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Later Wittgenstein on 'Truth' and Realism in Mathematics

PHILIP BOLD

Abstract

I show that Wittgenstein's critique of G.H. Hardy's mathematical realism naturally extends to Paul Benacerraf's influential paper, 'Mathematical Truth'. Wittgenstein accuses Hardy of hastily analogizing mathematical and empirical propositions, thus leading to a picture of mathematical reality that is somehow akin to empirical reality despite the many puzzles this creates. Since Benacerraf relies on that very same analogy to raise problems about mathematical 'truth' and the alleged 'reality' to which it corresponds, his major argument falls prey to the same critique. The problematic pictures of mathematical reality suggested by Hardy and Benacerraf can be avoided, according to Wittgenstein, by disrupting the analogy that gives rise to them. I show why Tarskian updates to our conception of 'truth' discussed by Benacerraf do not answer Wittgenstein's concerns. That is, because they merely presuppose what Wittgenstein puts into question, namely, the essential uniformity of 'truth' and 'proposition' in ordinary discourse.

> What a mathematician is inclined to say about the objectivity and reality of mathematical facts, is not a philosophy of mathematics, but something for philosophical *treatment*. (Wittgenstein, 2009, §254)

1. Introduction

In an article titled 'Mathematical Proof', to which Wittgenstein regularly referred in his *Lectures on the Foundations of Mathematics*, G.H. Hardy offers one of several criteria 'that a philosophy must satisfy if it is to be at all sympathetic to a working mathematician':

> It seems to me that no philosophy can possibly be sympathetic to a mathematician which does not admit, in one manner or another, the immutable and unconditional validity of mathematical truth. Mathematical theorems are true or false; their truth or falsity is absolute and independent of our knowledge of them. In *some* sense, mathematical truth is part of objective reality. (Hardy, 1929, p. 4)

Hardy then provides the following elaboration on what he admits is at most a rough idea:

> 'Any number is the sum of 4 squares'; 'any number is the sum of 3 squares'; 'any even number is the sum of 2 primes'. These are not convenient working hypotheses, or half-truths about the Absolute, or collections of marks on paper, or classes of noises summarizing reactions of laryngeal glands. They are, in one sense or another, however elusive and sophisticated that sense may be, theorems concerning reality, of which the first is true, the second is false, and the third is either true or false, though which we do not know. (*ibid*.)[1]

Commenting on these remarks from Hardy, Wittgenstein is reported to have said the following in his lectures:

> Consider Professor Hardy's article ('Mathematical Proof') and his remark that 'to mathematical propositions there corresponds—in some sense, however sophisticated—a reality'. (The fact that he said it does not matter; what is important is that it is a thing which lots of people would like to say.)
> (Wittgenstein, 1976, p. 239)

Indeed, similar things have been said by a wide variety of philosophers and mathematicians, including (as likely occurred to Wittgenstein) Frege, Russell (for a time), and Gödel. And Hardy's idea, a passionate expression of so-called 'mathematical realism', is neither quaint nor dated. As Putnam (2001) has rightly noted, a version of this idea – that mathematical truths are truths concerning objective mathematical reality – is elegantly crystallized and defended in what is often taken to be a staple in contemporary philosophy of mathematics, Paul Benacerraf's

[1] Compare Hardy (1967, pp. 123–24): 'I believe that mathematical reality lies outside us, that our function is to discover or *observe* it, and that the theorems which we prove, and which we describe grandiloquently as our "creations", are simply our notes of our observations'.

'Mathematical Truth'.[2] Wittgenstein in the passage above is thus addressing a general idea that, however controversial it might be, has exerted a great influence on philosophical thought about mathematics.

But there is a crucial difference between Hardy's and Benacerraf's invocations of an objective mathematical reality. Rather than merely offering such an idea as a constraint on any sympathetic account of mathematical practice, Benacerraf argues that such a conception of mathematical reality is a necessary outfall of a sufficiently general theory of truth. To illustrate this, he offers the following two sentences:[3]

(1) There are at least three large cities older than New York.
(2) There are at least three perfect numbers greater than 17.

According to what he takes to be the 'standard account' of the semantics of these two sentences, they are both of the form

(3) There are at least three FG's that bear R to a.

and thus they have parallel truth conditions. The result of this, which Benacerraf grapples with in his paper, is a version of Hardy's idea – namely, that true mathematical claims are claims concerning objective reality – since mathematical claims make reference to entities which, if those claims are indeed true, must exist.

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[2] Compare Shapiro (2000, p. 31): 'Paul Benacerraf's "Mathematical Truth" (1973) [is] an article that continues to dominate contemporary discussion in the philosophy of mathematics'; and Linnebo (2017, p. 12, fn. 8): 'Recent discussions of the challenge often focus on the version developed in Benacerraf (1973)', though Linnebo confesses he, 'find[s] this focus unfortunate'. It is a common focus all the same. Compare also Hacking (2014, p. 216): 'Benacerraf's superb paper, "Mathematical Truth" (1973), is a fundamental benchmark for philosophical platonism/nominalism debates'.

[3] See Benacerraf (1973, p. 663).

However, Benacerraf acknowledges that this leads to problems. Given that mathematical entities are abstract and thus do not have the causal properties possessed by their empirical counterparts, it is mysterious how mere mortals in the causal realm might acquire knowledge of such things. The issue then, in short, is that it is quite difficult to square up mathematical truth – which implies the existence of an a-causal realm of entities – with mathematical knowledge – which seems to require causal access to its subject matter. Resolving Benacerraf's dilemma is generally regarded as a condition on any adequate philosophy of mathematics today. Whereas some philosophers might resolve the issue by offering a 'non-standard' conception of mathematical truth (or, radically deny that such claims are literally true), Hardy and Benacerraf would agree that a sympathetic rendering of mathematical practice requires that we maintain the standard view and not simply abandon it on epistemological grounds.[4]

How does Wittgenstein respond to these (allegedly standard or natural) claims about mathematical reality? As reported by students who attended his lectures, Wittgenstein suggests that Hardy's version, at least, is at best unclear and at worst meaningless.

> Taken literally, this seems to mean nothing at all – *what* reality? I don't know what this means. – But it is obvious what Hardy compares mathematical propositions with: namely physics. (Wittgenstein, 1976, p. 239)

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[4] Some have argued that Benacerraf's dilemma as it stands is unpersuasive, but that there are related problems for what Benacerraf calls a 'standard account of mathematical truth' and the 'platonistic' view of reality it seems to imply. See especially Field (1988, pp. 25–30) for discussion. My purpose is simply to note that the conception of mathematical truth suggested by Benacerraf naturally leads to (and is intended to lead to) metaphysical and epistemological problems. This simpler point does not depend crucially on Benacerraf's own formulation of the problem, which requires a controversial appeal to a causal theory of knowledge. That said, it is worth noting that Wittgenstein, too, would have found the assimilation of all scientific (including mathematical) knowledge to 'causal' knowledge deeply suspect. I will discuss some of his grounds for suspicion in section 5 by emphasizing differences between uses of 'know' in empirical and mathematical contexts.

