A Reassessment of Cantorian Abstraction based on the ε-operator

Nicola Bonatti*

Ludwig Maximilian Universität,
Munich Center for Mathematical Philosophy

nicola.bonatti@campus.lmu.de

July 29, 2022
Forthcoming in Synthese

Abstract

Cantor's abstractionist account of cardinal numbers has been criticized by Frege as a psychological theory of numbers which leads to contradiction. The aim of the paper is to meet these objections by proposing a reassessment of Cantor's proposal based upon the set theoretic framework of Bourbaki – called BK – which is a First-order set theory extended with Hilbert's ε-operator. Moreover, it is argued that the BK system and the ε-operator provide a faithful reconstruction of Cantor's insights on cardinal numbers. I will introduce first the axiomatic setting of BK and the definition of cardinal numbers by means of the ε-operator. Then, after presenting Cantor's abstractionist theory, I will point out two assumptions concerning the definition of cardinal numbers that are deeply rooted in Cantor's work. I will claim that these assumptions are supported as well by the BK definition of cardinal numbers, which will be compared to those of Zermelo-von Neumann and Frege-Russell. On the basis of these similarities, I will make use of the BK framework in meeting Frege's objections to Cantor's proposal. A key ingredient in the defence of Cantorian abstraction will be played by the role of representative sets, which are arbitrarily denoted by the ε-operator in the BK definition of cardinal numbers.

Keywords: Georg Cantor, Nicolas Bourbaki, ε-operator, abstraction, arbitrary reference.

1 Introduction

Cantor (1887) develops a theory of numbers based on the process of abstraction from an initial set of objects, by which the characteristic properties of these objects are omitted in order to form a set of *pure units*. More precisely, Cantor (1887) conceives abstraction as

*Earlier drafts of the paper were presented at the Twelfth French PhilMath Workshop and at the Genericity and Arbitrariness Seminar organized by the University of Campinas. I would like to thank the members of the respective audiences for their useful comments. I am especially indebted to Claudio Ternullo, Giacomo Guglieri, Giorgio Venturi, Francesco Berto, Marco Panza, Norbert Gratzl and Tom Kaspers for helpful comments on previous drafts of the paper. I also wish to thank three anonymous referees for their comments and suggestions that have helped to improve this paper.

a two-step process: first, abstraction from the characteristic properties of the elements of the set, which leads to a set of well-ordered units – corresponding to the ordinal number of the set. Secondly, abstraction from the order of the so-obtained units, which results in the size (or power) of the initial set – corresponding to the cardinal number. Cantor (1887) further claims that the cardinal set obtained in this way is equinumerous with the given set. Based on this definition, Cantor (1887) proves the fundamental property of cardinality, namely that equipotent sets are equinumerous, i.e. two sets have the same cardinal number *iff* there exists a one-to-one correspondence between them – see §1.2 for the comparison with the *Hume's Principle* formulated by Frege (1884). Apart from being widespread during Cantor's time – as Frege (1884) objections show (see §3) – the abstractionist theory of number described by Cantor (1887) is one of the major influences on Husserl (1891), who analyses the phenomenological processes to which, he believed, the concept of number owed its genesis – see Hill (1997).

Nevertheless, Cantorian abstraction has been discredited since Frege (1884) made his criticisms of it as a *psychological* theory of numbers which leads to contradiction. For example, Dummett asserts that the sections §§29-44 of Frege (1884) refute Cantor's proposal "brilliantly, decisively and definitively" (Dummett, 1991, p. 82). Even Hallett (1984) and Tait (1996) – who's work will be discussed later in the defence of Cantorian abstraction – consider Cantor's proposal as a ill-motivated step in the development of Cantor's theory of transfinite numbers. Only Fine advances a reassessment and defence of Cantorian abstraction, one based on his theory of arbitrary objects – see (Fine, 1985) – making use of the assumption that "abstraction, as Cantor and Dedekind conceive it, is ontologically innovative: it leads to objects which are genuinely new" (Fine, 1998, p. 601). Even if Fine's interpretation of abstraction as ontologically innovative poses a challenge to the one based on the BK framework – see §3.2 – Fine strongly diverges from Cantor's assumption concerning ordinal numbers and the well-ordering principle– see §2.1. That is why I will argue that the present proposal is closer to the leading motivations behind Cantor (1887) abstractionist theory.

The paper intends to overturn the disaffection for Cantorian abstraction in the literature. More precisely, the aim of the paper is twofold. On the one hand, the paper resists Frege (1884) objections by arguing for the coherence and plausibility of Cantorian abstraction. The defence of Cantorian abstraction will be built upon the set theoretic framework of Bourbaki (1968) – called BK – which is formulated in First-order Logic extended with the ε-operator. A key element in the defence of Cantorian abstraction will be played by the notion of arbitrary reference, as formalized by the ε-operator. On the other hand, the paper argues that the BK framework and the ε-operator provide a faithful reconstruction of Cantor's central ideas. Specifically, after presenting Cantor's abstractionist theory of ordinal and cardinal numbers, the paper spells out two leading assumptions in Cantor's work. Then, using a comparison with the Frege-Russell and Zermelo-von Neumann definitions of cardinal numbers, I argue that both assumptions are met by the BK framework. That is why the BK system is faithful to the Cantorian framework, even if the ε-operator and the notion of arbitrary reference were extraneous to Cantor's thought.

