of how certain kinds of problems arise – not to mention a sensitivity to the particular facts about the relevant "patient" or participant in therapy. Below I explain some of Wittgenstein's heuristics, techniques, and his general sense of how *philosophical* problems arise, which, again, will very clearly differ from those relied on in standard medical practice (e.g., doctors don't look to details about *language* to find the sources of leg-pain).

3. Diagnostics

It will help to begin by attending to diagnostics, which will reveal in broad outline the problematic nature of philosophical questions (akin to illnesses) and the methods one might use to counteract them (akin to therapies).[6] According to Wittgenstein, the major source of philosophical questions and problems is to be found in misunderstandings about the uses of words, "problems arising

[6] The distinction here, though not a sharp one, is easily overlooked and leads to a common misconception about Wittgenstein's understanding of "therapy". For instance, Maddy (2014) characterizes Wittgenstein's response to confusion about rule-following as follows: "the troubled philosopher is confused about rule-following because he insists on the priority of sense; he's cured when attention to our ordinary rule-following practices reveals that they adequately support attributions of correctness and incorrectness entirely on their own […] This would be the end of the story; the troubled philosopher is free of his perplexities" (Maddy, 2014: 108). This makes it seem as if a philosophical problem is answered *merely* by insisting on the adequacy of our ordinary practices (roughly: "You *think* the ordinary descriptions are not enough to answer the question – but they are!"). On my reading, one first needs to diagnose the source of the problem in order to know what sorts of descriptions will serve as the relevant "therapy" (e.g., what in the first place made the so-called "priority of sense" seem like a requirement for rule-following? – though we will find in Chapter 2 that Wittgenstein's own diagnosis of his interlocutor in the PI is more nuanced). The importance of diagnosis ("present [a problem] as it arises with most power") is beautifully expressed by Wittgenstein in AWL: "You must not try to avoid a philosophical problem by appealing to common sense; instead, present it as it arises with most power. You must allow yourself to be dragged into the mire, and get out of it. Philosophy can be said to consist of three activities: to see the commonsense answer, to get yourself so deeply into the problem that the commonsense answer is unbearable, and to get from that situation back to the commonsense answer. But the commonsense answer in itself is no solution; everyone knows it. One must not in philosophy attempt to short-circuit problems" (AWL 108-9). Maddy, in other words, characterizes philosophical therapy as an attempt to "short-circuit" problems about rule-following, that is, by shifting immediately to descriptions of rule-following practices (as "commonsense answers") without proper attention to the sources of the problems themselves.

through a misinterpretation of our forms of language" (PI 111). The main source of such misunderstandings is that we lack a proper overview (*übersicht*) of the use of our words.

> A main source of our failure to understand is that we don't have *an overview* (*nicht übersehen*) of the use of our words. – Our grammar is deficient in surveyability (*Übersichtlichkeit*). A surveyable representation (*Die übersichtlichkeit Darstellung*) produces precisely that kind of understanding which consists in 'seeing connections'. Hence the importance of finding and inventing intermediate links. (PI 122)

Language itself is thus partly responsible for a variety of these misunderstandings ("Our grammar is deficient in surveyability"). The problems, "are as deeply rooted in us as the forms of our language" (PI 111), stemming from "superstition[s] produced by grammatical illusions" (PI 110). Such misunderstandings or superstitions are often expressed by pre-philosophical "pictures" taking hold of us.

> A *picture* held us captive. And we couldn't get outside it, for it lay in our language, and language seemed only to repeat it to us inexorably. (PI 115)

Indeed, the most central picture of language and meaning critically addressed in the PI is the so-called "Augustinian picture" – that the meaning of a word is the object for which it stands (PI 1).[7] It is important to note that Augustine's quote was selected by Wittgenstein to open the PI for at least two reasons: (1) if someone as brilliant as Augustine was gripped by such a picture of language, that picture must be quite powerful, seductive, and important,[8] and (2) since Augustine comes from a different historical epoch, this reveals the perennial significance of such a picture, which rears its head wherever we find language, rather than resting completely on some particular individual's

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[7] The importance of Wittgenstein's discussion of "the Augustinian picture" for the philosophy of mathematics is highlighted by Gerrard (1991).

[8] See Monk (1990: 478): "[H]e told Malcolm that he used the quotation from Augustine to begin the *Investigations* because: 'the conception must be important if so great a mind held it'."

contingent thoughts about language.[9] This latter point, about the perennial significance and sources of the problems Wittgenstein grapples with, is made explicit in a passage from CV:

> People say again and again that philosophy doesn't really progress, that we are still occupied with the same philosophical problems as were the Greeks. But the people who say this don't understand why it has to be so. It is because our language has remained the same and keeps seducing us into asking the same questions. As long as there continues to be a verb 'to be' that looks as if it functions in the same way as 'to eat' and 'to drink', as long as we still have the adjectives 'identical', 'true', 'false', 'possible', as long as we continue to talk of a river of time, of an expanse of space, etc., etc., people will keep stumbling over the same puzzling difficulties and find themselves staring at something which no explanation seems capable of clearing up. (CV 15)

(This also makes clear that Wittgenstein isn't pinning philosophical problems on, say, German, Greek, Latin, or English, but on language quite generally.) In short then, and in broad strokes, misunderstandings about language and the seductive pictures leading to them are the major sources of philosophical problems. A proper diagnostic procedure (or, investigation, we might say) will have to attend to some particular question or problem and see which misunderstandings of language have led to it.

But what is it in language that leads to the problems of philosophy? So far, the answer is that the uses of our words and the rules for such usage (which Wittgenstein sometimes calls "grammar" as a term of art[10]) are not easily surveyable. Putting philosophy aside for a moment, we can grant Wittgenstein this much at least: there is no point in life at which one's language is simply and definitively grasped; it requires constant work and exploration, not to mention regular updating as language changes and expands (c.f., PI 18, 79, & 354).[11] Further, it is a commonplace that questions

[9] See Baker & Hacker (2005: 49-50): "For, as he remarked […], this conception (*Auffassung*) is significant for us precisely because it belongs to a naturally clear-thinking person, temporarily far removed from us, who does not belong to our cultural milieu."

[10] See especially PI 496-7.

[11] Notwithstanding technical senses of "grasping a language" on which, say, a language is grasped when the basic principles of grammar have been sufficiently internalized. The sense intended here is the sense according to which, e.g., Virginia Woolf had a more profound grasp of the English language than, say, George W. Bush.

(outside philosophy) do sometimes arise from confusions about the meanings of words. To slightly modify an example offered by Wittgenstein, a child might ask in great astonishment, "How can you possibly *sew* a dress?". The source of the child's question is a confused picture: the child is supposing that "a dress [is] produced by sewing alone, by sewing one thread on to another" (PI 195). Their astonishment is diminished by showing them that this is not how "sewing a dress" is to be understood – fabrics are produced via weaving or other means and then used as material for sewing with needle and thread. The child might still like to see how one sews a dress, but their curiosity would then lack the sense of mystery and astonishment that it previously had; the sense that sewing a dress amounted to doing something either spectacular or impossible.[12]

Astonishment at how one can possibly name an object, or a sensation, or attend to the shape of a color, or *mean* something with their words is, for Wittgenstein, not unlike the astonishment of the child in this example – astonishment which is dissolved by careful attention to the actual uses of our words, e.g., 'name' (PI 27ff), 'sensation' (PI 244ff), 'attend' (PI 33ff), 'saying and *meaning* something' (PI 507ff), and so on. That is, "here as in countless similar cases, we must look at what really happens *in detail*, as it were from close up" (PI 51). The point is nicely summarized in Wittgenstein's rich metaphor of PI 52: "If I am inclined to suppose that a mouse comes into being by spontaneous generation out of grey rags and dust, it's a good idea to examine those rags very closely to see how a mouse could have hidden in them, how it could have got there, and so on." Perplexity is thus resolved via careful examination of details, not (as is often thought by philosophers) via a general theory of the relevant concepts: "[W]hat it is in philosophy that resists such an examination of details, we have yet to come to understand" (*ibid*).

[12] Compare also Wittgenstein's example of the boy, "who racked his brains over the question whether the verb "to sleep" … meant something active or passive", when having to, "say whether the verbs in certain sentences were in the active or passive voice" (PI 47). These analogies will be especially relevant to my discussion of mathematical determination in Chapter 2.

Consider another analogy from Wittgenstein, which illustrates a further sense in which language itself can cause misunderstandings. Someone might be shown a picture of a line sitting next to a circle (e.g., / O) and told, "This line cuts the circle at imaginary points". Wittgenstein is reported to have said that this example, "has a certain charm, now only for schoolboys and not for those whose whole work is mathematical" (LFM 16). The sense of mystery or perplexity (the "charm") is dissolved when one points out that the use of "cut" here departs from a common usage of "cut": it is a matter of applying the equation of the circle over the domain of complex numbers, not a matter of the line "going through" the circle in the visual sense of cutting. The charm (or perplexity or astonishment) is a result of mixing these uses together, an example of what Wittgenstein refers to as "a crossing of different pictures" (PI 191). In the LFM, he provides the following diagnosis, noting that this is a "kind of misunderstanding" (albeit not of a particularly important or threatening sort):

> "Cut" has the ordinary meaning: ∅ [by contrast with / O]. But we prove that a line always cuts a circle – even when it doesn't. Here we use the word "cut" in a way it was not used before. We call both "cutting" – and add a certain clause: "cutting in imaginary points, as well as real points". Such a clause *stresses a likeness*. – This is an example of the assimilation to each other of two expressions.
>
> (LFM 16)

In the PI, we see that Wittgenstein took this sort of phenomenon – viz., confusions stemming from the assimilation of two apparently uniform expressions – to be one of the general sources of philosophical confusion.

> Of course, what confuses us is the uniform appearance of words when we hear them in speech, or see them written or in print. For their *use* is not that obvious. Especially when we are doing philosophy. (PI 11)
>
> Our inquiry is therefore a grammatical one. And this inquiry sheds light on our problem by clearing misunderstandings away. Misunderstandings concerning the use of words, brought about, among other things, by certain analogies between forms of expression in different regions of our language. (PI 90)
> A simile that has been absorbed into the forms of our language produces a false appearance which disquiets us. (PI 112)[13]

[13] Likewise, in a section of BT titled, "Philosophy Points Out the Misleading Analogies in the Use of our Language", Wittgenstein is also quite explicit that philosophical mistakes arise from taking up an analogy without recognizing that

So, a regular source of confusion in philosophy stems from the uniform appearances of certain words and expressions which nonetheless have importantly different uses (e.g., the different uses of 'cutting' mixed together lead to perplexity). Despite differences in usage, the uniform appearance of words encourages us to assimilate expressions in a way which leads to misbegotten pictures, problems, and questions (e.g., "how can one possibly *sew* a dress?"; "how can *this line* really be cutting *this circle*? → [/ O]"). Although this is just one respect in which language can encourage the confused questions of philosophers, we will find later that this sort of diagnosis is fundamental to Wittgenstein's engagement with philosophical ideas regarding mathematical reality, determination, and infinity.[14] More generally speaking, philosophical problems arise from a lack of clarity about, and departure from, the ordinary uses of words: "philosophical problems arise when language *goes on holiday*" (PI 38).

4. Therapy

With some of the major sources of philosophical confusion to hand, it is not difficult to understand Wittgenstein's proposed antidote: the careful and persistent description of the uses of words.[15] If

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one has done so: "If I rectify a philosophical mistake and say that this is the way it has always been conceived, but this is not the way it is, I must always point out an analogy according to which one had been thinking, but which one did not recognize as an analogy. […] The effect of a false analogy accepted into language: it means a constant battle and uneasiness (a constant irritant, as it were). […] What the other person acknowledges is the analogy I'm presenting to him as the source of his thought" (BT 302-303).

[14] In Chapter 2, we will find that G.H. Hardy incautiously assimilates mathematical to empirical propositions, in Chapter 3 that Wittgenstein's interlocutor assimilates mathematical and empirical senses of 'determination', and in Chapter 4 that seductive pictures of 'infinity' arise from crossing the uses of words (e.g., 'big', 'series', 'collection', etc.) across mathematical and non-mathematical contexts.

[15] Pears (1988) likewise emphasizes the importance of description, by contrast with theory, in Wittgenstein's later conception of philosophy: "There is really no need to settle precisely how far Wittgenstein pushes his new conception of philosophy. The important thing is that he is moving it away from theorizing and towards plain description of the phenomenon of language. The description is intended to make us see how our own linguistic devices work, simply by putting them in their place in our lives without using any technical terms. If we object, that nothing so familiar or banal can possibly yield philosophical understanding, he will reply that, on the contrary, it gives us all the insight that we need, and that the attempt to go beyond it and theorize can only produce misunderstanding. We think that we need nourishment when what we really need is clear water" (Pears, 1988: 218-19).

philosophical questions and problems largely arise from unnoticed features of language, many of which come from assimilating expressions with significantly different uses, then we can undermine questions arising from such hasty assimilations by attending to the uses of words and describing their differences in detail. No wonder then that we find such recommendations again and again throughout the PI. The list below is hardly exhaustive.

> It disperses the fog if we study the phenomena of language in primitive kinds of use in which one can clearly survey the purpose and functioning of the words. (PI 5)
>
> Now what do the words of this language *signify*? – How is what they signify supposed to come out other than in the kind of use they have? (PI 10)
>
> This odd conception springs from a tendency to sublimate the logic of our language – as one might put it. The proper answer to it is: we call *very different* things "names"; the word "name" serves to characterize many different variously related kinds of use of a word – but the kind of use that the word "this" has is not among them. (PI 38)
>
> It is important to note that it is a solecism to use the word "meaning" to signify the thing that 'corresponds' to the word. (PI 40)
>
> For a *large* class of cases of the employment of the word "meaning" – though not for *all* – this word can be explained in this way: the meaning of a word is its use in the language. (PI 43)
>
> The question "Is what you see composite?" makes good sense if it is already established what kind of compositeness – that is, which particular use of this word – is in question. (PI 47)
>
> [I]f the words "language", "experience", "world" have a use it must be as humble a one as that of the words "table", "lamp", "door". (PI 97)
>
> When philosophers use a word – "knowledge", "being", "object", "I", "proposition/sentence", "name" – and try to grasp the *essence* of the thing, one must always ask oneself: is the word ever actually used in this way in the language in which it is at home? – What *we* do is to bring words back from their metaphysical to their everyday use. (PI 116)
>
> Philosophy must not interfere in any way with the actual use of language, so it can in the end only describe it. (PI 124)

However, it is extremely important to understand that by describing use and making the workings of language surveyable, Wittgenstein is not proposing that we offer a general theory or a technical apparatus in hopes of revealing the hidden essence of language once and for all (PI 91-2). This is one of many mistakes he attributes to his earlier self in the TLP, connected with the attempt found there to reveal, once and for all, the general form of the proposition (see, e.g., TLP 4.5 & 6; by contrast with PI 23, 97, 108, 114, & 134-7). Instead, in order to counteract philosophical questions and problems, we are to *describe* (rather than theorize) the generally accepted uses of words – not a newfound technical apparatus designed by philosophers in hopes of displaying the (hidden) workings of language once and for all.

> But now it may come to look as if there were something like a final analysis of our linguistic expressions, and so a single completely analyzed form of every expression. That is, as if our usual forms of expression were, essentially, still unanalyzed; as if there were something hidden in them that had to be brought to light. As if, when this is done, the expression is completely clarified and our task accomplished.
>
> It may also be put like this: we eliminate misunderstandings by making our expressions more exact; but now it may look as if we were aiming at a particular state, a state of complete exactness, and as if this were the real goal of our investigation. (PI 91)

> Here it is difficult to keep our heads above water, as it were, to see that we must stick to matters of everyday thought, and not to get on the wrong track where it seems that we have to describe extreme subtleties, which again we are quite unable to describe with the means at our disposal. We feel as if we had to repair a torn spider's web with our fingers. (PI 106)

Wittgenstein's insistence on sticking to "matters of everyday thought" or speaking "the language of every day" (PI 120) is thus not a mere stylistic choice: it is directly connected with his diagnosis of the major pitfalls in philosophy. For instance, an attempt to uncover the general form of the proposition (via mathematical logic or other technical means) is to do the very thing he thought was a major source of confusion, namely, to assimilate a wide range of uses of the word 'proposition' and thereby overlook their fundamental differences, treating them as if they had the very same function, purpose, or general significance wherever they are to be found (PI 136). One must counteract the impulse to assimilate by articulating different uses of the word 'proposition', however much the uniform appearance of the word might tempt us to do otherwise. (We will return to this point about 'propositions' in Chapter 2, Section 4.)

The general lesson here is most famously illustrated in Wittgenstein's discussion of "games", the major upshot of which is that uses of the word 'game' reveal, "a complicated network of similarities overlapping and criss-crossing: similarities in the large and the small" (PI 66), which Wittgenstein calls "family resemblances" (PI 67) – *not* a rigidly bounded essence that is the same, always and everywhere, in all the things we call "games" (PI 68).[16] We undermine the inclination to think otherwise by enumerating and describing these myriad uses in detail – as is beautifully, albeit partially, done in PI 66. Though, to be clear, Wittgenstein was never aiming at *complete* descriptions of language-use for their own sake (PPF 202; Z 452 & 465) – but only to describe things in enough detail so that certain philosophical problems go away. How much detail will suffice is not something that can be decided in advance, as it crucially depends on the relevant question, the complexity of language surrounding it, as well as the subject of (i.e., participant in) therapy. This is in part why Wittgenstein expresses hope in the Preface of the PI that his work will, "stimulate someone to thoughts of his own": the method he articulates via examples would have to be applied by the reader in various ways depending on the questions, problems, or pictures that grip them. Hence, if the reader happens to be gripped by a question, problem, or picture which is *not* addressed by Wittgenstein, this is not *ipso facto* an objection to his work, but something to be taken up by that reader.

There is one major caveat to Wittgenstein's insistence on describing the uses of ordinary language. Throughout his investigations, he makes regular use of what he famously calls "language-games", an expression which sometimes refers to (i) parts of actual language, sometimes to (ii) the

[16] As Floyd (2021) puts it, ""Family resemblance" characterizes the generality of certain concepts. A single property, a fixed-for-all-cases criterion, an explicit set of grammatical rules – these are not required. A concept may hold together – like a family – with a variegated, evolving series of properties" (Floyd, 2021: 51).

whole of language as it is actually used, sometimes quite literally to (iii) "games" used, say, to teach someone the uses of words, but at other times to (iv) highly simplified fictional uses of language that can be easily described (PI 7).[17] It is this last sense, (iv), of "language game" that needs to be squared up with his insistence on the careful description of *actual* ordinary language. If we are concerned with actual language, why the turn to fiction? Wittgenstein makes clear that his highly simplified language-games are to be understood (especially when they are fictional) as objects of comparison, aimed at revealing aspects of *actual* use that might otherwise be overlooked.

> Our clear and simple language-games are not preliminary studies for a future regimentation of language – as it were, first approximations, ignoring friction and air resistance. Rather, the language-games stand there as *objects of comparison* which, through similarities and dissimilarities, are meant to throw light on features of our language. (PI 130)

For instance, the example of the shopkeeper (PI 1) displays a highly simplified use of words in which their functioning is crystal clear. Despite the fact that *our* use of the words 'five', 'red', and 'apples' will be more complicated than what we see in this simplified and artificial example,[18] it nonetheless highlights the fact that, however much more complicated some of our uses might be, we will best understand them by seeing how their use plays out in the relevant context.[19] Likewise,

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[17] Given Wittgenstein's various uses of "language game", I find the following remark by Floyd (1991) somewhat misleading: "structure is to be exhibited and elicited by "comparing" our uses of language to games with fixed rules (the simplified "language games" of, e.g., *Philosophical Investigations*). But this indicates that for Wittgenstein our language is not itself a game with fixed rules; that is, our language is not itself a "language game"" (Floyd 1991: 147).

[18] For instance, in such circumstances one does not ordinarily, "look[] up the word "red" in a chart and find[] a color sample next to it" (PI 1). Wittgenstein acknowledges this in PI 53: "We don't usually carry out the order "Bring me a red flower" by looking up the color red in a color chart and then bringing a flower of the color that we find in the chart". But he had good reasons to include this detail, as it sets up for discussions about understanding the meaning of color-words (and their relation to memory, mental images, samples, tools, and charts) later on (e.g., PI 51ff). Those discussions in turn set up for various considerations about what it means to follow a rule (see especially PI 54).

[19] This is at the heart of the long and detailed series of language-games offered in the BrB – which, much like in the PI, are successively augmented, leaving us to wonder at what point some grand, hidden magic is supposed to take place that creates problems regarding the essence of meaning. As I will say in the next paragraph, if descriptions of the workings of language (i.e., its myriad language-games) dissolve the alleged problems in simpler cases, then this shows what we should aspire to in more complicated cases (see again PI 122). This is quite at odds with the philosophical tradition of clarifying problems by offering general theories, definitions, or accounts of "meaning" (PI 118). Wittgenstein aims to dissolve philosophical perplexity by *describing* language, not by *theorizing* it.

with Wittgenstein's slab-game (PI 2). There are no deep mysteries about meaning when we display the activity of the builders. However much more complicated our uses of 'slab' or 'brick' might be, those complications and the differences with the simple game offered in PI 2 are to be displayed in the uses of the relevant terms and in their relevant contexts. Thus, by drawing out similarities and differences with the simplified, fictional language-games Wittgenstein offers us, we become more sensitive to the uses of our words – allowing us to see connections and contrasts where they might have otherwise been overlooked (PI 122 & 144). It is thus quite apt to consider the simple games as therapeutic devices, helping attune us to the nitty gritty details of use as it is displayed in our life's practices; counteracting one's natural impulses to do otherwise.

There is a somewhat deeper point, however, which is revealed by the simple language-games that Wittgenstein offers up as objects of comparison. These are situations in which traditional puzzles about meaning do not arise – notwithstanding philosophers who are especially resistant, and thus whose questions are addressed in turn (PI 2-38ff) (more on this in a moment). There is no question – "But what is the meaning of the word "five"?" – in the shopkeeper example, because we can see completely how the word "five" is being used.[20] (Regarding the specific question, "What does 'five' signify?", compare PI 10: "Now what do the words of this language *signify*? – How is what they signify supposed to come out other than in the kind of use they have?"). There is no question of how the builder knows that 'slab' refers to *this*, since his knowledge of meaning is adequately

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[20] "No such thing was in question here, only how the word "five" is used" (PI 1). Kripke aptly notes the relevance of PI 1 to the philosophy of mathematics: "Many philosophers of mathematics – in agreement with the Augustinian conception of 'object and name' – ask such questions as, "What entities ('numbers') are denoted by numerals? What relations among these entities ('facts') correspond to numerical statements?" […] As against such a 'Platonist' conception of the problem, Wittgenstein asks that we discard any *a priori* conceptions and *look* ("Don't think, look!") at the circumstances under which numerical assertions are actually uttered, and at what roles such assertions play in our lives" (Kripke, 1982: 75). For this reason, among others, I find Baker & Hacker's (2005) remark on the PI's Preface somewhat misleading: "Of the subjects W. mentions here, it is noteworthy that the foundations of mathematics is *not* discussed" (Baker & Hacker, 2005: 33). Especially, if, as Maddy (2014) emphasizes, "the material on logic [in the RFM], like the private language argument, is intended to grow out of the rule-following discussion" (Maddy, 2014: 74). Though perhaps this is just a matter of what Baker & Hacker mean by "the foundations of mathematics".

displayed in the activity of building as it is described in the example: "Since everything lies open to view, there is nothing to explain" (PI 126). If puzzles arise regarding the meanings of words in more complicated regions of our actual language, then the simple games show us what we lack, namely, "*an overview* of the use of our words" (PI 122), and thus what is needed to counteract such puzzles. We can only improve the surveyability of language by describing its actual workings, drawing comparisons between different regions of discourse (as well as any fictional constructions that might aid in illustration), with careful attention to the crucial differences that might otherwise be overlooked due to the uniform appearance of certain words. Such an activity is "therapeutic" to the extent that it counteracts the force of philosophical perplexity: "Problems are solved (difficulties eliminated), not a *single* problem" (PI 133). It does not aim at concocting a theory of 'meaning' or 'language', since it relies only on ordinary descriptions of word-usage, whether actual or fictional (PI 69, 75, 109).

This, I take it, is Wittgenstein's intention at least. I am not assuming that such an upshot is self-evident from the simple language-games taken on their own, especially on one's first reading of these particular texts or before reading the entire book. If one feels resistance to the therapeutic trajectory of Wittgenstein's language-games (which is not uncommon), this is something to be further explored. Further exploration will likely require careful attention to a range of relevant examples, not to mention a more direct engagement with the questions, problems, or pictures that grip the reader his or herself. As I mentioned earlier, Wittgenstein insists that his method is demonstrated by examples, which will need to be worked through in detail (PI 133). How much detail? That will, again, largely depend upon the reader herself.[21]

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[21] Compare, "The real discovery is the one that enables me to break off philosophizing when I want to. … [A] method is now demonstrated by examples, and the series of examples can be broken off" (PI 133). At what point the examples can be broken off is presumably not decidable *a priori*, but in response to the reader's own specific needs throughout their own investigations. (Again, the analogy with medical therapy is apt in this respect: a doctor likewise requires sensitivity to details about their patient when administering treatments or therapies.)

(The open-ended character of Wittgenstein's therapeutic methods creates a special problem for exegesis: the reader might feel disappointed by the specific examples presented, or feel that the problems for them have not gone away as promised. *I* do not promise to make the problems go away *for you* – my aim is only to explain Wittgenstein's own application of his therapeutic methods to a selection of topics in the philosophy of mathematics. In other words, I only intend to *present* and *clarify* the workings of Wittgenstein's methods in certain particular cases; I am not thereby *performing* philosophical therapy on the reader – which is something they would have to do on their own time. As Wittgenstein puts it in CV, "Working in philosophy … is really more a working on oneself. On one's own interpretation. On one's way of seeing things. (And what one expects of them.)" (CV 16). It will be helpful to keep in mind the personal and open-ended character of Wittgenstein's later conception of philosophy throughout the remaining essays.)

5. Did Wittgenstein give up on "serious philosophy"?

There is no doubt that Wittgenstein's therapeutic conception is quite antithetical to certain widespread and traditional ways of doing philosophy – which typically involve either treating some perennial philosophical question as innocent and answering it directly by way of an account, definition, or theory, or as guilty (e.g., unanswerable or confused) and showing that this is so via some general theory of language or cognition (and their necessary limits).[22] In his own words, "it seems only to destroy everything interesting", to which he famously retorts, "But what we are

[22] A paradigm of the former approach is Socrates; a paradigm of the latter Kant – though the latter tradition (broadly construed) would also include early Wittgenstein of the TLP as well as, e.g., A.J. Ayer's (1952) *Language, Truth & Logic*. That Wittgenstein's invocation of language-games and forms of life is not intended to be the forefront of a new branch of theory is emphasized in Stern's (2004) characterization of the "quietist" position: "Wittgenstein's invocation of forms of life is not the beginning of a positive theory of practice […] but rather is meant to help his readers get over their addiction to theorizing about mind and world, language and reality" (Stern, 2004: 169).

destroying are only houses of cards, and we are clearing up the ground of language on which they stood" (PI 118). However, we ought not to conclude with Russell that

> The later Wittgenstein […] seems to have grown tired of serious thinking and to have invented a doctrine which would make such an activity unnecessary.
> (Russell, 1959: 216-217)

On the contrary: Wittgenstein has offered up a radically new way of doing philosophy which requires a tireless and endless struggle against confusions stemming from the misinterpretation of language.[23] If "serious thinking" is simply equated with "engaging in the traditional game of proposing philosophical accounts, refuting them with counterexamples, offering new accounts to get around said counterexamples, and so on", then indeed Wittgenstein grew tired of this and proposed a new way of doing things which would make it unnecessary. But that is obviously a narrow conception of "serious thinking".[24] Likewise, we should be careful with the common sentiment that Wittgenstein's philosophy is completely "negative", that there is nothing "positive" in his new approach to things. Indeed, it is largely aimed at clearing, counteracting, and taking down confused forms of thought. But it is not without insights, e.g., into the sources of philosophical perplexity, or into the actual workings of our language (PI 109). If by "positive" we mean "revealing some general theory about a phenomenon of interest", then this is just as narrow a conception of "positive" as Russell's conception of "serious thinking" above. It would be unfair to measure the

[23] As McDowell (2009) nicely puts it (though I will personally refrain from using the label 'quietism', in part because of the misleading connotations and associations that McDowell himself notes): "Wittgensteinian quietism is absolutely not a recommendation of a kind of idleness, a practice of leaving necessary tasks to others, out of some distaste […] for the sorts of activity that go into performing them. Quietism does indeed urge us not to engage in certain supposed tasks, but precisely because it requires us to work at showing that they are not necessary. And it is indeed work" (McDowell, 2009: 371-2). The "radical" nature of Wittgenstein's later conception of philosophy (especially as it contrasts with more common practices in the philosophy of mathematics) is similarly emphasized by Chihara (1982: 105) and Monk (2007: 274).

[24] Compare Monk (1990: 308): "Wittgenstein's abandonment of theory was not, as Russell thought, a rejection of serious thinking, of the attempt to understand, but the adoption of a different notion of what it is to understand."

seriousness or value of Wittgenstein's new conception by the standards of the very intellectual traditions he is persistently acting against and thereby putting into question.[25]

However, a related suspicion about Wittgenstein's therapeutic conception is perfectly natural and worth holding onto. Namely: "Is this (as Russell describes it) really just a *doctrine* – a dogmatic insistence on a largely destructive way of doing philosophy?". If one reads Wittgenstein's remarks about his new methods and swallows them down uncritically, then this is certainly cause for suspicion. One should instead *take up* the method Wittgenstein offers, as it is repeatedly articulated with his rich series of examples, and adjudicate its value accordingly. The issue here thus cannot be decided *a priori* – we need to apply Wittgenstein's methods and see where they take us, whether with their help, "Problems are [really] solved" (PI 133).[26] Enough, then, with this schematic picture of Wittgenstein's later thought. Let's see what fruit it might bear in understanding Wittgenstein's remarks on mathematical reality, determination, and infinity.[27]

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[25] Those who were personally acquainted with Wittgenstein have emphasized this fact about his later work, against those who wish (often with charitable intentions, e.g., to save it from Russell's objection) to assimilate it with said traditions. See for instance Maurice Drury's remark about "well-meaning commentators" who "make it appear that his writings were now easily assimilable into the very intellectual milieu they were largely a warning against" (Rhees, 1984: 101). See also Monk (2007: 270).

[26] I thus recommend we approach Wittgenstein's therapeutic methods in the spirit of Aristotle's wonderful remark at the end of *Nicomachean Ethics*: "But while these sorts of considerations also carry a certain conviction, the truth in practical matters must be discerned from the facts of our life […] if it clashes, we should suppose it mere words" (Aristotle 2014: X, 7). Likewise, Wittgenstein's methods should neither be accepted nor discarded *a priori.*

[27] As I mentioned earlier, there is much more to say about Wittgenstein's therapeutic conception of philosophy than can be offered here. A more comprehensive discussion would include the following topics (among others): (i) Wittgenstein's insistence that philosophical problems are not to be addressed with empirical considerations or mathematical discoveries, (ii) his emphasis that his examples and explorations are aimed at changing our ways of *seeing* things, (iii) that, according to him, he relies only on details that we would all concede to, (iv) that his remarks are really just "reminders" of things with which we've long been familiar, or (v) the connection, noted by scholars, between Wittgenstein's conception of overview (*übersicht*) and Goethe's notion of the *urpflanze* in his studies of plants (on this last topic, see especially Baker (2004: 35-6)). I have not discussed these important and interesting aspects of his thought only in the interest of focusing on the details that matter most for the essays that follow.