Such a comparison – between mathematical propositions and the propositions of physics – is not unique to Hardy. Benacerraf motivates the existence of mathematical reality likewise on the grounds that there is a parallel between mathematical and empirical propositions (as illustrated by sentences (1) – (3) above). Though, again, in Benacerraf's case this is allegedly supported by the (desirable) uniformity of our theory of truth and in turn of our semantics. Thanks to Tarski, we now have the foundations of such a theory, the 'essential feature' of which 'is to define truth in terms of reference (or satisfaction) on the basis of a particular kind of syntactico-semantical analysis of the language' (Benacerraf, 1973, p. 667). The truth of mathematical propositions implies the existence of a special class of entities, i.e., those entities to which true mathematical claims refer.

Does Benacerraf's appeal to truth and Tarskian semantics clarify the matter – i.e., why it is that, in doing mathematics, we seem committed to the existence of a distinctive class of objects? Does it help to show what might be meant by a mathematical reality to which those claims refer? Wittgenstein's further comments on Hardy do not leave much room for hope.

> Suppose we said first, 'Mathematical propositions can be true or false.' The only clear thing about this would be that we affirm some mathematical propositions and deny others. If we then translate the words 'It is true …' by 'A reality corresponds to …' – then to say a reality corresponds to them would say only that we affirm some mathematical propositions and deny others. We also affirm and deny propositions about physical objects. – But this is plainly not Hardy's point. If this is all that is meant by saying that a reality corresponds to mathematical propositions, it would come to saying nothing at all, a mere truism: if we leave out the question of *how* it corresponds, or in what sense it corresponds. (Wittgenstein, 1976, p. 239)

It is no surprise then that Wittgenstein, characteristically for his later philosophical career, suggests that Hardy's idea results from a failure to attend to the uses of our words: 'We have here a thing which constantly happens. The words in our language have all sorts of uses; some very ordinary uses which come into one's mind immediately, and then again they have uses which are more and more remote' (*ibid*., p. 239). More specifically, in making such claims about mathematical 'truth' or the 'reality' to which it corresponds, one can easily forget the ordinary uses of these words – where they are 'really at home' (*ibid*., p. 240) – and (thereby) end up in philosophical confusion. Thus, it seems that Wittgenstein's concerns about Hardy's idea would just as easily extend to Benacerraf's version of it. That is, to the extent that it arises from an analogy between empirical and mathematical propositions without attention to their distinctive uses (as well as the distinctive uses of 'truth' and 'reality' accompanying them), Benacerraf's invocation of an objective mathematical reality would be equally suspect.

How can Wittgenstein accuse such folks of meaninglessness or lack of clarity? Or, even if Wittgenstein might have rightly objected to Hardy on such grounds (Hardy does admit his suggestion is vague), how could such a concern possibly carry over to Benacerraf's presentation, grounded in a Tarskian semantical framework which is judged by many to be a perfectly intelligible advancement in mathematics and philosophy? Perhaps this is one of many remarks made by Wittgenstein that can be put aside as relying on a false pessimism about the prospects of philosophical or formal theories in general, or, at the very least, as an opinion that ought to have been updated in light of later advances in the subject of analytic philosophy. Thanks to Tarski, we know much better what 'truth' is. With a better grasp of 'truth' to hand, and thus of the conditions on any adequate theory of 'meaning' (semantics), Benacerraf's suggestion that mathematical truth implies mathematical reality could hardly be rejected on grounds of unintelligibility. It might seem, then, that Wittgenstein's concerns here are obsolete, to the extent that they can so much as be understood.

My aim is to provide a reading of Wittgenstein on mathematical 'truth' and 'reality' that will help to address these concerns – showing how Wittgenstein's critique of Hardy naturally extends to Benacerraf's invocation of an objective mathematical reality. In section 2, I briefly explain the later Wittgenstein's therapeutic conception of philosophy, as I understand it – which will help to frame the reading I offer. In section 3, I explain Wittgenstein's basic therapeutic critique of Hardy and others (like Benacerraf) who accept a picture of mathematical reality that is somehow akin to empirical reality. In section 4, I examine Wittgenstein's remarks about Hardy regarding the significance of mathematical 'truth' in Lecture XXV. Finally, in section 5, I will conclude by showing how Wittgenstein's critique naturally extends to the major argument of Benacerraf's influential paper, 'Mathematical Truth'. More specifically, I show why Tarskian updates to one's theory of 'truth' do not suffice as a response to Wittgenstein's concerns. That is, because they merely presuppose what Wittgenstein puts into question, namely, the essential uniformity of 'truth' and 'proposition' in ordinary discourse.[5]

2. Wittgenstein's Therapeutic Conception of Philosophy

The reading that follows relies on Wittgenstein's later, therapeutic conception of philosophy, as I understand it. This 'conception' is not a theory or description of all the things we might happen to call 'philosophy', but Wittgenstein's own radical conception of how philosophy should be done – 'radical' in that it disrupts the traditional philosophical mode of treating certain questions either as innocent and answering them directly by way of an account, definition, or theory; or as guilty (e.g., unanswerable or confused) and showing that this is so via some theory of language or cognition (and their necessary limits).[6] By contrast, on Wittgenstein's conception of philosophy, philosophical questions are themselves an object of suspicion and require an investigation of their sources without any aspiration to theory (Wittgenstein, 2009, §§255, 109). They are treated 'like an illness' and thus submitted to diagnosis and therapy (*ibid.*, §§255, 133). Since these characterizations are intended to be analogical or metaphorical ('The philosopher treats a question *like* an illness' (*ibid.*, §255, my emphasis); 'there are indeed methods, different

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[5] My discussion will thus substantiate and elaborate on similar readings offered by Diamond (1996) and Conant (1997). If Gerrard (1991) is right that Wittgenstein's critique of 'the Hardyian picture' is fundamental to his later philosophy of mathematics, then my reading will shed significant light on this part of his later philosophy. See especially Bold (2022) for an extensive discussion of Wittgenstein's later, therapeutic philosophy of mathematics.

[6] A paradigm of the former approach is Socrates; of the latter Kant – though the latter tradition (broadly construed) would also include early Wittgenstein of the *Tractatus* as well as, e.g., Ayer (1952). That Wittgenstein's invocation of language-games and forms of life is not intended to be the forefront of a new branch of theory is emphasized in Stern's characterization of the so-called 'quietist' position: 'Wittgenstein's invocation of forms of life is not the beginning of a positive theory of practice […] but rather is meant to help his readers get over their addiction to theorizing about mind and world, language and reality' (Stern, 2004, p. 169).

therapies, *as it were*' (*ibid*., §133, my emphasis)), I unpack them as follows.[7] Wittgenstein's general 'diagnosis' of philosophical problems is that they stem from misunderstandings about the uses of words due to one's lacking a proper overview (*übersicht*) of our language (*ibid*., §§110, 111, 122). One major aspect of language that encourages such misunderstandings is the apparent similarity between different kinds of words (*ibid*., §11); confusions arise when they are assimilated despite important differences between their uses (*ibid*., §§90, 112; Wittgenstein, 2005, pp. 302–3). Such confusion often yields misbegotten 'pictures' of the meanings of those words (Wittgenstein, 2009, §§1, 115), which take hold of the philosopher's imagination and are effectively counteracted with reminders about how those words are ordinarily used (*ibid*., §§116, 126, 127).