More precisely, the paper is organized as follows. I will first introduce the axiomatic setting of BK (§1.1) and the definition of cardinal numbers adopting the ε-operator (§1.2). Then, I will present Cantor (1887) abstractionist theory, discussing in details the definitions of ordinal and cardinal numbers (§2). Based on this framework, I will point out two leading assumptions in Cantor's work, namely the logicality of the well-ordering principle (§2.1) and the representational account of numbers required by his foundational project (§2.2). I will argue that these assumptions are supported as well by the BK definition of cardinal numbers, which will be compared to the definitions of Zermelo-von Neumann and Frege-Russell. On the basis of these similarities, I will adopt the BK framework to resist the objections originally made by Frege (1884) to Cantor (1887) proposal, which we divide into those based on the charge of psychologism (§3.1) and those based on the charge of incoherence (§3.2). I will argue that abstraction should be interpreted as arbitrary reference to a representative set, specified in no other way than being equinumerous to the given set – as for Cantor's set of *pure units*.

1.1 Bourbaki's Theory of Sets

Bourbaki (1968) 'Theory of Sets' (BK) has been considered as an outdated theory with a cumbersome notations (Mathias, 1992). However, the increasing attention to the pioneering work of Bourbaki in Category Theory (Heinzmann and Petitot, 2020) and the implementation of BK for automated proof assistant (Grimm, 2010) demands further attention to their foundational axiomatic system. Yet, a comprehensive evaluation of Bourbaki's system is beyond the scope of the present discussion – see Anacona et al. (2014). Instead, I will focus below on the implementation of the ε-operator for the explicit definition of cardinal numbers, which resembles the one given by Cantor (1887). It is worth mentioning that Bourbaki's terminology differs from modern presentations of axiomatic set theory. That is why I will introduce first the BK system following Bourbaki's terminology and then compare it with the well-known system of Zermelo-Fraenkel set theory with the Axiom of *Choice* (ZFC).

BK is a first-order set theory with equality that distinguishes between sets and classes. Every well-formed formula in the language $\mathcal{L}_{BK}$ is either a *term* – which represents an object of the theory – or a *relation* – which represents a statement about these objects, i.e. a proposition. Bourbaki (1968) warns from the start that the word 'set' must be considered strictly as a synonym for term. If a and b are terms, then the well-formed formula $a \in b$ is a relation, called the *membership relation*. Relations determine classes, namely the class of objects satisfying the given relation – such as $R = \{x|x = x\}$. The sets are obtained from particular relations that comply with the condition of being *collectivizing*. Let R be a relation and let y be a variable not occurring free in R, then the relation $\exists y \forall x((x \in y) \leftrightarrow R)$ is denoted by $Coll_x R$. If $Coll_x R$ is a theorem of BK, R is said to be *collectivizing in x* in BK. If this happens, an auxiliary constant a can be introduced to obtain the relation $\forall x((x \in a) \leftrightarrow R)$. Therefore, to say that R is collectivizing in x is to say that there exists a set a such that the objects x which posses the property R are precisely the elements of a. In this manner, Bourbaki (1968) prevents the formation of contradictory sets such as the 'set of all sets that do not belong to themself'. Specifically, Bourbaki shows that the relation $A = \{x|x \notin x\}$ is not collectivizing in x[1]. In other words, $\neg Coll_x A$ is a theorem in BK. Therefore, the class $A = \{x|x \notin x\}$ does not determine a set, thus avoiding Russell's paradox.

The logical framework is First-order Logic, except that Bourbaki (1968) replaces the classical quantifiers $\forall$ and $\exists$ with Hilbert (1922) ε-operator[2]. The ε-operator is a variable-binding operator which forms terms from open sentences, like $\varepsilon_x \varphi(x)$, which is interpreted

[1] The proof in BK runs as follows. Suppose that the relation $x \notin x$ is collectivizing, then $\exists y \forall x((x \in y) \leftrightarrow (x \notin x))$ is provable. Let a be a term, then we have that $\forall x((x \in a) \leftrightarrow (x \notin x))$. Therefore, the relation '$((a \in a) \leftrightarrow (a \notin a))$' is true, which is a contradiction.

[2] More precisely, Bourbaki (1968) introduces the τ-operator as a notational variant of Hilbert (1922) ε-operator. However, given the subsequent literature on the ε-operator, I will adopt the latter to introduce the BK system.

as ‘an arbitrary x such that $\varphi(x)$, if any’. The ε-operator is defined by two axioms:

Ax.1 $\varphi(t) \rightarrow \varphi(\varepsilon_x \varphi(x))$ (Critical Formula)

Ax.2 $\forall x[\varphi(x) \leftrightarrow \psi(x)] \rightarrow [\varepsilon_x \varphi(x) = \varepsilon_x \psi(x)]$ (Extensionality)

Based on the ε-operator, the classical quantifiers are defined as follows:

$$\exists x \varphi(x) =_{Df} \varphi(\varepsilon_x \varphi(x))$$

$$\forall x \varphi(x) =_{Df} \varphi(\varepsilon_x \neg \varphi(x))$$

Concerning the existential quantifier, if there is an x that satisfies the predicate φ, then there is also a description $\varepsilon_x \varphi(x)$ that satisfies φ. To look at it the other way around: if there is a description $\varepsilon_x \varphi$ that satisfies φ, then there must be an x that satisfies φ. By contraposition and the interdefinability of $\forall$ and $\exists$, we obtain the definition of the universal quantifier based on the ε-operator.