Chapter 1

'Truth' and 'Reality' in Mathematics

> What a mathematician is inclined to say about the objectivity and reality of mathematical facts, is not a philosophy of mathematics, but something for philosophical *treatment*. (PI 254)

1.1 Introduction

In an article titled "Mathematical Proof", to which Ludwig Wittgenstein regularly referred in his LFM, G.H. Hardy offers one of several criteria, "that a philosophy must satisfy if it is to be at all sympathetic to a working mathematician":

> It seems to me that no philosophy can possibly be sympathetic to a mathematician which does not admit, in one manner or another, the immutable and unconditional validity of mathematical truth. Mathematical theorems are true or false; their truth or falsity is absolute and independent of our knowledge of them. In *some* sense, mathematical truth is part of objective reality. (Hardy, 1929: 4)

Hardy then provides the following elaboration on what he admits is at most a rough idea.

> 'Any number is the sum of 4 squares'; 'any number is the sum of 3 squares'; 'any even number is the sum of 2 primes'. These are not convenient working hypotheses, or half-truths about the Absolute, or collections of marks on paper, or classes of noises summarizing reactions of laryngeal glands. They are, in one sense or another, however elusive and sophisticated that sense may be, theorems concerning reality, of which the first is true, the second is false, and the third is either true or false, though which we do not know. They are not creations of our minds; Lagrange discovered the first in 1774; when he discovered it he discovered *something*; and to that something Lagrange, and the year 1774, are equally indifferent. (*ibid*)[28]

Commenting on a slightly inaccurate version of this quote,[29] Wittgenstein is reported to have said the following in his lectures:

[28] Compare Hardy (1967: 123-4): "I believe that mathematical reality lies outside us, that our function is to discover or *observe* it, and that the theorems which we prove, and which we describe grandiloquently as our "creations", are simply our notes of our observations."

> Consider Professor Hardy's article ("Mathematical Proof") and his remark that "to mathematical propositions there corresponds—in some sense, however sophisticated—a reality". (The fact that he said it does not matter; what is important is that it is a thing which lots of people would like to say.) (LFM 239)

Indeed, similar things have been said by a wide variety of philosophers and mathematicians, including (as likely occurred to Wittgenstein) Frege, Russell (for a time), and Gödel. And the idea is neither quaint nor dated: As Putnam (2001) has rightly noted, a version of Hardy's idea – that mathematical truths are truths concerning objective mathematical reality – is elegantly crystallized and defended in what is often taken to be a staple in contemporary philosophy of mathematics, Paul Benacerraf's "Mathematical Truth".[30] Wittgenstein in the passage above is thus addressing a general idea that, however controversial it might be, has exerted a great influence on philosophical thought about mathematics.

But there is a crucial difference between Hardy's and Benacerraf's invocations of an objective mathematical reality. Rather than merely offering such an idea as a constraint on any sympathetic account of mathematical practice, Benacerraf argues that such a conception of mathematical reality is a necessary outfall of a sufficiently general theory of truth. To illustrate this, he offers the following two sentences:[31]

(1) There are at least three large cities older than New York.

(2) There are at least three perfect numbers greater than 17.

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[29] The inaccuracy is that Hardy does not appeal to "correspondence" as Wittgenstein does in his version. Whether this inaccuracy is the fault of Wittgenstein or his notetaker will not concern us here. I will explain in Section 3 why the misquote does not affect Wittgenstein's argument in this lecture.

[30] Compare, e.g., Shapiro (2000: 31): "Paul Benacerraf's 'Mathematical Truth' (1973) [is] an article that continues to dominate contemporary discussion in the philosophy of mathematics"; and Linnebo (2017: 12, fn. 8): "Recent discussions of the challenge often focus on the version developed in Benacerraf (1973)", though Linnebo confesses he, "find[s] this focus unfortunate". It is a common focus all the same. Compare also Hacking (2014: 216): "Benacerraf's superb paper, 'Mathematical Truth' (1973), is a fundamental benchmark for philosophical platonism/nominalism debates."

[31] Benacerraf (1973: 663).

According to what he takes to be the "standard account" of the semantics of these two sentences, they are both of the form

(3) There are at least three FG's that bear R to a.

and thus have parallel truth conditions. That is to say, (1) is true just in case there are three (FG's) large cities (R) older than (a) New York, while (2) is true just in case there are three (FG's) perfect numbers (R) larger than (a) 17. The result of this, which Benacerraf grapples with in his paper, is a version of Hardy's idea – namely, that true mathematical claims are claims concerning objective reality – since mathematical claims make reference to entities which, if those claims are indeed true, must exist.

However, Benacerraf acknowledges that this leads to problems. Given that mathematical entities are abstract and thus do not have the causal properties possessed by their empirical counterparts, it is mysterious how mere mortals in the causal realm might acquire knowledge of such things. The issue then, in short, is that it is quite difficult to square up mathematical truth – which implies the existence of an a-causal realm of entities – with mathematical knowledge – which seems to require causal access to its subject matter. A standard account of mathematical truth comes at the cost of rendering mathematical knowledge either impossible or inexplicable, while a natural (causal) constraint on knowledge comes at the cost of sacrificing mathematical truth (according to a standard conception of truth). It is probably fair to say that resolving the dilemma which Benacerraf has laid out is generally regarded as a condition on any adequate philosophy of mathematics today. Whereas some philosophers might resolve the issue by offering a "non-standard" conception of mathematical truth (or, radically deny that such claims are literally true, or that their *value* is to be explained in terms of their "truth"), Hardy and Benacerraf would agree that a sympathetic rendering of mathematical practice requires that we maintain the standard view and not simply abandon it on epistemological grounds.[32]

How does Wittgenstein respond to these (allegedly standard or natural) claims about mathematical reality? As reported by students who attended his lectures, Wittgenstein suggests that Hardy's version, at least, is at best unclear and at worst meaningless.

> Taken literally, this seems to mean nothing at all – *what* reality? I don't know what this means. – But it is obvious what Hardy compares mathematical propositions with: namely physics. (LFM 239)

Such a comparison – between mathematical propositions and the propositions of physics – is not unique to Hardy. Benacerraf motivates the existence of mathematical reality likewise on the grounds that there is a parallel between mathematical and empirical propositions (as illustrated by sentences (1) – (3) above). Though, again, in Benacerraf's case this is allegedly supported by the (desirable) uniformity of our theory of truth and in turn of our semantics. Thanks to Tarski, we now have the foundations of such a theory, the "essential feature" of which, "is to define truth in terms of reference (or satisfaction) on the basis of a particular kind of syntactico-semantical analysis of the language" (Benacerraf, 1973: 667). The truth of mathematical propositions implies the existence of a special class of entities (i.e., those entities to which true mathematical claims refer). Does Benacerraf's appeal to truth and Tarskian semantics clarify the matter – i.e., why it is that, in doing mathematics, we seem committed to the existence of a distinctive class of objects? Does it help to show what might be meant by a mathematical reality to which those claims refer? Wittgenstein's further comments on Hardy do not leave much room for hope.

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[32] Some have argued that Benacerraf's dilemma as it stands is not completely persuasive, but that there are closely related problems for what Benacerraf calls a standard account of mathematical truth and the "platonistic" view of mathematical reality it seems to imply. See especially Field (1988: 25-30) for further discussion. My purpose here is simply to note that the conception of mathematical truth which Benacerraf puts forward naturally leads to (and is intended to lead to) metaphysical and epistemological problems. This simpler point does not depend crucially on Benacerraf's own formulation of the problem, which requires a controversial appeal to a causal theory of knowledge.

> Suppose we said first, "Mathematical propositions can be true or false." The only clear thing about this would be that we affirm some mathematical propositions and deny others. If we then translate the words "It is true …" by "A reality corresponds to …" – then to say a reality corresponds to them would say only that we affirm some mathematical propositions and deny others. We also affirm and deny propositions about physical objects. – But this is plainly not Hardy's point. If this is all that is meant by saying that a reality corresponds to mathematical propositions, it would come to saying nothing at all, a mere truism: if we leave out the question of *how* it corresponds, or in what sense it corresponds. (LFM 239)

It is no surprise then that Wittgenstein, characteristically for his later philosophical career, suggests that Hardy's idea results from a failure to attend to the uses of our words: "We have here a thing which constantly happens. The words in our language have all sorts of uses; some very ordinary uses which come into one's mind immediately, and then again they have uses which are more and more remote" (LFM 239). More specifically, in making such claims about mathematical "truth" or the "reality" to which it "corresponds", one can easily forget the ordinary uses of these words and (thereby) end up in philosophical confusion.

> So if you forget where the expression "a reality corresponds to" is really at home—
>
> What is "reality"? We think of "reality" as something we can *point* to. It is *this*, *that*. Professor Hardy is comparing mathematical propositions to propositions of physics. This comparison is extremely misleading. (LFM 240)

Thus, it seems that Wittgenstein's concerns about Hardy's idea would just as easily extend to Benacerraf's version of it. That is, to the extent that it arises from an analogy between empirical and mathematical propositions without attention to their distinctive uses (as well as the distinctive uses of 'truth' and 'reality' accompanying them), Benacerraf's invocation of an objective mathematical reality would be equally suspect.

We can anticipate frustration from contemporary philosophers (of mathematics, or otherwise) on a number of different fronts. As a matter of exegesis, the texts "from Wittgenstein" above seem corrupt. After all, he (or his notetaker) misquotes Hardy and thus we might naturally accuse Wittgenstein (or, his notetaker) of misunderstanding Hardy's view from the outset. Further, with how much confidence can we say that these are *Wittgenstein's* ideas quoted above? They are notes taken by his students during Wittgenstein's lectures. Few would recognize their students' notes as an accurate portrayal of their own philosophy. Further, one might say things in passing during lecture which have not yet been submitted to proper scrutiny (say, for publication). Thus, we might doubt whether these are a reliable guide to Wittgenstein's (considered) thoughts.[33] We might also doubt whether they make for a coherent objection against Hardy.

As a matter of philosophy (and ignoring the exegetical concerns above), how can Wittgenstein accuse such folks of meaninglessness or lack of clarity? Or, even if Wittgenstein might have rightly objected to Hardy on such grounds (Hardy does admit his suggestion is vague), how could such a concern possibly carry over to Benacerraf's presentation, grounded in a Tarskian semantical framework which is judged by many to be a perfectly intelligible advancement in mathematics and philosophy? Perhaps this is one of many remarks made by Wittgenstein which can be put aside as relying on a false pessimism about the prospects of philosophical or formal theories in general, or, at the very least, as an opinion that ought to have been updated in light of later advances in the subject of analytic philosophy. Thanks to Tarski, we know much better what 'truth' is. With a better grasp of 'truth' to hand, and thus of the conditions on any adequate theory of 'meaning' (semantics), Benacerraf's suggestion that mathematical truth implies mathematical reality could hardly be rejected on grounds of unintelligibility. It might seem then that Wittgenstein's concerns here are obsolete, to the extent that they can so much as be understood.

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[33] Cora Diamond (in her Editor's Preface to the LFM) is quite explicit about its limitations in 'giving Wittgenstein's views': "A great deal of caution must, however, be used before anything in the text here can be taken as 'giving Wittgenstein's views' or even as giving good evidence for some particular interpretation of what he says elsewhere. This is not merely on account of the inevitable inaccuracies. Much of the text given here *is* accurate; that is, Wittgenstein did say the words in the text or something very close. But he did not read the material; he did not correct it; he was not in a position to throw any of it away. *Much* here he would have discarded. In fact he often did point out in the lectures that something he had said was misleadingly put; he had no opportunity to say that of any of the rest" (Diamond, 1976: 9). For these reasons I treat the LFM and more official texts such as the PI and the RFM as having a different status: the latter can be used to substantiate and interpret the former, but the former cannot (without great caution) be used to substantiate or interpret the latter.

My aim is to provide a reading of Wittgenstein on mathematical 'truth' and 'reality' which will help to address these concerns. More directly, Wittgenstein's reported quote in the LFM is completely predictable and in keeping with his later philosophy. I will justify this by appeal to texts in the PI and the RFM.[34] Although my focus will primarily be one of clarifying Wittgenstein's (later) views on these subjects, I also hope to show along the way that they are far from obsolete. In particular, that they are not so easily debunked by the Tarskian updating that Benacerraf provides (or, of any other general theory of truth or semantics), in part because Wittgenstein provides reasons to think the philosophical significance of such a maneuver is deeply suspect. Through a careful reading of more official texts from Wittgenstein (i.e., by comparison with the LFM), we will uncover considerations that challenge the assumed intelligibility and importance of philosophical claims, questions, and problems regarding mathematical reality – such as those we find discussed by Hardy and Benacerraf.

The rest of the paper will proceed as follows. In Section 2, I will explain Wittgenstein's basic therapeutic strategy against Hardy's and Benacerraf's pictures of mathematical reality. In Sections 3-5, I discuss a series of passages from Wittgenstein that undermine the analogy between mathematical and empirical propositions deployed by Hardy and Benacerraf above – which is required to get their claims about "mathematical reality" (and the inevitable problems they engender) off the ground. In Section 3, I examine Wittgenstein's remarks about Hardy regarding the significance of mathematical 'truth' in LFM XXV. Although these remarks on their own leave some uncertainty about whether they are "Wittgenstein's own views", I will show in the following sections that the underlying ideas

[34] So, to be clear, I will not engage in a more documentary approach of comparing lecture notes, letters, or other historical documents in order to determine, say, whether Wittgenstein *really said* what he is reported to have said in the LFM. Instead, I will use the PI and RFM to interpret and clarify Wittgenstein's remarks about Hardy in the LFM (rather than vice versa) and show that they are in keeping with his later philosophy. In other words, the issue for me is not *what Wittgenstein said* during his lectures, but the extent to which *what he is reported to have said* adequately represents his considered philosophical views.

are well represented in more official texts. In Section 4, I give a reading of Wittgenstein's general remarks on 'truth' and 'propositions' in PI 134-7 and explain what bearing they have on Hardy's picture of mathematical reality. In Section 5, I will show that the very same considerations are applied more specifically to *mathematical* 'truth' and 'propositions' in RFM I, Appendix III. Finally, in Section 6, I will conclude by summarizing the major upshot of these remarks, and show why Tarskian updates to one's theory of 'truth' do not suffice as a response to Wittgenstein's concerns. That is, because they merely presuppose what Wittgenstein puts into question, namely, the essential uniformity of 'truth' and 'proposition' in ordinary discourse.[35]

1.2 Hardy's Picture of Mathematical 'Truth': The Basic Therapeutic Strategy

In the Introduction to these essays, I gave a broad outline of Wittgenstein's therapeutic conception of philosophy, as I understand it. This 'conception' is not a theory or description of all the things we might happen to call "philosophy", but Wittgenstein's own radical conception of how philosophy should be done – "radical" in that it disrupts the traditional philosophical mode of treating certain questions either as innocent and answering them directly by way of an account, definition, or theory; or as guilty (e.g., unanswerable or confused) and showing that this is so via some theory of language or cognition (and their necessary limits). By contrast, on Wittgenstein's conception of philosophy, philosophical questions are themselves an object of suspicion and require an investigation of their sources without any aspiration to theory. They are treated "like an illness" and thus submitted to diagnosis and therapy. Since these characterizations are merely analogical or metaphorical, I unpacked them as follows. Wittgenstein's general 'diagnosis' of philosophical problems is that they

[35] My discussion will thus substantiate and elaborate on similar readings offered by Diamond (1996) and Conant (1997). Like Gerrard (1991), I think that Wittgenstein's critique of "the Hardyian picture" is fundamental to his later philosophy of mathematics quite generally – and so this essay will serve as crucial background for the essays that follow.

stem from misunderstandings about the uses of words due to one's lacking a proper overview of our language. One major aspect of language which encourages such misunderstandings is the apparent similarity between different kinds of words; confusions arise when they are assimilated despite important differences between their uses.

Wittgenstein's 'diagnosis' of philosophical questions is at the core of his notion of philosophical 'therapy'. The misunderstandings about language which give rise to philosophical puzzlement can only be counteracted by *describing* the uses of words and drawing out differences between them – differences which might easily be overlooked due to their surface similarities. A common tool in Wittgenstein's philosophical therapy is his use of language-games, some of which are entirely fictional. The value of language-games (especially those that are fictional) is to serve as "objects of comparison", designed to throw light on features of our *actual* language. Language-games, whether actual or fictional, also articulate a kind of ideal (an "overview", the lack of which, according to Wittgenstein, is the major source of misunderstandings), because when language-games are described in sufficient detail traditional philosophical problems about meaning do not arise. Thus, generally speaking, philosophical 'illnesses' are confusions about the uses of words due to a lack of oversight (*übersicht*); philosophical 'therapy' counteracts such confusions by careful description of and attention to ordinary word-use, deploying (sometimes fictional) language-games to highlight aspects of use or features of our language that might otherwise be ignored. This is Wittgenstein's therapeutic conception of philosophy in a nutshell, the further clarification of which (by his own lights) must go by way of example and application.

With Wittgenstein's therapeutic conception of philosophy to hand, I will now provide a reading of Wittgenstein on mathematical 'truth', as revealed in his engagement with Hardy's (and by extension Benacerraf's) picture of mathematical reality. According to Hardy's picture, mathematical truth is "in *some* sense" part of objective reality. Such truths are immutable, unconditionally valid, and independent of our knowledge of them. The true theorems of mathematics are, "however elusive and sophisticated that sense may be", genuine objects of discovery and *not* mere creations of our minds. When Lagrange discovered that 'any number is the sum of 4 squares', he discovered *something*, i.e., something which was there to be discovered before anyone had anything to think or say about it. At a later stage of his paper, "Mathematical Proof", Hardy elaborates on his understanding of mathematical discovery with an analogy.

> I have myself always thought of a mathematician as in the first instance an *observer*, a man who gazes at a distant range of mountains and notes down his observations. His object is simply to distinguish clearly and notify to others as many different peaks as he can. There are some peaks which he can distinguish easily, while others are less clear. He sees A sharply, while of B he can obtain only transitory glimpses. At last he makes out a ridge which leads from A, and following it to its end he discovers that it culminates in B. B is now fixed in his vision, and from this point he can proceed to further discoveries. In other cases perhaps he can distinguish a ridge which vanishes in the distance, and conjectures that it leads to a peak in the clouds or below the horizon. But when he sees a peak he believes that it is there simply because he sees it. If he wishes someone else to see it, he *points to it*, either directly or through the chain of summits which led him to recognize it himself. When his pupil also sees it, the research, the argument, the *proof* is finished.
>
> The analogy is a rough one, but I am sure that it is not altogether misleading. (Hardy, 1929: 18)[36]

Hardy's picture of mathematical discovery is thus strongly analogized to the observation and discovery of peaks in a mountain range, some of which can be seen more or less clearly than others. Mathematical reality, according to Hardy, is akin to a mountain range, there to be discovered quite independently of anything we might have to say or think about it.

Versions of Hardy's picture have surfaced in a number of different places. For instance, Hardy's thoughts are reminiscent of Russell's (1919) passing remark that logic is akin to zoology:

[36] Though, taken to its extreme, Hardy does think it might lead to the "paradoxical conclusion" that "there is no such thing as mathematical proof; that we can, in the last analysis, do nothing but *point* [i.e., to the truths which are simply there to be observed by mathematicians and their students]" (Hardy, 1929: 18). Wittgenstein, as we'll see, would think of this analogy, and the picture of mathematical reality it suggests, as misleading for quite different reasons.

> Logic, I should maintain, must no more admit a unicorn than zoology can; for logic is concerned with the real world just as truly as zoology, though with its more abstract and general features. (Russell, 1919: 169)

Despite important differences between Hardy's and Russell's[37] understandings of logic and mathematics, they overlap in thinking of their fields as areas of discovery, concerning reality much like the geological reality observed by someone studying mountain peaks, or the biological reality studied by zoologists. Another famous instance of such a picture is from Gödel, who likewise conceives of mathematics as an independent reality which, "clearly [does] not belong to the physical world"; one which, on Gödel's view, is accessed via something like perception (thus quite reminiscent of the "seeing" of mountain peaks in Hardy's image above).

> [T]he objects of transfinite set theory […] clearly do not belong to the physical world and even their indirect connection with physical experience is very loose […].
>
> But, despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. (Gödel, 1984: 483-484)

Hardy's picture of mathematical reality is also clearly echoed more recently by the mathematical physicist, mathematician, and philosopher of science, Roger Penrose:

> How 'real' are the objects of the mathematician's world? From one point of view it seems that there can be nothing real about them at all. Mathematical objects are just concepts; they are the mental idealizations that mathematicians make, often stimulated by the appearance and seeming order of aspects of the world around us, but mental idealizations nevertheless. Can they be other than mere arbitrary constructions of the human mind? At the same time there often does appear to be some profound reality about these mathematical concepts, going quite beyond the mental deliberations of any particular mathematician. It is as though human thought is, instead, being guided towards some external truth – a truth which has a reality of its own, and which is revealed only partially to any one of us. (Penrose, 1989: 95-6)

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[37] That is, Russell's understanding *at the time of writing this particular book*. One relevant difference might be that Russell's picture does not invoke a *separate* reality for logic, but instead views logic as exploring the abstract features of a single world. This is especially relevant in comparing Russell's picture with Gödel's picture, which we'll see momentarily. Floyd (2006) also notes the connection between Hardy and Russell's pictures of mathematical reality: "Wittgenstein's emphasis on the image of the mathematician as *inventor* or fashioner of models, pictures, and concepts was, in the main, directed at the philosophical talk of those, like Hardy and Russell, who insisted on speaking of mathematical reality in a freestanding way, picturing the logician or mathematician as a zoologist embarked on an expedition to new, hitherto unseen lands, analogous to an empirical scientist" (Floyd, 2006: 112).

The idea that mathematics concerns an external reality which is only partially revealed to us is very much in keeping with Hardy's suggestions, e.g., that "There are some peaks which [a mathematician] can distinguish easily, while others are less clear". The examples of Hardy, Russell, Gödel and Penrose suffice to show that a suggestive picture of mathematical reality articulated (however roughly) by Hardy has attracted serious mathematicians and philosophers alike. I will not assume that their full-fledged views or theories of mathematical reality are all the same (indeed, they are not) – what is important for our purposes is a common picture at the root of each: that of a reality which is "out there" independently of mathematical practice and discovered by mathematicians in the process of doing mathematics, akin to the independent realities studied by geologists, zoologists, or physicists.

As we saw earlier, Hardy's picture is likewise echoed in Benacerraf's famous paper, "Mathematical Truth": an independent reality of entities akin to empirical objects (e.g., cities), but quite different in that they are a-causal and abstract. Although Benacerraf provides a more technical argument for this picture than Hardy, they both proceed by analogizing the truths of mathematics to empirical truths (e.g., those having to do with mountain peaks or cities).

What does Wittgenstein have to say about these pictures of mathematical reality? According to his therapeutic conception of philosophy, we should investigate the sources of Hardy's and Benacerraf's pictures as well as the philosophical questions and puzzlement they engender. The puzzlement here includes questions (more or less standard in the philosophy of mathematics) such as, "What *is* mathematical reality?", "How is mathematical knowledge *possible*?", "How can we make sense of the application of mathematical truths (which seem to inhabit their own kind of reality) to empirical reality?", among others. Since questions of this sort themselves hinge on a certain picture of mathematical reality – i.e., as *something* hanging out there, though perhaps beyond the causal realm – the main focus for Wittgenstein should be that very picture.

Where does this picture come from? More specifically, on the therapeutic conception, what is it about our language that has encouraged such a picture? Given the presentations offered by Hardy and Benacerraf, we thankfully don't have to look far and wide for an answer. The obvious source for Wittgenstein would be the assimilation of mathematical truths (or, propositions) to empirical truths (or, propositions), an assimilation that both authors make explicitly, albeit without much reflection or scrutiny. As we saw in the Introduction to these essays, Wittgenstein thinks that the assimilation of apparently similar expressions is a general source of philosophical confusion. A therapeutic response will have to go by way of studying the uses of these expressions so as to draw out their crucial differences. Studying the differences should, in turn, undermine the hasty analogy which led to Hardy's and Benacerraf's pictures of mathematical reality and thus the questions they engender.

We can see that this is the very approach Wittgenstein is reported to have taken up in the LFM – though I'll argue that his approach here is completely in keeping with his writings in the PI and the RFM. But since the LFM provides the most explicit engagement with Hardy's picture of mathematical reality, let's begin there and work our way into his more official writings.

1.3 Wittgenstein's Discussion of Hardy in LFM XXV

At the beginning of LFM XXV, Wittgenstein reportedly[38] addresses, "a false idea of the role which mathematical and logical propositions play" (LFM 239). He illustrates this false idea by examining Hardy's remark that, "to mathematical propositions there corresponds—in some sense, however sophisticated—a reality", though he emphasizes that *Hardy's* saying this is not crucial, since "it is a thing which lots of people would like to say" (*ibid*) – as we ourselves saw with the selection of

[38] For convenience, when discussing passages from the LFM hereon, I will drop the important qualification that these are things Wittgenstein is *reported* to have said.

philosophers and mathematicians in Section 2. Wittgenstein immediately complains that, "Taken literally, this seems to mean nothing at all—*what* reality?" (*ibid*). Although he claims not to understand Hardy's remark, he notes that, "it is obvious what Hardy compares mathematical propositions with: namely physics" (*ibid*). This is also something we have seen confirmed above in the cases of Hardy, Russell, Gödel, Penrose and Benacerraf – all of whom analogize the truths of mathematics to the truths of some empirical domain or other (e.g., geology, zoology, physics, or geography). But whereas the analogy, on their view, suffices to explain what they mean by an "independent mathematical reality", Wittgenstein insists that this produces a muddle.

Why does the analogy fail to clarify what is meant by "to mathematical propositions there corresponds a mathematical reality"? The major objection raised by Wittgenstein is that Hardy extrapolates a bewildering picture from "a mere truism", namely, that, "Mathematical propositions can be true or false" (LFM 239). But, according to Wittgenstein, this is just to say, "that we affirm some mathematical propositions and deny others" (*ibid*). We could paraphrase, "It is true …" by "A reality corresponds to …", but this is just to replace one set of words with another, which in turn can only state the obvious if it states anything at all: "that we affirm some mathematical propositions and deny others" (*ibid*). Of course, it's also true that, "We […] affirm and deny propositions about physical objects. — But this is plainly not Hardy's point" (*ibid*). That is, again, Hardy extracts an exotic picture from a trivial and obvious fact: that we affirm and deny both mathematical and empirical propositions. If Hardy were merely stating this trivial and obvious fact, "it would come to saying nothing at all, a mere truism", that is, "if we leave out the question of *how* it corresponds, or in what sense it corresponds" (*ibid*).

In other words, the mere fact that we say of both mathematical and empirical propositions that they are true (or false) does not imply anything about a special mathematical reality to which mathematical propositions correspond. To think otherwise would be a hasty and unjustified leap from (i) a superficial similarity between these kinds of expressions (that they can both be 'true' or 'false', or that we affirm and deny both) to (ii) a problematic metaphysical picture. That is, if one thinks on the basis of these superficial similarities that mathematical propositions correspond to reality in *exactly* the same way that empirical propositions do, then this encourages a highly misleading picture of mathematics – that of mathematics corresponding to a reality somehow akin to a mountain range, the animal kingdom, or particles in a cloud chamber. This would be akin to inferring from the analogy (i) "reading a book is like riding a bike" (i.e., if you've learned once before, it's easy to pick back up), that therefore (ii) reading a book is *exactly* like riding a bike: it requires pedaling, shifting gears, and wearing a helmet. One similarity between A and B does not imply total similarity between A and B – a truism about analogy which, if forgotten, can lead to philosophical confusion.

At this point, however, it is worth reckoning with the concern that Hardy himself does not speak of "correspondence" in his paper (as is acknowledged by the editor in LFM 239, fn. 1). Is Wittgenstein's objection based on a simple misreading of Hardy? Thus far, the considerations that Wittgenstein puts forth don't depend on the word "correspondence" in any crucial way. Presumably, even the claim Hardy *does* make, viz., that, "[Mathematical theorems] are, in one sense or another, however elusive and sophisticated that sense may be, theorems *concerning* reality" (Hardy 1929: 4, emphasis added), faces the very same line of questioning: viz., that if this says anything at all, it states the trivial point that mathematical propositions can be true or false; or, that we affirm and deny some and not others. In what sense does this mean that mathematical theorems are "theorems *concerning* reality"? And what "reality" does one have in mind here? All that we are left with to answer this question is an analogy: mathematical truths are truths concerning reality *in the same sense* that empirical truths are truths concerning reality; the 'reality' a mathematical theorem concerns, e.g., Fermat's Last Theorem, is (somehow) akin to the 'reality' with which "There is an apple in the fridge" is concerned. But this is exactly what Wittgenstein would challenge from the therapeutic angle. That is, he would diagnose this very maneuver as a problematic assimilation of expressions which have importantly different uses.

Wittgenstein's remarks in the LFM thus do not hinge specifically on the word "correspond", since whether we state Hardy's point in terms of mathematics "*corresponding to* reality" or "*concerning* reality", in either case Hardy has been taken in by a misleading analogy, itself encouraged by a trivial and superficial similarity between mathematical and empirical propositions. Such an analogy leads to a problematic picture of mathematical reality (i.e., as something "out there" yet beyond our causal reach) by overlooking all of the crucial differences between the uses of mathematical and empirical propositions.

Consider, for instance, that one could have proceeded in the opposite direction: mathematical and empirical propositions have quite different uses, including the conditions under which we take them to be 'true', 'known', 'believed', etc.; therefore, mathematical propositions do *not* concern or correspond to reality *in anything like the way* empirical propositions do. Wittgenstein's stance (if we can call it that) is somewhere between Hardy's direction of thought and its opposite. That is, there *are* crucial differences between mathematical and empirical propositions, though there *are* also similarities (however superficial they might be). The trick is not to get seduced by either the differences or similarities into a misleading picture of their roles or meanings.[39] Wittgenstein's tasks

[39] For instance, in the TLP Wittgenstein himself was so impressed by the *differences* between mathematical and empirical propositions that he was not even willing to call the former "propositions" but instead "pseudo-propositions": "The correct explanation of logical propositions must give them a peculiar position among all propositions" (TLP 6.112); "Mathematics is a logical method. The propositions of mathematics are equations, and therefore pseudo-propositions" (TLP 6.2). On my reading of his later remarks from LFM, PI, and RFM, Wittgenstein is perfectly willing to grant that there are 'propositions' in mathematics and that these can legitimately be called 'true' and 'false' (since there are indeed ordinary language-games in which these terms have a home). Problems arise, however, from a hasty assimilation of empirical and mathematical propositions on the basis of these superficial similarities. Thus, there is some continuity here with the TLP, to the extent that Wittgenstein still sees it as his task to *describe* the "peculiar position" of logical and mathematical propositions "among all propositions" – the major discontinuity lies in his no longer being interested in articulating these differences *theoretically* according to "the general form of the proposition", especially not so as to deny that there are really "propositions" in mathematics. My reading thus contrasts with Monk's (2007: 283), who argues that Wittgenstein's rejection of "propositions" in mathematics lasts into his later writings.

are thus (a) to display and 'diagnose' this leap in thought *from* superficial similarities among different kinds of 'propositions' *to* Hardy's picture of mathematical reality, as well as (b) to draw our attention to the differences between mathematical and empirical truths (or, propositions) in order to counteract the hold of this misleading picture.

Wittgenstein provides a diagnosis ("a thing which constantly happens") immediately after the foregoing passages.