Wittgenstein's 'diagnosis' of philosophical questions is at the core of his notion of philosophical 'therapy'. The misunderstandings about language that give rise to philosophical puzzlement can only be counteracted by *describing* the uses of words and drawing out differences between them (*ibid*., §§69, 75, 109) – differences that might easily be overlooked due to their surface similarities. A common tool in Wittgenstein's philosophical therapy is his use of language-games (*ibid*., §130). The value of language-games (especially those that are fictional) is to serve as 'objects of comparison', designed to throw light on features of our *actual* language (*ibid*.). Language-games, whether actual or fictional, also articulate a kind of ideal (an 'overview' (*übersicht*), the lack of which, according to Wittgenstein, is the major source of

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[7] To say that Wittgenstein's characterizations are analogical or metaphorical is not meant to undermine their importance. As Wittgenstein himself said elsewhere, 'A good simile refreshes the intellect' (Wittgenstein, 1980, p. 1). Given that these are analogies, one should not haphazardly import features of 'illness' or 'therapy' in, say, their medical senses into Wittgenstein's conception of philosophy. The most crucial analogy is simply this: just as a doctor investigates the sources of the pain in your leg in order to relieve it, likewise Wittgenstein studies the sources of philosophical questions in order to relieve them and make them go away. A crucial disanalogy: whereas a doctor may require some 'theory' or other in order to properly diagnose the pain in your leg or administer physical therapy, Wittgenstein intends to proceed by example alone (Wittgenstein, 2009, §133) – and thus (by his own lights) does not depend on any theory (say, a theory of 'meaning' or of 'language').

misunderstandings), because when language-games are described in sufficient detail traditional philosophical problems about meaning do not arise (*ibid*., §§1, 126). Thus, generally speaking, philosophical 'illnesses' are confusions about the uses of words due to a lack of oversight (*übersicht*); philosophical 'therapy' counteracts such confusions by careful description of and attention to ordinary word-use, deploying (sometimes fictional) language-games to highlight aspects of use or features of our language that might otherwise be ignored.

This is Wittgenstein's therapeutic conception of philosophy in a nutshell, the further clarification of which (by his own lights) must be 'demonstrated by examples' (*ibid*., §133). The application of Wittgenstein's therapeutic method articulated throughout this paper will serve as one such example.

3. Hardy's Picture: The Basic Therapeutic Strategy

With Wittgenstein's later, therapeutic conception of philosophy to hand, I will now provide a reading of Wittgenstein on mathematical 'truth', as revealed in his engagement with Hardy's (and by extension Benacerraf's) picture of mathematical reality. According to Hardy's picture, mathematical truth is 'in *some* sense' part of objective reality. Such truths are immutable, unconditionally valid, and independent of our knowledge of them. The true theorems of mathematics are, 'however elusive and sophisticated that sense may be', genuine objects of discovery and *not* mere creations of our minds. At a later stage of his paper, 'Mathematical Proof', Hardy elaborates on his understanding of mathematical discovery with an analogy:

> I have myself always thought of a mathematician as in the first instance an *observer*, a man who gazes at a distant range of mountains and notes down his observations. His object is simply to distinguish clearly and notify to others as many different peaks as he can. There are some peaks which he can distinguish easily, while others are less clear. […] But when he sees a peak he believes that it is there simply because he sees it. If he wishes someone else to see it, he *points to it*, either directly or through the chain of summits which led him to recognize it himself. […]

> The analogy is a rough one, but I am sure that it is not altogether misleading. (Hardy, 1929, p. 18)[8]

Hardy's picture of mathematical discovery is thus strongly analogized to the observation and discovery of peaks in a mountain range, some of which can be seen more or less clearly than others. Mathematical reality, according to Hardy, is akin to a mountain range, there to be discovered quite independently of anything we might have to say or think about it.

Versions of Hardy's picture have surfaced in a number of different places. For instance, Hardy's thoughts are reminiscent of Russell's passing remark that logic is akin to zoology:

> Logic, I should maintain, must no more admit a unicorn than zoology can; for logic is concerned with the real world just as truly as zoology, though with its more abstract and general features. (Russell, 1919, p. 169)

Despite important differences between Hardy's and Russell's understandings of logic and mathematics,[9] they overlap in thinking of their fields as areas of discovery, concerning reality

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[8] Though, taken to its extreme, Hardy does think it might lead to the 'paradoxical conclusion' that 'there is no such thing as mathematical proof; that we can, in the last analysis, do nothing but *point* [i.e., to the truths that are simply there to be observed by mathematicians and their students]' (Hardy, 1929, p. 18). Wittgenstein, as we'll see, would think of this analogy, and the picture of mathematical reality it suggests, as misleading for quite different reasons.

[9] That is, Russell's understanding at the time of writing this particular book. One relevant difference might be that Russell's picture does not invoke a *separate* reality for logic, but instead views logic as exploring the abstract features of a single world. This is especially relevant in comparing Russell's picture with Gödel's picture, which we'll see momentarily. Floyd (2006) also notes the connection between Hardy and Russell's pictures of mathematical reality: 'Wittgenstein's emphasis on the image of the mathematician as *inventor* or fashioner of models, pictures, and concepts was, in the main, directed at the philosophical talk of those, like Hardy and Russell, who insisted on speaking of mathematical reality in a freestanding way, picturing

much like the geological reality observed by someone studying mountain peaks, or the biological reality studied by zoologists. Such a conception of mathematical discovery is akin to Frege's remark that 'the mathematician cannot create things at will, any more than the geographer can' (Frege, 1980, p. 108), which likewise analogizes mathematics to the (empirical) science of geography.[10] Another famous instance is from Gödel, who likewise conceives of mathematics as an independent reality that 'clearly [does] not belong to the physical world'; one which, on Gödel's view, is accessed via something like perception (thus quite reminiscent of the 'seeing' of mountain peaks in Hardy's image above).

> [T]he objects of transfinite set theory […] clearly do not belong to the physical world and even their indirect connection with physical experience is very loose […].
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> But, despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. (Gödel, 1984, pp. 483–84)

Hardy's picture of mathematical reality is also clearly echoed more recently by the mathematical physicist, mathematician, and philosopher of science, Roger Penrose:

> How 'real' are the objects of the mathematician's world? […] [T]here often does appear to be some profound reality about these mathematical concepts, going quite beyond the mental deliberations of any particular mathematician. It is as though human thought is, instead, being guided towards some external truth – a truth which has a reality of its own, and which is revealed only partially to any one of us. (Penrose, 1989, pp. 95–96)

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the logician or mathematician as a zoologist embarked on an expedition to new, hitherto unseen lands, analogous to an empirical scientist' (Floyd, 2006, p. 112).

[10] See Wittgenstein (1979, p. 7) for a specific critique of the geographical analogy in an earlier lecture.