For what concerns the other set theoretical axioms, BK comprehends the ZFC Axioms of *Extensionality*, *Paring*, *Powerset* and *Infinity.* The Axiom of the *Empty set* is deduced in BK as a theorem, namely Bourbaki shows that the relation $\exists y \forall x(x \notin y)$ is functional, i.e. single-valued. By the Axiom of *Extensionality*, such set y is unique, and thus defined as the empty set $\emptyset$ – see Bourbaki (1968, p. 72). BK also includes the Axiom schema of *Selection and Union*, which is formulated as:

$$\forall x \exists y \forall z(R \rightarrow (z \in y) \rightarrow \forall x \exists w \forall z(z \in w \leftrightarrow \exists y(y \in x) \wedge R)) \qquad \text{(Selection and Union)}$$

where R is a relational formula such as $R(x_1, x_2, ..., x_n)$. If, for any x, there is a set y, such that R implies $z \in y$ then for any x there is a set w whose elements are all z such that R is true for at least one $y \in x$. Clearly, the Axiom of *Selection and Union* resembles the ZFC Axiom of *Replacement*, which asserts that if the relation $R(x, y)$ is single-valued in y – i.e. $R(x, y) = R(x, y')$ implies that $y = y'$ – then the consequent of *Selection and Union* holds[3]. However, unlike for the Axiom of *Replacement*, both the ZFC Axioms of *Union* and *Separation* follow in BK from the Axiom of *Selection and Union* – see Anacona et al. (2014, pp. 4078-4079).

Therefore, we are left to discuss the assumption of the Axiom of *Choice* in BK. It is remarkable that in BK the Axiom of *Choice* is presented as a theorem rather than an axiom. Indeed, the BK Axiom of *Selection and Union* is not restricted, namely the formula R might contain ε-terms. Let R be $z = \varepsilon_u(u \in y)$ and replace the ε-term in the consequent of *Selection and Union* holds that:

$$\forall x \exists w \forall z(z \in w \leftrightarrow \exists y(y \in x) \wedge z = \varepsilon_u(u \in y)) \tag{1}$$

Which asserts that for any x, there exists a set w whose members are the selected entities from each member of x. That is why the Axiom of *Choice* is derivable in BK. More precisely, by adopting the definition of ordered sets and the ε-axioms Bourbaki (1968, pp. 152-153) proves the well-ordering theorem, stating that all sets can be well-ordered – which is logically equivalent to the Axiom of *Choice.* I shall discuss this issue more extensively in §2.1, pointing out that the ε-operator is logically equivalent to the Axiom of *Global Choice.* To conclude, all the axioms of ZFC – except the Axiom of *Foundation* – are present in BK as either axioms or theorems. Moreover, (Anacona et al., 2014, p. 4080) further proves that the BK Axiom of *Selection and Union* is verified in ZFC as a theorem. That is why BK is logically equivalent to ZFC^{-}, namely ZFC without the Axiom of *Foundation.*

[3]More precisely, the Axiom of *Replacement* if formulated as: $\forall x \forall y \forall y'[(R(x, y) \wedge R(x, y') \rightarrow y = y')] \rightarrow \forall x \exists w \forall z(z \in w \leftrightarrow \exists y(y \in x) \wedge R(x, y))$

1.2 The BK definition of Cardinal Numbers

I will now turn to consider the BK definition of cardinal numbers – or ‘power’ in Cantor’s terminology – which will be the focus for the rest of the paper. Even if Bourbaki (1968) makes use of the ε-operator to define cardinal sets, the idea actually tracks back to Ackermann, who motivates the assumption of the ε-operator as follows:

> There is a certain vagueness in the assignment of a cardinal number to a set, for it is not explained how cardinal number and set are to be understood. In order to fix this vagueness, one can take different ways. [...] The other possibility is the one that one understands under the cardinal number a certain set that is equivalent to the given set. [...] The advantage then is that you do not have any needs of special axioms of abstraction, but the relevant formulas become provable. [...] From the axiomatic point of view, the mentioned indefiniteness is not disturbing, since all properties of the cardinal numbers can also be derived in this way. Furthermore, the concept of the cardinal number seems to me to be afflicted with this indeterminacy. (Ackermann, 1938, pp. 16-17) [Translation by the Author[4]]

I will discuss below what the indefiniteness mentioned by Ackermann corresponds to. Even if the details of Ackermann’s framework are not relevant to the present discussion, it is remarkable that both Ackermann (1938) and Bourbaki (1968) can be seen to define the cardinal of a set based on the resources of the ε-operator, as:

Def.1 Let t be a term denoting a set and x a variable not occurring free in t, then the cardinal of t is defined as:

$$|t| =_{Df} \varepsilon_x(x \approx t)$$

The idea is that, for any equivalence relation $\approx$, **Def.1** can be used to specify a representative element from the equivalence class $\approx$. In order to define cardinal sets, Bourbaki (1968) takes the equivalence relation $\approx$ to be the equinumerosity relation, i.e. one-to-one correspondence between two sets. Ackermann (1938) and Bourbaki (1968) further agree on the theoretical advantages of assuming **Def.1**, by which the fundamental property of cardinality follows. This property – namely that equipotent sets are equinumerous – was singled out by Cantor (1887) as:

$$\forall x \forall y(|x| = |y| \leftrightarrow x \approx y) \qquad \text{(Cantor Principle)}$$

Where x and y range over sets. However, Cantor’s Principle (CP) should not be confused with the principle introduced by Frege (1884) – known as the *Hume’s Principle* – by which the number of the F is equal to the number of the G if and only if F and G are equinumerous, where F and G are monadic second order variables that range over

[4] “In dieser Zuordnung von Menge und Kardinalzahl liegt eine gewisse Unbestimmtheit, da nitcht erklärt wird, was eigentlich unter der Kardinalzahl einer Menge zu verstehen ist. Um diese Unbestimmtheit zu beheben, hat man verschiedene Wege eingeschlagen. [...] Die andere Möglichkeit ist die, dass man unter der Kardinalzahl einer Menge eine bestimmte, zu der gegebenen Menge äquivalente Menge versteht. [...] Der Vorteil ist dann, dass man kaine besonderen Abstraktionsaxiome braucht, sondern die betreffenden Formeln beweisbar werden. [...] Vom Standpunkt der Axiomatik aus ist die erwähnte Unbestimmtheit nicht störend, da sich alle Eigenschaften der Kardinalzahlen auch so ableiten lassen. Ferner scheint mir der Begriff der Kardinalzahl an und für sich mit dieser Unbestimmtheit behaftet zu sein.”

concepts[5]. Clearly, if concepts are constructed as sets, then CP and HP are logically equivalent – Ternullo and Zanetti (2021) provide a common justification to both principles. I shall discuss further in §2 the role played by CP in Cantor's theory of cardinal numbers. Indeed, Bourbaki (1968, pp. 157-158) proves that:

Th.1 Let $\approx$ be the equivalence relation of equinumerosity, then CP follows from Def 1.

Proof. Let s and t be any two terms (i.e. sets). Since $t \approx t$, then $\exists z(z \approx t)$, and consequently, $\varepsilon_z(z \approx t) \approx t$, i.e. $|t| \approx t$ **(1)**. Similarly, we get $|s| \approx s$ **(2)**. From (1) and (2) and the fact that $\approx$ is an equivalence relation we obtain $|s| = |t| \to s \approx t$ **(3)**. On the other hand, the fact that $\approx$ is an equivalence relation implies $s \approx t \to \forall z(z \approx s \leftrightarrow z \approx t)$ **(4)**. By the EC axiom of Extensionality, we obtain $\forall z(z \approx s \leftrightarrow z \approx t) \to \varepsilon_z(z \approx s) = \varepsilon_z(z \approx t)$ **(5)**. Therefore, (4) and (5) yield $s \approx t \to |s| = |t|$ **(6)**. Consequently, from (3) and (6) we get $s \approx t \leftrightarrow |s| = |t|$ **(7)**. CP clearly follows from (7). This proof is adapted from (Leisenring, 1969, pp. 104-105)

$\square$

Before continuing, I should make the reader aware of a possible misunderstanding concerning the interpretation of ε-terms. It should be remarked that this kind of concerns does not strictly regard BK – which is a formal theory of sets with a fixed interpretation. However, it is plausible to ask what the arbitrary denotation of ε-terms amounts to, i.e. what it means for $\varepsilon_x\varphi(x)$ to select an arbitrary x such that $\varphi(x)$, if any. In model theoretic terms, the extensional semantics defined by **Ax.1** and **Ax.2** above evaluates ε-terms relative to a model $\mathfrak{M}$ equipped with a total choice function $f : \mathcal{P}(|\mathfrak{M}|) \to |\mathfrak{M}|$ such that for any non-empty set $A \in |\mathfrak{M}|$, $f(A) \in A$ and $f(A) \in |\mathfrak{M}|$ if $A = \emptyset$ – see Zach (2017). However, the crucial point is that any choice function $f : \mathcal{P}(|\mathfrak{M}|) \to |\mathfrak{M}|$ would do exactly as well as any other as the interpretation of ε-terms. Namely, it is semantically indeterminate what the global choice function should be. That is why I take arbitrary reference to be a *primitive* form of reference which should not be glossed in terms of canonical reference. Woods supports this claim as related to the interpretation of the ε-operator:

> The arbitrariness of the intended interpretation of indefinite expressions like [the ε-operator] is not merely epistemic. We do not understand $\varepsilon_x F(x)$ as being some particular F whose identity is determined by its domain of application in some way we are blocked from knowing. Rather, it is essential to understanding an indefinite expression like $\varepsilon_x F(x)$ that we recognize that its value really is arbitrary in the sense that facts about the domain do not determine which F, if any, it denotes. (Woods, 2014, p. 290)

In this sense, the cardinal set $|t|$ denoted by $\varepsilon_x(x \approx t)$ in **Def.1** is a representative set of the equivalence class. Nothing can be said about the cardinal set $|t|$ in **Def.1** except that it is equivalent to t and that it equals the cardinal number of any set which is equivalent to t. More precisely, a representative set is *neither* a specific yet unknown set – as for the epistemic account of arbitrariness of Breckenridge and Magidor (2012) – *nor* a new object added to the domain – as for the ontological account of arbitrariness of Fine (1985). I will further discuss below why the epistemic and ontological accounts of arbitrariness do not provide, respectively, a coherent and faithful reconstruction of Cantorian abstraction.