> We have here a thing which constantly happens. The words in our language have all sorts of uses; some very ordinary uses which come into one's mind immediately, and then again they have uses which are more and more remote. For instance, if I say the word 'picture', you would think first and foremost of something drawn or painted and, say, hung up on the wall. You would not think of Mercator's projection of the globe; still less of the sense in which a man's handwriting is a picture of his character. A word has one or more nuclei of uses which come into everybody's mind first; so that if one says so-and-so is also a picture—a map, or *Darstellung* in mathematics—in this lies a comparison: as it were, "Look at this as a continuation of that."
>
> So if you forget where the expression "a reality corresponds to" is really at home—
>
> What is "reality"? We think of "reality" as something we can *point* to. It is *this*, *that*.
>
> Professor Hardy is comparing mathematical propositions to propositions of physics. This comparison is extremely misleading. (LFM 239-240)

Thus, as we saw before, Hardy is misled by a comparison between mathematical and empirical propositions (or, "propositions of physics"). Why is this comparison misleading? Given the above passage, it is misleading because it haphazardly generalizes from the features of *one* use of 'a reality corresponds to' to *all* uses of this expression, viz., from the use of that phrase as applied to empirical propositions to its use as applied to mathematical propositions. The use of this phrase, "a reality corresponds to", as applied to empirical propositions is taken as central and dictates the sense of all other uses because it is more natural or common. Wittgenstein likens this to taking a picture hung up on the wall as a central use of 'picture' because it is most natural and common, which is bound to mislead if one assumes that this instance (however natural or common) has the very same essence as

"Mercator's projection of the globe" as a 'picture' of the globe, or "a man's handwriting" as a 'picture' of his character.

The same goes for the comparison between mathematical and empirical propositions. Propositions like, "There are at least three cities larger than New York", "There is a small chair in the living room", or, "The Sun is approximately 94.389 million miles away from the Earth", (which are themselves quite different from one another!) might be the most common and natural instances when we think of a reality (e.g., cities, chairs, the sun) corresponding to a proposition, but it would be a mistake to generalize all of their features to a proposition like, "There are at least three perfect numbers greater than 17", and assume that it "corresponds to reality" in exactly the same way as the others.

Wittgenstein notes that in the most common or natural cases, we think of "reality" as "something we can *point* to", "It is *this*, *that*." For instance, if I say, "There is a small chair in the living room", and someone expresses doubts about this, we can walk to the living room together and I can point at the small chair, "Aha! See! I was right". If there are doubts about whether "There are at least three perfect numbers[40] greater than 17", then we need to perform some calculations (e.g., we might check one at a time the numbers greater than 17 to see whether any is a perfect number – which would take quite some time as the next in line after 6 are 28, 496, and 8,128! – more plausibly we'd verify by relying on someone else's work, as I did). I might "point" to the results of our calculations, but this is not much like walking into a room and pointing at a chair – nor is it like, as Hardy suggests, looking at a distant mountain peak and pointing to it. To think otherwise would amount to being misled by the uniform appearance of the word 'pointing' across quite different uses – i.e., to mistakenly assume that its use in the mathematical case is exactly like its use in the empirical case because the word 'pointing' is used both times.

[40] A perfect number is a positive integer that is equal to the sum of its positive divisors.

Notice that, by drawing out such differences, Wittgenstein is not making a sophisticated philosophical suggestion so much as noting obvious facts about the ordinary uses of our expressions. The assimilation suggested by Hardy gains its power by overlooking such facts. If there is any sense to be made of "a reality corresponding to" mathematical propositions, it will need to pay heed to these obvious facts. Going back to where we started, this is the, "false idea of the role which mathematical and logical propositions play" (LFM 239): namely, that they play the same essential role as do empirical propositions and thus "correspond to reality" in essentially the same manner.

The considerations we find reported in the LFM are a predictable and natural extension of remarks about 'truth' and 'propositions' that we find in the PI and the RFM. The fact that we find the same kinds of warnings – about "something that constantly happens" – in these more official texts should settle most concerns we might have about the connection between Wittgenstein's reported remarks about Hardy in the LFM and the rest of his later philosophy. Noting the connections between these texts should also help to substantiate and clarify some of the points he makes therein. I turn now to Wittgenstein's discussion of 'truth' and 'propositions' in PI 134-7, which we'll then see extended to mathematics in RFM I, Appendix III.

1.4 'Truth' and 'Propositions' in PI 134-7

In PI 134-7, Wittgenstein addresses a temptation to unveil the general form, nature, and essence of propositions – a temptation Wittgenstein himself succumbed to in the TLP. It is especially relevant to Wittgenstein's discussion of Hardy because, as we saw earlier, Hardy's picture depended crucially on assimilating mathematical and empirical propositions, treating them as if they have the same general form, nature, or essence.[41] PI 134 begins,

> Let's examine this sentence "This is how things are". – How can I say that this is the general form of propositions? – It is first and foremost *itself* a sentence, an English sentence, for it has a subject and a predicate. But how is this sentence applied – that is, in our everyday language? For I got it from *there*, and nowhere else. (PI 134)

The issue with stating that "This is how things are" is the general form of propositions is that this itself is just a sentence taken from our ordinary language – thus leaving open the question of how *it* is used and thus what *it* means. It does not serve as an expression the meaning of which is so self-evident that it explains in general terms what a proposition is or does. But these considerations apply quite generally to other schemata that one might be tempted to use, e.g., that propositions "represent reality", or that they "can be true or false", or that they, if true, "tell us how the world is", and so on. These expressions, just as much as "this is how things are", are ordinary expressions from our language and leave us the task of answering how they themselves are used – what sense they themselves have if they have any sense at all. Wittgenstein continues by describing an ordinary use of the phrase "This is how things are", a clear instance of "bring[ing] words back from their metaphysical to their everyday use" (PI 116).

> We say, for example, "He explained his position to me, said that this was how things were, and that therefore he needed an advance". So far, then, one can say that this sentence stands for some statement or other. It is employed as a propositional *schema*, but *only* because it has the construction of an English sentence. (PI 134)

In this ordinary English sentence, "this is how things are" refers to some claim or collection of claims made by the person who explained their position. Thus it stands in as a "propositional *schema*" because this is the use of that ordinary English sentence. But its use in that instance is not to

[41] These passages are also relevant to Hardy because he expresses a desire for a general theory of propositions at the end of "Mathematical Proof", which he finds only partly achieved by Wittgenstein in the TLP. See especially Hardy (1929: 25): "I can find nothing, in Wittgenstein's theory [of the TLP], that is common to all the ways in which I can say that something is true and is not common also to many of the ways in which I can say that it is false". Indeed, to think otherwise would be a severe misreading of the TLP, since Wittgenstein there attempts to show that mathematics does *not* contain "propositions" (in the strict sense) *at all,* but instead mathematical equations which are deemed "pseudo-propositions" (see fn. 39 above). As we'll see in a moment, according to Wittgenstein in the PI, the very search for something "common to all the ways in which I can say that something is true" is bound, at best, to yield trivial results and, at worst, confused assimilations of quite different kinds of 'truths' and 'propositions'.

unveil the essential inner-workings or role of propositions, but instead merely to point toward something that has already been said. Again, the phrase "This is how things are", is not special in this regard, as we could have used other expressions to play the same role here.

> One could easily say instead "such-and-such is the case", "things are thus-and-so", and so on. One could also simply use a letter, a variable, as in symbolic logic. But surely no one is going to call the letter "p" the general form of propositions. To repeat: "This is how things are" had that role only because it is itself what one calls an English sentence. But though it is a sentence, still it gets used as a propositional variable. To say that it agrees (or does not agree) with reality would be obvious nonsense, and so it illustrates the fact that one feature of our concept of a proposition is *sounding* like one. (*ibid*)

So, for instance, we could rephrase the sentence earlier as, "He explained his position to me, said that p and that therefore he needed an advance", which could achieve the same effect (amongst logicians or analytic philosophers, presumably!). But if "p" is playing that role here, then "no one is going to call the letter "p" the general form of propositions". Why? Because its function in this context is simply to refer back to something already said rather than to reveal the essential inner-workings or role of propositions. Thus, "to say that it agrees (or does not agree) with reality would be obvious nonsense", since such a claim is completely disconnected from the ordinary role of "p" or "this is how things are" in the ordinary sentence above. This then "illustrates the fact that one feature of our concept of a proposition is *sounding* like one" because the variety of schemata which we think are akin to one another ("This is how things are", "such-and-such is the case", "things are thus-and-so") reveal the ways a proposition should *sound* in order to count as a proposition. However, these phrases do nothing on their own to reveal the general nature, form, essence, or role of propositions (we'll see this point again in PI 137 momentarily).

The major upshot of PI 134 is that explaining what all propositions have in common with a bit of philosophical jargon (or, rather, with an ordinary expression co-opted for metaphysical purposes) only serves to highlight superficial similarities between all the things we call 'propositions'. Going back to Hardy: The same applies to the expression "corresponds to reality", which, if it can meaningfully be applied to both mathematical and empirical propositions, highlights only a superficial similarity between these two kinds of expressions, viz., that we can say of both that they are "true" or that they "correspond to reality". The work is left for us to describe the distinctive uses of "correspond to reality" in each context; the expression taken alone does not wear its meaning (that is, its use) on its sleeve.

But even if "This is how things are" (etc.) does not reveal the general form of propositions, shouldn't we look for something better? That is, shouldn't we look for a better definition or account of propositions? PI 135 works to deflate this natural philosophical temptation by connecting Wittgenstein's famous discussion of games and family resemblances to propositions.

> But haven't we got a concept of what a proposition is, of what we understand by "proposition"? — Indeed, we do; just as we also have a concept of what we understand by "game". Asked what a proposition is – whether it is another person or ourselves that we have to answer – we'll give examples, and these will include what one may call an inductive series of propositions. So, it is in *this* way that we have a concept of a proposition. (Compare the concept of a proposition with the concept of a number.) (PI 135)

So, it is of course true: we have a concept of what we understand by 'proposition' – just as we have the concepts 'game' and 'number'. But earlier on in PI 66, Wittgenstein explains that there is no one thing that games all have in common, but instead are connected by "a complicated network of similarities overlapping and criss-crossing: similarities in the large and small". This fact about how the various things we call 'games' are related to one another is famously summarized as consisting in a series of family resemblances.

> I can think of no better expression to characterize these similarities than "family resemblances"; for the various resemblances between members of a family […] overlap and criss-cross in the same way. – And I shall say: 'games' form a family.
> (PI 67)

The comparison with "the concept of number" in PI 135 makes it all but certain that Wittgenstein is referring back to these passages.[42] PI 67 continues:

> And likewise the kinds of number, for example, form a family. Why do we call something a "number"? Well, perhaps because it has a – direct – affinity with several things that have hitherto been called "number"; and this can be said to give it an indirect affinity with other things that we also call "numbers". And we extend our concept of number, as in spinning a thread we twist fiber on fiber. And the strength of the thread resides not in the fact that some one fiber runs through its whole length, but in the overlapping of the many fibers. (PI 67)

Thus, aside from the fact (as in PI 134) that a generalization such as "This is how things are" tells us nothing informative or interesting about the role of propositions (since that is just an ordinary English sentence, the role of which would still need to be described), it is a mistake to look for *the general form* of propositions in the first place. Why? Because there are many different things we call "propositions" – 'proposition' is a family resemblance concept. Given that there is no *one thing* that is true of all propositions, at least aside from superficial similarities like *sounding* this way or that, our best hope of clarifying what 'propositions' are must go by way of providing various examples.

Such a list of examples (albeit, including some 'non-propositions') is found most directly in PI 23, at which point "the author of the *Tractatus Logico-Philosophicus*" is directly criticized for overlooking their variety. It is especially worth noting for our purposes that the list includes, "Solving a problem in applied arithmetic", which is thus meant to be compared and contrasted with the other examples in the list.

> Consider the variety of language-games in the following examples, and in others:
>
> > Giving orders, and acting on them –
> > Describing an object by its appearance, or by its measurements –

[42] Compare Floyd (2021): ""Family resemblance" characterizes the generality of certain concepts. A single property, a fixed-for-all-cases criterion, an explicit set of grammatical rules – these are not required. A concept may hold together – like a family – with a variegated, evolving series of properties (PI §§6, 236 suggests the concepts of *number* and *calculator* as examples)" (Floyd, 2021: 51).

> Constructing an object from a description (a drawing) –
> Reporting an event –
> Speculating about the event –
> Forming and testing a hypothesis –
> Presenting the results of an experiment in tables and diagrams –
> Making up a story; and reading one –
> Acting in a play –
> Singing rounds –
> Guessing riddles –
> Cracking a joke; telling one –
> Solving a problem in applied arithmetic –
> Translating from one language into another –
> Requesting, thanking, cursing, greeting, praying.
>
> – It is interesting to compare the diversity of the tools of language and of the ways they are used, the diversity of kinds of word and sentence, with what logicians have said about the structure of language. (This includes the author of the *Tractatus Logico-Philosophicus*.) (PI 23)

There should be no presumption that all such examples will be used in the very same ways or obey the very same rules, just as there should be no such presumption about all of the things we call "games" (e.g., the rules and workings of chess are quite different from the rules of solitaire, even if we might also find similarities). As emphasized in PI 69-71, this does not reveal a deficiency in our concepts of 'game' or 'proposition', it is just an obvious fact about their uses that does not lead to any special difficulties in practice. (Notice that, at least prior to Wittgenstein's work, no one was especially concerned to find a rigid and timeless definition of the word 'game'.) Whether a more precise definition of 'game' or 'concept' or 'number' is needed will ultimately depend on one's purposes and the situation at hand (c.f., PI 17). It would be a mistake to suggest that, in the absence of a precise definition, we do not ordinarily know what 'game' means (PI 70). If we *insist* on an artificial uniformity across all uses in the name of precision-for-its-own-sake, then this is bound to lead to confusions about how 'game', 'proposition', or 'number' are ordinarily used. This, again, bears directly on the assimilation of mathematical propositions and empirical propositions found in Hardy's picture of mathematical reality – since it assumes that these propositions, despite their clear differences in use, both "correspond to reality" *in the very same way*. This is partly rooted in a mistake about the essential uniformity of our concept of 'proposition', which is better understood to be a family resemblance concept.[43]

PI 136 continues Wittgenstein's thought about using "This is how things are" as giving the general form of propositions, but this time with the variant, "Such-and-such is true".

> At bottom, giving "This is how things are" as the general form of propositions is the same as giving the explanation: a proposition is whatever can be true or false. For instead of "This is how things are", I could just as well have said "Such-and-such is true". (Or again, "Such-and-such is false".) But
>
> 'p' is true = p
> 'p' is false = not-p
>
> And to say that a proposition is whatever can be true or false amounts to saying: we call something a proposition if *in our language* we apply the calculus of truth functions to it. (PI 136)

Thus, if someone were to generalize that all of those things we call "propositions" tell us that "Such-and-such is true", then this will be as unhelpful as assimilating them to the schema "This is how things are". One important reason for the unhelpfulness of *this* expression ("Such-and-such is true") is that there is no special difference between saying p and saying 'p' is true (or between saying not-p and 'p' is false). If saying p is just a way of saying 'p' is true, and vice versa, then it is difficult to see how "Such-and-such is true" would reveal the essential, inner-workings of propositions. At best, it reveals the trivial and uninteresting fact that, "we call something a proposition if *in our language* we apply the calculus of truth functions to it."

However, it might have looked like we were on to some kind of explanation about what propositions *really* are.

> Now it looks as if the explanation – a proposition is whatever can be true or false – determined what a proposition was, by saying: what fits the concept 'true', or what the concept 'true' fits, is a proposition. So it is as if we had a concept of true and

[43] Wittgenstein, however, does not himself infer that there is an essential uniformity among empirical and mathematical propositions in the TLP. Instead, he uses his theory of the general form of propositions to argue that mathematical "propositions" are not really propositions, but instead "pseudo-propositions". See fn. 39 above.

> false, which we could use to ascertain what is, and what is not, a proposition. What *engages* with the concept of truth (as with a cog-wheel) is a proposition. (*ibid*)

So, in other words, it might seem as if we had an antecedent and independent grasp of 'true', which we could then use to reveal what a proposition really is. A 'proposition' is what fits the concept 'true', or vice versa. But Wittgenstein then explains why a temptation to unify propositions (or truth) in this manner rests on "a bad picture".

> But this is a bad picture. It is as if one were to say "The chess king is *the* piece that one puts in check". But this can mean no more than that in our game of chess only the king is put in check. Just as the proposition that only a *proposition* can be true can say no more than that we predicate "true" and "false" only of what we call a proposition. And what a proposition is, is in *one* sense determined by the rules of sentence formation (in English, for example), and in another sense by the use of the sign in the language-game. And the use of the words "true" and "false" may also be a constituent part of this game; and we treat it as *belonging* to our concept 'proposition', but it doesn't '*fit*' it. As we might also say, check *belongs* to our concept of the chess king (as, so to speak, a constituent part of it). To say that check did not *fit* our concept of the pawns would mean that a game in which pawns were checked, in which, say, the player who lost his pawns lost the game, would be uninteresting or stupid or too complicated or something of the kind. (*ibid*)

To explain what propositions fundamentally are by appeal to the fact that they are the things to which we apply 'true' and 'false' is misguided. Why? Because this leaves completely open how one is to use a particular proposition, or 'true' or 'false' as applied to it, in any given case. We don't have an antecedent understanding of those uses prior to seeing them, describing them, or acting them out. To explain the concept of 'proposition' with the concept 'true' is thus akin to explaining what the chess king is by saying, "It's the piece that one puts in check", as if the concept of 'check' were somehow prior to or more fundamental than the concept of 'king' in chess.[44] But what does it mean to put something in check? The meaning of 'check', just as the meaning of 'king', needs to be

[44] Of course, if one understands what it means to put a piece in check, and also that *this* piece needs to be put in check, but doesn't yet know that *this* piece is called "the king", then such an explanation makes perfect sense and is quite useful. But by that token, so would the reverse explanation make sense for someone who knew that *this* piece is called the "king", but didn't yet know what it meant to put a piece in check, or that "the king" needs to be put in check. The mistake is to think that one of these concepts ('the king', 'check') is somehow metaphysically or conceptually more fundamental than the other. Compare PI 31.

explained by providing the rules of the game or (if those leave room for confusion) showing how the game is played via example. Thus, 'king' and 'check' are (conceptually) on a par – they *both* need to be explained by showing how the game in which they are used works; one does not "reveal the essence" of the other. These concepts "belong" to one another, but one does not "fit" the other (in some metaphysical sense of the word). The same goes for 'true', 'false', and 'proposition': to understand any of these concepts, we need to understand the language-games in which they are used. There should be no presumption (as we saw in PI 135) that the language-games will be *exactly* the same across various uses of 'proposition', 'true', or 'false'. And thus we shouldn't assume that the roles of 'proposition', 'true', or 'false' is exactly the same across such language-games either.[45]

Of the things we do not ordinarily say are 'true' or 'false' (e.g., a boat, a shoe, a headache – unless in the sense of real, authentic, or genuine), we might insist that 'true' does not *fit* them. But that is just to say that a game in which we said such things were 'true' (in the same sense that propositions can be true) would be "uninteresting or stupid or too complicated or something of the kind". Wittgenstein continues this thought about 'fitting' in PI 137, bringing us back to the considerations in PI 134 regarding the expression "This is how things are".

> What about learning to determine the subject of a sentence by means of the question "Who or what …?" – Here, surely, there is such a thing as the subject's '*fitting*' this question; for otherwise how should we find out what the subject was by means of the question? We find it out much as we find out which letter of the alphabet comes after 'K' by saying the alphabet up to 'K' ourselves. Now in what sense does 'L' fit this series of letters? – In *that* sense "true" and "false" could be said to fit propositions; and a child might be taught to distinguish propositions from other expressions by being told "Ask yourself if you can say 'is true' after it. If these words fit, it's a proposition". (And in the same way one might have said: Ask yourself if you can put the words "*This* is how things are:" in front of it.) (PI 137)

[45] This explains why it would be misleading, at best, to label Wittgenstein as a "deflationist" about truth. The reason is that, whereas deflationists tend to offer what they take to be the general role of the truth-predicate (e.g., disquotation), Wittgenstein's passage above indicates that there is no single role for 'true' or 'proposition' – the role in any given instance will depend on the language-game and context at hand. That said, PI 136 is certainly deflation*ary*, in that it undermines an attempt to explain 'proposition' in terms of 'truth' (*and*, crucially, vice versa).

The role of "true", "false", or "This is how things are" as determining whether something is a proposition (i.e., whether these concepts *fit* the construction at hand) is likened to determining the subject of a sentence by asking "Who or what…?" or to determining whether 'L' follows 'K' by reciting the alphabet. 'L' fits this series of letters in the sense that it is conventionally taught: 'L' comes after 'K' in the alphabet. There's nothing strange or mysterious about this particular language-game. Likewise, there's nothing strange or mysterious about the way in which 'true' *fits* a proposition (at least in an ordinary sense of 'fitting'): we conventionally only apply 'true' to things we call propositions (again, notwithstanding 'true' in the sense of real, authentic, or genuine). An exercise like reciting the alphabet can help a child determine whether 'L' fits or where it fits. Similarly, an exercise like asking oneself whether one can say "is true" after it can help us determine whether something is a proposition. But grade school exercises like these are not what a philosopher is after when seeking a fundamental explanation of what a proposition *really* is. As in the other passages, these remarks help to deflate the picture on which propositions are all fundamentally or essentially the same. The application of "This is how things are: …", "… is true", or "… is false" to them might serve as a common thread amongst various kinds of propositions, but to infer from this that they are all *fundamentally* similar and that this similarity reveals their *essential* role (e.g., saying "This is how things are") would be another instance of being misled by an analogy.[46]

[46] My reading of PI 134-7 shows why the following remark from Floyd (2006) is misleading, as Wittgenstein's later remarks on "truth" are not merely a result of long-held commitments (or a mere inheritance from Kant and Frege), but instead stem from ways of thinking that are special to the PI (especially as they relate to his famous discussion of "games" in PI 66-7): "Of course, his resistance to making a definition of *truth* central to philosophy was, as we have seen, based on purely philosophical considerations that predated by over a decade his encounter with Gödel's work; in general historical terms it was a hallmark of the Kantian tradition—shared by Frege, among others—that philosophers could learn nothing important from analyzing or defining the concept of *truth*" (Floyd, 2006: 112).

1.5 'Truth', 'Assertion', and 'Proposition' in RFM I, Appendix III

Although Wittgenstein does not discuss *mathematical* propositions explicitly in PI 134-7, we find the very same considerations applied to mathematics in RFM I, Appendix III: namely, that the meanings of 'true' and 'proposition' in the context of mathematics need to be clarified by describing the language-games of mathematics in which 'true' and 'proposition' play a role. It would be a mistake to assume that these roles are exactly the same as their empirical counterparts. But Wittgenstein uses a unique maneuver in the RFM, involving the use of fictional language-games, to make his point about the misleading effects of superficial similarities among all of the things we call "propositions". Appendix III begins with an analogy to warm us up.

> It is easy to think of a language in which there is not a form for questions, or commands, but question and command are expressed in the form of statements, e.g. in forms corresponding to our: "I should like to know if …" and "My wish is that …".
>
> No one would say of a question (e.g. whether it is raining outside) that it was true or false. Of course it is English to say so of such a sentence as "I want to know whether …". But suppose this form were always used instead of the question?—
>
> (RFM I, Appendix III, 1)

This is an example of a *fictional* language-game that is designed to serve as an object of comparison – as discussed in the Introduction to these essays. In *our* ordinary language, questions and commands are distinctive forms of sentences which differ from statements in that we do not apply 'true' or 'false' to them. So, in the grade school exercise we encountered earlier in PI 137, the question "How many dogs are in the yard?" would not count as a statement or proposition because it does not make sense to place "It is true that…" in front of it. However, the fictional language-game Wittgenstein describes is one in which 'questions' and 'commands' are expressed in the form of statements like "I should like to know if …" and "My wish is that …". One *can* meaningfully place "It is true that …" in front of these. So, in *this* fictional language-game, it does make perfect sense to say of commands or questions that they are 'true' or 'false'. Thus, the fact that this does not make sense *in our language* is a product of a superficial and contingent detail about the language-games we just so happen to play.

Why does this matter for Wittgenstein's purposes? In this particular case, it shows that the misapplication of 'true' and 'false' to questions and commands is not a deep (say, metaphysical) fact about questions and commands but a superficial feature of *our* language-games of 'question' and 'command'. (Which is just to say that it is easy to imagine equally legitimate alternatives, not that it is unimportant for us to follow these conventions, once established. Compare the conventions of driving on either the right or left side of the road in the US and UK, respectively.) Likewise, it would be misguided to think that these superficial features tell us something deep about the nature of truth, e.g., that the nature or essence of truth (or propositions) determines that a question or command can never be true or false. Again, this is just a matter of our contingent language-games of 'question' and 'command'. This same point will be made about the presence (or absence) of 'propositions' in the context of mathematics, viz., that this is a contingent and superficial feature of our language-games, one which does not imply any deep facts about, say, the nature of truth and propositions. Further, the fact that the concept 'proposition' finds a home in mathematics and thereby establishes a connection with all other 'propositions' is likewise superficial and apt to mislead.

The second passage of Appendix III turns immediately to the themes of 'truth' and 'propositions' that we saw earlier in the PI.

> The great majority of sentences that we speak, write and read, are statement sentences.
>
> And—you say—these sentences are true or false. Or, as I might also say, the game of truth-functions is played with them. For assertion is not something that gets added to the proposition, but an essential feature of the game we play with it. Comparable, say, to that characteristic of chess by which there is winning and losing in it, the winner being the one who takes the other's king. Of course, there could be a game in a certain sense very near akin to chess, consisting in making the chess moves, but without there being any winning and losing in it; or with different conditions for winning. (RFM I, Appendix III, 2)

To say that, "Statement sentences are true or false", is just to say that, "The game of truth-functions is played with them". But as we saw in PI 136, this does not mean that truth or assertion is something that gets *added* to a proposition in order to make it a statement – 'true' and 'assertion' do not *fit* a proposition as independent concepts which are simply latched on, but instead *belong* to our concept of a proposition, i.e., "assertion is … an essential feature of the game we play with [propositions]" (*ibid*). And again, just as in PI 136, Wittgenstein deploys an analogy with chess to make his point. "Winning" and "losing" are essential to chess, but they are not *added* to the game, as if they were wholly independent, self-standing concepts simply latched on to it. "Winning" and "losing" *belong* to chess and are to be found in the game of chess itself, *not* in an antecedent concept of "winning". If one wants to understand what "winning" means in the context of chess, they need to learn the rules, watch the game being played, or play for themselves. We can imagine a game "very near akin to chess" that either doesn't include "winning" and "losing" or has different conditions for "winning". In either case, we will again need to learn the rules, watch the game being played, or play for ourselves if we hope to understand how the game works. Likewise, if one wants to understand what 'truth' means in the context of mathematics, one needs to either learn the rules of mathematics, examine people engaged in it, or (most importantly) do some mathematics for oneself. All that said, we can imagine a language-game "very near akin" to our mathematics that does not include 'propositions', 'truth', or 'falsity'; or with very different conditions for the application of 'truth'. To understand these games too, one will need to learn the rules, watch the game, or play it for oneself.

Wittgenstein then considers directly what we should make of the presence of 'proposition' and 'truth' in mathematics. As in the first passage regarding 'question' and 'command', Wittgenstein imagines a fictional language-game, this time of an arithmetic in which 'proposition' and 'truth' are *not* present.

> Might we not do arithmetic without having the idea of uttering arithmetical *propositions*, and without ever having been struck by the similarity between a multiplication and a proposition? (RFM I, Appendix III, 4)

So, not only is this a language-game in which 'truth' and 'proposition' do not occur, but (presumably because of this fact) one is not at all struck by the similarity between, say, a multiplication and a proposition. In other words, there would be no temptation like Hardy's or Benacerraf's: an analogy between mathematical and empirical 'truth' would not be made because their similarities are not so much as noticed. Wittgenstein provides further details about this fictional scenario that nonetheless allow us to draw out some similarities, though none that establish the kind of analogy Hardy and Benacerraf were impressed by.

> Should we not shake our heads, though, when someone showed us a multiplication done wrong, as we do when someone tells us it is raining, if it is not raining?—Yes; and here is a point of connection. But we also make gestures to stop our dog, e.g. when he behaves as we do not wish. (*ibid*)

That is, although 'true' and 'false' are not present in this language-game, there is still a practice of communicating that someone has made an error, e.g., by shaking one's head. This is similar to shaking one's head if someone says, "It's raining", when in fact it's not, and thus establishes "a point of connection" with 'proposition' and 'truth' outside of the fictional language-game of arithmetic described above. But this is cold comfort for someone wishing to show that there is a fundamental connection between multiplications (in the fictional game) and propositions, since shaking one's head is also quite similar to the gestures one might use to stop our dog from behaving "as we do not wish". In other words, shaking one's head is similar to the use of 'false' applied to propositions, but it is also similar to the non-propositional use of gestures to make our dog stop misbehaving. Thus, in this fictional language-game, there should not be so much as a temptation to think that there is a fundamental connection between multiplications and, say, empirical propositions (e.g., "It's raining"). Since the fictional language-game of arithmetic is in all other respects similar to *our* language-game of arithmetic, it is difficult to see why we should be so impressed by the mere fact that 'proposition' and 'true' appear both in the arithmetical and in non-arithmetical contexts. These are merely contingent and superficial facts about our language-games which in turn establish superficial similarities between multiplications and empirical propositions – not ones that should mislead us into thinking that they have the same nature, essence, form, or role.

But why did we think that mathematics includes (or should include) propositions in the first place? Surely – one might think – there was some feature of arithmetic that justified the use of 'proposition' therein, and thus some reason to establish such a connection with empirical propositions. Wittgenstein (partially) diagnoses the *felt* need for 'propositions' in arithmetic as follows.

> We are used to saying "2 times 2 is 4", and the verb "is" makes this into a proposition, and apparently establishes a close kinship with everything that we call a 'proposition'. Whereas it is a matter only of a very superficial relationship. (*ibid*)

The use of "is" in both contexts might suggest to us that "2 times 2 is 4" is (fundamentally) similar to "Jones is tall", "Kelly is the first woman to win a professional bowling tournament", or "There is nothing in the fridge". We might naturally conclude on these grounds that all such expressions are 'propositions' and that they have the same nature, essence, form, or role. But "2 times 2 is 4" has a very different role from, say, "Jones is tall". We learn the former by rote in grade school (it does not require the slightest thought or calculation); the latter we could only determine by taking a look at Jones and deciding, either by measurement or rough eye-balling, whether his height merits calling him "tall". More complicated arithmetic requires more careful thought and calculation. More complicated questions of height and tallness require more complicated observations and clearer standards of tallness. To think that "2 times 2 is 4" (or other equations) and "Jones is tall" (or other claims about height) are somehow fundamentally similar is to overlook all of these crucial differences between them. Thus, the presence of "is" in both is "a matter only of a very superficial relationship", *not* a fundamental similarity which would imply that they possess the same nature, form, essence, or role.

Wittgenstein rounds off his discussion of 'truth' and 'proposition' in arithmetic with a passage that is very similar to PI 136.

> For what does a proposition's *'being true'* mean? '*p*' *is true* = *p*. (That is the answer.)
> So we want to ask something like: under what circumstances do we assert a proposition? Or: how is the assertion of the proposition used in the language-game? And then 'assertion of the proposition' is here contrasted with the utterance of the sentence e.g. as practice in elocution,—or as *part* of another proposition, and so on. (RFM I, Appendix III, 6)

Since 'p' is true = p, neither 'truth' nor 'proposition' reveal the essence of the other concept – they *belong* together, but do not *fit* one another (PI 136). To understand 'truth' or 'proposition' in mathematics, we need to ask under what circumstances we assert a proposition. More generally: we need to ask how the assertion of the proposition is used in the relevant language-game. This is by contrast with simply imposing a role, function, or meaning of 'truth' and 'proposition' in mathematics by analogizing these concepts with 'truth' and 'proposition' in empirical contexts. As Wittgenstein puts it in the final passage of Appendix III, "the mere *ring of a sentence* is not enough to give these connections of signs any meaning" (RFM I, Appendix II, 20). To understand the propositions of mathematics or logic (i.e., to understand their *meanings*), we need to examine their distinctive uses and roles rather than being taken in by their surface similarities with other types of propositions.