The idea that mathematics concerns an external reality that is only partially revealed to us is very much in keeping with Hardy's suggestions, e.g., that 'There are some peaks which [a mathematician] can distinguish easily, while others are less clear'. The examples of Hardy, Russell, Frege, Gödel, and Penrose suffice to show that a suggestive picture of mathematical reality articulated (however roughly) by Hardy has attracted serious mathematicians and philosophers alike. I will not assume that their full-fledged views or theories of mathematical reality (to the extent that each has one) are all the same (indeed, they are not) – what is important for our purposes is a common picture at the root of each: that of a reality which is 'out there' independently of mathematical practice and discovered by mathematicians in the process of doing mathematics, akin to the independent realities studied by geologists, geographers, zoologists, or physicists.[11]

As we saw earlier, Hardy's picture is likewise echoed in Benacerraf's famous paper, 'Mathematical Truth': that of an independent reality of entities akin to empirical objects (e.g., cities), but quite different in that they are a-causal and abstract. Although Benacerraf provides a more technical argument for this picture than Hardy, they both proceed by analogizing the truths of mathematics to empirical truths (e.g., those having to do with mountain peaks or cities).

What does Wittgenstein have to say about these pictures of mathematical reality? According to his therapeutic conception of philosophy, we should investigate the sources of Hardy's and Benacerraf's pictures as well as the philosophical questions and puzzlement they engender. The puzzlement here includes questions (more or less standard in the philosophy of mathematics) such as, 'What *is* mathematical reality?', 'How is mathematical knowledge *possible*?', 'How can we make sense of the application of mathematical truths (which seem to inhabit their own kind of reality) to empirical reality?', among others. Since questions of this sort themselves hinge on a certain picture of mathematical reality – i.e., as *something* hanging out

[11] It is worth noting that this common picture can arise from sources other than those discussed in this article. For instance, Bold (2023) highlights that, for Wittgenstein, such a picture of mathematical reality can arise from misunderstandings about 'rules' and 'determination', while Bold (2022: Ch. 3) investigates sources in our talk of 'infinity' and related expressions.

there, though perhaps beyond the causal realm – the main focus for Wittgenstein should be that very picture.[12]

Where does this picture come from? More specifically, what is it about our language that has encouraged such a picture? Given the presentations offered by Hardy and Benacerraf, we thankfully don't have to look far and wide for an answer. The obvious source for Wittgenstein would be the assimilation of mathematical truths (or, propositions) to empirical truths (or, propositions), an assimilation that both authors make explicitly, albeit without much reflection or scrutiny. As we saw in section 2, Wittgenstein thinks that the assimilation of apparently similar expressions is a general source of philosophical confusion. A therapeutic response will go by way of studying the uses of these expressions so as to draw out their crucial differences. Studying the differences should, in turn, undermine the hasty analogy that led to Hardy's and Benacerraf's pictures of mathematical reality and thus the questions they engender.

We can see that this is the very approach Wittgenstein is reported to have taken up in his lectures – which provide the most explicit engagement with Hardy's picture of mathematical reality.[13]

4. Wittgenstein's Discussion of Hardy in Lecture XXV

At the beginning of *Lectures on the Foundations of Mathematics* XXV, Wittgenstein reportedly[14] addresses 'a false idea of the role which mathematical and logical propositions play' (Wittgenstein, 1976, p. 239). He illustrates this false idea by examining Hardy's remark that 'to mathematical propositions there corresponds—in some sense, however sophisticated—a reality', though he emphasizes that *Hardy's* saying this is not crucial, since 'it is a thing which lots of people would like to say' (*ibid.*) – as we ourselves saw with the selection of philosophers and mathematicians in section 3. Wittgenstein immediately complains that 'Taken literally, this

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[12] Cf., 'It is in such pictures that most problems of philosophy arise' (Wittgenstein, 1976, p. 190).

[13] Although they provide Wittgenstein's most explicit engagement with Hardy on this particular topic, this is not the only place where Wittgenstein engages with Hardy philosophically. See especially Floyd & Mühlhölzer (2020).

[14] For convenience, when discussing passages from the lectures hereon, I will drop the important qualification that these are things Wittgenstein is *reported* to have said.

seems to mean nothing at all—*what* reality?' (*ibid*.). Although he claims not to understand Hardy's remark, he notes that 'it is obvious what Hardy compares mathematical propositions with: namely physics' (*ibid*.). This is also something we have seen confirmed above in the cases of Hardy, Russell, Frege, Gödel, Penrose, and Benacerraf – all of whom analogize the truths of mathematics to the truths of some empirical domain or other (e.g., geology, zoology, physics, or geography). But whereas the analogy, on their view, suffices to explain what they mean by an 'independent mathematical reality', Wittgenstein insists that this produces a muddle.

Why does the analogy fail to clarify what is meant by 'to mathematical propositions there corresponds a mathematical reality'? The major objection raised by Wittgenstein is that Hardy extrapolates a bewildering picture from 'a mere truism', namely, that 'Mathematical propositions can be true or false' (*ibid*., p. 239). But, according to Wittgenstein, this is just to say 'that we affirm some mathematical propositions and deny others' (*ibid*.). We could paraphrase 'It is true …' by 'A reality corresponds to …', but this is just to replace one set of words with another, which in turn can only state the obvious if it states anything at all: 'that we affirm some mathematical propositions and deny others' (*ibid*.). Of course, it's also true that 'We […] affirm and deny propositions about physical objects. — But this is plainly not Hardy's point' (*ibid*.). That is, again, Hardy extracts an exotic picture from a trivial and obvious fact: that we affirm and deny both mathematical and empirical propositions. If Hardy were merely stating this trivial and obvious fact, 'it would come to saying nothing at all, a mere truism', that is, 'if we leave out the question of *how* it corresponds, or in what sense it corresponds' (*ibid*.).

In other words, the mere fact that we say of both mathematical and empirical propositions that they are true (or false) does not imply anything about a special mathematical reality to which mathematical propositions correspond. To think otherwise would be a hasty and unjustified leap from (i) a superficial similarity between these kinds of expressions (that they can both be 'true' or 'false', or that we affirm and deny both) to (ii) a problematic metaphysical picture. That is, if one thinks on the basis of these superficial similarities that mathematical propositions correspond to reality in *exactly* the same way that empirical propositions do, then this encourages a highly misleading picture of mathematics – that of mathematics corresponding to a reality somehow analogous to a mountain range, the animal kingdom, or particles in a cloud chamber. This would be akin to inferring from the analogy (i) 'reading a book is like riding a bike' (i.e., if you've learned once before, it's easy to pick back up), that therefore (ii) reading a book is *exactly* like riding a bike: it requires pedalling, shifting gears, and wearing a helmet. One similarity between A and B does not imply total similarity between A and B – a truism about analogy that, if forgotten, can lead to philosophical confusion.

Thus, according to Wittgenstein, Hardy has been taken in by a misleading analogy, itself encouraged by a trivial and superficial similarity between mathematical and empirical propositions. Such an analogy leads to a problematic picture of mathematical reality (i.e., as something 'out there' yet beyond our causal reach) by overlooking all of the crucial differences between the uses of mathematical and empirical propositions.