[5]Note that for Frege sets and concepts are essentially different entities. While sets are extensional mathematical objects, concepts unlike objects are 'unsaturated entities', i.e. intensional ones.

Therefore, the semantic function of ε-terms does not involve its referring to an object in a canonical way. In particular, we do not model the referential nature of an ε-term by assigning it a particular object in a model. It is worth mentioning that there are two different ways to model arbitrary reference as involved with the evaluation of ε-terms. According to the first one, ε-terms are evaluated relative to an arbitrary choice function, thus pushing the arbitrariness back into the metalanguage – see Leitgeb (2022). In this sense, the denotation of ε-terms is arbitrary because there is no semantic fact which determines what member of the set denoted by φ is 'chosen' by the ε-term $\varepsilon_x\varphi(x)$ – unless φ denotes a singleton set, in which case $\varepsilon_x\varphi(x)$ denotes the only object such that $\varphi(x)$. According to the second one, we can model arbitrary reference in terms of a supervaluational semantics where properties had by all individual choices of referent are had by the denotation of the ε-term – see Woods (2014) and Boccuni and Woods (2020). Roughly, if every precisification of a model assigning a particular member of the domain to the ε-term in the formula Φ agrees that Φ is true, then Φ is true and false otherwise. While the higher-order account of semantic indeterminacy maintains classical logic, the supervaluational view weaken classical logic by adding truth value gaps. More precisely, while the supervaluational account replaces classical truth with super-truth – i.e. truth according to all the precisifications of ε-terms – the higher-order interpretation states the existence of an arbitrary interpretation which satisfies the ε-term, thus maintaining classical truth as formalized by Tarskian semantics. However, choosing between the two interpretations requires a conceptual analysis of arbitrary reference which goes beyond the scope of the present work – i.e. the reassessment of Cantorian abstraction. That is why in §3, I will make use of arbitrary reference in order to resist Frege's objections without taking a stand on whether arbitrary reference should be understood in higher-order or supervaluational terms.

2 Cantorian Abstraction

The abstractionist theory of numbers is mainly discussed in Cantor (1887), which lays down the groundwork for the last work of Cantor (1895). As I will point out below, the abstractionist definitions of cardinal and ordinal numbers diverges from the ones proposed in the early work of Cantor (1883). Cantor (1887) was first published between 1887 and 1888 in a philosophical journal and it represents the attempt of the author to provide the epistemological foundation for his theory of transfinite numbers. Indeed, Cantor regards his abstractionist theory as part of the foundation of his broader work in transfinite numbers, asserting that the transfinite numbers follow "from the logical power of proofs, based upon definitions which are neither arbitrary nor artificial, but which arise naturally and regularly through the process of abstraction" (Cantor et al., 1991, p. 136).

Abstraction is considered by Cantor as a two-step process by which we disregard, first, the characteristic properties of a set of objects, retaining only their order, and then also from their arrangement so as to obtain a set of *pure units* corresponding to the size of the initial sets. I will explain below how Cantor (1887) attempts to provide a definition of ordinal and cardinal numbers based on the abstractionist theory. Even if abstraction of cardinal numbers follows the abstraction of ordinal numbers, Cantor (1887) introduces first the definition and properties of cardinal numbers. I shall motivate Cantor's choice in §2.1, but for the sake of clarity I will follow Cantor's presentation below.

The second step of abstraction, indicated by a double bar $\bar{\bar{M}}$, requires one to abstract from the order of the elements of $\bar{M}$ so as to obtain a set of *pure units* corresponding to the size (or power) of the initial set M. The idea is that pure units, through lacking any characteristic properties of the objects they correspond to, contribute only to the size of the set – which is thus defined as the cardinal number. Indeed, Cantor describes the second step of abstraction as follows:

> We denote the cardinal number or power of [a set] M, the result of this twofold abstraction by $\bar{\bar{M}}$. Since each individual element m if we disregard its nature becomes a 'one', the cardinal number $\bar{\bar{M}}$ *is itself a definite set* composed of nothing but ones which exists in our mind as the intellectual image or projection of the given set M. (Cantor et al., 1991, p. 136) [Italics added]

It is remarkable that Cantor considers cardinal numbers to be themself sets. Section §2.2 further discusses Cantor's assumption behind his reduction of cardinal numbers to set-sized objects. Moreover, Cantor claims that the cardinal set $\bar{\bar{M}}$ is equivalent to the initial set M:

> For, as we saw, $\bar{\bar{M}}$ grows, so to speak, out of M in such a way that from every element m of M a special unit of M arises. (Cantor, 1887, p. 88)