1.6 Hardy's and Benacerraf's Pictures: The Major Diagnosis

Wittgenstein's discussions of 'truth' in general and mathematical 'truth' in particular go a long way toward deflating the pictures of mathematics offered by Hardy and Benacerraf. According to the therapeutic perspective, a primary source of Hardy's picture and others that resemble it is the assimilation of mathematical propositions with empirical propositions like, "Jones is tall", "There are at least three cities larger than New York", or "There are no apples in the fridge". Granted: these propositions sound quite similar to "3 is prime", "There are at least three perfect numbers greater than 17", and "There are no prime numbers in the set of even numbers (except 2)", respectively. We can also (by contrast with the fictional language-game described in Section 5) attach "It is true: …" to the front of each without any confusion in practice. But for all that, it would be a mistake to infer any deeper similarity on the basis of these parallels in sound and syntax.

Further, the fact that Hardy's and Benacerraf's pictures rely on such an argument from these similarities (superficial, by Wittgenstein's lights) to a deeper similarity in role, form, function, or nature shows a subtle movement in thought that is quite easy to overlook, yet, once it is pointed out, is obviously suspect. One person might observe the similarities between propositions noticed by Hardy and Benacerraf and think, "That's impressive! This must reveal a fundamental similarity", and yet another might (either independently or upon reading Wittgenstein's remarks) respond, "So what? We use the word "true" in both contexts – why is this supposed to lead us to an exotic metaphysical picture of mathematics?". Putting the therapeutic perspective aside for just a moment: an adequate philosophical defense of Hardy's or Benacerraf's pictures would have to tell us why one reaction is somehow privileged over the other. But hopefully, if Wittgenstein's therapeutic strategy has had any effect, by this point one will feel the push toward a quite different attempt at gaining clarity: to describe in some detail how the various things called "propositions" are actually used, without any presumption that they are somehow fundamentally similar in their nature, form, role, or function.

Although Benacerraf's paper came many years after Wittgenstein's discussion of Hardy in the LFM, it is easy to see that his distinctive contributions to the discussion of mathematical 'truth' and 'reality' are equally vulnerable to Wittgenstein's therapeutic strategies. Benacerraf argues that a uniform theory of truth, such as that provided by Tarski, shows that mathematical sentences have the very same kinds of truth conditions as empirical sentences. I repeat his examples here for clarity.

(1) There are at least three large cities older than New York.

(2) There are at least three perfect numbers greater than 17.

Sentences (1) and (2) have the very same (syntactic) form, namely:

(3) There are at least three FG's that bear R to a.

We are meant to infer from this similarity in syntactical form that (2), just as much as (1), says something about a collection of objects. However, the objects referred to in (2) – "out there" in some sense – are beyond our causal grasp. But why should we infer that (2) refers to objects *in the very same way* that (1) does? Or, similarly, why should we think that the "objects" referred to in (2) are somehow similar to the objects of (1) – i.e., as being "out there", yet beyond our causal reach? (As if a number were like an apple or a chair, but somehow ghostly and transparent – passing through all things without touching them.) Benacerraf's argument is that an adequate theory of truth shows them to have the very same kind of truth conditions.

Wittgenstein's response however would be the following: you've pointed out an interesting syntactic and phonetic similarity between (1) and (2), but for all that, you haven't given us any reason to think these propositions are therefore exactly the same, either in "saying how things are", or "referring to objects *in the very same way*", or "implying an independent reality "out there" for us to "discover"". A similarity in syntax does not all on its own reveal any deeper similarity, that is, not unless there is somehow a fundamental similarity between the roles or uses of these sentences.

Their roles or uses, however, are quite different. We can venture out to study the various cities of the world – whether by foot, car, boat, or plane – study historical records in order to determine their ages by some acceptable standard, collect our data, and make a report about the findings, "OK, given our findings, there *are* at least three large cities older than New York". By contrast, as was mentioned earlier, grappling with (2) will require calculation or proof (or, again, the reliance on someone else's work): it certainly won't require "venturing out", studying historical records, or doing anything that can't be performed on a blackboard or with pencil, paper, and a decent calculator. It might involve programming a computer to go through the series of numbers for us to determine whether a perfect number has been found – and however "experimental" that might be, it is still quite different from determining whether (1) is true. These clear and obvious differences in roles make it difficult to motivate the assumptions that Benacerraf's argument requires, namely, (i) that empirical and mathematical propositions are true in the very same sense of 'truth' and (ii) that this sense is captured fully by Tarski's formalism. Again, there is indeed a similarity in syntax, but the differences between the uses of these propositions make it difficult to see why an exotic mathematical reality, somehow resembling empirical reality, should be inferred from such similarities.

It is quite important, however, to recognize that Wittgenstein's aim is not simply to remove from our language the expressions, "(1) and (2) have similar truth conditions", or "(1) and (2) refer to objects", or "(1) and (2) both (in some sense) correspond to reality".[47] After all, these are just English sentences – ones which might have, or might be given, a perfectly ordinary and acceptable use in practice. The first sentence, "(1) and (2) have similar truth conditions", for instance, might just be a way of saying that (1) and (2) have the syntax represented by (3). Wittgenstein doesn't need to dispute this trivial and obvious fact. Compare Wittgenstein's famous remark, "Say what you please, so long as it does not prevent you from seeing how things are" (PI 79). The threat, then, is in being misled by such expressions in such a way that we overlook clear differences between the uses of mathematical and empirical propositions; thereby being prevented "from seeing how things are". Such a threat is explicitly brought up in Benacerraf's paper and thus requires our attention: namely,

[47] Compare Mülholzer's (2014) reading of the PI on which Wittgenstein does not deny that we "refer" to numbers, but that an understanding of such "reference" must be sensitive to the distinctive uses of mathematical expressions.

an inference from the syntactic similarities of empirical and mathematical propositions to a special epistemological problem for mathematics.

Recall that Benacerraf concluded from the similarity in truth-conditions between (1) and (2) (i.e., their syntactical similarities as emphasized in the Tarskian framework) that they both equally refer to objects (and in the same sense of 'object'), yet the objects referred to in (2) are beyond our causal reach and thus create a special problem about how they can so much as be known. So, the invocation of, "(1) and (2) have similar truth conditions", in Benacerraf's paper immediately leads to philosophical questions and problems (e.g., "What *is* a mathematical object *really*?"[48], or "How is knowledge of a mathematical theorem *possible*?"). These questions and problems, in turn, rely on overlooking the crucial differences between (1) and (2) – or, more specifically, overlooking their differences *at the wrong time* in this movement of thought, as I will explain now.

For consider that one could completely flip Benacerraf's reasoning around. (We discussed a similar strategy earlier in Section 2.) (2) is 'known' in an entirely different manner than (1) – that is, the circumstances in which we say that we 'know' (2) are very different from the circumstances in which we say that we 'know' (1). 'Knowing' (1) requires *literally* venturing out, studying documents, and reporting the results found (or otherwise relying on some folks who have done this work). 'Knowing' (2) requires either proof or calculation; it certainly doesn't require venturing out, pointing to objects in the literal sense of 'pointing', making certain observations with our eyes or ears, studying historical documents, and so on. These are obvious differences between the uses of 'know' with respect to each of (1) and (2). Given these obvious differences, why should we think that their truth-conditions (in some sense independent of their syntax) are "fundamentally the same"? More specifically, given that 'knowing' (2) doesn't involve anything like literally venturing out or literally

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[48] Which is pursued most directly in Benacerraf (1965), his other classic in the philosophy of mathematics.

pointing to objects, etc., why should we think that each of these sentences refers to objects in the very same manner? It seems that the sole motivation of Benacerraf's picture is the syntactic similarity between (1) and (2), codified and made explicit by the Tarskian framework. But those are just similarities in sound and syntax – they shouldn't lead us to think that there is some special problem of knowledge for basic arithmetical propositions. Of course, Benacerraf *does* recognize crucial differences between mathematical and empirical propositions, albeit not early enough in his reasoning to disrupt the picture of reality that is thereby concocted. He *first* infers a fundamental similarity, i.e., that (1) and (2) are true *in the very same sense* (since they both have the form of (3)). Only after this does he note *crucial differences*, e.g., that one does not literally look at or point to, say, the number 2, which then creates a special epistemological problem ("since I don't have causal access to the number 2, how can I *know* anything about it?"). Wittgenstein's strategy is to shift the order of considerations here:[49] one does not literally look at or point to, say, the number 2 when determining whether '2 + 2 = 4'; so it would be highly misleading to think that 'Jones is tall' and '2 + 2 = 4' are true *in the very same way*. They involve distinctive uses of 'true' which take part in distinctive language-games. Likewise, they involve distinctive uses of 'know'. So much is obvious once we examine the differences between their uses or roles in ordinary life.

A defender of Benacerraf might point out that in his paper he is also concerned to undermine attempts to identify 'truth' with proof-conditions or justification-conditions more

[49] Compare Wittgenstein's remark that language becomes "*surveyable* through a process of ordering" (PI 92). The *order* in which we put forth ordinary considerations is thus crucially important for Wittgenstein, which is not to say that there is a *single* order, but rather, "an order for a particular purpose, one out of many possible orders, not *the* order" (PI 132). Benacerraf notes *some* obvious facts about the role of numbers and mathematical propositions (numbers are not 'seen' – at least not in the sense that one 'sees' a tree), albeit in an order that misleads and encourages one to overlook other obvious facts (one does not, in any ordinary sense, need to 'see' or otherwise be in a causal relation with the Fundamental Theorem of Arithmetic to 'know' that it is 'true'). The purpose of Wittgenstein's re-ordering of considerations is thus to diagnose the errors in this line of thought and emphasize distinctions that have been overlooked in the process: "For this purpose we shall again and again *emphasize* distinctions which our ordinary forms of language easily make us overlook", that is, without making it, "our task to *reform* language" (*ibid*, emphasis added), but instead to *describe* it.

generally.[50] It might seem thus far that Wittgenstein is doing just this. But the line of thought considered throughout this paper has nothing to do with *identifying*, say, the 'truth' of mathematical claims with their justification-conditions. After all, 'true' and 'proposition' are considered by Wittgenstein to be conceptually on a par – one does not stand independently and somehow illuminate the other all on its own. Instead, we need to examine the language-games to which they both equally belong; what is important above all is how 'true' and 'proposition' are *used* in mathematics. It thus would not make much sense for Wittgenstein to seek a "theory of truth" in the first place, such as one according to which truth is identified with proof or justification. Just as 'true' and 'proposition' belong to the language-games of arithmetic, so do 'know', 'certain', 'justified', and the like. Wittgenstein is merely pointing to obvious differences between these concepts as they play out in arithmetical and non-arithmetical language-games (such as determining how many cities are older than New York). For all that Wittgenstein says, there might very well be conditions for 'knowing' in some language-games that do *not* require any proof or calculation whatsoever. Our 'knowledge' that 2 + 2 = 4 would be an obvious example in ordinary life, since it is learned by rote and never requires further justification. (Notwithstanding mathematical logicians who play a distinctive language-game in which this too requires proof, a language-game in which distinctive rules for 'knowing' and 'justification' are deployed). Attention to the obvious differences between mathematical and empirical propositions disrupts the analogy that Hardy and Benacerraf rely on in their invocation of a special mathematical reality.

[50] See especially Benacerraf's (1973: 665) discussion of what he calls "the combinatorial view of truth", on which, "the truth conditions for arithmetic sentences are given as their formal derivability from specified sets of axioms", a view which Benacerraf claims was, "torpedoed by the incompleteness theorems". As I explain above, there is no reason to attribute such a view to the later Wittgenstein. See especially Shanker (1988) and Floyd (1995) for discussions of Wittgenstein's later stances on both formalism of the sort Benacerraf criticizes here as well as the incompleteness theorems. In short, since Wittgenstein does not endorse the former, the latter does not present a special problem for his later views. More generally, the discussion of this paper shows that Wittgenstein had no interest in offering a unified theory of truth that would simply identify it with the "formal derivability from specific sets of axioms", since such a theory is bound to overlook the diverse uses of 'truth' and 'propositions' in various kinds of language-games.

In short, then, Wittgenstein's approach is entirely different from the proof-theoretic strategy Benacerraf considers and rejects in his paper and does not fall prey to his arguments against it. His approach is also not disrupted by the introduction of Tarski's formal work on 'truth' (the novel contribution of Benacerraf) into this dialectic. Tarski's formal work systematizes and makes explicit syntactic similarities across various different kinds of propositions; it does not thereby (without confusion) imply a distinctive and mysterious reality, somehow akin to empirical reality, to which mathematical claims refer. To be clear, this does not require a rejection of Tarski's *mathematical* work, it merely challenges Benacerraf's application of Tarski's formal framework and his attempt to show that it leads to special epistemological and metaphysical problems for mathematics.[51]

[51] This is central to Wittgenstein's strategy in the philosophy of mathematics quite generally, as is nicely stated in RFM II: "What I am doing is, not to show that calculations are wrong, but to subject the *interest* of calculations to a test. I test e.g. the justification for still using the word … here. Or really, I keep on urging such an investigation. I show that there is such an investigation and what there is to investigate there. Thus I must say, not: "We must not express ourselves like this", or "That is absurd", or "That is uninteresting", but: "Test the justification of this expression in this way". You cannot survey the justification of an expression *unless you survey its employment* [emphasis added]; which you cannot do by looking at some facet of its employment, *say a picture attaching to it* [emphasis added]" (RFM II, 62). Wittgenstein's therapeutic perspective thus allows us to challenge what Benacerraf takes to be the *interest* of Tarski's mathematical work and the use of a certain expression to interpret that work, e.g., "This shows that mathematical propositions refer to objects *in the very same way* that empirical propositions do." The error is in being seduced by a "picture attaching to" Tarski's framework, rather than the deployment of that framework and the sentences to which it is applied. See Floyd (2001) for similar arguments against the philosophical significance of Tarski's work on truth from the perspective of later Wittgenstein.

Chapter 2

Crossing Pictures of 'Determination'

> When you get the picture of "being determined" out of your mind, then you get rid of the puzzle. – But still one can say the algebraic expression determines his actions – and perfectly correctly. But now you have got rid of the cramp. (LSD 24)

2.1 Introduction

In PI 188, Wittgenstein addresses a certain conception, adopted by an imagined interlocutor in the text, of one's meaning something or other by a simple arithmetical rule such as '+2'.

> Here I'd like to say first of all: your idea was that this *meaning the order* had in its own way already taken all those steps: that in meaning it, your mind, as it were, flew ahead and took all the steps before you physically arrived at this or that one. (PI 188)

For instance, on this conception, when a teacher orders their student to count through the series 0, 2, 4, 6, 8, 10, and so on, the teacher's meaning this particular series as opposed to some other consists in their mind taking all of these steps before they or the pupil speaks or writes down any particular number in that series. This conception leads the interlocutor to describe the process of teaching and learning such a rule in a rather surprising way.

> So you were inclined to use such expressions as "The steps are *really* already taken, even before I take them in writing or in speech or in thought". And it seemed as if they were in some *unique* way predetermined, anticipated – in the way that only meaning something could anticipate reality. (*ibid*)

The interlocutor's expression here is an instance of what Wittgenstein soon after calls being, "seduced into using a super-expression. (It might be called a philosophical superlative.)" (PI 192). Part of the "super-ness" of this expression comes through in the emphasis "*really*", i.e., it is not enough for the interlocutor to merely say that "The steps are already taken", but that instead "The steps are *really* already taken". This emphasis signifies that there is something powerful and mysterious happening when one gives the order to add 2 ("as if [the steps] were in some *unique* way predetermined"), by contrast with the mundane fact that someone or other has (of course, many people have) already counted through this series (though not *all* of it!) before the student goes on to do so (c.f., RFM I, 22). Instead, it is assumed that *the teacher's mind* has taken *all* of these steps in advance once they understand '+2' and mean something specific by it in giving the order.

It is not difficult to see why such an idea would seem perplexing to the interlocutor, since if the mind *really* does take all these steps in advance, then our presumably finite minds are capable of containing infinite quantities and performing an infinite number of steps "in a flash", or as the interlocutor puts it, "It is as if we could grasp the whole use of the word [or rule] at a stroke" (PI 191). The exotic nature of such an idea should give us pause. Where does this idea and its attending expressions come from? Why is the interlocutor tempted to use such expressions in describing something as mundane as teaching and learning the simple rule '+2'? According to Wittgenstein,

> [T]his mode of expression suggests itself to us [e.g., "It is as if we could grasp the whole use of the word at a stroke" or "The steps are *really* already taken"]. As a result of the crossing of different pictures. (*ibid*)

These expressions are thus not to be taken with much authority, but rather seen as the product of "crossing different pictures". The interlocutor's idea – which might have inspired one to seek out a philosophical or scientific theory as to how the mind can perform such profound actions – is instead diagnosed by Wittgenstein as the product of a certain kind of confusion. They have been, "*seduced* into using a super-expression" (PI 192, my emphasis).

However, it might seem (to the interlocutor or anyone sympathetic with them) that Wittgenstein is here *denying* the very possibility of meaning something by a rule or of a rule's determining its applications.[52] In other words, it can seem as if the phenomenon of following a rule

[52] Compare Kripke (or, rather, "Kripkenstein"): "There can be no such thing as meaning anything by any word. Each new application we make is a leap in the dark; any present intention could be interpreted so as to accord with anything we may choose to do. So there can be no accord, nor conflict. This is what Wittgenstein said in §201" (1982: 55). See also Fogelin (1987) and Wright (1980). The general philosophical problem that arises here (with varying degrees of connection to Wittgenstein's own discussion of it) has been a matter of continuing discussion. See especially Miller &

just is mysterious, crying out for a special kind of explanation, and that if Wittgenstein diagnoses such an idea as confused, he is denying the very possibility of following a rule. This is why the interlocutor asks him, after Wittgenstein begins to note the "expressions" that the interlocutor is (merely) "inclined to use": "But are the steps then *not* determined by the algebraic formula?" (PI 189) That is, if Wittgenstein is not willing to say along with the interlocutor that, "The steps are *really* already taken", then it seems he is controversially denying that the steps are determined by the algebraic formula (e.g., '+2') intended by the teacher. "But the formula *must* determine these steps!", thinks the interlocutor, since to say otherwise would imply that one could legitimately count in whatever way one likes when given the rule, or that there is nothing whatsoever to explain why folks go on as they do when they are given the order to follow it. Wittgenstein responds, however, that, "The [interlocutor's] question contains a mistake" (PI 189).

What is the mistake "contained in the question", according to Wittgenstein? Wittgenstein's intentions in PI 189 are not exactly transparent.[53] Rather than saying explicitly what is the "mistake" in the question, he instead transitions into a discussion of how one might use the expression "The steps are determined by the formula . . .". On the one hand, Wittgenstein suggests, it can be used to describe how people react or respond to an arithmetical formula such as $y = x^2$. On the other hand, it can be used as a way of categorizing different kinds of formulae, e.g., to differentiate formulae that 'determine' a value of y given x (e.g., $y = x^2$) from those that do not (e.g., $y \neq x^2$). But how exactly are these uses of the expression meant to illustrate the "mistake" contained in the interlocutor's question – especially if the interlocutor had an entirely different use of 'determine' in mind? (If that

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Wright (2006) and Kusch (2006) for more recent discussions of the Kripkensteinian paradox. For a variety of reactions to Kripke's exegesis of Wittgenstein, see Baker & Hacker (1984), Goldfarb (1985), Cavell (1990), Floyd (1991), Minar (1991, 1994), Steiner (1996), and Wright (2001, Ch. 7).

[53] As noted by Floyd (1991: 151).

were so, Wittgenstein's elaboration would appear to be a complete non-sequitur.) Wittgenstein apparently left this matter as an exercise for the reader (c.f., "I should not like my writing to spare other people the trouble of thinking" (PI, Preface)).

Juliet Floyd (1991: 155-157) has proposed that PI 189 should be read in the following manner. The interlocutor presumes that they understand what they mean by 'determine' when they ask, "But are the steps then *not* determined by the algebraic formula?". In fact, there is no antecedently fixed notion of 'determination' (or of 'necessity' as it is used to describe the truths of logic or mathematics). When Wittgenstein lists some possible uses of 'determine', he is placing a burden on the interlocutor to explain, more precisely, what they have in mind. At the moment, however, this is not at all clear. Until the interlocutor can tell us more precisely what it means for a rule to 'determine' its applications, Wittgenstein cannot say one way or another whether this is something he would deny. This is disruptive of a tempting (and not uncommon) philosophical trajectory that assumes there is a unified notion of 'determination' or 'necessity' in mathematics that is amenable to philosophical theorizing.[54] The "mistake" is thus, on Floyd's reading, that the interlocutor presumes they mean something specific by "The steps are determined by the formula …" when in fact they do not. (Again, the more general point being that philosophers tend to theorize mathematical or logical 'necessity' or 'determination' as if they had something clear in mind at the outset.) They can select amongst one of the uses Wittgenstein has offered, or (if none of those will suffice) they should tell us more clearly what they have in mind.

This reading saves Wittgenstein's elaboration in PI 189 from being a non-sequitur, though it does make his case against the interlocutor rather indecisive. Such indecision does not, on the face

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[54] Although not uncommon, according to Floyd, "Frege, Moore, and Russell had [also] each rejected as unclear the notion of *necessity*, replacing it with that of *universality*" (Floyd 2006: 84). But for the later Wittgenstein, to say that a concept (like 'game') is not amenable to philosophical theorizing does not imply that there's something *wrong* with the concept or that it needs to be replaced; rather, there is something wrong with the inclination to submit that concept to strict definition or systematic theorizing.

of it, seem to match the tone we find in the succeeding remarks (or perhaps even of the accusation that the interlocutor's question contains a "mistake"). Here are some examples to illustrate.

> You are *seduced* into using a super-expression. (PI 192, my emphasis)
>
> When does one have the thought that a machine already contains its possible movements in some mysterious way? – Well, when one is doing philosophy. (PI 194)
>
> "[I]n a strange way, the use itself is in some sense present." – But of course it is, 'in *some* sense'! Really, the only thing *wrong* [my emphasis] with what you say is the expression "in an odd way". (PI 195)
>
> In *misunderstanding* [my emphasis] the use of the word, one takes it to signify an odd *process*. (As one thinks of time as a strange medium, of the mind as an odd kind of being.) (PI 196)

And most dramatically,

> Though we do pay attention to the way we talk about these matters, *we don't understand it*, but *misinterpret it*. When we do philosophy, we are like savages, primitive people, who hear the way in which civilized people talk, *put a false interpretation on it*, and then draw the oddest conclusions from this. (PI 194, my emphasis)

Given that in PI 188, Wittgenstein is criticizing the interlocutor's "idea" that, "meaning the order has in its own way already taken all those steps", it seems most natural to see Wittgenstein as committing his interlocutor to something *specific enough* to be labelled in the following ways: a "result of crossing pictures", encouraging a "super-expression", or a "philosophical superlative", a "misunderstanding of the use of a word" which inclines them to take it as signifying "an odd kind of process", and, last but certainly not least, a "false interpretation" put on the talk of "civilized people" from which the interlocutor draws "the oddest conclusions". Thus there is *some* idea here (or a thought, or a picture) that the interlocutor has in mind, however vague it may be; an idea that inspires him to seek out a deeper explanation of the allegedly mysterious phenomenon of following a rule.[55] Wittgenstein seeks to remove, undermine, or dissolve this idea by diagnosing it as a product of confusion, i.e., a result of "crossing different pictures".

This is not to suggest against Floyd's reading that the "idea" the interlocutor has in mind is clear enough, by Wittgenstein's lights, to justify initiating a systematic theory of mathematical 'determination' or 'necessity'. Floyd is completely right that Wittgenstein would deny this and it is entirely plausible that he would criticize a philosopher on such grounds. For instance, to go beyond Floyd's suggestion a bit, it is very natural in this context to invoke the Wittgensteinian staple that 'determine' (among most other concepts in ordinary language) is a "family-resemblance concept", not a rigidly unified category that adheres to strict rules and boundaries (c.f., PI 66-7).[56] If that were so, then one is better off describing examples of 'determination' and noting likenesses (as well as differences) between them ("All *explanation* must disappear, and description alone must take its place" (PI 109); "Philosophy must not interfere in any way with the actual use of language, so it can in the end only describe it" (PI 124)).[57] I won't deny, then, that the considerations Floyd raises are

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55 Compare McDowell's reading of PI 195: "What this suggests is something we might anyway have expected: that Wittgenstein's target is not the very idea that a present state of understanding embodies commitments with respect to the future, but rather a certain seductive misconception of that idea" (McDowell, 1998: 223).

56 As Floyd (2021) puts it, ""Family resemblance" characterizes the generality of certain concepts. A single property, a fixed-for-all-cases criterion, an explicit set of grammatical rules – these are not required. A concept may hold together – like a family – with a variegated, evolving series of properties" (Floyd, 2021: 51).

57 McDowell (1998) rightly accuses Kripke (1982) of failing to note the latter parts of PI 201, which reveal that Wittgenstein intends to diagnose a misunderstanding about rule-following rather than to deny the very possibility of it (McDowell, 1998: 229). But I think McDowell makes a similar mistake related to Wittgenstein's emphasis on "description" above. According to McDowell, the most crucial detail of PI 201 (missed by Kripke) is the following: "The right response to the paradox, Wittgenstein in effect tells us, is not to accept it but to correct the misunderstanding on which it depends: that is, to realize "that there is a way of grasping a rule which is *not* an *interpretation*"" (*ibid*). This removes the very last clause – "but which is exhibited in what we call "obeying the rule" and "going against it" *in actual cases*" (PI 201) – which illustrates a shortcoming of both Kripke's and McDowell's readings: a failure to clarify matters by describing actual cases in which we say of someone that they are "obeying a rule" (a shortcoming I hope to avoid in later sections). By contrast with Wittgenstein's descriptive aim, McDowell (1998) sets out to generally characterize rule-following (or what makes it possible) in such a way that "a familiar intuitive notion of objectivity" (McDowell 1998: 222) is preserved, which Wittgenstein nowhere in the PI expresses any interest in. Pears (1988) likewise notes the fundamental importance of description for Wittgenstein's later conception of philosophy: "The important thing is that he is moving it away from theorizing and towards plain description of the phenomenon of language. The description is intended to make us see how our own linguistic devices work, simply by putting them in their place in our lives without using any technical terms" (Pears, 1988: 218-19).

the *sorts of criticisms* Wittgenstein would have issued against philosophers of mind or mathematics, and I myself will appeal to some of these considerations in developing the reading that follows.[58]

However, I will offer a reading of PI 189 on which the intended critique goes somewhat deeper than this. Although the interlocutor's idea is vague and unsystematic, it is surprising and exotic enough to invoke philosophical perplexity of the very sort that Wittgenstein would like to diagnose and remove ("The philosopher treats a question; like an illness" (PI 255)). It is the sort of "picture" that holds one "captive" while they are doing philosophy (PI 115). To reveal the confusions inherent in such an idea is thus apt to diminish the interlocutor's motivation to theorize 'meaning' something or other by a rule – it removes the sort of wonder that Aristotle famously regarded as being the root of philosophy: "[I]t is because of wondering at things that humans, both now and at first, began to do philosophy" (*Metaphysics,* 982b).

The rest of the paper will proceed as follows. In the next section, I will illustrate more generally what Wittgenstein means by "crossing pictures" and how it factors into his therapeutic conception of philosophy. In Section 3, I will take a close look at what I call "the machine analogy" in PI 193, which is meant to shed light on the interlocutor's earlier confusion about 'determination'.

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[58] The considerations that Floyd attributes to Wittgenstein suffice to explain why Wittgenstein is not a "conventionalist" as Dummett (1978) characterizes the view. "Conventionalism", as much as "platonism", assumes that "necessity" is a unified phenomenon which allows for a general account such as the following: "all necessity is imposed by us not on reality, but upon our language; a statement is necessary by virtue of our having chosen not to count anything as falsifying it" (Dummett, 1978: 169). But, as noted by Floyd, Wittgenstein would have denied that there is a unified phenomenon which would allow for such a systematic account. This should be borne in mind when I highlight Wittgenstein's appeals to "conventions" and "stipulations" later in Sections 3 & 4. See Baker (1988: 259) for a similar assessment of Dummett's reading of Wittgenstein. On a separate note: It is surprising, given the considerations Floyd raises here, that in later work she seems to treat "the necessities of logic and mathematics" as unified phenomena which allow for the following theoretical generalizations: "In Wittgenstein's later philosophy, [we find] a naturalistic, evolving web of contingencies on which ride the necessities of logic and mathematics. However, it is important that pointing to any particular natural fact will not explain or justify the development of a mathematical technique *as* the technique that it is" (Floyd, 2021: 57). The reader should bear in mind that the considerations I agree with above may not express Floyd's current view (either on later Wittgenstein or in general).

Lastly, in Section 4, I will use the machine analogy to explain how the interlocutor's "mistake" in PI 189 is a mistaken conception of 'determination' that results from crossing different pictures.[59]

2.2 'Crossing Pictures' in Wittgenstein's Therapeutic Method

The reading I will provide gives pride of place to Wittgenstein's distinctive methodology for dealing with philosophical problems, which I refer to as his "therapeutic conception of philosophy". This 'conception' is not a theory or description of all the things we might happen to call "philosophy", but Wittgenstein's own radical conception of how philosophy should be done – "radical" in that it disrupts the traditional philosophical mode of treating certain questions as innocent and answering them directly by way of an account, definition, or theory. By contrast, on Wittgenstein's conception of philosophy, philosophical questions are themselves an object of suspicion and require an investigation of their sources without any aspiration to theory. Philosophical questions are treated "like an illness" and thus submitted to diagnosis and therapy. Wittgenstein's general 'diagnosis' of philosophical problems is that they stem from misunderstandings about the uses of words due to one's lacking a proper overview of our language.

According to Wittgenstein, one major aspect of language that encourages such misunderstandings is the apparent similarity between different kinds of words; confusions arise when they are assimilated despite important differences between their uses. In the PI, we see that Wittgenstein took this sort of phenomenon – viz., confusions stemming from the assimilation of two apparently uniform expressions – to be one of the general sources of philosophical confusion (PI 11, 90, 112). Despite differences in usage, the uniform appearance of certain words encourages

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[59] The importance of "crossing pictures" for the diagnosis of the interlocutor's mistake is noted briefly in Baker & Hacker's exegesis (Baker & Hacker, 2009: 103-110). The reading offered in this paper can be seen as an elaboration and explanation (with some minor departures from Baker & Hacker) of how such a diagnosis is meant to work in these texts.

us to assimilate expressions in a way that leads to misbegotten pictures, problems, and questions. These assimilations in usage are also referred to as instances of "crossing pictures" (in some translations "crossing similes"), especially in cases where disparate uses of the same word are naturally associated with vivid images or ways of speaking. Wittgenstein provides many examples of this phenomenon as they occur outside philosophy in order to (more surprisingly) illustrate what he takes to be a common source of philosophical perplexity. – I will offer just three of his examples for illustration.

The first comes from PI 195 in the course of his remarks on rule-following.