Consider, for instance, that one could have proceeded in the opposite direction: mathematical and empirical propositions have quite different uses, including the conditions under which we take them to be 'true', 'known', 'believed', *etc*.; therefore, mathematical propositions do *not* concern or correspond to reality *in anything like the way* empirical propositions do. Wittgenstein's stance (if we can call it that) is somewhere between Hardy's direction of thought and its opposite. That is, there *are* crucial differences between mathematical and empirical propositions, though there *are* also similarities (however superficial they might be). The trick is not to get seduced – either by their differences or similarities – into a misleading picture of their roles or meanings (Wittgenstein, 1976, p. 15).[15] Wittgenstein's tasks are thus (a)

[15] For instance, in the *Tractatus* Wittgenstein himself was so impressed by the *differences* between mathematical and empirical propositions that he was not even willing to call the former 'propositions' but instead 'pseudo-propositions' (Wittgenstein, 1961, §6.112, §6.2). On my reading of his later remarks, Wittgenstein is perfectly willing to grant that there are 'propositions' in mathematics and that these can legitimately be called 'true' or 'false' (since there are indeed ordinary language-games in which these terms have a home). Problems arise, however, from a hasty assimilation of empirical and mathematical propositions on the basis of these superficial similarities. Thus, there is some continuity here with the *Tractatus*, to the extent that Wittgenstein still sees it as his task to *describe* the 'peculiar position' of logical and mathematical propositions 'among all propositions' – the major discontinuity lies in his no longer being interested in articulating these differences *theoretically* according to 'the general form of the proposition', especially not so as to deny that there are really 'propositions' in

to display and 'diagnose' this leap in thought *from* superficial similarities among different kinds of 'propositions' *to* Hardy's picture of mathematical reality, as well as (b) to draw our attention to the differences between mathematical and empirical truths (or, propositions) in order to counteract the hold of this misleading picture.

Wittgenstein provides a diagnosis ('a thing which constantly happens') immediately after the foregoing passages.

> We have here a thing which constantly happens. The words in our language have all sorts of uses; some very ordinary uses which come into one's mind immediately, and then again they have uses which are more and more remote. For instance, if I say the word 'picture', you would think first and foremost of something drawn or painted and, say, hung up on the wall. You would not think of Mercator's projection of the globe; still less of the sense in which a man's handwriting is a picture of his character. A word has one or more nuclei of uses which come into everybody's mind first; so that if one says so-and-so is also a picture—a map, or *Darstellung* in mathematics—in this lies a comparison: as it were, 'Look at this as a continuation of that.'
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> So if you forget where the expression 'a reality corresponds to' is really at home—
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> What is 'reality'? We think of 'reality' as something we can *point* to. It is *this*, *that*.
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> Professor Hardy is comparing mathematical propositions to propositions of physics. This comparison is extremely misleading. (Wittgenstein, 1976, pp. 239–40)

Thus, as we saw before, Hardy is misled by a comparison between mathematical and empirical propositions (or, 'propositions of physics'). The comparison is misleading because it haphazardly generalizes from the features of *one* use of 'a reality corresponds to' to *all* uses of

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mathematics. My reading thus contrasts with Monk's (2007, p. 283), who argues that Wittgenstein's rejection of 'propositions' in mathematics lasts into his later writings.

this expression, *viz*., from the use of that phrase as applied to empirical propositions to its use as applied to mathematical propositions. Propositions like 'There are at least three cities larger than New York', 'There is a small chair in the living room', or 'The Sun is approximately 94.389 million miles away from the Earth' (which are themselves quite different from one another!) might be the most common and natural instances when we think of a reality (e.g., cities, chairs, the sun) corresponding to a proposition, but it would be a mistake to generalize all of their features to a proposition like 'There are at least three perfect numbers greater than 17' and assume that it 'corresponds to reality' in exactly the same way as the others.

Wittgenstein notes that in the most common or natural cases, we think of 'reality' as 'something we can *point* to', 'It is *this*, *that*'. For instance, if I say, 'There is a small chair in the living room', and someone expresses doubts about this, we can walk to the living room together and I can point at the small chair, 'Aha! See! I was right'. If there are doubts about whether 'There are at least three perfect numbers[16] greater than 17', then we need to perform some calculations (e.g., we might check one at a time the numbers greater than 17 to see whether any is a perfect number – which would take quite some time as the next in line after 6 are 28, 496, and 8,128! – more plausibly we'd verify by relying on someone else's work, as I did). I might 'point' to the results of our calculations, but this is not much like walking into a room and pointing at a chair – nor is it like, as Hardy suggests, looking at a distant mountain peak and pointing to it. To think otherwise would amount to being misled by the uniform appearance of the word 'pointing' across quite different uses – i.e., to mistakenly assume that its use in the mathematical case is exactly like its use in the empirical case because the word 'pointing' is used both times.

Wittgenstein repeatedly emphasizes elsewhere in the lectures that mathematical propositions often function as *rules* governing our descriptions of empirical reality, and are thus also for this reason importantly different from empirical propositions themselves (Wittgenstein 1976, pp. 33, 44, 82, 98, 112, 246, 256, 292; see also Wittgenstein, 1978, pp. 363, 324, 98–99). For instance, if we say 'there are (only) 7 apples in this box' (an empirical proposition), then the truth of this is sensitive to what we find in the box. If we find that there are 8 apples in the box, we will reject the initial claim that there are (only) 7. By contrast, if we say 'if there are (only) 4

[16] A perfect number is a positive integer that is equal to the sum of its positive divisors.

apples in Box A and (only) 3 apples in Box B, then there are (only) 7 apples in Boxes A & B' (i.e., 4 apples + 3 apples = 7 apples), then this stands as a fixed rule for intelligibly describing the situation, rather than being vulnerable to any 'findings' that might otherwise appear to controvert it (Wittgenstein, 1976, pp. 33, 200). A discrepancy between the arithmetical rule and our 'findings' is an immediate reason to think that there is something wrong with our 'findings', rather than with the rule itself (cf., *ibid*., pp. 44, 257). That is, if Thom counts 4 apples in Box A, Jónsi counts 3 apples in Box B, and PJ counts 8 apples in Boxes A & B, we will *not* thereby conclude, 'So, I guess that 4 apples + 3 apples = 8 apples after all!'. We would instead conclude either that one of Thom, Jónsi, or PJ has miscounted, or that an apple was added to one of the boxes before PJ performed their count, or cite some other reason that preserves the arithmetical rule that 4 apples + 3 apples = 7 apples. (Although none of us would need an independent arithmetical justification of this particular obvious rule, we could imagine more complex analogues, say, using larger numbers, where one might desire a proof or calculation for extra assurance. But the justification would be a matter of arithmetical proof or calculation, rather than empirical investigation.) It is in this sense that mathematical propositions regularly function as rules regulating certain areas of our (empirical) discourse, and are thus importantly different from empirical propositions themselves.