The equivalence relation is defined by Cantor as one-to-one correspondence (or equinumerosity) between the elements of two sets, thus $\bar{\bar{M}} \approx M$. This explains why Cantor considers the *pure units* of $\bar{\bar{M}}$ to be representatives of the elements of M. Concerning the identity condition for cardinal sets, Cantor adopts the criterion endorsed since Cantor (1883) – and presented above as CP – namely that equipotent sets are equinumerous:

> Every well-defined set M has a power, such that two sets have the same power when it is possible to correlate one to the other according to determinate laws so that to each element of M belongs an element of N and conversely to each element of N belongs one of M. (Cantor, 1883, p. 141)

That is why two sets x and y have the same numbers of elements if and only if there is a function $f : x \to y$ which is one-to-one and onto between them. Based on the identity condition for cardinal sets, Cantor (1887) goes on to explain the relations of 'greater than' and 'less than' for cardinal numbers in terms of the comparability of their corresponding powers. Given $\mathfrak{a} = \bar{\bar{M}}$ and $\mathfrak{b} = \bar{\bar{N}}$, then if there is no proper subset $M' \subset M$ such that $M' \approx M$, and if there is no proper subset $N' \subset N$ such that $N' \approx N$, then either $\mathfrak{a} < \mathfrak{b}$ or $\mathfrak{b} > \mathfrak{a}$. However, Cantor (1887) further claims that whenever two sets M and N can be mapped (in a one-to-one fashion) to proper subsets of each other, so that $M \approx N' \subset N$ and $N \approx M' \subset M$, then M and N are necessarily equivalent. While Cantor had already shown that, given any two cardinal numbers $\mathfrak{a}$ and $\mathfrak{b}$, only one of the order relations could hold, he was unable to prove that *exactly* one is always valid. Consequently, Cantor (1887) fails to guarantee the necessary comparability of all cardinal numbers, finite and infinite. The matter is of critical importance because, it they are not, it would be impossible to arrange all cardinal numbers in an ordered sequence. In §2.1, I will point out the ordinal assumption on which Cantor's (1887) definition of cardinal numbers relies.

The first step of abstraction brings about a generalisation of the theory of ordinal numbers presented in Cantor (1883). Indeed, Cantor (1883) focuses solely on well-ordered sets such as $\mathcal{N}$, defining ordinal numbers as the numerals representing the paradigmatic arrangement of a well-ordered set – more on this in §2.1. A set is said to be well-ordered if all of its elements are ordered by some relation such that, given any two, one could always be said to precede the other. Cantor (1887) realises that the well-ordered sequence of natural numbers represents but one type of order, an important yet limited case. That is why Cantor (1887) introduces the notion of *order-type*, which is defined for a set t if for any $a, b \in t$ then either $a < b$, or $a = b$, or $a > b$. Consider for example Q, namely the set of all rational numbers, which produces different order-types depending on how the elements are arranged. Indeed, Cantor (1887, p. 111) shows that Q can be ordered according to the magnitude of the sum of the numerator and denominator, thus following the natural order-type of ω. However, Q could also be arranged in a sequence determined by the sum of each numerator and denominator. Thus, given two fractions $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$, the lesser of the two sums $p_1 + q_1$, $p_2 + q_2$ determines which is to be taken as the lesser under this particular ordering. Then, while the order-type of natural numbers is still indicated as ω, the order-types of rational and real numbers is represented as, respectively, η and θ. Moreover, while for Cantor (1883) numerals are introduced through counting, order-types are obtained from sets of objects by abstracting from the characteristic properties of its elements, indicated by a single bar $\bar{M}$, as stated by Cantor:

> By this [namely, the order-type of M] we understand the general concept which arises from M when we abstract only from the nature of the elements of M, retaining the order of precedence among them. [...] Thus, the order type [...] is itself an ordered set whose elements are pure units. (Cantor, 1887, p. 297)

Indeed, for finite sets, the ordinal number α is univocally defined as the order-type of the predecessor of α, since the rearrangement of elements does not affect the type of order. That is why the cardinal and ordinal numbers of finite sets are isomorphic and represented by the familiar sequence of $1, 2, 3, ...$ Instead, for infinite sets, the definition is not straightforward, given that for one and the same ordinal number β might correspond to several order-types – as shown by the example above. That is why Cantor (1887) defines ordinal numbers differently from his early work, as classes of equivalent order-types – denoted as $[\alpha]$. Two ordered sets have the same order-type if they can be put into a one-to-one correspondence, as for the cardinal sets discussed above. Therefore, sets of equal order-type are always of equal cardinality, namely if $\bar{M} \approx \bar{N}$ then $\bar{\bar{M}} \approx \bar{\bar{N}}$, though clearly the converse is not generally true.