> Someone once told me that as a child he had been amazed that a tailor could 'sew a dress' – he thought this meant that a dress was produced by sewing alone, by sewing one thread on to another. (PI 195)

We can imagine the child in this example asking in great astonishment, "How can you possibly *sew* a dress?". The source of the child's question is a confused picture: the child is supposing that, "a dress [is] produced by sewing alone, by sewing one thread on to another" (PI 195). Their astonishment is diminished by showing them that this is not how "sewing a dress" is to be understood in this context – fabrics are produced via weaving or other means and then used as material for sewing with needle and thread. The child might still like to see how one sews a dress, but their curiosity would then lack the sense of mystery and astonishment that it previously had; the sense that sewing a dress amounted to doing something either spectacular or impossible. The source of perplexity in this case arises from crossing two ever so slightly different uses of 'sewing', viz., 'sewing' understood as something one does with needle and thread by contrast with 'sewing' understood as a general project that one can "start from scratch" (e.g., one can "sew a dress from scratch"). In other words, the child has a picture of sewing, viz., as something one does with needle and thread, but illicitly projects this picture onto the action of 'sewing a dress' – thus thinking that this is done with nothing other than needle and thread (i.e., without pre-made fabric). According to Wittgenstein, then, the sentence ‘so-and-so is sewing the dress’, “seems odd only when one imagines it to belong to a different language-game from the one in which we actually use it” (PI 195).

The second example comes from Wittgenstein’s LFM. In this example, someone is shown a picture of a line sitting next to a circle (e.g., / O) and told, “This line cuts the circle at imaginary points”. Wittgenstein is reported to have said that this example, “has a certain charm, now only for schoolboys and not for those whose whole work is mathematical” (LFM 16). The sense of mystery or perplexity (the “charm”) is dissolved when one points out that the use of “cut” here departs from a common usage of “cut”: it is a matter of applying the equation of the circle over the domain of complex numbers, not a matter of the line “going through” the circle in the visual sense of cutting. The charm (or perplexity or astonishment) is a result of mixing these uses together, i.e., an example of what Wittgenstein refers to as “a crossing of different pictures”. In the LFM, he provides the following diagnosis, noting that this is a “kind of misunderstanding” (albeit not of a particularly important or threatening sort):

> “Cut” has the ordinary meaning: ∅ [by contrast with / O]. But we prove that a line always cuts a circle – even when it doesn’t. Here we use the word “cut” in a way it was not used before. We call both “cutting” – and add a certain clause: “cutting in imaginary points, as well as real points”. Such a clause *stresses a likeness*. – This is an example of the assimilation to each other of two expressions. (LFM 16)

To use the idiom of “crossing pictures”, any perplexity about how a line could cut a circle at imaginary points stems from crossing this technical use of “cut” with the ordinary picture of a line going through a circle. One gets confused about the sense of “cutting at imaginary points” if one assumes that all “cutting” must adhere to the ordinary picture of a line going through a circle.

The last example comes from Wittgenstein’s discussion of ‘simples’ and ‘composites’ in the PI.

> Asking "Is this object composite?" *outside* a particular game is like what a boy once did when he had to say whether the verbs in certain sentences were in the active or passive voice, and who racked his brains over the question whether the verb "to sleep", for example, meant something active or passive. (PI 47)

The boy in this example is confused because they are crossing different uses or pictures of "active" and "passive", roughly: the grammatical[60] senses of these terms with what we might call their physical senses. "To sleep" has the grammatical appearance of the active voice, but the child is perplexed because *sleeping itself* is a passive state – one is surely not *doing anything* when they sleep! Perhaps the child is comparing the verb "to sleep" with the verb "to run", where "to run" is paradigmatically active in both the physical and the grammatical senses – they get confused because sleeping does not bear the physical qualities of running. Their confusion is resolved by distinguishing these disparate uses or pictures of "active" and "passive" (i.e., the grammatical from the physical senses) and noting that their confusion arose from illicitly crossing them. They simply need to be instructed or reminded that "active" and "passive" play special roles in the language-game of grammar, which shouldn't be confused with other language-games in which these words might appear.

In each of these examples, confusions arise as a result of crossing different uses or pictures of a word, which is encouraged by their superficial similarities. For instance, in each case "the same word" ('sew', 'cut', 'active') is used and thus misleads one into crossing disparate senses of that word. Confusion is resolved, and perplexity diminished, when one distinguishes these uses and notes that the subject's initial state of perplexity depended on illicitly crossing them. Although these examples all pertain to basic errors that might be made by schoolchildren, Wittgenstein's point is that some of the errors made by philosophers are comparable: that philosophical perplexity can likewise rest on illicitly crossing disparate uses of a word or pictures naturally associated with it (i.e., "imagining it to

[60] Not in Wittgenstein's idiosyncratic use of "grammar", but in the more typical sense.

belong to a different language-game from the one in which we actually use it" (PI 195)). As we saw earlier, Wittgenstein notes in the remarks on rule-following that his interlocutor is, "seduced into using a super-expression", or, "philosophical superlative" (PI 192), "As a result of the crossing of different pictures" (PI 191). In the rest of the paper, I will attempt to explain how this might be so, and how it reveals a crucial aspect of Wittgenstein's attribution of a "mistake" to the interlocutor's question in PI 189.

2.3 The Machine Analogy in PI 193

However, something that seems to upset my reading of PI 189 as an instance of "crossing pictures" is that Wittgenstein does not mention this at all in the passage. As we noted earlier, he simply lists some uses of the expression "The steps are determined by the formula . . ." and then pithily mentions that, "it is not clear offhand what we are to make of the question, "Is $y = x^2$ a formula which determines y for a given x?"". This seems to support Floyd's reading that Wittgenstein's only intention here is to show that the interlocutor does not have anything clear in mind, thus placing a burden on them to explain more precisely what they mean by 'determine' in this context.

An indication that there is something more to Wittgenstein's attribution of a "mistake" to the interlocutor comes from the fact that Wittgenstein's discussion of the word 'determine' does not end in PI 189, but is taken up explicitly in PI 193. In this later passage, Wittgenstein discusses, "A machine as a symbol of its mode of operation" and compares it with what we might call an "actual" (say, physical) machine. PI 193 is nestled into Wittgenstein's remarks on rule-following and more specifically the idea that the meaning of a word determines its applications. It is thus highly unlikely that Wittgenstein presented his machine example for its own sake – rather, the example is presumably an analogy that is meant to shed light on the interlocutor's perplexity about rule-following. The intended connection between PI 189 and PI 193 is also clearly indicated by the fact that immediately following Wittgenstein's discussion of the machine, his interlocutor responds with the following.

> "But I don't mean that what I do now (in grasping the whole use of a word) determines the future use *causally* and as a matter of experience, but that, in a *strange* way, the use itself is in some sense present." (PI 195)

Further, given that PI 193 explores the idea that, "if we know the machine, everything else – that is the movements it will make – seem already completely determined", and that, "The machine seems already to contain its own mode of operation", it is plausible to assume that Wittgenstein here intends us to compare this example with the interlocutor's earlier idea that, "The steps [following from the rule '+2'] are *really* already taken", as if they were, "in some *unique* way predetermined" (PI 188). For these reasons, I will refer to the example in PI 193 as "the machine analogy", i.e., an analogy that is meant to shed light on the "mistake" contained in the interlocutor's question of PI 189. In this section I will focus on the machine analogy. In the next section I will show how this indeed sheds light on the interlocutor's earlier mistake alluded to in PI 189.

First, let's get clear on what the example in the machine analogy is supposed to be an example of. Wittgenstein opens the passage as follows.

> A machine as a symbol of its mode of operation. The machine, I might say for a start, seems already to contain its own mode of operation. What does it mean? – If we know the machine, everything else – that is the movements it will make – seem to be already completely determined. (PI 193)

Wittgenstein has us consider "a machine as a symbol of its mode of operation", or, put another way, a machine as a symbol of the way it operates. What does Wittgenstein mean by this? Wittgenstein elaborates somewhat in the next paragraph.

> We use a machine, or a picture of a machine, as a symbol of its mode of operation. For instance, we give someone such a picture and assume that he will derive the successive movements of the parts from it. (Just as we can give someone a number by telling him that it is the twenty-fifth in the series 1, 4, 9, 16, . . .). (*ibid*)

So, the kind of picture Wittgenstein has in mind is (at least in one instance) something like a machine blueprint or diagram. This is the sort of picture that a teacher could give an engineering student on a test and ask them to explain its successive movements from this initial state given basic kinematic principles. It is, as Wittgenstein notes, akin to giving, "someone a number by telling him that it is the twenty-fifth in the series 1, 4, 9, 16 …", thereby likening their use of the "machine as symbol" to a simple arithmetical exercise. (To jump ahead a bit, he thereby likens the successive movements of the machine as symbol to "the steps determined by a formula", since in this series it is easy to notice that the pattern is most plausibly represented by $y = x^2$; thus allowing the student to easily find the "twenty-fifth" in the series by calculating 25^2.)

But Wittgenstein says that this could be either "a picture of a machine" or "a machine" (e.g., one made of steel and not mere scribbles on paper), the intention being that this would have to be a machine *taken in a certain way*, i.e., "as a symbol of its mode of operation". For instance, the same teacher could give their engineering student the following exercise: Open up the machine sitting on your desk, examine its inner parts without turning its crank, and explain what the successive movements of its parts will be given the structure of its initial state – you can assume for the exercise that the parts in the machine are perfectly rigid. As the machine is being used in this exercise, this is likewise an instance of, "a machine as its mode of operation", i.e., a machine taken as a symbol of how it will operate, where this can be inferred (as it were *a priori*) from what we might call its initial state: "We might say that a machine, or a picture of it, is the first in a series of pictures which we have learnt to derive from this one" (*ibid*).

"A machine as a symbol of its mode of operation" is contrasted in this section with what Wittgenstein refers to as "an actual machine": "[I]t may now look as if the way it moves must be contained in the machine *qua* symbol still more determinately than in the actual machine" (*ibid*). In the machine *qua* symbol, we were able to derive its successive movements simply by looking at the structure of its initial state (the sort of thing that can be represented in a blueprint or diagram). The "actual machine", however, is subject to, "the possibility of [its parts] bending, breaking off, melting, and so on" (*ibid*). In the case of the actual machine, "we do not in general forget the possibility of a distortion of the parts and so on" (*ibid*), i.e., "when this is a matter of predicting the actual behavior of a machine" (*ibid*).

For instance, the engineering student might have a blueprint of a machine from which (as we saw earlier) they can derive its successive movements according to basic kinematics. Their teacher might then place a physical machine on their desk and show them that its parts correspond exactly with the blueprint, but ask them to decide whether *this* machine will have the same successive movements as we saw in our earlier derivations. The student hastily answers, "Yes!", and they fail the test. Why? Because they didn't check to see what the parts were made of. As it turns out, one of the gears was made out of soft clay and thus when the crank was turned, that gear was crushed and the entire machine collapsed. Given the mismatch between the successive movements of the blueprint and the physical machine, the teacher explains, "We assumed in our earlier derivations that the parts of the machine were perfectly rigid, but we can see now that the parts in *this machine* were not." In Wittgenstein's less familiar way of putting things (though making essentially the same point), the student has failed to recognize the distinction between the machine *qua* symbol of its mode of operation from the physical machine on his desk, which can bend, break, melt, and so on.

With a clearer sense of this distinction to hand, it is easier to see what the intended upshot is meant to be in this passage. Let's go back to the beginning.

> The machine, I might say for a start, seems already to contain its own mode of operation. What does that mean? – If we know the machine, everything else – that is the movements it will make – seem to be already completely determined. (*ibid*)

This idea, however, that the machine, "seems already to contain its own mode of operation", soon leads to confusion.

> We talk as if these parts could only move in this way, as if they could not do anything else. Is this how it is? Do we forget the possibility of their bending, breaking off, melting, and so on? (*ibid*)

In other words, we have the beginnings of a paradox on our hands: the movements seem to be completely determined (given its initial state), and yet, at the same time, we know that the machine can bend or break. So, the machine is completely determined and yet it is also *not* completely determined! But this rests on crossing two importantly different uses of "the machine" and thus different senses in which its successive movements might be 'determined'. Indeed, there is a use of "the machine" where, "in many cases, we don't think of [the machine bending or breaking] at all" (*ibid*). This is "the machine as its mode of operation".

> We use a machine, or a picture of a machine, as a symbol of a particular mode of operation. For instance, we give someone such a picture and assume that he will derive the successive movements of the parts from it. (Just as we can give someone a number by telling him that it is the twenty-fifth in the series 1, 4, 9, 16, . . .) (*ibid*)

It is the use of "the machine" in this sense (*qua* symbol) that encourages a seemingly odd expression.

> "The machine seems already to contain its own mode of operation" means: we are inclined to compare the future movements of the machine in their definiteness to objects which have been lying in a drawer and which we now take out. (*ibid*)

However, Wittgenstein chimes in to say that this odd way of putting things is not encouraged when we consider "the machine" in the sense of an actual, physical mechanism.

> — But we don't say this kind of thing when it is a matter of predicting the actual behavior of a machine. Then we do not in general forget the possibility of a distortion of the parts and so on. (*ibid*)

The idea that, "the future movements of the machine are akin to objects which have been lying in a drawer and which we now take out", might nonetheless be encouraged when we consider the machine as a symbol of its mode of operation.

> We *do* talk like that, however, when we are wondering at the way we can use a machine as a symbol of some way of moving – since [the actual, physical machine], can, after all, also move quite differently.
>
> We might also say that a machine, or a picture of it, is the first of a series of pictures which we have learnt to derive from this one. (*ibid*)

So, the machine *qua* symbol encourages a picture on which its future movements are like objects lying in a drawer that simply need to be pulled out. But Wittgenstein insists that it is not at all fitting to consider a physical machine in these terms, "since it can, after all, also move quite differently". To say that its movements were already sitting in a drawer, simply needing to be pulled out, would be to ignore this possibility.

This clarification, on its own, however, does not resolve the confusion at hand, since it only leads us to wonder at how the machine *qua* symbol can involve a yet more powerful kind of determination than we might find in the actual physical machine.

> But when we reflect that the [actual, physical] machine could have moved differently, it may now look as if the way it moves must be contained in the machine *qua* symbol still more determinately than in the actual machine. As if it were not enough for the movements in question to be empirically predetermined, but they had to be really – in a mysterious sense – already *present*. (*ibid*)

Thus, the distinction between 'determination' as we find it in the machine *qua* symbol by contrast with the mere empirical predetermination of the movements in the actual, physical machine leads us to wonder at a kind of 'super-determination' involved in the machine as symbol. How can the machine *qua* symbol determine its future movements in this way – as if its movements were contained in it at the outset – as if its movements were like objects sitting in a drawer waiting to be taken out? Wittgenstein's sober response brings us back down to earth.

> And it is quite true: the movement of the machine *qua* symbol is predetermined in a different way from how the movement of any given actual machine is. (*ibid*)

There is nothing exotic in this difference, since it is easily seen in the contrasting uses of "the machine *qua* symbol" and "the physical machine". With respect to the machine *qua* symbol, the student can crank out its successive movements with the assumption of perfect rigidity, and thus can completely disregard how the make-up of its parts might lead to complications. Their results are derived from basic kinematics and inferred in a way akin to the operation of mathematical rules (e.g., as we saw earlier, calculating the successive steps in the pattern 1, 4, 9, 16, and so on). As Wittgenstein puts it in the RFM,

> "If the parts were perfectly rigid this is how they would move", is that [an empirical] hypothesis? It seems not. For when we say: "Kinematics describes the movements of the mechanism on the assumption that its parts are perfectly rigid", on the one hand we are admitting that this assumption never squares with reality, and on the other hand it is not supposed to be in any way doubtful that completely rigid parts would move in this way. But whence this certainty? The question here is not really one of certainty *but of something stipulated by us*. (RFM I, 120, my emphasis)

By contrast, the physical machine is understood to depend crucially on its physical make-up and other (physical) conditions surrounding its operation. The future movements of an actual, physical machine are thus not stipulated by us or a matter of convention. The distinction between the machine *qua* symbol and the actual, physical machine nonetheless allows for useful comparisons between them, not to mention predictions made on the basis of the machine *qua* symbol albeit with caution and perhaps some relevant qualifiers about its probability.

PI 193 thus illustrates the resolution of at least three important confusions that might arise, each of which involves "crossing different pictures" of "the machine" as well as different senses of 'determine' associated with them.

First off, there is a relevant distinction between the machine *qua* symbol and an actual physical machine. The successive movements of the former are determined as a matter of mathematical convention or stipulation (RFM I, 120), whereas the successive movements of the latter depend on complicated physical conditions which allow for the possibility of bending, breaking, and so on. This resolves completely the apparent paradox of how the machine's movements can *only* move this or that way (i.e., in the sense of the machine *qua* symbol) while also being subject to bending or breaking (i.e., in the sense of an actual, physical machine).

Second, given that an actual, physical machine is subject to the possibility of bending or breaking, it is misleading at best to consider the successive movements of the actual machine as akin to objects, "lying in a drawer waiting to be taken out". Why? Because physical conditions are complicated and various things might happen – they are not merely "contained" in the initial state. To think otherwise would be to impose a natural picture of the machine *qua* symbol onto the actual, physical machine, which does not simply "contain" its future movements.

Thirdly, and lastly, any resulting perplexity about the mysterious way in which a machine *qua* symbol determines its movements can be resolved by noting the obvious differences in use between 'determine' as it pertains to "the machine *qua* symbol" and "the actual machine". The former is 'determined' as a matter of mathematical convention or stipulation, the latter is 'determined' as a matter of complicated physical conditions. Ignoring this distinction might incline one to (mistakenly) think that the machine *qua* symbol is a machine made from, "material harder and more rigid than any other", (RFM I, 119) which obeys quasi-physical principles – principles somehow akin to those of the physical machine, but lying in some outer and more perfect realm, involving perfectly rigid machines with their successive movements timelessly awaiting discovery (Plato's heaven for machines and engineering, as it were). Crossing the disparate uses and pictures of "the machine" thus risks seducing one into super-expressions or philosophical superlatives. Teasing these pictures apart allows us to see this as misbegotten, thereby undermining their seductive quality.

2.4 Crossing Pictures of 'Determination' in PI 189: The Interlocutor's "Mistake"

Let's return now to PI 189. Wittgenstein's machine analogy – which illustrates confusions that might arise from crossing different senses of 'determine' in our talk about "a machine" – is, on my reading, intended to shed light on the "mistake" contained in the interlocutor's earlier question, "But are the steps then *not* determined by the algebraic formula?". As we saw earlier, the interlocutor's idea that "…the steps are really already taken" is diagnosed by Wittgenstein as a super-expression that results from crossing different pictures. PI 189, however, only provides a list of different uses of ‘determine’ as it might apply to an algebraic formula, thus initially making it unclear what the mistake is supposed to be and how this might involve “the crossing of different pictures”. But with the help of the machine analogy, it is not difficult to see how this might go.

Generally put, the two major uses of ‘determine’ as it applies to an algebraic formula are being crossed and thus result in a misbegotten picture of a formula’s determining its successive steps. Just as in the machine analogy, we here find a distinction drawn between two uses of ‘determine’, but in this case as it pertains to “the algebraic formula”: what we might call a “behavioral” and a “mathematical” use of ‘determine’.[61] The former “behavioral” sense pertains to the behavior of some people, i.e., how they in fact respond to a specified formula, say, uttered by a teacher or written with chalk. The latter “mathematical” sense pertains to basic conventions regarding the classification of different formulae, i.e., those that determine a value of y given x by contrast with those that do not. The “behavioral” sense is akin to our talk about the “behavior” of an actual physical machine, whereas the “mathematical” sense is akin to our talk about the machine *qua* symbol of its mode of operation. Following the lessons from the machine analogy, the interlocutor’s confusions arise from crossing the behavioral and mathematical senses of ‘determine’, just as confusion might arise from crossing the machine as a mode of its operation with the actual, physical machine. In this section I will explain how this is so.[62]

[61] Compare Baker & Hacker (2009)’s exegesis according to which Wittgenstein was concerned with the crossing of “empirical” and “grammatical” senses of ‘determine’: “An example of such ‘crossing of pictures’ was in effect given in §189, where the interlocutor’s misconception of ‘determining the steps to be taken’ derived from crossing the empirical sense of ‘determines’ with the grammatical sense” (Baker & Hacker, 2009: 103).

[62] A discussion of how these considerations might bear on “Kripkenstein’s” skeptical argument would require a separate paper. But one immediate connection to Kripke’s presentation of the so-called “skeptical paradox” relates to the following crucial premise of that argument: “An answer to the sceptic must satisfy two conditions. First, it must give an account of what fact it is (about my mental state) that constitutes my meaning plus, not quus. But further, there is a condition that any putative candidate for such a fact must satisfy. It must, in some sense, show how I am justified in giving the answer ‘125’ to ‘68+57’. The ‘directions’ mentioned in the previous paragraph, that *determine* [my emphasis] what I should do in each instance, must somehow be ‘contained’ in any candidate for the fact as to what I meant” (Kripke, 1982: 11). So, “Kripkenstein” requires a “fact” which *both* determines “what I mean” and why I am “justified” in going on as I do. For Wittgenstein, “what I mean” is found in the general use of an expression, i.e., the behavior of some people. Matters of “justification” or “giving reasons” for one’s answer to a mathematical problem are part of the

After mentioning the interlocutor's question and claiming that it contains a mistake, Wittgenstein proceeds to examine some uses of the expression "The steps are determined by the formula".

> We use the expression "The steps are determined by the formula . . .". *How* is the expression used? – We may perhaps mention that people are brought by their education (training) so to use the formula $y = x^2$, that they all work out the same number for y when they substitute the same number for x. Or we may say: "These people are so trained that they all take the same step at the same point when they receive the order '+3'." We might express this by saying "For these people the order '+3' completely determines every step from one number to the next". (By contrast with other people who do not know what they are to do on receiving this order, or who react to it with perfect certainty, but each one in a different way.) (PI 189)

As Wittgenstein puts it in the LFM, the general sense of 'determine' in this paragraph (even with its slight variations) is a matter of the, "description of the behavior of people" (LFM 29) and serves as an answer to the question, "Do most people act in the same way in this connection?" (LFM 28). It is not an issue for this sense of 'determine' that people can and do sometimes count differently, skip a step, make what they would call a "miscalculation", and so on. These are exceptions that, as we might say, prove the rule – as shown in the various behaviors people engage in to "correct" such deviations from the general practice. It is also possible in this sense of 'determine' that, despite the fact that this is how all or most people respond to '+3', someone *might* still misinterpret the rule in a variety of ways that would be corrected, e.g., by an ordinary school teacher. This use of 'determine' is thus akin to the "actual, physical machine" discussed in PI 193, which might bend or break despite the fact that, generally speaking, it won't do so when it is properly set up. Human beings in their use

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language-game of mathematics. Hence, assuming the reading offered in this paper is correct, Wittgenstein would have likely accused Kripkenstein of crossing pictures of 'determination': i.e., a behavioral ("what I mean") with a mathematical ("justification") sense of 'determine'. Crossing these senses of 'determine' is what allows the "paradox" to get off the ground. Since whatever fact one cites to explain their behavior does not seem adequate as a mathematical justification ('68+57=125' is not true *because* I was trained to say so (c.f., PI 241)); and whatever fact one cites to mathematically justify the answer given, this does not seem adequate for determining – with mathematical necessity, as it were – their behavior (since it is logically possible that they might see or hear the mathematical 'directions' Kripke refers to above, but nonetheless go on differently). Undermining this crucial premise in the skeptical argument (viz., "An answer to the sceptic must satisfy two conditions", etc.) would thus dissolve the famous Kripkensteinian paradox.

of "the formula" might also "bend or break" when given an algebraic formula and told to calculate a series of steps from it. This generally doesn't happen, i.e., people generally do not disagree in their algebraic calculations; if they do, they re-calculate until they get the same result (or simply assume that one or the other made an error). This is a simple empirical matter of how people contingently behave, i.e., how they respond to a formula provided in writing or speech. We might just as easily call these matters of "anthropological fact".[63]

Wittgenstein then proceeds to describe a different family of uses of the expression, "The steps are determined by the formula . . .".

> On the other hand, we may contrast different kinds of formula, and the different kinds of use (different kinds of training) appropriate to them. Then we *call* formulae of a particular kind (with the appropriate method of use) "formulae which determine a number y for a given value of x", and formulae of another kind, ones which "do not determine the number y for a given value of x." ($y = x^2$ would be of the first kind, $y \neq x^2$ of the second.) (*ibid*)

These uses of 'determine' pertain to the formulae themselves and are a matter of mathematical stipulation or convention.[64] They have nothing to do, *per se*, with the behavior of people as in the

[63] As Goldfarb (1985) highlights, the alleged inadequacy of such empirical details is a crucial element of the interlocutor's perplexity: "We give a rule, some examples of its application, and perhaps some further explanations. Yet, for all that, a person 'could' go on in different ways and take himself to be going on the same. This seems to indicate that what we give is insufficient to tell, or to justify, how to go on; and we demand something more. The demand is not for that which in fact succeeds in showing a person, in particular circumstances, how to go on. It is rather for that which picks out the correct continuation in some unconditioned way, by giving that in which the same really consists" (Goldfarb, 1985: 105). Or as Maddy (2014) puts it, "The philosopher requires […] an account of the sense of our words that doesn't depend on any contingencies" (Maddy, 2014: 72). The interlocutor's demand for "something more" that "doesn't depend on any contingencies" is, on my reading, a result of crossing pictures – which we'll see momentarily.

[64] Wittgenstein in the LFM notes a similarity between these uses of 'determine' and a related distinction that might apply to different senses in which 'pointing' can determine how one goes (i.e., how people happen to behave vs. a convention of distinguishing different kinds of pointing):

> Does my pointing determine the way he goes?—Do people normally go in one way? Yes. Or *we might have a convention* [my emphasis] by which we distinguish pointing which determines the way from pointing which does not. [Includes a diagram with a single arrow and one with two arrows going in different directions.] Pointing in the second way indicates that it does not matter in which of the directions one goes. "Did your pointing determine the way he was to go?" might then mean "Did you point in one direction or in two?" (LFM 29).

sense of 'determine' discussed in the previous paragraph. To the extent that they are relevant to the behavior of people, this is a matter of which kinds of training are "*appropriate*", i.e., consistent with pre-established mathematical conventions. In this "mathematical" sense of determine, different formulae are categorized either as determining a value of y given x or as not. More directly, as Wittgenstein puts it in the LFM, this is a matter of "a description of the formula" (LFM 29) and serves as an answer to the question, "Is it a formula of this kind or that?" (LFM 29). These categorizations, by contrast with a description of the behavior of people, are not ones that, as it were, allow for "bending or breaking". $y = x^2$ does not determine y given a value for x on Tuesday, but not on Wednesday. It does not determine y given x for Paul, but not for Suzy. (If it does, then we might say that they were actually talking about different formulae, c.f., PI 190.)

The sense of 'determine' as it applies to the categorization of mathematical formulae is thus akin to "the machine as a symbol of its mode of operation". Just as the steps of the machine qua symbol can be derived without attention to the steps taken by an actual, physical machine, the steps of the formula and its classification as either determining a value for y or not can be derived without attention to how Paul or Suzy happens to apply or categorize it (i.e., unless we are taking them as perfectly demonstrating the convention). These are matters of "mathematics", and, as Wittgenstein often puts it in other writings, they are "non-temporal" and thus not subject to change (see, e.g., RFM I, 23, 27, 101-103, & RFM VI, 2). By contrast, descriptions about the behavior of people are "temporal", regarding how people behave at some time or other, and are subject to the possibility of variation, e.g., with the obvious possibility of miscalculation and even the possibility of alternative mathematical practices in other cultures.[65]

[65] The two senses of 'determine' at work here – and the fact that they are all too easily crossed – might help to resolve (or, rather, dissolve) the disagreement between Dummett (1978) and Stroud (1965) over whether Wittgenstein was a "full-blooded conventionalist". When faced with a mathematical proof, Dummett thinks "at each step we are free to choose to accept or reject the proof" (Dummett, 1978: 171) and thus the "necessity" of a proof is always a matter of our own (personal) decision. Stroud, by contrast, attributes to Wittgenstein the view that certain ways of going on in

So far, then, we have a distinction between at least two different uses of 'determine': one use that pertains to the behavior of people, and another use that pertains to the classification of different formulae and which takes place in the context of doing or talking about mathematics. Now we have to consider how crossing these different uses might yield the interlocutor's confusion.

Recall that in PI 188, the interlocutor's idea was that

> this *meaning the order* had in its own way already taken all those steps: that in meaning it, your mind, as it were, flew ahead and took all the steps before you physically arrived at this or that one. (PI 188)

This idea inclined them to, "use such expressions as "The steps are *really* already taken, even before I take them in writing or in speech or in thought" (*ibid*). According to this idea, it seemed that the steps, "were in some *unique* way pre-determined, anticipated – in the way that only meaning something could anticipate reality" (*ibid*). Thus the interlocutor has a suspicious picture of 'determination': one on which a formula determining its steps seems to require that the steps are *really* already taken by one's mind when they give the order to add 2.

The interlocutor's picture is completely analogous to the super-expression about the machine according to which, "the machine seems already to contain its own mode of operation" (PI 193), which means, "we are inclined to compare the future movements of the machine in their definiteness to objects which have been lying in a drawer and which we now take out" (*ibid*). This confusion about the machine resulted from crossing pictures of the "machine qua symbol" and the "actual, physical machine". The interlocutor's confusion at PI 188 is likewise a result of crossing

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mathematics are constitutive of "mankind" or "human nature": "What facts does he have in mind here, and what role do they play in his account of logical necessity? The reason for calling them "facts of natural history" is to emphasize both what I have called their contingency—that is, that they might not have obtained—and the fact that they are somehow "constitutive" of mankind—that is, that their obtaining is what is responsible for human nature's being what it is" (Stroud, 1965: 514). But in both of their attempts to account for "necessity" or "determination", what sense of 'determine' do they have in mind? In the behavioral sense, it *is* true that one can go on however one likes – notwithstanding the obvious consequences in a standard educational facility. In the mathematical sense, the convention *does not* allow for variation in whether a formula does or does not determine *y* given *x*. Is there some other sense of 'determination' that Dummett and Stroud are sparring over and that they think Wittgenstein is "providing an account" of? See also fn. 58 above.

different pictures, this time regarding "the formula" and the different senses in which it can 'determine its steps' – as articulated in PI 189. There is the behavioral sense according to which the teacher writes the formula on the chalkboard, and thereby 'determines' the steps taken in that those steps are then written uniformly by his students in response. Alternatively, if (some or all of) the students struggle to give the desired response, the teacher might write down the series for them on the chalkboard, erase it, and then request that they do the same from memory (c.f., RFM I, 22). The picture of "the steps *really* already being taken" is inappropriate in this context, since it is clear that human beings are imperfect and the situation could go a variety of different ways (just as "the movements being contained in the machine" was inappropriate when talking about the actual, physical machine, which can bend or break). The teacher might himself make a couple of missteps as the result of a severe migraine, or the students might go on to write a different, unintended series and so be instructed to write down what the teacher deems to be "correct" (with the reminder that if they fail to do so on their next test, they will have points marked off). To think that such possibilities are incompatible with "the formula's determining its steps" would be to cross the behavioral sense of 'determine' with the non-temporal and mathematical sense of 'determine', which does not admit of exceptions or variations. This is a matter of mathematical stipulation and takes place in the language-games of doing and discussing mathematics – it does not have anything to do *per se* with the sense in which the formula might determine a student's behavior as a result of their training in a classroom, or the teacher's behavior as a result of their past training. We can imagine the interlocutor insisting, "'+2' determines its steps without any possibility of variation! So it must be true that the steps are *really* already taken in the teacher's mind!". The response from Wittgenstein would be that this picture might be encouraged by the mathematical sense of determine, but it is not at all appropriate in the behavioral sense.