Notice that, by drawing out such differences, Wittgenstein is not making a sophisticated philosophical suggestion so much as noting obvious facts about the ordinary uses of our expressions (Wittgenstein, 1976, p. 22). The assimilation suggested by Hardy gains its power by overlooking such facts. If there is any sense to be made of 'a reality corresponding to' mathematical propositions, it will need to pay heed to these obvious facts. Going back to where we started, this is the 'false idea of the role which mathematical and logical propositions play' (*ibid*., p. 239): namely, that they play the same essential role as do empirical propositions and thus 'correspond to reality' in essentially the same manner.

5. Application to Benacerraf's, 'Mathematical Truth'

Wittgenstein's discussion of mathematical 'truth' in his lectures goes a long way towards deflating the pictures of mathematics offered by Hardy and Benacerraf. According to the therapeutic perspective, a primary source of Hardy's picture and others that resemble it is the assimilation of mathematical propositions with empirical propositions like 'Jones is tall', 'There are at least three cities larger than New York', or 'There are no apples in the fridge'. Granted: these propositions sound quite similar to '3 is prime', 'There are at least three perfect numbers greater than 17', and 'There are no prime numbers in the set of even numbers (except 2)', respectively. We can also attach 'It is true: …' to the front of each without any confusion in practice. But for all that, it would be a mistake to infer any deeper similarity on the basis of these parallels in sound and syntax.

Further, the fact that Hardy's and Benacerraf's pictures rely on such an argument from these similarities (superficial, by Wittgenstein's lights) to a deeper similarity in nature or function shows a subtle movement in thought that is quite easy to overlook, yet, once it is pointed out, is obviously suspect. One person might observe the similarities between propositions noticed by Hardy and Benacerraf and think, 'That's impressive! This must reveal a fundamental similarity', and yet another might (either independently or upon reading Wittgenstein's remarks) respond, 'So what? We use the word "true" in both contexts – why is this supposed to lead us to an exotic metaphysical picture of mathematics?' Putting the therapeutic perspective aside for just a moment: an adequate philosophical defence of Hardy's or Benacerraf's pictures would have to tell us why one reaction is somehow privileged over the other. But hopefully, if Wittgenstein's therapeutic strategy has had any effect, by this point one will feel the push towards a quite different attempt at gaining clarity: to describe in some detail how the various things called 'propositions' are actually used, without any presumption that they are somehow fundamentally similar in their nature or function.

Although Benacerraf's paper came many years after Wittgenstein's discussion of Hardy in his lectures, it is easy to see that his distinctive contributions to the discussion of mathematical 'truth' and 'reality' are equally vulnerable to Wittgenstein's therapeutic strategies. Benacerraf argues that a uniform theory of truth, such as that provided by Tarski, shows that mathematical sentences have the very same kinds of truth conditions as empirical sentences. I repeat his examples here for clarity.

(1) There are at least three large cities older than New York.
(2) There are at least three perfect numbers greater than 17.

Sentences (1) and (2) have the very same (syntactic) form, namely:

(3) There are at least three FG's that bear R to a.

We are meant to infer from this similarity in syntactical form that (2), just as much as (1), says something about a collection of objects. However, the objects referred to in (2) – 'out there' in some sense – are beyond our causal grasp. But why should we infer that (2) refers to objects *in the very same way* that (1) does? Or, similarly, why should we think that the 'objects' referred to in (2) are somehow similar to the objects of (1) – i.e., as being 'out there', yet beyond our causal reach? (As if a number were like an apple or a chair, but somehow ghostly and transparent – passing through all things without touching them.) Benacerraf's argument is that an adequate theory of truth shows them to have the very same kind of truth conditions.

Wittgenstein's response however would be the following: you've pointed out an interesting syntactic and phonetic similarity between (1) and (2), but for all that, you haven't given us any reason to think these propositions are therefore exactly the same, either in 'saying how things are', or 'referring to objects *in the very same way*', or 'implying an independent reality "out there" for us to "discover"'. A similarity in syntax does not all on its own reveal any deeper similarity, that is, not unless there is somehow a fundamental similarity between the roles or uses of these sentences.

Their roles or uses, however, are quite different. We can venture out to study the various cities of the world – whether by foot, car, boat, or plane – study historical records in order to determine their ages by some acceptable standard, collect our data, and make a report about the findings. By contrast, as was mentioned earlier, grappling with (2) will require calculation or proof (or, again, the reliance on someone else's calculations): it certainly won't require 'venturing out', studying historical records, or doing anything that can't be performed on a blackboard or with pencil, paper, and a decent calculator (Wittgenstein, 1976, p. 249). It might involve programming a computer to go through the series of numbers for us to determine whether a perfect number has been found – and however 'experimental' that might be, it is still quite different from determining whether (1) is true. These clear and obvious differences in roles make it difficult to motivate the assumptions that Benacerraf's argument requires, namely, (i) that empirical and mathematical propositions are true in the very same sense of 'truth' and (ii) that this sense is captured fully by Tarski's formalism. Again, there is indeed a similarity in syntax, but the differences between the uses of these propositions make it difficult to see why an exotic mathematical reality, somehow resembling empirical reality, should be inferred from such similarities.

It is quite important, however, to recognize that Wittgenstein's aim is not simply to remove from our language the expressions '(1) and (2) have similar truth conditions', or '(1) and (2) refer to objects', or '(1) and (2) both (in some sense) correspond to reality'.[17] After all, these are just English sentences – ones that might have, or might be given, a perfectly ordinary and acceptable use in practice. The first sentence, '(1) and (2) have similar truth conditions', for instance, might just be a way of saying that (1) and (2) have the syntax represented by (3). Wittgenstein doesn't need to dispute this trivial and obvious fact. Compare Wittgenstein's famous remark, 'Say what you please, so long as it does not prevent you from seeing how things are' (Wittgenstein, 2009, §79). The threat, then, is in being misled by such expressions in such a way that we overlook clear differences between the uses of mathematical and empirical propositions, thereby being prevented 'from seeing how things are' (cf., Wittgenstein, 1976, p. 251). Such a threat is explicitly brought up in Benacerraf's paper and thus requires our attention: namely, an inference from the syntactic similarities of empirical and mathematical propositions to a special epistemological problem for mathematics.

Recall that Benacerraf concluded from the similarity in truth-conditions between (1) and (2) (i.e., their syntactical similarities as emphasized in the Tarskian framework) that they both equally refer to objects (and in the same sense of 'object'), yet the objects referred to in (2) are beyond our causal reach and thus create a special problem about how they can so much as be known. So, the invocation of '(1) and (2) have similar truth conditions' in Benacerraf's paper immediately leads to philosophical questions and problems (e.g., 'What *is* a mathematical object *really*?'[18], or 'How is knowledge of a mathematical theorem *possible*?'). These questions and problems, in turn, rely on overlooking the crucial differences between (1) and (2) – or, more

[17] Compare Mülhölzer's (2014) reading on which Wittgenstein does not deny that we 'refer' to numbers, but that an understanding of such 'reference' must be sensitive to the distinctive uses of mathematical expressions.