So far, I have carefully reconstructed Cantor (1887) definitions of (in)finite cardinal and ordinal numbers, as arising from his abstractionist framework. In the next two sections, I will point out on what assumptions relies Cantor (1887) in his abstractionist theory of numbers, namely: i) the well-ordering assumption, which is required for the definition of cardinal numbers and ii) the representational assumption, by which cardinal numbers are defined as sets rather than classes. As explained below, these assumptions are deeply rooted in, respectively, Cantor's conception of sets and his foundational program. That is why I will consider the well-ordering and representational assumptions as the *desiderata* for a correct interpretation of Cantorian abstraction. I will further argue that these desiderata are met by the BK definition of cardinal numbers, thus providing in this respect a faithful reconstruction of Cantor (1887) abstractionism. I will support my claim by comparing the BK definition of cardinal numbers with the ones of Frege-Russell and Zermelo-von Neumann. While the Frege-Russell approach defines cardinal numbers as classes of equinumerous sets or concepts[6], the Zermelo-von Neumann approach defines cardinal numbers as sets equinumerous with the smallest ordinal number[7] – see Incurvati (2020). The comparison

[6] I am here disregarding the different constructions of classes as either Fregean concepts or Russellian propositional functions.

[7] I am here disregarding on whether ordinal numbers are defined as Zermelo's or von Neumann's ordinals.

with Cantor's proposal is supported as well by the literature. Indeed, while Hallett (1984) argues for the development of Cantor's work within the axiomatic framework of ZFC as endorsed by Zermelo-von Neumann, Tait (1996) stresses the differences between the logicist project of Frege-Russell and the set theoretic one of Cantor.

2.1 The Ordinal Assumption

Several Cantor's scholars – such as Hallett (1984) and Dauben (2020) – agree that one of the aim of the abstractionist theory is to free Cantor (1887) definition of powers from ordinal assumptions. Cantor's aim might be better appreciated by considering his early definitions of ordinal and cardinal numbers. For instance, Hallett (1984) describes Cantor (1883) work as an ordinal theory of powers, which had a major influence on the successive development of axiomatic set theory. More precisely, as mentioned above, in (Cantor, 1883, p. 168) ordinal numbers are characterized as the "numerals of a well-ordered set." However, Cantor (1883) offers no proof of the existence of a numeral for each well-ordered set. As stated by Hallett:

> Thus Cantor believes and asserts that the step from the well-ordered set to the numeral is short and immediate. But he does not say why; we are left with the vagueness of "obtaining immediate representation in our inner intuition". Hallett (1984, p. 54)

Yet, numerals are required by the definition of cardinal numbers, which are represented by number-classes that gather together all ordinal numbers up to a certain point[8] – more on this in §2.2. The assumption lying in the background is the Axiom of *Choice*, by which the well-ordering theorem follows. Then, it is customary practice in ZFC to prove that every well-ordered set is isomorphic to a unique ordinal number (Jech, 2013). As explained below, Cantor fails to formulate the well-ordering principle (which is logically equivalent to the Axiom of *Choice*) as an independent assumption because he considers it as a constitutive principle of the concept of set. By the time of Cantor (1887), missing a proof of the well-ordering theorem, the author intends to provide a definition of cardinal numbers independent of ordinal assumptions. That is why Cantor supposes that the abstractionist account of cardinal numbers presented in §2 avoids such assumption, by being applicable to ordinal numbers as well.

However, Cantor (1887) dramatically fails in freeing cardinal numbers from ordinal assumptions. Indeed, as pointed out by Hallett (1984), the definition of ordinal numbers as classes of order-types requires a much stronger assumption than the one adopted in Cantor (1883). In Cantor (1883), any number-class contains many sets determined by an ordinal, that is to say of the form $\{\alpha \; \alpha < \beta\}$ for some ordinal β. Then, based on the Axiom of *Choice*, for each number-class we can choose representative set determined by the smallest ordinal – as done in ZFC by the *von Neumann cardinal assignment*, see Incurvati (2020). However, this is not the case if ordinals themselves are defined as classes of equinumerous order-types – as for Cantor (1887), see §2. Indeed, in order to choose representative sets for all the isomorphism classes one is faced with the problem of choosing simultaneously from a class of non-empty sets. The only known way of doing this is to assume as given a well-ordering of the universe $\mathcal{V}$ – which is equivalent to the Axiom of *Global Choice* (AGC). Indeed, AGC implies that there is a bijection between $\mathcal{V}$ and the class of all ordinal numbers.

[8]Number-classes are sets of ordinal numbers rather than proper classes, i.e. $\aleph_0$ = the set of all finite ordinal numbers.

The implicit assumption of AGC further supports the plausibility of the BK framework for the reassessment of Cantor (1887) abstractionist theory. Indeed, as explained in §1.1, the Axiom of *Choice* follows in BK from the assumptions of **Ax.1**, **Ax.2** and the unrestricted Axiom of *Selection and Union*. Actually, the ε-operator has a greater inferential power than the Axiom of *Choice*, as pointed out by Wang, comparing the BK and ZFC systems:

> There are also cases where, although the ε-rule yield the desired result, the axiom of choice would not. For example, in the Zermelo theory we can infer '$\forall x R(x, \varepsilon_y Rxy)$' from '$\forall x \exists y R(x, y)$' by the ε-rule, but we cannot infer '$\exists f \forall x R(x, f(x))$' from '$\forall x \exists y R(x, y)$' by the axiom of choice, on account of the absence of a universal set in Zermelo's theory. (Wang, 1957, pp. 66-67)