Alas, this might leave us with a residual worry about the mathematical sense of 'determine'. As we saw in the machine analogy, even when one grants that an actual, physical machine might bend and break, and thus that the picture of its steps being contained in advance is inappropriate, one might then be mystified by the sense in which a machine as symbol can determine its steps in advance, "it may now look as if the way it moves must be contained in the machine *qua* symbol still more determinately than in the actual machine" (PI 193). An analogous confusion might undoubtedly result for the interlocutor regarding the 'mathematical' sense of determine, which seems to contain its steps "still more determinately" than in the teacher's and the student's responses to the formula, say, as it is written on the blackboard. This confusion pertains not to a perplexity about how human beings respond to formulae in the ways that we do, but how the *must* of logic or mathematics is so much as possible.[66] Wittgenstein's answer here is that this is a matter of convention or stipulation – thus there is no special mystery regarding the distinctive way in which the steps might be mathematically pre-determined.[67]

[66] This topic is explored in detail by Maddy (2014). Pears puts what I'm calling the residual worry in the following way, "the application of any general word might well have been different from what it now is, and it is often easy to imagine circumstances in which it really would have been different. […] But logic [and mathematics] seems to be made of harder stuff" (Pears, 2006: 65).

[67] Maddy (2014) might insist that there is still a further question, even for Wittgenstein, of what "grounds" such stipulations or conventions (see especially Maddy (2014: 90-100)). If, as Maddy sometimes emphasizes, such a further inquiry would be answered by a collection of obvious and mundane facts about "our interests and motivations, our natural inclinations, and very general features of the world" (Maddy, 2014: x; examples of "very general features" from the PI and RFM include that marbles don't suddenly pop in and out of existence, that cheese doesn't suddenly become lighter or heavier, and the like), then the further question doesn't require a deep philosophical investigation of the sort that Wittgenstein is attempting to dismantle. But Maddy sometimes treats such answers as if they were interchangeable with a "Second Philosophical" account such as the following: "logic is grounded in the structure of our contingent world […] our representational systems reflect that structuring" (Maddy, 2014: ix). The "structure" of our world is referred to by Maddy as "Kant-Frege" or "KF-structure", as it obeys Kantian logic with the addition of several Fregean "updates" (Maddy, 2014: 17-18). The metaphysical tone of the latter phrasing by contrast with the former is obvious (especially given the terms "grounded", "structure", "representational systems", with their various philosophical connotations, not to mention the intended association with Kant and Frege!); and it seems plausible that Wittgenstein would recoil from a general, metaphysical characterization of such obvious and mundane facts. Further, the very idea that logic or mathematics "represents", "corresponds to", or "concerns" some independent reality (e.g., the "KF-structure" of the world) seems to fall prey to a crossing of empirical and mathematical uses of 'truth' as was discussed in Chapter 1 (see also Gerrard (1991)). In short, if there is a further question here, it is not a matter of deep philosophy, though perhaps of scientific journalism; but if someone insists that it *is* a matter of deep philosophical inquiry (say, regarding the logical structure of the world, or the reality that is reflected by logic and mathematics), then they are likely operating with a

However, if one projects the behavioral sense of 'determine' onto the mathematical, then it can appear as if what the teacher does on the chalkboard is somehow performed timelessly in mathematical reality. Whereas the teacher can write out the series 2, 4, 6, 8, 10, and so on with chalk and subsequently erase it, the series "written out", as it were, in abstract mathematical reality persists timelessly and cannot be erased. (And now we really are talking about Plato's heaven in the traditional sense.) This is, again, a crossing of different pictures of 'determine'. Such a crossing of the behavioral into the mathematical, resulting in an ethereal picture of mathematical reality, is noted explicitly in the RFM. Wittgenstein first describes pictures on which logical and mathematical applications are somehow "already completed" before any one of us performs them.

> In his fundamental law Russell seems to be saying of a proposition: "It already follows—all I still have to do is, to infer it". Thus Frege somewhere says that the straight line which connects any two points is really already there before we draw it; and it is the same when we say that the transitions, say in the series +2, have really already been made before we make them orally or in writing—as it were tracing them. (RFM I, 21)

Wittgenstein's response is that someone seduced by these expressions (such as Russell or Frege perhaps) is illicitly projecting a picture of behavior onto logic and mathematics.

> One might reply to someone who said this: Here you are using a picture. One *can determine* [in the behavioral sense] the transitions which someone is to make in a series, by doing them for him first. E.g. by writing down in another notation the series which he is to write, so that all that remains for him to do is to translate it; or by actually writing it down very faint, and he has to trace it. In the first case we can also say that we don't write down *the* series that he has to write, and so that we do not ourselves make the transitions of that series; but in the second case we shall certainly say that the series which he is to write *is already there* [my emphasis]. We should also say this if we *dictate* what he has to write down, although then we are producing a series of sounds and he a series of written signs. It is at any rate a sure way of *determining* [albeit in the behavioral sense] the transitions that someone has to make, if we in some sense make them first. (RFM I, 22)

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picture of logical or mathematical 'truth' and 'reality' that Wittgenstein would find deeply suspect (for reasons discussed in Chapter 1).

Thus when one projects the behavioral sense of 'determine' onto the mathematical sense, this suggests a picture of the mathematical realm where there is a series somehow akin to that written on the chalkboard by the teacher, but different in that the objects of the series are timeless, eternal, unchanging, and awaiting discovery by mathematicians in some special realm.

> Here what is before our minds in a vague way is that this reality is something very abstract, very general and very rigid. Logic is a kind of ultra-physics, the description of the 'logical structure' of the world, which we perceive through a kind of ultra-experience (with the understanding e.g.). (RFM I, 8)

A seductive picture, indeed, as the history of the philosophy of mathematics has shown. Wittgenstein's method aims to dismantle it by revealing one of its sources, namely, an illicit crossing of two disparate uses of 'determine'.

The interlocutor's mistake is thus akin to that made by the boy who wonders at how one can possibly "sew a dress", or the one who is perplexed by how a line can "cut" a circle at imaginary points, or how the verb "to sleep" could possibly be active. The example of sewing a dress is in fact used directly in response to a follow-up from the interlocutor.

> "But I don't mean that what I do now (in grasping the whole use of a word) determines the future use *causally* and as a matter of experience, but that, in a *strange* way, the use itself is in some sense present." (PI 195; c.f., Z 296)

The interlocutor thus infers from the machine analogy that Wittgenstein is claiming that the 'determination' involved in one's understanding of the formula is merely "causal" or "a matter of experience" (akin to the 'determination' involved in the actual, physical machine). The interlocutor insists to the contrary that on *his* idea of determination, the use is determined "in a strange way", in such a way that it makes sense to say "the steps are already taken". Wittgenstein's response makes clear, yet again, that the interlocutor's idea relies on an illicit crossing of pictures.

> —But of course it is, 'in *some* sense'! Really, the only thing wrong with what you say is the expression "in an odd way". The rest is right; and the sentence seems odd only when one imagines it to belong to a different language-game from the one in which we actually use it. (Someone once told me that as a child he had been amazed that a

> tailor could 'sew a dress' – he thought this meant that a dress was produced by sewing alone, by sewing one thread onto another.) (*ibid*)

Wittgenstein has already provided at least two different senses in which the formula can indeed determine its successive steps: a behavioral sense and a mathematical sense. Neither of these are odd. The behavioral sense of 'determination' is illustrated quite easily via our earlier descriptions of an ordinary classroom setting. The mathematical sense of 'determination' takes place within the language-game of doing or discussing mathematics, a fairly banal distinction between different types of algebraic formulae. The interlocutor apparently takes issue with having his idea of 'determination' lumped into the behavioral sense (as it is merely "causal" or "a matter of experience"). Wittgenstein's diagnosis is that the interlocutor is imagining the expression "the formula determines its steps" to take place in a language-game different from any of those in which it is actually used, namely, a language-game in which a formula 'determines' its steps both in the behavioral and mathematical senses. This is at best puzzling and at worst paradoxical: behavior takes place in time, and yet mathematics is timeless, implying a "strange" sense of 'determination' on which my grasping the use *in time* involves the *timeless* performance of all the steps in an infinite series (c.f., PI 138: "[W]e grasp the meaning at a stroke, and what we grasp in this way is surely something different from the 'use' which is extended in time!"). This leads to a major conclusion about the interlocutor in PI 196.

> In misunderstanding the use of the word, one takes it to signify an odd *process*. (As one thinks of time as a strange medium, of the mind as an odd kind of being.)
> (PI 196)

Thus we have a fairly decisive diagnosis of the interlocutor's misbegotten idea. Their idea – on which the steps are *really* already taken in one's understanding of the rule and thus 'determined' in a strange and special sense – is the result of crossing different uses of 'determine'. Crossing these uses leads to a fantastic picture of an individual's mental life, one which is counteracted via an ordinary description of people's behavior when they specify and respond to a mathematical formula. It also leads to a fantastic picture of the mathematical realm, in which something like our "counting" takes place perfectly, timelessly, completely, and so on, a picture which is counteracted by the reminder that, within mathematics, the series following from '+2' is a matter of stipulation or convention. The wonder that might have inspired the interlocutor to further theorizing or speculation is thereby diminished.

When Wittgenstein begins to diagnose the interlocutor's idea in PI 188 by noting the expressions they are merely inclined to use, the interlocutor takes this to mean that Wittgenstein is denying that a formula can determine its steps – hence his question that opens PI 189. The mistake in the interlocutor's question is a mistaken conception of 'determination', one which leads to perplexity and paradox. The source of this mistaken idea is, according to Wittgenstein, an illicit crossing of different uses of the word 'determine', which are laid out in PI 189. It is worth noting that this reading is compatible with the idea that there are far more uses of the word 'determine' than are laid out in this passage – so the point is not that this list is somehow exhaustive and that the interlocutor must pick one from the list. Indeed, Wittgenstein describes uses of 'determine' as it might apply to a machine in PI 193 (and as it might apply to 'pointing' in LFM 29). Among the broad family (or families) of uses, however, there are at least two which the interlocutor has illicitly crossed: a behavioral and a mathematical use as they might take place respectively in our talk about a mathematical formula. The crossing of these uses leads to their super-expressions about following a simple mathematical rule. Without this mistaken conception of 'determine', the interlocutor's question dissolves – along with the downstream puzzles and paradoxes their idea might have encouraged ("Problems are solved (difficulties eliminated), not a single problem" (PI 133)).

Although the following text comes from notes taken by Rush Rhees of Wittgenstein's lectures, it seems to usefully summarize the situation we find in PI 188-189:

> When you get the picture of "being determined" out of your mind, then you get rid of the puzzle. – But still one can say the algebraic expression determines his actions – and perfectly correctly. But now you have got rid of the cramp. (LSD 24)

Thus, "A picture held [the interlocutor] captive", and they, "couldn't get outside it, for it lay in our language, and language seemed only to repeat it to us inexorably" (PI 115). Given the occurrence of "the same word", viz., 'determine', in both the mathematical and behavioral contexts, it is all too easy to lump them together and thereby get seduced by the interlocutor's misbegotten conception of determination. But now, hopefully, they can see it for what it is: "a result of the crossing of different pictures" (PI 191).

Chapter 3

"The Infinite": From Pictures to Employments

> On mathematics: "Your idea [*Begriff*] is wrong.—However, I cannot illumine the matter by fighting against your words, but only by trying to turn your attention away from certain expressions, illustrations, images, and *toward the employment* of the words." (Z 463)

3.1 Introduction

In this chapter, I will discuss some of the later Wittgenstein's most important commentary on infinity. The major point of this commentary is that talk of the infinite naturally takes hold of one's imagination due to the sway of verbal pictures and analogies suggested by our words. However, one's imagination in these instances does not determine the meanings of 'infinity' and its surrounding expressions – but rather their meanings are ultimately to be found in their uses and applications. Thus, when the verbal pictures associated with our expressions take us far beyond their uses or perhaps even conflict with them, then the conditions are ripe for confusion. By Wittgenstein's lights, to avoid such confusion, we need to be reminded of the applications of these expressions so as to weaken the hold of said pictures and analogies on our imagination.

In Section 2, I will explain this general critical perspective on 'infinity' and its attending confusions as Wittgenstein describes it in Z. In Section 3, I will show how this critical perspective is deployed in the LFM – which will illustrate more explicitly the various misleading pictures that arise from analogizing 'infinity' across different mathematical and non-mathematical contexts. In Section 4, I will provide some useful background for Wittgenstein's remarks on Cantor, namely, his general suspicion about the tendency of philosophers to be taken in by the methods and results of science. In Section 5, I will show how Wittgenstein's discussion of diagonalization in MS 117 (as it is presented in RFM II) aims at counteracting the misleading effects it might have on one's imagination via a sober redescription of Cantor's proof.[68]

3.2 'Infinity' and Imagination in Z

In a series of passages from Z, Wittgenstein discusses the temptation to think that the meaning of 'endlessness' or 'infinity' is determined by what takes place in one's imagination.

> "It makes sense to speak of an endless row of trees; I can surely imagine a row of trees going on without end." That means something like: If it makes sense to say the row of trees comes to an end here, then it makes sense to say [it never comes to an end.][69] Ramsey used to reply to such questions: "But it *just is* possible to think such a thing." As, perhaps, one says: "Technology achieves things nowadays which you can't imagine at all." (Z 272)

Wittgenstein's interlocutor in this passage thinks it makes sense to speak of an endless or infinite row of trees because they are able to imagine such a thing. Wittgenstein considers justifying their idea with the following rationale: since we can say that the row of trees comes to an end, we can surely also say its negation, i.e., that the row of trees does not come to an end (c.f., PI 344-5). He also considers the idea, allegedly expressed by Frank Ramsey, that our ability to think such a thing is so obvious that it needs no justification: "But it *just is* possible to think such a thing." Both reactions are suspect, according to Wittgenstein.

> —Well here one has to find out *what* you are thinking. (Your asseveration that this phrase can be *thought*—what can I do with that? For that's not the point. Its purpose is not that of causing a fog to rise in your mind.) (*ibid*)

The interlocutor's appeal to their imagination and Ramsey's "asseveration" that it *just is* possible to think "the row of trees is endless" give us little indication of *what* either person is thinking. Neither the insistence that something can be imagined nor the asseveration that it can be thought is the

[68] Compare Monk's (2007: 285) reading on which Wittgenstein aims to "redescribe" mathematics so as to dislodge certain pictures of it.

[69] Anscombe's and von Wright's conjecture.

purpose of such a sentence: “Its purpose is not that of causing a fog to rise in your mind”. Although not the purpose of these words, Wittgenstein (as we’ll see more clearly later) recognizes the fact that “endlessness” fires one’s imagination, or causes “a fog to rise in one’s mind”, as a major source of these ideas. Namely, that the meaning of ‘endless’ – quite independently of its use – is either determined by our imagination or by brute powers of thought. What is needed is an antidote to these tempting yet misbegotten ideas, which Wittgenstein offers immediately.

> *What* you mean—how is that to be discovered? We must patiently examine how this sentence is supposed to be applied. What things look like *round about it*. Then its sense will come to light. (*ibid*)

What one means by an expression (“the row of trees is endless”) is to be discovered by patiently examining how it is applied and what things look like in the context of its application. Such an examination will bring its sense to light, if anything will. This suggestion is not at all surprising coming from the philosopher who famously suggested that if we want to understand the meaning of a word, we should (“for a *large* class of cases”) look to its use.

> For a *large* class of cases of the employment of the word “meaning” – though not for *all* – this word can be explained in this way: the meaning of a word is its use in the language. (PI 43)

Wittgenstein’s general reaction to talk of “endlessness” or “infinity” is thus to draw us away from tempting pictures of their meaning (manifested by the “fog” they cause to rise in our minds) and back to their uses or applications. This methodological program is familiar enough from the PI.

> A *picture* held us captive. And we couldn’t get outside it, for it lay in our language, and language seemed only to repeat it to us inexorably. (PI 115)

> What *we* do is to bring words back from their metaphysical to their everyday use. (PI 116)

In the next passage of Z, Wittgenstein shows more explicitly that he is dealing with mystification about “the infinite” in mathematics (not just sentences about endless rows of trees) and vividly describes his intended methods for demystification.

> Hardy: "That 'the finite cannot understand the infinite' should surely be a theological and not a mathematical war-cry." True, the expression is clumsy (*ungeshickt*). But what people are using it to try and say is: "We mustn't have any juggling! How comes this leap from the finite to the infinite?" Nor is the expression all that nonsensical—only the 'finite' that can't conceive the infinite is not 'man' or 'our understanding', but the calculus. And *how* this conceives the infinite is well worth an investigation. (Z 273)

Hardy objects to a certain phrase by comparing it to theology and thereby distancing it from any serious mathematical inquiry. Wittgenstein grants that the phrase is clumsy or awkward, but he also thinks that the expression calls out for a certain kind of investigation that Hardy is hastily ruling out. The investigation he suggests is not, as one might be tempted to think, an inquiry into how "finite minds" or "finite beings" are capable of comprehending something as vast as "the infinite"[70], but instead a matter of how our mathematical calculus "conceives" what we call 'the infinite'. Wittgenstein grants that *this* "is well worth an investigation", especially when that tends to alleviate the mystification which gave rise to the clumsy expression. The expression is not "all that nonsensical" (however mistaken it might be) when it is understood as saying "our calculus cannot conceive the infinite". We can address this concern by examining the calculus and seeing just how "the infinite" is conceived there. Such an examination will show the concern to be based on a superficial understanding of how this occurs.

> This may be compared to the way a chartered accountant precisely investigates and clarifies the conduct of a business undertaking. The aim is a synoptic comparative account (*eine übersichtliche, vergleighende Darstellung*) of all the applications, illustrations, conceptions of the calculus. The complete survey (*vollkommene Ubersicht*) of everything that may produce unclarity. And this survey (*diese Übersicht*) must extend over a wide domain, for the roots of our ideas reach a long way.—"The finite cannot understand the infinite" means here: It cannot work *in the way* you, with characteristic superficiality, are presenting it. (*ibid*)

[70] Compare Nagel (1997: 71-2): "Though our direct acquaintance with and designation of specific numbers is extremely limited, we cannot make sense of it except by putting them, and ourselves, in the context of something larger, something whose existence is independent of our fragmentary experience of it. […] When we think about the finite activity of counting, we come to realize that it can only be understood as part of something infinite. […] [T]he description of what happens when we count must include the relation of that activity to the infinite series of natural numbers, since that is part of what our operation with the concept of number makes evident."

Again, we find a natural connection with the methodological aspirations of the PI, in which Wittgenstein suggests a general source of, as well as an effective antidote to, philosophical misunderstanding: "A main source of our failure to understand is that we don't have *an overview* (*nicht übersehen*) of the use of our words"; this failure can be addressed by providing, "A surveyable representation (*Die übersichtliche Darstellung*) [that] produces precisely that kind of understanding which consists in 'seeing connections'" (PI 122). Hence, as in Z 273 above, a call for a "synoptic comparative account (*eine übersichtliche, vergleighende Darstellung*)" of all the applications, illustrations, and conceptions of the calculus that conceives "the infinite", which will allow us to clear away misunderstandings by revealing "everything that may produce unclarity" on this matter. Unlike the interlocutor's initial feeling (in Z 272) that the meaning of 'endlessness' manifested itself directly in their imagination or Ramsey's insistence that 'endlessness' *just can* be thought (end of story!), Wittgenstein suggests that a survey of "the infinite" needs to extend over "a wide domain, for the roots of our ideas reach a long way" (c.f., PI Preface: "the very nature of the investigation […] compels us to travel crisscross in every direction over a wide field of thought"). As we'll see later, it requires a comparison with a variety of related terms ('big', 'series', 'collection', 'order', 'denumerable', etc.) as they occur both within and outside the mathematical calculus under investigation – a wide domain indeed.

Wittgenstein ends Z 273 by indicating what has led to the various mistakes described thus far: that the meaning of 'infinity' is grounded in our imagination, that it *simply can* be thought, and that 'the finite cannot conceive the infinite'. In each case, we imagine that our thoughts can fly far beyond what might take place in the mere applications of a word. And so we can *think about* "the infinite", even if our mere 'finite' uses are incapable of "conceiving" such a thing.

> Thought [which can conceive "the infinite"] can as it were *fly*, it doesn't have to walk [as do our uses or applications]. (*ibid*)

But this idea, that our thought can so thoroughly outstrip the applications of words, arises from one's lacking a synoptic view of our practice with those words.

> You do not understand your own transactions, that is to say you do not have a synoptic view of them, and you as it were project your lack of understanding into the idea of a medium in which the most astounding things are possible. (*ibid*)

We mistakenly project our ability to "think" about "the infinite" into "a medium in which the most astounding things are possible". We then compare our "thoughts" of the infinite to what we find in the calculus, which does not seem to contain the vast thing that comes directly to our minds. Seen in this way, it is easy to notice the major source of misunderstanding: namely, that our thoughts – in this medium ("the mind") which contains the most astounding possibilities – are capable of outstripping the uses that we make of 'the infinite' in our humdrum mathematical applications. One then looks back at the calculus and sees it as inadequate, as unable to fully comprehend the vast reality that is the object of our thoughts. But the object of our thoughts (say, "the fog that rises in one's mind") has no meaning without the calculus and its applications. It is thus an illusion to think that we grasp something with our minds that cannot be comprehended by the calculus.

The succeeding passages state the foregoing diagnosis and intended cure very directly.

> The 'actual infinite' is a 'mere word'. It would be better to say: for the moment this expression merely produces a picture—which still hangs in the air: you owe us an account of its application. (Z 274)

> An infinitely long row of marbles, an infinitely long rod. Imagine these coming in in some kind of fairy tale. What application—even though a fictitious one—might be made of this concept? Let us ask now, not "Can there be such a thing?" but "What do we imagine?" So give free rein to your imagination. You can have things now just as you choose. You only need to *say* how you want them. So (just) make a verbal picture, illustrate it as you choose—by drawing, comparisons, etc.! Thus you can—as it were—prepare a blueprint.—And now there remains the question how to work from it. (Z 275)

The negative aspect of Wittgenstein's critique of "the infinite" described here has been seen already: viz., that the pictures that come to mind when we think about various kinds of infinity do not tell us on their own how such illustrations are to be applied – however tempting it might be to treat them as the true arbiter of meaning. At best such a picture provides a "blueprint" which leaves us with the question "how to work from it" (c.f., PI 139: "The picture of the cube did indeed *suggest* a certain use to us, but it was also possible for me to use it differently"). One still owes us "an account of its application." But the positive aspect of Wittgenstein's commentary is well worth noting: against the temptation to think that, say, 'actual infinity' is a 'mere word', Wittgenstein is granting (against a common brand of finitism) that this is perfectly meaningful in the context of its application. So, by critiquing the hold that *pictures* of 'the infinite' might have on our imagination, Wittgenstein is not thereby denying that there is a meaningful concept of 'infinity' – the instances of which presumably form a family (PR 304, PI 67)[71] – but insisting that, if there is such a thing, it is to be found in its various uses and applications.

All of this links up neatly with Wittgenstein's general thought "On mathematics", expressed later in Z.

> On mathematics: "Your idea [*Begriff*] is wrong.—However, I cannot illumine the matter by fighting against your words, but only by trying to turn your attention away from certain expressions, illustrations, images, and *towards* the *employment* of the words." (Z 463)

The hope is that, rather than engaging in a direct theoretical confrontation with someone who endorses (c.f., realism) or denies (c.f., finitism) some evocative picture of 'infinity', the force of such pictures over our imaginations will be weakened by examining the employments of 'infinity' and related words where they are at home. The ideas that such pictures give rise to (e.g., that "the finite cannot comprehend the infinite"), can then be seen as mistaken. Thus, to the extent that realism and finitism both rely on such pictures – the former to affirm, the latter to explain what they deny – Wittgenstein is best seen as neither a realist nor a finitist on the topic of "infinity".[72] Rather, he is

[71] As noted by Moore (2011: 108). Compare Floyd (2021): ""Family resemblance" characterizes the generality of certain concepts. A single property, a fixed-for-all-cases criterion, an explicit set of grammatical rules – these are not required. A concept may hold together – like a family – with a variegated, evolving series of properties" (Floyd, 2021: 51).

undermining a presupposition of their debate: a conception of 'infinity' rooted in misbegotten pictures, which is affirmed by the former and denied by the latter. (C.f., RFM II, 61: "Finitism and behaviorism are quite similar trends. Both say, but surely, all we have here is … Both deny the existence of something, both with a view to escaping from a confusion"[73]).

3.3 The Picture of Infinity as something "Colossal" in LFM XXVI

We find the very same critical strategy articulated in LFM XXVI, but with greater attention to the more specific confusions that might arise when thinking haphazardly about 'the infinite'. Wittgenstein's discussion continues a theme from the previous lecture regarding Hardy's (1929) temptation to think mathematics describes or corresponds to a special, external reality (as discussed in Chapter 1).[74]

> If you have a mathematical proposition about $\aleph_0$, and you imagine you are talking about a realm of numbers,—I would reply that you aren't as yet talking about a realm of anything, in the most important sense of "about". You are only giving rules of the use of "$\aleph_0$".
>
> You are developing the mathematics of it. And you have now to ask: in which *non*-mathematical propositions is it used? If you want to know the realm to which it points, you have to see in what sentences we use it. (LFM 251-2)

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72 My reading thus contrasts with those on which Wittgenstein endorses some brand of "finitism". See especially Ambrose (1935), Dummett (1997), Kielkopf (1970), Kreisel (1958), Marion (1998), Wang (1958), and Wright (1980).

73 My reading of this passage from RFM is thus comparable to one offered by Kripke: "Mathematical finitists and psychological behaviorists […] make parallel unnecessary moves when they deny the legitimacy of talk of infinite mathematical objects or inner states. Behaviorists either condemn talk of mental states as meaningless or illegitimate, or attempt to define it in terms of behavior. Finistists similarly regard the infinitistic part of mathematics as meaningless. Such opinions are misguided: they are attempts to repudiate our ordinary language game" (Kripke, 1982: 107). My reading differs from Kripke's in its emphasis on "realists" and "finitists" equally relying on a misbegotten picture of "the infinite". See also Monk (2007): "in his manuscripts, his lectures and his private conversations, Wittgenstein repeatedly denied being either a finitist or an intuitionist, upon both of which he poured the utmost scorn" (Monk, 2007: 276). Klenk (1976) likewise argues persuasively that Wittgenstein should not be seen as a finitist (Klenk, 1976: 18-24, 92-123). See LFM 111 & 141 for explicit disavowals of "finitism".

74 See Gerrard (1991: 126) for discussion of how the critique of such a picture (which Gerrard calls "the Hardyian picture") is a fundamental and recurring theme in Wittgenstein's later philosophy of mathematics.

To the extent that our use of '$\aleph_0$' (conventionally used to represent "the number of elements in the (infinite) set of all integers") points or corresponds to some reality, this is to be seen in its non-mathematical uses. Looking at its applications *within* mathematics is apt to mislead, as it suggests a picture according to which '$\aleph_0$' somehow points or corresponds to a special realm of numbers. Looking to the use of '$\aleph_0$' in *non*-mathematical propositions will give us a different picture.

> As soon as you do this, you get an entirely different picture of what you have been doing. At first, we picture ourselves flying to the end of the cardinal number series and beyond; this comes from thinking of mathematical propositions as the *application* of numbers. We get an entirely different picture if we consider it the other way: the statement that John has mastered $\aleph_0$ multiplications will mean he has mastered a certain technique of multiplying. And now we see we haven't been flying anywhere.
> (LFM 251)

The example of John mastering $\aleph_0$ multiplications brings our understanding of infinity back down to earth by showing that nothing vast or extraordinary is involved in our use of $\aleph_0$ or "the reality to which it corresponds". Such a mastery is simply that of knowing the integers and knowing how to multiply – not something we would typically marvel at. So, looking to this particular non-mathematical use of '$\aleph_0$' both weakens the feeling of mystery about the vastness of infinite cardinalities as much as it weakens the feeling that '$\aleph_0$' must point to some special, external reality.

But why not think that John's mastery really *is* something to marvel at – that there really is something vast and mysterious involved in John's understanding? (Compare my discussion of the alleged mysteries of rule-following in Chapter 2.) Contrary to Wittgenstein's suggestion, one might be sympathetic to a concern raised by a class member at the end of the lecture.

> *A member of the class:* Even when one says that a child has mastered an infinite technique, there is even there an element of hugeness and one has the idea of something huge. (LFM 255)

Wittgenstein diagnoses this feeling that "something huge" is involved in John's mastery, however, as a merely verbal suggestion from the word "infinite".

> Yes, but the idea of hugeness in that case comes only from the word "infinite" and not from what it's used for. By watching his work, we shouldn't get the idea of anything huge. The teacher does not say to himself, "Ah, fancy these boys of ten and eleven having such vast knowledge!" (LFM 255-6).

The example might then help to bring us back down to Earth – or at the very least show that we would get a very different picture of "infinity" by looking at certain non-mathematical uses of '$\aleph_0$'. However, it does still leave open the question of how the word "infinite" has come to be associated with a special kind of hugeness, thus encouraging the class member's feeling that even a child mastering an infinite technique involves "an element of hugeness and [that] one has the idea of something huge". So, to fully understand Wittgenstein's response to the concern, we first need to see how he diagnoses the source of these misleading associations with the word 'infinite'.

Just as in Z and PI, Wittgenstein insists that we must look to the use of a word in order to understand its meaning, rather than the pictures with which it might be associated.

> If you want to understand a word, we always say: "You have to know its use." It is immensely important that to the great majority of words there correspond certain pictures which in some sense or other *represent* for us the meaning of the word. – In one sense this is clear: a picture of a chair may stand for the word "chair" and so on. In the case of "chair" that picture is of enormous use; it is actually used to explain the word—or to explain what a 'Chippendale' is, for instance [i.e., a piece of furniture in the 18th century English style of Thomas Chippendale]. Once we are shown this, we are sure to use the word in the same way. (LFM 252)

So Wittgenstein is granting at the outset that the meanings of some words are usefully represented by pictures, as in the case of a picture of a chair representing the meaning of the word "chair" or more specifically a "Chippendale". Problems arise, however, when we overgeneralize the usefulness of pictures in illustrating the meanings of other words.

> But in other cases these pictures are more or less misleading or useless. For instance, the picture of a 'particle' can be extremely misleading—where the expression is no longer applied in such a way that this picture is of any use. You cannot explain how "particle" is used in physics by pointing to, say, a grain of sand; indeed, if you did that, you'd make a hash of it. (LFM 252)

Perhaps one could illustrate a certain activity of particles by creating a show involving the movement of grains on a projector, but Wittgenstein is surely right that one does not explain what a particle is, what 'particle' means in physics, or how we use 'particle' in such contexts merely by pointing to a grain of sand. So, if the word 'particle' suggests a picture of a grain of sand to some person or other, this would hardly constitute an understanding of the meaning of this word. A much better sign of their understanding would be their effective use of that word in physics. Again, the general point being made here is clear enough: in *many* cases, a picture associated with a word is misleading and a better source of understanding comes from examining the uses of the relevant word.

Wittgenstein suggests that there are many such misleading pictures in mathematics. One major source of such pictures comes from hasty analogies between distinctive uses of some symbol or operation in different mathematical calculi (such as arithmetical operations symbolized by '+' and 'x' used in finite arithmetic by contrast with their uses in transfinite arithmetic).

> There are many such pictures in mathematics.—A calculus for us is something we have learned. Every one of us, whether he is a mathematician or not, first learned to multiply in this way—as we do when we get 30 x 30 = 900. So that is what is first and foremost in everyone's mind, directly he hears of multiplication. And similarly with addition. So now if someone says $\aleph_0 + 1 = \aleph_0$, we get the old picture of adding something to something. Whether this picture is misleading or not will depend on the further use one makes of, say, $\aleph_0$. (*ibid*)

So, in a typical educational trajectory one begins by learning multiplication and sums such as 30 x 30 = 900 and 3 + 4 = 7. These calculations are themselves associated with various illustrations and images – the most obvious being collections of blocks or other simple objects. We can create a grid with 5 rows and 5 columns and count out 25 to illustrate 5 x 5 = 25. We can take a group of 3 blocks and a group of 4 blocks and put them together to illustrate 3 + 4 = 7. One could go on for quite a while surveying the various pictures and illustrations associated with these calculations as they feature in early mathematical education. In later life perhaps one learns about Cantor and transfinite arithmetic. As soon as they see $\aleph_0 + 1 = \aleph_0$, they can't help the thought that this is an addition just like the addition they learned in grade school. Perhaps they even bring to mind something like an "$\aleph_0$ collection of blocks" (whatever that might be) combined with 1 block to make sense of the result. (Maybe they imagine something like the single block being "swallowed up" by the vast and unruly collection of "$\aleph_0$ blocks", making it remain the same "size" as before – akin to there being no discernible difference to an ocean when a tiny drop is added to it.) But Wittgenstein thinks these associations are more or less misleading depending on the further use we make of '$\aleph_0$'. Its use is not simply read off of grade school arithmetic and its associated illustrations – it requires first developing the mathematics around it (as in Cantor's case) and then surveying what has been developed (as in our post-Cantorian situation).