[18] A question that is pursued most directly in Benacerraf (1965), also widely considered to be a classic paper in the philosophy of mathematics.

specifically, overlooking their differences *at the wrong time* in this movement of thought, as I will explain now.

For consider that one could completely flip Benacerraf's reasoning around. (We discussed a similar strategy earlier in section 4.) (2) is 'known' in an entirely different manner than (1) – that is, the circumstances in which we say that we 'know' (2) are very different from the circumstances in which we say that we 'know' (1). 'Knowing' (1) requires *literally* venturing out, studying documents, and reporting the results found (or otherwise relying on folks who have done this work). 'Knowing' (2) requires either proof or calculation; it certainly doesn't require venturing out, pointing to objects in the literal sense of 'pointing', making certain observations with our eyes or ears, studying historical documents, and so on. These are obvious differences between the uses of 'know' with respect to each of (1) and (2). Given these obvious differences, why should we think that their truth-conditions (in some sense independent of their syntax) are 'fundamentally the same'? More specifically, given that 'knowing' (2) doesn't involve anything like literally venturing out or literally pointing to objects, *etc*., why should we think that each of these sentences refers to objects in the very same manner? It seems that the sole motivation of Benacerraf's picture is the syntactic similarity between (1) and (2), codified and made explicit by the Tarskian framework. But those are just similarities in sound and syntax – they shouldn't lead us to think that there is some special problem of knowledge for basic arithmetical propositions. Of course, Benacerraf *does* recognize crucial differences between mathematical and empirical propositions, albeit not early enough in his reasoning to disrupt the picture of reality that is thereby concocted. He *first* infers a fundamental similarity, i.e., that (1) and (2) are true *in the very same sense* (since they both have the form of (3)). Only after this does he note *crucial differences*, e.g., that one does not literally look at or point to, say, the number 2, which then creates a special epistemological problem: 'since I don't have causal access to the number 2, how can I *know* anything about it?' Wittgenstein's strategy is to shift the order of considerations:[19]

[19] Compare Wittgenstein's remark that language becomes '*surveyable* through a process of ordering' (Wittgenstein, 2009, §92). The *order* in which we put forth obvious considerations is thus crucially important for Wittgenstein, which is not to say that there is a *single* order, but rather 'an order for a particular purpose, one out of many possible orders, not *the* order' (Wittgenstein, 2009, §132). The purpose of Wittgenstein's re-ordering of considerations in this

one does not literally look at or point to, say, the number 2 when determining whether '2 + 2 = 4'; so it would be highly misleading to think that 'Jones is tall' and '2 + 2 = 4' are true *in the very same way*. They involve distinctive uses of 'true' that take part in distinctive language-games. Likewise, they involve distinctive uses of 'know'. So much is obvious once we examine the differences between their uses or roles in ordinary life.

A defender of Benacerraf might point out that in his paper he is also concerned to undermine attempts to identify 'truth' with proof-conditions or justification-conditions more generally.[20] It might seem thus far that Wittgenstein is doing just this. But the line of thought considered throughout this paper has nothing to do with *identifying*, say, the 'truth' of mathematical claims with their justification-conditions. After all, 'true' and 'proposition' are considered by Wittgenstein to be conceptually on a par – one does not stand independently and somehow illuminate the other all on its own (Wittgenstein, 2009, §§134–37).[21] Instead, we need to examine the language-games to which they both equally belong; what is important above all is

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context is to diagnose the errors in this line of thought and emphasize distinctions that have been overlooked in the process: 'For this purpose we shall again and again *emphasize* distinctions which our ordinary forms of language easily make us overlook', that is, without making it 'our task to *reform* language' (*ibid*., emphasis added), but instead to *describe* it.

[20] See especially Benacerraf's (1973, p. 665) discussion of what he calls 'the combinatorial view of truth', on which 'the truth conditions for arithmetic sentences are given as their formal derivability from specified sets of axioms', a view that Benacerraf claims was 'torpedoed by the incompleteness theorems'. As I explain above, there is no reason to attribute such a view to the later Wittgenstein. See especially Shanker (1988) and Floyd (1995) for discussions of Wittgenstein's later stances on both formalism of the sort Benacerraf criticizes here as well as the incompleteness theorems.

[21] See Bold (2022, pp. 37–46) for a more detailed discussion of this connection with Wittgenstein's remarks on 'truth' and 'proposition' in the *Philosophical Investigations*. As I read those passages (Wittgenstein, 2009, §134–37), Wittgenstein is suggesting that 'truth' and 'proposition' are akin to 'game' in that they are family resemblance concepts – the instances of which are thus connected by overlapping and evolving similarities across their language-games, rather than adhering to a strict or unified essence (*ibid*., §66ff.).

how 'true' and 'proposition' are *used* in mathematics. It thus would not make much sense for Wittgenstein to seek a 'theory of truth' in the first place, such as one according to which truth is identified with proof or justification. Just as 'true' and 'proposition' belong to the language-games of arithmetic, so do 'know', 'certain', 'justified', and the like. Wittgenstein is merely pointing to obvious differences between these concepts as they play out in arithmetical and non-arithmetical language-games (such as determining how many cities are older than New York). For all that Wittgenstein says, there might very well be conditions for 'knowing' in some language-games that do *not* require any proof or calculation whatsoever. Our 'knowledge' that 2 + 2 = 4 would be an obvious example in ordinary life, since it is learned by rote and never requires further justification. (Notwithstanding mathematical logicians who play a distinctive language-game in which this too requires proof, a language-game in which distinctive rules for 'knowing' and 'justification' are deployed.) Attention to the obvious differences between mathematical and empirical propositions disrupts the analogy that Hardy and Benacerraf rely on in their invocation of a special mathematical reality.

In short, then, Wittgenstein's approach is entirely different from the proof-theoretic strategy Benacerraf considers and rejects in his paper and does not fall prey to his arguments against it. His approach is also not disrupted by the introduction of Tarski's formal work on 'truth' (the novel contribution of Benacerraf) into this dialectic. Tarski's formal work systematizes and makes explicit syntactic similarities across various different kinds of propositions; it does not thereby (without confusion) imply a distinctive and mysterious reality, somehow akin to empirical reality, to which mathematical claims refer. To be clear, this does not require a rejection of Tarski's *mathematical* work, it merely challenges Benacerraf's application of Tarski's formal framework and his attempt to show that it leads to special epistemological and metaphysical problems for mathematics.[22, 23]

References

A.J. Ayer, *Language, Truth & Logic* (Dover Publications, Inc., 1952).

P. Benacerraf, 'What Numbers Could Not Be', *The Philosophical Review*, 74:1 (1965), 47–73.

P. Benacerraf, 'Mathematical Truth', *Journal of Philosophy*, 70:19 (1973), 661–79.

P. Bold, Three Essays on Later Wittgenstein's Philosophy of Mathematics: Reality, Determination, & Infinity, PhD diss., UNC, Chapel Hill, (2022).