The problem is that however different are the uses and applications of terms in grade school arithmetic by contrast with their counterparts in transfinite arithmetic, they are couched in a similar vocabulary. These similarities encourage misleading associations and pictures of what is going on in the transfinite operations.

> If I say "the cardinal number of all cardinal numbers" or "the cardinal number of the concept 'cardinal number'" – this is an expression which reminds us of *ten* thousand other expressions, like "the cardinal number of all the chairs in this room". It conjures up all sorts of pictures—for example, the picture of an *enormous colossal* number—which gives it a great charm. And to say that there is a subject treating of this number and of greater numbers—we are dazzled by the thought. (It is not only children that ask, "What's the greatest so-and-so?") (LFM 253)

Ignoring the many differences between finite and transfinite arithmetic, we might see the expressions "the cardinal number of all cardinal numbers" and think it is analogous to an expression like "the cardinal number of all the chairs in this room". In other words, they are both about a collection of objects (cardinal numbers, chairs) but they are different in that the former number is *colossal* and refers to a *massive* collection of objects. The expression thus conjures up "all sorts of pictures" which give it a "great charm". Perhaps we imagine a row of chairs as massive and endless as the cardinal numbers (c.f., Z 272: "an endless row of trees"), which gives us the sense that something incredibly vast is referred to by '$\aleph_0$' and so we are "dazzled by the thought". But these are just pictures *suggested* by the expression "the cardinal number of cardinal numbers" when we unwittingly compare it to, say, "the cardinal number of chairs". When we note that they take place in different calculi and thus have quite different uses and applications (both within and outside mathematics), we see that we have no right to use the same imagery in both cases.

> Then if you realize that forming the expression "cardinal number of the concept 'cardinal number'" you haven't yet given it any application at all, you see that you have as yet no right to have any image. Because the imagery is connected with a *different* calculus, 30 x 30 = 900. (*ibid*)

The major diagnosis of our wonder at the vastness of '$\aleph_0$' is thus that we unwittingly analogize it to a different calculus which is itself associated with various pictures. The antidote is to examine the calculus in which '$\aleph_0$' is at home and to draw out the many differences between this and other calculi.

Wittgenstein then returns to his earlier suggestion for getting "the right image for $\aleph_0$".

> If you want the right image for $\aleph_0$, you musn't form it from mathematics. If you say "How terrific!", if your head reels—you can be sure it is the wrong image. It's not terrific at all. (*ibid*)

Simply looking at '$\aleph_0$' within mathematics can encourage various pictures of a colossal mathematical reality – in part because it is all too easy to analogize it to other similar looking branches of mathematics (such as finite arithmetic) and the various pictures that are associated with them. We can get a better picture by looking at its non-mathematical applications.

> If we say of a child who has learned to multiply that he has learned $\aleph_0$ multiplications, then we have the right imagery. But not if we have the image of a line of $\aleph_0$ lime trees, which we cannot see the end of. (LFM 253-4)

This clearly connects up with the discussion in Z 272 of "an endless row of trees". This is a suggestive image, but one which does not wear its application on its sleeve. (What exactly does one imagine here? How does one identify a row of trees as one that is 'endless' in the relevant sense? We can "give free rein to our imagination" (Z 275), but then it remains how the picture that comes to mind is supposed to be applied.) A better image comes from considering someone who has learned how to multiply and thus "has learned $\aleph_0$ multiplications" – a situation that is fairly ordinary, involves nothing "vast", and certainly doesn't inspire wonder. (Notwithstanding the objection considered earlier which we will return to at the end of this section.)

But why is it so easy – even for mathematicians who are quite familiar with the complex applications of '$\aleph_0$' in transfinite arithmetic – to get caught in the web of associated pictures that Wittgenstein thinks are misleading? (C.f., RFM II, 15: "A clever man got caught in this net of language! So it must be an interesting net.") Wittgenstein answers this question by having us consider "Professor Littlewood".

> I know what I'll say now will sound awful. Certain parts of mathematics tend to be regarded as specially deep. Professor Littlewood in one of his books talks like this: The part of mathematics with which he is dealing is extraordinary, not in that it contains complicated calculations, *but that the depth lies in what is said* [emphasis added]. The beauty of the subject lies not in the calculations, which are as simple as anything, but in the *meaning* which they have.
>
> Now I say that the only meaning they have in mathematics is what the calculation gives them. And if it's simple, it's simple. (LFM 254)

Wittgenstein expresses a theme here which is frequently repeated in the RFM: Do not look to the verbal descriptions of mathematical *conclusions* to understand what they mean, but instead look to the calculations, proofs, and applications which together make up their use.[75] Littlewood is pointed out as an example of someone who thinks that the depth and meaning of mathematics is to be found "*in what is said*" – e.g., "the set of real numbers is *uncountable*" – rather than in the calculations surrounding these expressions. The verbal descriptions of these results have a certain charm about them and tend to fire one's imagination. But, according to Wittgenstein, the relevant calculations are

[75] This contrasts with readings on which Wittgenstein simply identifies the meaning of a mathematical proposition (or its "truth") with its proof – which, as emphasized above, is just one aspect of its use. See especially Dummett (1978) and Moore (2016).

simple and unmysterious. We can thus mitigate the misleading effects of "what is said" on our imagination by looking to these simple and unmysterious uses of the relevant expressions. Treating these as merely secondary to the real depth of mathematics – as Littlewood allegedly does – makes it appear "As though we had to see through the calculations to a depth beyond" (LFM 254).

> —This I want to say is most misleading. The calculus (system of calculations) is what it is. It has a use or it hasn't. But its use consists either in the mathematical use—(a) in the calculus which Littlewood gives, or (b) in other calculi to which it may be applied—or in a use outside mathematics. It is as pedestrian as any calculus […]. If you think you're seeing into unknown depths—that comes from a wrong imagery. The metaphor is only exciting as long as it is fishy. (*ibid*).

Wittgenstein's major point, then, is that (even for sophisticated mathematicians) the conclusions and descriptions of mathematical results can seem to wear their meanings on their sleeves, inviting us to look into the great depths of mathematical reality independently of the calculations which lead us to them. But these come from a misleading imagery, which arises from looking at these conclusions and descriptions in isolation from their uses (calculations, proofs, and applications within and outside mathematics). This is "most misleading" as it allows one's imagination to whirl around various pictures associated with the verbal expressions and their apparently analogous counterparts ('counting', 'adding', 'multiplying') in quite different calculi. This in turn encourages a misbegotten sense that one is thereby looking into depths beyond the calculations and thus the true (however hidden) meanings of mathematical expressions.

Wittgenstein illustrates the importance of distinguishing between the applications of a calculus from the pictures it might suggest with an instructive example from the history of mathematics.

> Where does a calculus (system of calculation) take its interest from? It may be (a) from an application of it, (b) from the pictures which go with it and which arise from certain analogies which this calculus has to other calculi.
>
> Take the infinitesimal calculus. The idea that it deals with infinitely small things gave it a charm quite apart from its usefulness. And although this idea has now been abandoned, it still has a charm.

> When people criticized the earlier idea, they sometimes said, "When we look into the calculus, we find that there is nothing infinitely small there." But what did they expect to find? Why were they disappointed? What does it mean to say that the calculus doesn't treat of anything infinitely small? Or that "there isn't such a thing as anything infinitely small"? or: "We look at these calculations and we don't see anything infinitely small"?—This might be contradicted. Why shouldn't you say *dx* was infinitely small? (LFM 254-5)

If one takes whatever picture they associate with the expression "infinitely small" as prior to its applications and ultimately determining its meaning, they can get puzzled about how the infinitesimal calculus could possibly be telling us about anything "infinitely small". (Compare our earlier discussion of Z 273: "Thought can as it were *fly*, it doesn't have to walk.") But the idea that it deals with "infinitely small things" has since been abandoned, especially given that the enormous usefulness of the infinitesimal calculus has swamped out such scholastic concerns regarding its alleged subject matter. According to Wittgenstein, the initial charm that arose from considering whether the infinitesimal calculus is about "infinitely small things" arose from a verbal play of images surrounding this phrase, rather than anything to do with the work that is actually performed with this calculus. Since the pictures one happens to associate with the expression "infinitely small" say little if nothing about its application, we can just as easily look at *dx* and say, "Here is an example of something infinitely small", i.e., "In *this context*, this is what we are going to call "infinitely small"".[76] The development of this branch of mathematics was thankfully neither dictated nor undermined by whatever prior images one might have had of "infinite smallness".

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[76] My reading thus contrasts significantly with Moore's (2011): "It may be possible […] to say how we have arrived at the grammar, why it takes the form it does, what it enables us to do, and suchlike, in thoroughly finite terms, that is in terms of our finite capacities, our facility with manipulating finite symbols, our skill in both giving and understanding finite instructions, the applications that we make of our grammar in characterizing finite objects, and suchlike. […] It is scarcely surprising, then, that time and again we find Wittgenstein discussing our grammar of the infinite in just this way, with a self-conscious and pointed emphasis on its several *finite* features. […] So great is Wittgenstein's concern to characterize the grammar of the infinite without redeploying it, in particular without mention of any infinite subject matter, that he does at times visibly struggle to maintain his grip on the grammar" (Moore, 2011: 113 & 115). In response to Moore's claim that Wittgenstein is explaining "the infinite" via "*finite* features", Wittgenstein would rather ask, "Why not call these features "*infinite*"?". Moore imposes his own antecedent conception of "the infinite" versus "the finite" on Wittgenstein's remarks, whereas Wittgenstein would grant that "the infinite" is whatever is so-called in the language-game under discussion. Wittgenstein's point is *not* that "the infinite" can be *explained* in "finite terms", but rather that looking to its ordinary applications (e.g., to John's mastery of $\aleph_0$ multiplications) shows that "the infinite" is

> Similar confusions arise for "the infinite", which lend themselves to similar resolutions.
>
> Similarly with "the infinite". "We aren't talking of anything you would call *big*, and therefore not of anything infinite."—But as long as you try to point out that we are not treating of anything infinite, this means nothing, because why not say that this *is* infinite? What is important is that it is nothing *big*. (*ibid*)

To say that transfinite arithmetic does not *really* include anything "infinite" because it contains nothing which we should call "big" is, once again, to allow ourselves to put a suggestive picture of infinity ahead of its applications. We might legitimately say in response, "This *just is* what we call "infinite" in this context. But, sure, you are right to say that it is nothing "big" in the way we might use that word to talk about bigger or smaller bags of potatoes." With only its associated picture and in isolation from whatever applications the word "infinite" might have, it "means nothing" to say "we are not treating of anything infinite here", since we have (at this point) yet to decide how we are using the word "infinite" in this context. When those applications have been decided, we are right to say, "That is what we *call* "infinite" in *this* context (i.e., the context of *this* calculus)." The associations are difficult to shake only because "infinity" is so often explained as something huge – a picture which one is then tempted to carry around and use as a measure of its rightful application in various quite different contexts.

> When one is a child, "infinite" is explained as something huge. The difficulty is that the picture of its being huge adheres to it. But if you say that a child has learned to multiply, so that there is an infinite number of multiplications he can do—then you no longer have the image of something huge. (*ibid*)

This conclusion then inspires the class member's concern about the hugeness involved in one's mastery of '$\aleph_0$' multiplications: "… there is even there an element of hugeness and one has the idea of something huge." But now we can fully make sense of Wittgenstein's dismissal of this concern, as

nothing to marvel at. Another way of putting the point is that Moore equates the unmysterious with "the finite" *by contrast with* "the infinite" (and sees Wittgenstein as explaining the latter in terms of the former), whereas on my reading Wittgenstein is showing more simply that "the infinite" is unmysterious. It is also possible that Moore is subject to the very confusion that Wittgenstein diagnoses in LFM, namely, that of thinking "the infinite", *unlike* John's mastery of $\aleph_0$ multiplications, must be something "huge".

arising "only from the word "infinite" and not from what it's used for" (LFM 256). To think that John's mastery of multiplication involves something mysterious and colossal is to project a picture of "infinity" onto that situation. Since, after all, before one happened to be thinking about "infinity", it is difficult to see why one would be puzzled about a child learning how to multiply or why one should think of the child's understanding as something "vast". The word 'infinite', with its various associated pictures, more or less misleading, might encourage one to see their achievement as "huge" or "colossal", but (if Wittgenstein is right) we can now see that for what it is. That is, the example does not involve a mysterious instance of someone's *vast* intellect, but only appears so due to these more or less misleading associations with the word 'infinite'.

Wittgenstein's critique of "the infinite" as it features in Z and LFM can thus be summarized as follows. The words 'endless' and 'infinity' tend to fire our imagination as a result of the various pictures they are associated with. The mystifying pictures which arise, however, tell us nothing about the uses of these words. Looking to the uses of them (especially through their related calculations, proofs, and applications in and outside mathematics) tends to demystify one's sense of "the infinite" – especially when the analogies unwittingly made across different calculi are highlighted and dislodged by reminding ourselves of the many differences between them. "The infinite" is nothing mysterious when it is seen in its specific applications. It only seems mysterious when we pretend that the pictures associated with this word are the sole arbiter of its meaning. Hence Wittgenstein's hope that "by trying to turn your attention away from certain expressions, illustrations, images, and *toward the employment* of the words" (Z 463), such marvel, mystery, and puzzlement regarding "the infinite" can be put to rest.

3.4 "Philosophers constantly see the method of science before their eyes"

Thus far, we have seen that according to Wittgenstein talk of "the infinite" takes a hold on one's imagination via suggestive yet misleading analogies between different areas of discourse (within and outside mathematics). As we'll see in Section 5, MS 117 repeats this general theme. But a full appreciation of that text (as much as the foregoing) requires highlighting an important element of Wittgenstein's later philosophy. For consider that while the concerns discussed in the previous sections highlight the effects of word play on our imagination, they do not necessarily help to explain why one might take such imaginings so seriously. Reading fantasy novels presumably has a greater power of firing one's imagination, but we do not ordinarily take such imaginings as having any particular bearing on our understanding of "reality". Indeed, we are perfectly comfortable treating "unicorn talk" in the context of a novel as "mere fantasy", rather than suggesting a special realm of entities, i.e., "unicorns", which requires philosophical explanation. Why then do the effects of "infinity talk" on one's imagination appear to demand explanation in a way that "unicorn talk" does not?

The most obvious answer seems to be contained in the very statement of the problem. "Unicorn talk" is a matter of fantasy; "infinity talk" (in some contexts, such as the work of Cantor) is a matter of rigorous science. Science purportedly tells us about reality; fantasy novels do not. The claims made in the context of science deserve a kind of respect that is not owed to claims made in the context of fantasy. (This general respect for science presumably informs Professor Littlewood's claims in the previous section about the special "depth" of mathematical "results".)

However innocuous such an answer might initially seem, the widespread veneration of science and its methods is marked as a common source of philosophical confusion in Wittgenstein's later philosophy. In the BlB, Wittgenstein claims that "Philosophers constantly see the method of science before their eyes" and cites this phenomenon as "the real source of metaphysics".

> Our craving for generality has another main source: our preoccupation with the method of science. I mean the method of reducing the explanation of natural phenomena to the smallest possible number of primitive natural laws; and, in mathematics, of unifying the treatment of different topics by using a generalization. Philosophers constantly see the method of science before their eyes, and are irresistibly tempted to ask and answer questions in the way science does. This tendency is the real source of metaphysics, and leads the philosopher into complete darkness. I want to say here that it can never be our job to reduce anything to anything, or to explain anything. Philosophy really *is* 'purely descriptive'. (BlB 18)

Philosophers respect the methods of science. (And why shouldn't they? Scientific methods have yielded astounding progress!) By their lights, science operates by reducing complex phenomena to the smallest number of laws. In mathematics, this goes by way of unifying different topics via definition and generalization. Science understood in this way encourages philosophers to do the same: to aim at an understanding of some apparently complex topic by unifying it with generalizations and reducing it to the smallest number of principles. Wittgenstein encourages us to replace such an attempt at unification with a "purely descriptive" characterization of the relevant phenomena – however disunified those descriptions might turn out to be.

The contrast intended here, as well as its relevance to the philosophy of mathematics, is sharpened in the next paragraph.

> Instead of "craving for generality" I could also have said "the contemptuous attitude towards the particular case". If, e.g., someone tries to explain the concept of number and tells us that such and such a definition will not do or is clumsy because it only applies to, say, finite cardinals I should answer that the mere fact that he could have given such a limited definition makes this definition extremely important to us. (Elegance is *not* what we are trying for.) For why should what finite and transfinite numbers have in common be more interesting to us than what distinguishes them? Or rather, I should not have said "why should it be more interesting to us?"—it *isn't*; and this characterizes our way of thinking. (BlB 18-19)

Philosophy, for Wittgenstein, is "purely descriptive" in that it attends to particular examples in detail – with no aim to generalize or reduce said examples to a small number of principles (c.f., PI 109). Whereas a philosopher preoccupied with "the method of science" might want to attend to the similarities between finite and transfinite numbers in order to reduce them both to the smallest number of principles (e.g., a single, unified definition of 'number' à la Frege), Wittgenstein encourages us to describe each type of number on its own (perhaps providing definitions of each that need not apply to the other) and give just as much attention to the differences between them. I discussed in the previous sections why drawing out differences between various examples (say, of different kinds of number, or different uses of arithmetical operations such as '+' and 'x') is methodologically crucial for Wittgenstein, as it allows us to dislodge suggestive yet misleading analogies between them. A philosopher "preoccupied with the method of science", however, "craves generality" and thereby has a "contemptuous attitude towards the particular case". So, the misleading assimilation of disparate phenomena is, by Wittgenstein's lights, especially encouraged and taken seriously when this is seen as part and parcel of doing either rigorous science or the kind of philosophy that is modeled after it. This is not to take issue with science *per se*, but instead with the uncritical acceptance of its alleged results simply on the grounds that they are based on "the method of science" and with the overextension of that method into philosophy. (It is one thing to grant some success for "the method of science", and quite another to grant that it will be successful everywhere – or to accept a "result" at face value simply on the grounds that it was produced via "the scientific method".)

The foregoing serves as helpful background for my reading of MS 117. In this text, Wittgenstein attends more specifically to the details of Cantor's proofs regarding infinity (especially what has come to be known as the method of "diagonalization") as a source of misbegotten pictures of the infinite. Cantor's work is widely regarded as a serious contribution to our understanding of infinity in mathematics as well as philosophy – showing us to some extent what "infinity" is *really* like. Indeed, the mathematical rigor of his work is itself often seen as putting certain philosophical qualms about "the infinite" to rest – legitimating "the infinite" as amenable to rigorous scientific inquiry. As A.W. Moore puts it,

> If [a rigorous, systematic, unified, mathematical theory of the infinite] is [available], that is if we can talk about the infinite in a way that is both coherent and precise, then we shall have the most effective possible vindication of the actual mathematical infinite against the weight of skeptical tradition.
>
> […] Georg Cantor (1845-1918) […] was a mathematician of genius who devised a theory of precisely this kind. Only Aristotle and Kant are as important in the history of thought about infinity. In terms of his mathematical contribution to the topic, nobody can hold a candle to him. He was years ahead of his time. He single-handedly paved the way for a whole new branch of mathematical enquiry and research. (Moore, 1990: 111-112)

Moore praises Cantor's work (in a way I presume is not uncommon[77]) on the grounds that it provides a "rigorous, unified, systematic, mathematical theory of the infinite". It is thus first and foremost the form that Cantor's work takes (rigorous, unified, and systematic) which sets it apart from other attempts at understanding infinity and thereby makes it especially deserving of our respect. It is the preoccupation with such a form (i.e, a "preoccupation with the method of science", "constantly seeing the method of science before one's eyes") that Wittgenstein regarded as "the real source of metaphysics". Such a concern is applied directly to Cantor's work in Wittgenstein's "Lecture on Aesthetics":

> C.f. the expression "The Cardinal number of all Cardinal numbers".
>
> C.f. Cantor wrote how marvelous it was that the mathematician could in his imagination transcend all limits.
>
> I would do my utmost to show it is this charm that makes one do it. [*In Rhees' notes:* I would try to show that it is this charm which makes the proof attractive.] *Being mathematics or Physics* [my emphasis] it looks incontrovertible and this gives it a still greater charm. If we explain the surroundings of the expression we see that the thing could have been expressed in an entirely different way. I can put it in a way in which it will lose its charm for a great number of people and certainly will lose its charm for me. [*In Rhees' notes:* If I describe the surroundings of the proof, then you may see that the thing could have been expressed in an entirely different way; and then you see that the similarity of $\aleph_0$ and a cardinal number is very small. The matter can be put in a way which loses the charm it has for many people.] (LC 28)

[77] Consider also David Hilbert's praise of Cantor in his famous lecture "On The Infinite": "[A]nalysis alone does not provide us with the deepest insight into the nature of the infinite. This insight is procured for us by a discipline which comes closer to a general philosophical way of thinking and which was designed to cast new light on the whole complex of questions about the infinite. This discipline, created by Georg Cantor, is set theory. In this paper, we are interested only in that unique and original part of set theory which forms the central core of Cantor's doctrine, viz., the theory of *transfinite* numbers. This theory is, I think, the finest product of mathematical genius and one of the supreme achievements of purely intellectual human activity" (Hilbert, 1926: 188).

The fact that one treats Cantor's work as authoritative – qua rigorous science about the infinite – can make its effects on our imagination appear innocuous, as themselves scientifically legitimate and not in need of questioning. Or, worse and perhaps more common, the scientific credentials of Cantor's work might blind us to the effects that it has on our imagination as well as the seductive qualities of those effects. In either case, Wittgenstein's aim in MS 117 is to mitigate these effects through a redescription of Cantor's work – one which "changes the aspect of Cantor's discovery" (RFM II, 17).[78] One might initially see Cantor's proofs as a paradigm of scientific rigor that thus demands our attention and respect; perhaps even as providing insight into the vast realms of mathematical infinity: allowing us to "compare the 'set' of real numbers in magnitude with that of the cardinal numbers" (RFM II, 22). Wittgenstein, by contrast, "believe[s], and hope[s], that a future generation will laugh at this hocus pocus" (*ibid*).

Wittgenstein's orientation to Cantor's work described thus far, however, might seem inconsistent with his earlier methodological pronouncements. In the earlier sections, we saw that Wittgenstein recommended dampening the effects of infinity talk on our imagination by looking to the role of 'infinity' in mathematical proofs and calculations. But Cantor's results, e.g., that "the set of real numbers is bigger than the set of cardinal numbers", are themselves backed by proofs and calculations. How then can Wittgenstein take issue with Cantor's work without taking issue with the very proofs and calculations that he thinks will clear things up for us?

The general answer, which can only be clarified by looking at particular examples of his strategy at work, is that Wittgenstein takes issue with misleading *descriptions* of Cantor's proofs, which often stem from disproportionate attention to the *verbal statements* of their so-called "results" rather

---

[78] As Monk (2007) puts it, "[Wittgenstein's] aim is to get us to *see* mathematics differently, to see it under a different aspect. This will 'leave mathematics as it is' in the sense that switching from seeing the duck to seeing the rabbit leaves the picture as it is. In another sense, though, it will have a radical effect, since, Wittgenstein believes, the "charm" of, for example, Cantor's diagonal proof, *depends* upon seeing mathematics in a particular way" (Monk, 2007: 285).

than the actual details of the proofs themselves. Further, to emphasize what has already been stated previously, Wittgenstein takes issue with misleading comparisons that are often made unwittingly between words and phrases used within Cantor's proofs (such as 'big', 'number', 'series', 'countable', 'collection', 'denumerable', among others) and outside those proofs – i.e., in other parts of mathematics as well as in non-mathematical contexts. Wittgenstein aims to mitigate these sources of confusion through more careful descriptions of the proofs, ones which attend to the broader mathematical surroundings of Cantor's "results" and which clearly draw out the differences between its uses of certain key vocabulary with uses of that vocabulary in other contexts. We will see that he also attempts to disrupt the sorts of analogies that fire one's imagination by replacing them with other analogies which make Cantorian infinity appear quite mundane, perhaps even boring. This is all in the service of showing, as he famously puts it in the LFM, that Cantor's work "is not a paradise—so that you'll leave of your own accord" (LFM 103).

In the next section, we'll see how Wittgenstein attempts to invoke such a change in aspect by looking at some of his descriptive and comparative interventions in action.

3.5 Diagonalization Redescribed: MS 117

Cantor is widely considered to have shown that not all infinite sets are the same size. However vast the set of natural numbers might have seemed, its vastness can be exceeded, e.g., by the set of real numbers. As A.W. Moore puts it,

> So, *are* all infinite sets the same size? […] Cantor was able to prove that they are not. All infinite sets are big, but some, it transpires, are bigger than others.
>
> (Moore, 1990: 118-19)

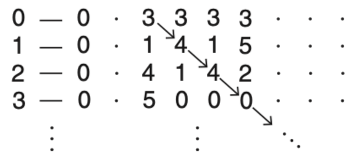
More specifically, while the natural numbers are enumerable – i.e., "can be arranged in a single list with a first entry, a second entry, and so on, so that every member of the set appears sooner or later on the list" (Boolos, Burgess, & Jeffrey, 2007: 3) – the real numbers are *not* enumerable. The reason typically given is that the set of reals is "too big" to be counted. Boolos, Burgess, & Jeffrey (BBJ) open their textbook chapter on the subject saying, "Not all sets are enumerable: some are *too big*" (2007: 16, emphasis added). These general upshots – about the relative sizes of different infinities – are included in Moore's presentation of Cantor's proof via the famous method of diagonalization. His presentation (which is more or less standard) will serve as a helpful comparison with Wittgenstein's remarks and so I reproduce it here (emphasis is added to phrases that are particularly relevant to Wittgenstein's critique of diagonalization).

> *Cantor's Diagonal Argument:* Each real number between 0 and 1 can be expressed by means of an infinite decimal expansion, beginning with a 0 and a decimal point; this allows for the case of an expansion that terminates with a never-ending sequence of 0's. Thus
>
> 1/3 = 0.3333 …;
> π – 3 (that is, the decimal part of π) = 0.1415 …;
> $\sqrt{2}$ – 1 (that is, the decimal part of $\sqrt{2}$) = 0.4142 …;
>
> and
>
> ½ = 0.5000 … .
>
> Now consider *any* pairing off of the natural numbers with a selection of these reals. For the sake of argument let us suppose that 0, 1, 2, and 3 have been paired off with the four reals just mentioned, in that order. Such a pairing off can be represented as an 'infinite square', as shown in [Figure 3.1].

```
0 — 0 · 3 3 3 3 · · ·
1 — 0 · 1 4 1 5 · · ·
2 — 0 · 4 1 4 2 · · ·
3 — 0 · 5 0 0 0 · · ·
  ⋮       ⋮
```

[Figure 3.1]

> Suppose next that we start with the first digit in the decimal expansion of the first real in this 'square', then move to the second digit in the decimal expansion of the second real, and so on indefinitely down the 'square diagonal', as shown in [Figure 3.2].



[Figure 3.2]

> If, each time we arrive at a new digit, we write down a 3 if that digit is a 4, and a 4 if that digit is anything other than a 4, then we shall find ourselves writing down *what may itself be regarded as the decimal expansion of a real number between 0 and 1* [emphasis added]. In this particular example the expansion will begin with a 4, two 3s, and a 4. That is, the resultant real will be 0.4334 …. Now, has *this* real been paired off with any natural number? That is, is it itself one of the reals listed in the 'square'? No. It has been so defined that it differs from the first of the reals in its first decimal place, from the second in its second, from the third in its third, and so on *ad infinitum*. It cannot itself be any one of them. What this shows is that, *regardless* of what pairing off we start out with, at least one real must inevitably be passed over. It will always be possible to define such a real by means of this kind of 'diagonalization'. The natural numbers cannot after all be paired off with the real numbers between 0 and 1. *There are more of the latter* [emphasis added]. The set of real numbers between 0 and 1, and *a fortiori* the set of all real numbers, *is bigger than the set of natural numbers* [emphasis added]. (Moore, 1990: 119-20)[79]

As noted by BBJ (2007), the proof lends itself to imaginative extravagance (much in the ways we have seen highlighted by Wittgenstein). They indulge in such extravagances for explicitly pedagogical reasons in the following manner – distinguishing a sober characterization of the proof (though their emphasis on "existence" would certainly be grist for the Wittgensteinian mill[80]) with a "theological" description of its significance.

> Of course, it is rather strange to say that the members of an infinite set 'can be arranged' in a single list. By whom? Certainly not by any human being, for nobody has that much time or paper; and similar restrictions apply to machines. In fact, to call a set enumerable is simply to say that it is the range of some total or partial function of positive integers. … [T]here is no need to refer to the list, or to a

[79] Compare also Papineau (2012): "The example of the reals shows that infinite sets come in *different sizes*. There is the size shared by all the denumerable sets. But the real numbers are *bigger* again" (27, emphasis added); "There is the infinite number that characterizes the denumerable sets, and the distinct and *bigger* infinite number that characterizes all the sets whose members can be paired up with the real numbers" (32, emphasis added). The fact that these are all more or less standard textbooks for philosophy students helps to illustrate the ubiquity of this verbal imagery.

[80] Further, Wittgenstein would have seen appeals to "gods" and the like to be idle devices for explanation, since "Even God can determine something mathematical only by mathematics" (RFM VII, 41).

> superhuman enumerator: anything we need to say about enumerability can be said in terms of the functions themselves; for example, to say that the set *P** is not enumerable is simply to deny the existence of any function of positive integers which has *P** as its range.
>
> Vivid talk of lists and superhuman enumerators may still aid the imagination, but in such terms the theory of enumerability and diagonalization appears as a chapter in mathematical theology. To avoid treading on any living toes we might put the whole thing in a classical Greek setting: Cantor proved that there are sets which even Zeus cannot enumerate, no matter how fast he works, or how long (even, infinitely long). (BBJ, 2007: 19)

Such a "chapter in mathematical theology" may indeed serve important pedagogical purposes. For instance, keeping students engaged in otherwise dry, technical material might be aided by fanciful illustrations such as this. The details of such a metaphor might also allow the teacher (in fanciful terms) to illustrate connections that "diagonalization" or "enumerability" has to other parts of science (perhaps this can be said of BBJ's theological elaboration that follows the quote above). Whatever the pedagogical merits of this "theological" approach, the authors are more or less in agreement with Wittgenstein here in thinking that one should not be misled by these merely pedagogical flourishes and remember that the results can be stated in more sober and less misleading terms, however less exciting those terms might seem. But, as we will see throughout the remainder of this section, Wittgenstein would say that neither Moore nor BBJ have flagged a major source of imaginative extravagance in Cantor's work as it is typically characterized: namely, the emphasis on the "bigness" of infinite sets.[81]

In MS 117, Wittgenstein discusses the method of diagonalization in order to undermine the various misleading effects it might have on one's imagination. He begins by comparing the method of diagonalization as it is typically deployed – i.e., as above, to show that some infinite set is "bigger" than another – with an atypical deployment of that method.