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[22] This is central to Wittgenstein's strategy in the philosophy of mathematics quite generally, as is nicely stated in Wittgenstein (1978): 'What I am doing is, not to show that calculations are wrong, but to subject the *interest* of calculations to a test. I test e.g. the justification for still using the word … here. Or really, I keep on urging such an investigation. I show that there is such an investigation and what there is to investigate there. Thus I must say, not: "We must not express ourselves like this", or "That is absurd", or "That is uninteresting", but: "Test the justification of this expression in this way". You cannot survey the justification of an expression *unless you survey its employment* [emphasis added]; which you cannot do by looking at some facet of its employment, *say a picture attaching to it* [emphasis added]' (Wittgenstein, 1978, p. 142). Wittgenstein's therapeutic perspective thus allows us to challenge what Benacerraf takes to be the *interest* of Tarski's mathematical work and the use of a certain expression to interpret that work, e.g., 'This shows that mathematical propositions refer to objects *in the very same way* that empirical propositions do'. The error is in being seduced by a 'picture attaching to' Tarski's framework, rather than the deployment of that framework and the sentences to which it is applied. See Floyd (2001) for similar arguments against the philosophical significance of Tarski's work on truth from the perspective of later Wittgenstein.

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P. Bold, 'Crossing Pictures of "Determination" in Wittgenstein's Remarks on Rule-Following', *Philosophical Investigations*, 00:0 (2023), 1–21.

J. Conant, 'On Wittgenstein's Philosophy of Mathematics', *Proceedings of the Aristotelian Society*, 97:2 (1997), 195–222.

C. Diamond, 'Wittgenstein, Mathematics, and Ethics: Resisting the Attractions of Realism', in H.D. Sluga and D.G. Stern (eds.), *The Cambridge Companion to Wittgenstein* (Cambridge: Cambridge University Press, 1996), 226–60.

H. Field, *Realism, Mathematics, and Modality* (Oxford: Blackwell, 1988).

J. Floyd, 'On Saying What You Really Want to Say: Wittgenstein on Gödel and The Trisection of the Angle', in J. Hintikka (ed.), *From Dedekind to Gödel: Essays on the Development of the Foundations of Mathematics* (Dordrecht: Kluwer Academic Publishers, 1995), 373–425.

J. Floyd, 'Prose *versus* Proof: Wittgenstein on Gödel, Tarski and Truth', *Philosophia Mathematica*, 9:3 (2001), 280–307.

J. Floyd, 'Wittgenstein on Philosophy of Logic and Mathematics', in S. Shapiro (ed.), *Oxford Handbook of Philosophy of Mathematics* (Oxford: Oxford University Press, 2006), 75–128.

J. Floyd and F. Mühlhölzer, *Wittgenstein's Annotations to Hardy's Course of Pure Mathematics: An Investigation of Wittgenstein's Non-Extensionalist Understanding of the Real Numbers*, Nordic Wittgenstein Studies 7 (Switzerland: Springer, 2020).

G. Frege, *The Foundations of Arithmetic*, 2nd revised edition, J.L. Austin (trans.), (Evanston, Ill.: Northwestern University Press, 1980).

S. Gerrard, 'Wittgenstein's Philosophies of Mathematics', *Synthese*, 87:1 (1991), 125–42.

K. Gödel, 'What is Cantor's Continuum Problem?', reprinted in P. Benacerraf and H. Putnam (eds.), *Philosophy of Mathematics: Selected Readings* (Cambridge: Cambridge University Press, 1984), 470–485.

I. Hacking, *Why Is There Philosophy of Mathematics at All?* (Cambridge: Cambridge University Press, 2014).

G.H. Hardy, 'Mathematical Proof', *Mind*, 38:149 (1929), 1–25.

G.H. Hardy, *A Mathematician's Apology* (Cambridge: Cambridge University Press, 1967).

Ø. Linnebo, *Philosophy of Mathematics* (Princeton & Oxford: Princeton University Press, 2017).

R. Monk, ‘Bourgeois, Bolshevist, or Anarchist?: The Reception of Wittgenstein’s Philosophy of Mathematics’, in G. Kahane, E. Kanterian, and O. Kuusela (eds.), *Wittgenstein and his Interpreters: Essays in Memory of Gordon Baker* (Oxford: Wiley-Blackwell, 2007), 269–294.

F. Mülhölzer, ‘How Arithmetic is About Numbers: A Wittgensteinian Perspective’, *Grazer Philosophische Studien*, 89 (2014), 39–59.

R. Penrose, *The Emperor’s New Mind: Concerning Computers, Minds, and the Laws of Physics* (Oxford: Oxford University Press, 1989).

H. Putnam, ‘Was Wittgenstein *Really* an Anti-Realist about Mathematics?’, in T. McCarthy and S.C. Stidd (eds.), *Wittgenstein in America* (Oxford: Oxford University Press, 2001), 140–194.

B. Russell, *Introduction to Mathematical Philosophy* (New York: Dover Publications, 1919).

S.G. Shanker, ‘Wittgenstein’s Remarks on the Significance of Gödel’s Theorem’, in S.G. Shanker (ed.), *Gödel’s Theorem in Focus* (London: Croom Helm, 1988), 176–77.

S. Shapiro, *Thinking About Mathematics* (Oxford: Oxford University Press, 2000).

D. Stern, *Wittgenstein’s Philosophical Investigations* (Cambridge: Cambridge University Press, 2004).

L. Wittgenstein, *The Big Typescript: TS* 213, C.G. Luckhardt and M.A.E. Aue (trans. and eds.), (Oxford: Blackwell, 2005).

L. Wittgenstein, *Culture and Value*, G.H. von Wright and H. Nyman (ed.), P. Winch (trans.), (Oxford: Blackwell, 1980).

L. Wittgenstein, *Philosophical Investigations,* revised 4th edition, P.M.S. Hacker and Joachim Schulte (eds.), G.E.M. Anscombe, P.M.S. Hacker, and Joachim Schulte (trans.), (Oxford: Wiley-Blackwell, 2009).

L. Wittgenstein, *Remarks on the Foundations of Mathematics,* revised edition, Ed. G.H. von Wright, R. Rhees, and G.E.M. Anscombe (Oxford: Blackwell, 1978).

L. Wittgenstein, *Tractatus Logico-Philosophicus*, D.F. Pears and B.F. McGuinness (trans.), (London: Routledge and Kegan Paul, 1961).

L. Wittgenstein, *Wittgenstein’s Lectures: Cambridge 1932–1935,* A. Ambrose (ed.), (Totowa, NJ: Rowman and Littlefield, 1979).

L. Wittgenstein, *Wittgenstein's Lectures on the Foundations of Mathematics, Cambridge 1939*, C. Diamond (Hassocks, Sussex: Harvester Press, 1976).

PHILIP BOLD (philipjbold@gmail.com) *is a lecturer in the Department of Philosophy at UMN, Twin Cities. His recent publications include 'Crossing Pictures of "Determination" in Wittgenstein's Remarks on Rule-Following'* in Philosophical Investigations *(2023) and* Three Essays on Wittgenstein's Later Philosophy of Mathematics: Reality, Determination, & Infinity (*PhD Diss, 2022*).