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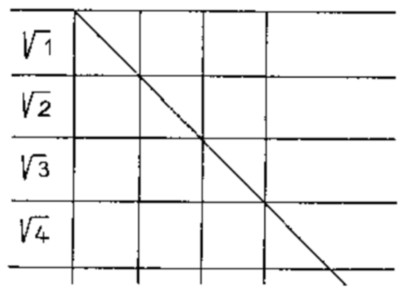
[81] Though Moore himself admits this elsewhere – or at least recognizes that this is a concern of Wittgenstein's. See Moore (2016: 326): "We feel a certain heady pleasure when we are told that some infinite sets are bigger than others. We feel considerably less pleasure when we are told that certain one:one correlations yield elements that are not in their ranges."

> How far does the diagonal method prove that there is a number which – let's say – is not a square root [of a natural number][82]? It is of course extremely easy to show that there are numbers that aren't square roots – but how does *this* method show it [as shown in Figure 3.3]?



[Figure 3.3]

> Have we a general concept of what it means to show that there is a number that is not included in this infinite set? (RFM II, 1)

Wittgenstein's question at the end of the quote shows what the comparison is meant to illustrate. Namely, that there is something misleading in saying that there is a *general* concept of "showing that there is a number not included in some infinite set".[83] On the one hand, we might say, "Sure, there's a general concept: we can use this general expression for a variety of different cases". On the other hand, if we consider the variety of examples falling under this "concept", we will find that they do not have all that much in common, aside from the trivial feature of "showing that there's a number not included in some infinite set". One would be hard pressed to come up with a singular rule, definition, or essence that somehow governed in each and every instance the application of this phrase. Thus, Wittgenstein has us consider several different cases in order to break one's temptation

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[82] This addition strikes me as the simplest way to make sense of the passage and does nothing to interrupt the major philosophical point, which I will explain in a moment. More specifically, it avoids the natural objection that one can take the square root of any number they like, not just the natural numbers that are listed in the figure below. If the "proof" is merely intended to "show" that "there is a number which is not a square root *of a natural number*", then the objection is irrelevant.

[83] I think (by his own lights) it would be a merely verbal disagreement whether Wittgenstein intended for us to answer "No" to this question. Is there a general concept? Sure, but for Wittgenstein this is akin to saying that 'game' is a general concept (PI 66-67): a common word used for a variety of different examples that are not unified by a strict definition or essence. Maybe for some this means that there is no *general* concept. But that depends on what one calls a "general concept".

to think otherwise.[84] Independently of the diagonal method, it is "extremely easy to show that there are numbers that aren't square roots". So, we can compare the odd use of diagonalization above with methods which more easily come to mind. Wittgenstein describes more natural methods at the end of the passage.

> Let us suppose that someone had been given the task of naming a number different from every $\sqrt{n}$; but that he knew nothing of the diagonal procedure and had named the number $\sqrt[3]{2}$; and had shown that it was not a value of $\sqrt{n}$. Or that he had said: assume that $\sqrt{2} = 1.4142\ldots$ and subtract 1 from the first decimal, but have the rest of the places agree with $\sqrt{2}$. 1.3142 cannot be a value of $\sqrt{n}$. (*ibid*)

Both of these proof strategies are more natural in part because they deploy elementary methods of calculation to get the desired "result". The first strategy, involving the $\sqrt[3]{2}$, gives us a specific and familiar number which we can easily see is not included in the list of $\sqrt{n}$'s above. The second likewise gives a specific number that involves a basic subtraction from $\sqrt{2}$ (and then a similar comparison with the list). The diagonal method, especially the bare bones sketch above, gives us a "number", but it is defined in terms that are quite different from numbers we might ordinarily encounter in arithmetic, i.e., $\sqrt[3]{2}$ and $\sqrt{2}$ with the subtraction of 1 from the first decimal. To think that each proof strategy "shows the very same thing" would be to overlook their important differences, which illustrate different senses of "showing that there is a number not included in some infinite set" as much as they illustrate different senses of "number". These differences

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[84] This seems to address Moore's (2016) objection against Wittgenstein that if mathematicians are engaged in the "*modification* of [pre-existing] concepts" (e.g., 'big'), then "the use of the relevant vocabulary will after all be essential to what mathematicians are doing" (Moore, 2016: 327). The objection is irrelevant if all Wittgenstein hopes to do is to highlight the differences between the uses of certain vocabulary in order to upset the temptation to think they share a singular rule, definition, or essence. One might suspect, on Wittgenstein's behalf, that Moore is thinking of "concepts" in such a way that its instances (i.e., the things we call, e.g., "big") share some underlying essence (c.f., Floyd (1991: 157): "The logical necessity we think of as inhering, in traditional terminology, in the relation between a concept and its instances, and indeed, in the unity of a judgment itself, begins to slip from our grasp"). If that is not true, it is difficult to see why or in what sense the use of such vocabulary is "essential" to mathematical work, as Moore puts it. Though it might be essential to someone's *interest* in the relevant part of mathematics that it is commonly used to describe – and if that's true Wittgenstein's critique is perfectly apt (c.f., RFM II, 62: "What I am doing is, not to show that calculations are wrong, but to subject the *interest* of calculations to a test").

between the various proof strategies make it reasonable to ask, as Wittgenstein does at the beginning of the passage, "*How* does *this* method show that there is such a number?". An answer to that question requires a comparison with other ways of doing so. Further, if someone were to say, "I've shown that there is a number not included in this set!", Wittgenstein's comparison shows why it would make sense for someone to respond, "I do not know yet *what* you have shown until you show me the proof." We should have a similar reaction to someone who tells us that they have shown that "there is a number which is not included in this infinite series of real numbers" – since that could mean various different things depending on the method used; the bare expression alone does not tell us which is the intended method.

But there is another sense in which it makes sense to ask "*How* does *this* method show … ?" At first glance, one might have thought that a method of proof was a mere means of showing something which was independently intelligible, e.g., conveying something about mathematical reality that can be understood independently of the proof. Then there might be a substantive question how the proof succeeds at representing this independent fact about reality, which might just as easily be shown by other means. The next passage, however, indicates that Wittgenstein thinks we should answer the "How" question quite differently.

> "Name a number that agrees with $\sqrt{2}$ at every second decimal place." What does this task demand? The question is: is it performed by the answer: It is the number got by the rule: develop $\sqrt{2}$ and add 1 or -1 to every second decimal place?
>
> It is the same as the way the task: *Divide an angle into three* can be regarded as carried out by laying 3 equal angles together. (RFM II, 2)

The comparison with "dividing an angle into three" gives us another example that is atypical relative to standard mathematics. As it is ordinarily understood by mathematicians, one does *not* "divide an angle into three" or (as it is more commonly put) "trisect the angle" merely by putting 3 equal angles together. This is because, as a matter of stipulation, "trisecting the angle" requires doing so using only an unmarked straightedge and a compass. It was famously proved (by Pierre Wantzel in 1837) that this cannot be done in general. So, to the extent that one can "divide an angle into three" by laying 3 equal angles together, this depends on a stipulation (or lack thereof) regarding what it means to do such a thing. The connection between *this* "proof" and its result is thus a matter of convention, i.e., regarding how we use the words "divide an angle into three". If we asked, "*How* does *this* method show … ?", the answer would be that we defined things in such a way that this is what we should call "dividing an angle into three". What is the connection, then, between "Naming a number that agrees with $\sqrt{2}$ at every second decimal place" and defining a number by the rule: develop $\sqrt{2}$ and add 1 or -1 to every second decimal place? Likewise, a stipulation about what we call "naming such-and-such a number" in this context.

The importance of stipulation and convention in connecting "methods" (of proof) with their "results" is further illustrated with Wittgenstein's depiction of an oddly resistant interlocutor.

> If someone says: "Show me a number different from all these", and is given the rule of the diagonal for answer, why should he not say: "But I didn't mean it like that!"? What you have given me is a rule for the step-by-step construction of numbers that are successively different from each of these. (RFM II, 3)

To be clear, Wittgenstein himself is not to be identified with the imagined interlocutor – but uses their resistance to illustrate something about the relationship between methods and results in mathematics, as the remainder of this section shows.

> "But why aren't you willing to call this too a method of calculating a number?"–But what is the method of calculating, and what the result here? You will say that they are *one*, for it makes sense now to say: the number *D* is bigger than . . . and smaller than . . .; it can be squared etc. etc.
>
> Is the question not really: What can this number be *used* for? True, that sounds queer.—But what it means is: what are its mathematical surroundings? (*ibid*)

The resistant interlocutor is met with a reaction from someone (perhaps a teacher) who insists on the legitimacy of Cantor's proof ("But why aren't you willing to call this too a method … ?"). What both imagined interlocutors seem to overlook is that there isn't a sharp line between "method" and "result" here, or more specifically between the "method" and the "number" it is intended to generate. Wittgenstein suggests we will say that "the method of calculating" and "the result here" are *one*, as this then allows us to perform the sorts of calculations and comparisons that are standard for other "numbers". And given the previous passage, the fact that they are *one* is a matter of stipulation – it is not a substantive or empirical issue how they are connected. The diagonal *as a matter of stipulation* is what we take to be a "number" not included in the provided "list" of reals. The interlocutor's resistance is thus confused because they treat "method" and "result" as separate and their connection as something other than a matter of convention (allowing them to question whether the "method" is *really* successful in showing the "result"). And, as we saw in Sections 2 & 3, it was a mistake to think that the result (stated verbally) could be made intelligible independently of its application – e.g., via a picture that the interlocutor happens to associate with it.

The philosophical upshots of these early passages are summed up nicely in a flurry of succeeding remarks. Remark 4 expresses the important theme that we cannot intelligibly separate "method" and "result" in mathematics – and that if we do we are bound to end up in confusion ("every opportunity of twisting and turning the meanings").[85]

> So I am comparing methods of calculating—only here there are certainly very different ways of making comparisons. However, I am supposed in some sense to be comparing the *results* of the methods with one another. But this is enough to make everything unclear, for in one sense they don't each have a single result, or it is not clear in advance what is to be regarded here as *the* result in each case. I want to say that here we are afforded every opportunity of twisting and turning the meanings.
> (RFM II, 4)

Remark 5 highlights that there are a variety of cases to consider in this territory that are not easily assimilated to one another as long as we give proper attention to their differences.

[85] This contrasts with readings on which Wittgenstein simply and crudely identifies the meaning of a mathematical proposition (or its "truth") with its proof. See especially Dummett (1978) and Moore (2016). My reading of RFM II's philosophical upshots, as highlighted in the series of remarks that follow, should also be contrasted with Floyd (2021): "The issue is how far diagonalization allows us to "transcend the limits" of a system. Wittgenstein's answer, influenced by Turing, is that it does not always do so, but that even when it does not, it may reveal a new aspect of a collection" (Floyd, 2021: 58). The "issue", on my reading, is more importantly and more simply the imaginative extravagance that Cantor's proof naturally encourages – which is tamed via description and comparison.

> Let us say—not: "This method gives a result", but rather: "it gives an infinite series of results". How do I compare infinite series of results? Well, there are very different things that I may call doing that. (RFM II, 5)

Remark 6 emphasizes that in order to avoid these confusions we need to look to the general context in which mathematical "results" are established.

> The motto here is always: Take a *wider* look round. (RFM II, 6)

Remark 7 more specifically encourages us not to be seduced by calculations expressed verbally but to the calculations themselves if we would like to understand the meaning of some mathematical result. Wittgenstein illustrates this lesson with a particularly vivid analogy.

> The result of a calculation expressed verbally is to be regarded with suspicion. The *calculation* illumines the meaning of the expression in words. It is the *finer* instrument for determining the meaning. If you want to know what the verbal expression means, look at the calculation; not the other way about. The verbal expression casts only a dim general glow over the calculation: but the calculation a brilliant light on the verbal expression. (As if you wanted to compare the heights of two mountains, not by the technique of measurement of heights, but by their apparent relation when looked at from below.) (RFM II, 7)

And, finally, Remark 8 illustrates the theme we saw earlier that the connection between "method" and "result" in mathematics is a matter of convention or stipulation (rather than a quasi-empirical connection, as it were, between a "method of verification" and a self-standing and independently intelligible "fact").

> "I want to show you a method by which you can serially *avoid* all these developments." The diagonal procedure is such a method.—"So it produces a series that is different from all of these." Is that right?—Yes; if, that is, you want to apply these words to the described case. (RFM II, 8)

These lessons together inform how Wittgenstein would like us to think about Cantor's proof regarding the relative "sizes" of "infinities". First, don't take the meaning of a mathematical result (e.g., "the set of real numbers is *bigger* than the set of natural numbers") stated verbally as wearing its meaning on its sleeve. Look to the calculation or the proof, along with the broader context of its use, to see what it means. Compare that calculation with others, especially those that, expressed verbally, might on the face of it be seen as having the same or similar results. Otherwise, we will be misled by the merely verbal similarities between those results as well as with other expressions in or outside mathematics – a mistake akin to comparing the heights of mountains via their apparent relation from below rather than the actual techniques of measurement. And most importantly, whatever words are used to describe the result of a mathematical proof, recognize that this requires a stipulation or convention about the use of those words, rather than simply following from some antecedent condition on them (say, via their underlying definition or essence). This means, more surprisingly, that to a large extent the connection between a method and its result (which are not sharply distinguished in the first place) is a matter of convention – rather than a "method" being a mere means to "verifying" some self-standing result (say, about an independent mathematical realm).

This last point – about the role of stipulation and convention in connecting "method" and "result" – is expressed very sharply in the next passage.

> It means nothing to say: "*Therefore* the X numbers are not denumerable". One might say something like this: I call number-concept X non-denumerable *if it has been stipulated that* [emphasis added], whatever numbers falling under this concept you arrange in a series, the diagonal number of this series is also to fall under that concept. (RFM II, 10)

Why does it "mean nothing to say" this? – The use of "*Therefore*" in this expression makes it appear as if "the X numbers are not denumerable" is intelligible independently of whatever precedes the use of "*Therefore*" (i.e., the method of proof). Is it independently intelligible? Well, perhaps we form a picture of "the X numbers being non-denumerable" somehow from examples of things that we (conventionally) call "denumerable". The cardinal numbers are denumerable. We can count through them. To say the X numbers are *not* denumerable is to say they are *not* like that. Of course this is something that we can bring to mind!—But as we saw earlier in Z, it's an illusion to think that we understand a mathematical expression about infinity in this way (Z 272: "Its purpose is not that of causing a fog to rise in your mind"). Whatever one "brings to mind" when they consider "the X numbers are not denumerable", the meaning of *that* will have to be clarified by looking to its use. Hence Wittgenstein's question in RFM II, 12 (my emphasis): "What can the concept 'non-denumerable' *be used for*?". The use is most straightforwardly found in what comes before: i.e., the method. So the "method" (in large part) reveals the meaning of the "result" ("they are *one*"), rather than being a mere means to an independently intelligible proposition. A more adequate description of what is going on here, which removes the potentially misleading use of "*Therefore*", is thus that "*I call* number-concept X non-denumerable *if it has been stipulated that*, whatever numbers falling under this concept you arrange in a series, the diagonal number of this series is also to fall under that concept" (my emphasis).[86]

Perhaps one might object that, even if we can't quite articulate it, we do have an independently intelligible conception of "non-denumerability", given that we have an independent grasp of "denumerability" via specific examples (say, the "denumerability" of the cardinal numbers). But according to Wittgenstein, this is exactly where "the mistake begins".

> The mistake begins when one says that the cardinal numbers can be ordered in a series. For what concept has one of this ordering? One has of course a concept of an infinite series, but here that gives us at most a vague idea, a guiding light for the formation of a concept. For the concept itself is *abstracted* from this and from other series; or: the expression stands for a certain analogy between cases, and it can e.g. be used to define provisionally a domain that one wants to talk about. (RFM II, 16)

The mistake alluded to in this paragraph is the mistake in thinking that we have a concept of "ordering" or "infinite series" clear enough that it would determine precisely what we mean by the expressions "the real numbers can/cannot be ordered". But the analogy one forms from the cardinal numbers and other infinite series (the even numbers, the odd numbers, the negative numbers, etc.)

[86] It is thus difficult to square up RFM II, 10 with Floyd's (2021) insistence that Wittgenstein does *not* think these are matters of stipulation: "the gloss that Cantor proved the uncountability of the real numbers is, so far as the diagonal argument technique goes, neither a logical must nor a mere "stipulation"" (Floyd, 2021: 60).

at most gives us a vague idea as to what "the real numbers can/cannot be ordered" might mean. Further stipulations need to be made in order to bridge the gap from the examples to the application of "orderability" applied to the real numbers. Hence, Wittgenstein thinks that an analogy from these examples does not give a clear sense to the question whether *the real numbers* can be "ordered".

> That, however, is not to say that the question: "Can the set R be ordered in a series?" has a clear sense. For this means e.g.: Can one do something with these formations, corresponding to the ordering of the cardinal numbers in a series? Asked: "Can the real numbers be ordered in a series?" the conscientious answer might be: "For the time being I can't form any precise idea of that".—"But you can order the roots and the algebraic numbers for example in a series; so you surely understand the expression!" [c.f., Ramsey's "asseveration" in Z 272]—To put it better, I *have got* certain analogous formations, which I call by the common name 'series'. But so far I haven't any certain bridge from these cases to that of 'all real numbers'. Nor have I any general method of trying whether such-and-such a set 'can be ordered in a series'. (*ibid*)

Compare our discussion earlier of the question "Have we a general concept of what it means to show that there is a number that is not included in this infinite set?". The same considerations are applied here to the question "Have we general concepts of "ordered", "series", "infinite series", etc.?" Well, we have a "common name" applied to certain analogous formations. Perhaps that is enough to say that we have a "general concept". But one should not mistake this with thinking that, from these analogous formations, we have a general and well-defined method of showing whether *any given set* "can be ordered in a series" – such as the set of real numbers. Such a method would have to be invented, certain conventions and stipulations set in place in order to make such an application of those words intelligible. As Wittgenstein puts it in RFM II, 38, "Such an employment is not: yet to be discovered, but: still to be *invented*" and in RFM I, 168: "The mathematician is an inventor, not a discoverer."

RFM II, 16 concludes with the major punchline for the relationship between the "method" of diagonalization and the "result" that the real numbers are non-denumerable.

> Now I am shown the diagonal procedure and told: "Now here you have the proof that this ordering can't be done here". But I can reply: "I don't know—to repeat—what it is that *can't be done here*". (*ibid*)

One can reply in this way because, merely given the analogous formations of "series" or "orderings", one does not yet know what it would even mean to say that "the real numbers cannot be ordered" or that "they do not form an infinite series". The resistance here is to the idea that the statement of this result is intelligible *all on its own*, i.e., independently of the method of diagonalization. This is why Wittgenstein emphasizes that what the method really reveals is a difference between the stipulated uses of words such as "root", "algebraic number", "real number", and the like, rather than a self-standing and independently intelligible result.

> Though I can see that you want to show a difference between the use of "root", "algebraic number", etc. on the one hand, and the "real number" on the other. Such a difference as e.g. this: roots are *called* [my emphasis] "real numbers", *and so too* is the diagonal number formed from the roots. And similarly for all series of real numbers. For this reason it makes no sense to talk about a "series of all real numbers", just because the diagonal number for each series is also *called* [my emphasis] a "real number". (*ibid*)

So now – given the stipulated connection between the diagonal method and the result that "the real numbers cannot be ordered" – it makes no sense to talk of "the series of real numbers". This illustrates a difference between how we (will) use the words "cardinal number" and "real number", rather than a deep fact about "real numbers" which was simply awaiting our discovery. As he puts it in two later remarks,

> The dangerous, deceptive thing about the idea: "The real numbers cannot be arranged in a series", or again "The set … is not denumberable" is that it makes the determination of a concept—concept formation—look like a fact of nature.
>
> (RFM II, 19)

> The following sentence sounds sober: "If something is *called* a series of real numbers, then the expansion given by the diagonal procedure is also *called* a 'real number', and is moreover *said to be* different from all members of the series".
>
> (RFM II, 20, my emphasis)

And in a separate manuscript discussing related themes,

> One would like to say of it e.g.: it introduces us to the mysteries of the mathematical world. *This* is the aspect against which I want to give warning. (RFM II, 40)

The general lesson – that these are matters of stipulation rather than discovery of some independent realm – is summed up in RFM II, 16 with an analogy.

> —Would this not be as if any row of books were itself ordinarily called a book, and now we said: "It makes no sense to speak of 'the row of all books', since this row would itself be a book." (*ibid*)

We can imagine one being surprised or mystified by the "result" that "there can never be a row of all books" – which on the face of it might appear to be a striking empirical fact about books or rows of objects. The mystification dissipates when one realizes that this "result" falls out of an (uninteresting) stipulation governing what we call "a row of books". One's mystification about the "un-orderability of the real numbers" as revealed in Cantor's work should likewise be diminished.

I mentioned earlier that to understand an expression such as "the real numbers are non-denumerable", one might look to the method which yields it (by stipulation). But one might also look to what, as it were, comes *after* the statement of this result. Wittgenstein describes some such downstream applications of the diagonal method in order to give a sobering picture of its general significance.

> Surely—if anyone tried day-in day-out 'to put all irrational numbers into a series' we could say: "Leave it alone; it means nothing; don't you see, if you established a series, I should come along with the diagonal series!" This might get him to abandon his undertaking. Well, that would be useful. And it strikes me as if this were the whole and proper purpose of this method. It makes use of the vague notion of this man who goes on, as it were idiotically, with his work, and it brings him to a stop by means of a picture. (But one could get him to resume his undertaking by means of another picture.) (RFM II, 13)

The description of this scenario is akin to Wittgenstein's discussion in LFM of "John mastering $\aleph_0$ calculations" (LFM 251). When one hears the "result" of diagonalization, it can make it appear exciting and surprising – it can, as Wittgenstein puts it, "fire our imagination most strongly" (RFM II, 11). Just as when one considers the vastness of $\aleph_0$, it can make "your head reel" (LFM 253). But these expressions are less apt to excite when we look at their humdrum roles in ordinary life. There is nothing particularly exciting about a child who can "perform $\aleph_0$ multiplications"; nor is there anything particularly exciting about making someone stop "as it were idiotically" attempting to put the real numbers into a list. Such a result is "useful", albeit not particularly exciting. The same lesson is repeated by describing a child's use of diagonalization in a classroom setting.

> For this kind of calculation is itself useful. The question set would be perhaps: write down a decimal number which is different from the numbers:
>
> 0.1246798 . . .
> 0.3469876 . . .
> 0.0127649 . . .
> 0.3426794 . . .
> ………… (Imagine a long series.)
>
> The child thinks to itself: how am I to do this, when should I have to look at all the numbers at once, to prevent what I write down from being one of them? Now the method says: Not at all: change the first place of the first number, the second of the second one etc. etc., and you are sure of having written down a number that does not coincide with any of the given ones. The number got in this way might always be called the diagonal number. (RFM II, 18)

These passages make clear that Wittgenstein's concern is not – as some authors have thought – that Cantor's work is uninteresting because it *lacks* any non-mathematical applications.[87] Rather, his concern seems to be that none of the downstream applications are particularly *interesting*. Hence his famous remark that Cantor's work is not a "paradise". And rather than showing that there is anything wrong *per se* with the stipulations linking up the method of diagonalization with its "results" (after all, "this calculation is itself useful"), Wittgenstein's remarks seem instead to diminish the sort of wonder or awe that a presentation of Cantor's proofs might have initially inspired.

[87] See especially Moore (2011: 119).

Thus, Wittgenstein's remarks are largely aimed at effecting a certain change in aspect – a different way of looking at Cantor's proofs than a mere verbal statement of its results might have initially inspired. Cantor's "discovery" might have seemed to consist in revealing a deep fact about the infinite realms of numbers ("a fact of nature" (RFM II, 19), "the mysteries of the mathematical world" (RFM II, 40)). But a simple thought experiment (along with all of the foregoing) might help to dislodge such a reaction.

> Here is it very useful to imagine the diagonal procedure for the production of a real number as having been well-known before the invention of set theory, and familiar even to school-children, as indeed might very well have been the case. For this changes the aspect of Cantor's discovery. The discovery might very well have consisted *merely* in the interpretation of this long familiar elementary calculation.
> (RFM II, 17)

Why is this "very useful to imagine"? The main reason is that it allows us to look at the method of diagonalization and its more banal applications on their own terms – isolated from the usual descriptions of the proof's significance. In the alternative, fictional history Wittgenstein asks us to imagine, diagonalization was originally a simple classroom exercise which did not inspire much wonder, but allowed us a simple and clever answer to a certain mathematical puzzle ("show me a number which is different from all those in this very long list"). Such an exercise, as most elementary exercises in mathematics do, would then appear quite mundane. This in turn would make it much harder for someone to get excited by any pronouncement by Cantor or his acolytes suggesting that, say, the calculation reveals a deep fact about the nature of "the infinite". (Just as it would be difficult these days to get someone excited about the infinitesimal calculus revealing deep facts about "infinite smallness" (LFM 255)). But perhaps this also allows us more specifically to separate diagonalization from Cantor himself, who viewed the significance of his mathematical work in explicitly theological and metaphysical terms.[88] In short, Wittgenstein's thought experiment allows us

[88] See for instance Moore (1990: 127): "There is one exceedingly important fact about [Cantor] that I have not yet mentioned. He was deeply religious. He believed he had a God-given gift to effect a mathematical study of the infinite

to disassociate, at least for a moment, the method of diagonalization from its usual imaginative extravagances.

MS 117 concludes with a particularly forceful passage, explaining why Wittgenstein hopes that "a future generation" will laugh at the "hocus pocus" expressed in the common verbal descriptions of Cantor's method of diagonalization.

> The usual expression creates the fiction of a procedure, a method of ordering which, though applicable here, nevertheless fails to reach its goal *because* of the number of objects involved, which is greater even than the number of all cardinal numbers.
> (RFM II 22, emphasis added)

This bears directly on the explanations of diagonalization that we saw earlier from A.W. Moore and BBJ. By their lights, the major upshot is that "the set of real numbers is *bigger* than the set of natural (or cardinal) numbers". The fact that they think of this as the major conclusion to draw from Cantor's proof might encourage one to think that we have an independent grasp of what this means, prior to looking at the method of diagonalization itself. It might encourage one to think that "the set of real numbers" is a vast reality existing independently outside our talk of it. Nowhere in their explanations is it emphasized that the "result" of Cantor's proof requires a collection of stipulations regarding the uses of 'series', 'ordering', 'countable', 'denumerable', and most importantly 'bigness'.[89] However technical and rigorous their presentations of Cantor's mathematics might otherwise be,

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and thereby, in a way, to vindicate certain cherished views about the divine against the charge of incoherence. Both when he was working on the infinite and subsequently, when he had eventually been forced away from that work by opposition and despair and was able to devote more time to religious studies and theology, he believed that he was in the hands of God. And this Absolute that had revealed itself in his own formal work, in a way that was so reminiscent of more traditional views of the infinite, was embraced by Cantor as a vital part of his conception of God."

[89] This crucial point is missed in Moore's (2011: 117) & (2016: 324) reading of RFM II, leading to his accusation that Wittgenstein is here, contrary to his own philosophical prescriptions (PI 124), *denying* a mathematical claim, (e.g., "that the set of real numbers is bigger than the set of natural numbers"). The point is, rather, (1) that the mathematical claim should not be confused with other uses of 'big', etc., and (2) that the use of 'big' in this context is a matter of stipulation, rather than an inevitable extension from other uses of 'big'. (1) & (2) are of course mutually supporting; failure to appreciate them can lead to a kind of excitement about Cantor's work that Wittgenstein (for that very reason) finds suspect. Compare Fogelin (1987: 220): "Cantor's argument shows that the cardinality of the class of real numbers must be greater than the cardinality of the class of integers (or rational numbers). But surely nothing forces us to extend our concepts in these ways, and thus the idea that Cantor has proved the existence of a hierarchy of transfinite cardinals is simply an exaggeration."

Wittgenstein would still think that their descriptions of the proof encourage the "fiction of a procedure" that "fails to reach its goal *because* of the number of objects involved" (my emphasis). He would insist on a yet more sober description, e.g.: when one set of numbers admits of diagonalization, and another set does not, we will *call* the former set "bigger" than the other". And now it is more difficult to get excited about Cantor's proofs on the grounds that it reveals something about the vast and incredible realm of infinity. Likewise,

> If it were said: "Consideration of the diagonal procedure shows you that the *concept* 'real number' has much less analogy with the concept 'cardinal number' than we, being misled by certain analogies, are inclined to believe", that would have a good and honest sense. But just the *opposite* happens: one pretends to compare the 'set' of real numbers in magnitude with that of cardinal numbers. The difference in kind between the two conceptions is represented, by a skew form of expression, as difference of extension. I believe, and hope, that a future generation will laugh at this hocus pocus. (RFM II, 22)

Wittgenstein thus offers a redescription of Cantor's proof – emphasizing the roles of conceptual innovation, stipulation, and convention – in order to counteract a kind of "hocus pocus" that it tends to inspire, a feeling that Cantor's work reveals a "fact of nature" regarding the relative "sizes" of various infinite "magnitudes".[90]

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[90] This passage might appear to be strong evidence that Wittgenstein was in fact a "finitist". However, Monk (2007) explains persuasively why such a label would be misleading: "I think that the term is more likely to mislead than to shed light on Wittgenstein's position. For the word 'finitism' is generally used to characterize, not just the above general view about the meaninglessness of statements of the existence of infinity in extension, nor the importantly different *denial* of the existence of infinity in extension, but also a view about the consequences this rejection of the actual infinite has for mathematics itself. [What characterizes 'finistism' as it is typically used] was not that they thought (philosophical) statements about the existence of infinite quantities to be nonsense, but that they inferred from that certain restrictions on what a mathematician can do in a mathematical proof […] Wittgenstein was never a [finitist] in this sense. He had no wish to lay down the law about correct method in mathematics" (Monk, 2007: 277). There is a slight difference, however, between my and Monk's readings of the later Wittgenstein. On my reading, rather than *denying* the existence of "infinity in extension" or "the actual infinite", Wittgenstein is insisting that the meaning of such notions does not go beyond the calculus that such phrases are used to describe – and when we look to that, we don't find anything especially intriguing, i.e., unless we succumb to tempting analogies across different uses of 'quantity', 'extension', 'size', 'order', and the like. The "hocus pocus" that Wittgenstein warns us against is that of thinking that when we are "comparing the sizes of different infinities" we are doing something perfectly analogous to, say, "comparing the sizes of two bags of potatoes". Here, as in many other places, I think Wittgenstein's motto "Say what you please, so long as it does not prevent you from seeing how things are. (And when you see that, there will be some things you won't say)" (PI 79) is especially helpful.

Thus, Z, LFM, and MS 117 together reveal a persistent style of critique applied to "the infinite" from the later Wittgenstein. As we have seen in each of these texts, Wittgenstein thinks that talk of the infinite naturally takes hold of one's imagination due to the sway of verbal pictures and analogies suggested by our words. However, one's imagination in these instances does not determine the meanings of 'infinity' and its surrounding expressions – but rather their meanings are ultimately to be found in their uses and applications. Thus, when the verbal pictures associated with our expressions take us far beyond their uses or perhaps even conflict with them, then the conditions are ripe for confusion. By Wittgenstein's lights, to avoid such confusion, we need to be reminded of the applications of these expressions so as to weaken the hold of said pictures and analogies on our imagination. When we do this, a certain romantic quality that tends to be inspired by talk of "the infinite" is diminished. More specifically, the picture of "the infinite" as a self-standing realm admitting of exploration or discovery by mathematicians is shown to be a product of illusion. Instead, *talk* of "the infinite" is just that – a collection of language-games governed by certain conventions rather than independent facts of nature.[91]

[91] Just as Stern (2004) characterizes the "quietist" position, to end on this note is not to suggest that, "Wittgenstein's invocation of forms of life is […] the beginning of a positive theory of practice […] but rather is meant to help his readers get over their addiction to theorizing about mind and world, language and reality" (Stern, 2004: 169).

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