Indeed, the ε-operator is logically equivalent to a global choice operator, whose existence is asserted by AGC of the von Neumann-Bernays-Gödel set theory (NBG). Indeed, both operators allow one to choose an arbitrary element x from each non-empty set t, such that $x \in t$. This can be checked by comparing the ε-axioms presented in §1.1 with the ones for the global choice operator σ introduced by Bernays (1991):

Ax.1* $a \in C \to \sigma(C) \in C$
Ax.2* $(A \leftrightarrow B) \to \sigma(A) = \sigma(B)$

While **Ax.1*** asserts that for every non-empty set C, the value of $\sigma(C)$ is a member of C, **Ax.2*** states that if two sets A and B are co-extensional, then the values of $\sigma(A)$ and $\sigma(B)$ is the same individual. It is clear that **Ax.1*** is equivalent to the ε-axiom of Critical Formulas and **Ax.2*** to the one of Extensionality. Therefore, the ε-operator of BK is logically equivalent to the AGC of the NBG system. Given that AGC is required by Cantor (1887) definition of ordinal and cardinal numbers, the equivalence of the ε-operator with AGC helps to support the plausibility of the BK system as a faithful reconstruction of the Cantorian framework.

Therefore both Cantor (1883) and Cantor (1887) theories of cardinal numbers rely on strong ordinal assumptions, respectively, the well-ordering principle and the well-ordering of the set theoretic universe $\mathcal{V}$. This assumption further explains why the reassessment of Cantorian abstraction based on the BK framework is closer to Cantor (1887) insights than Fine (1998) proposal, which relies on his theory of arbitrary objects (Fine, 1985). Even if an accurate presentation of Fine's proposal is beyond the scope of the present work, it is remarkable that in the introduction the author states that: "I am therefore inclined to think that, if Cantor's account can be made to work, then so can an account that takes unordered sets as its starting point" (Fine, 1998, p.603). Therefore, while Fine proposal might be taken to asserts the coherence of a abstractionist theory of cardinal numbers devoid of ordinal assumptions, the BK framework provides a more faithful reconstruction of Cantor (1887) original system.

Before continuing, it is worth clarifying the claim here at issue[9]. I am not arguing that, given the logical equivalence of the ε-operator (or, equivalently, AGC) with the well-ordering of $\mathcal{V}$, the former underlies the definition of cardinal numbers proposed by Cantor (1887). Indeed, the ε-operator belongs to the formal language of the axiomatic set theory BK, both of which were extraneous to Cantor's thought. Moreover, the intuitions behind these two assumptions are different. While the well-ordering of $\mathcal{V}$ concerns the

[9]Thanks to one of the anonymous reviewer for pointing this out.

arrangement of the elements into a well-ordered set, the ε-operator concerns the choosability of an element as representative of the set. Rather, I am claiming that, given the leading role of the well-ordering principle in Cantor's thought, **Def.1** provides a faithful reconstruction of Cantor (1887) insights on the definition of cardinal numbers. I will support my claim through a comparison with the Zermelo-von Neumann definition of cardinal numbers. I have already explained above why both Cantor (1883) and Cantor (1887) definitions of cardinal numbers rely on ordinal assumptions. The assumption of the well-ordering principle constitutes the structure providing the arithmetical scale for the comparability of powers. Even if Cantor fails to explicitly formulate such an assumption, he realizes that the well-ordering principle is tied up with the concept of set:

> The concept of *well-ordered set* is fundamental for the whole theory of manifolds. It is a basic law of thought, rich in consequences and particularly remarkable for its general validity, that it is possible to bring any *well-defined* set into the form of a *well-ordered* set. (Cantor, 1883, p. 169)

Cantor's remarks on the well-ordering principle had a major influence on the later development of axiomatic set theory, particularly on the work of Zermelo. Indeed, Zermelo (1908) complains about Cantor's psychological description of the well-ordering principle, which rests on how we conceive sets and what operations we perform on them. In order to avoid the intrusion of psychological notions within mathematics, Zermelo (1908) formulates the Axiom of *Choice*, which states that for every set x of non empty sets there exists a choice function f which maps each set of x to an element of that set. On the assumption of the Axiom of *Choice*, the well-ordering theorem follows, stating that every set can be well-ordered. Then, for any well-ordered set t, the von Neumann cardinal assignment defines its cardinal number $|t|$ to be the smallest ordinal number equinumerous with t. In this sense, the definition and properties of cardinal numbers as specified by the Zermelo-von Neumann account follows from the assumption of the Axiom of *Choice* – which is logically equivalent to the well-ordering theorem. That is why Hallett (1984) concludes that, even if the Axiom of *Choice* was not anticipated by Cantor, the Zermelo-von Neumann definition is an accurate reconstruction of Cantor's ordinal theory of powers – see also §2.2. But then a similar conclusion can be made concerning **Def.1**: even if the ε-operator is extraneous to the Cantorian framework, **Def.1** rests on the same ordinal assumption underlying Cantor's 1887 definition of cardinal numbers. Therefore, the BK framework provides a faithful reconstruction of Cantor's central ideas concerning ordinal and cardinal numbers.

2.2 The Representational Assumption

As for ordinal numbers, also Cantor (1887) definition of cardinal numbers departs from the precedent one of Cantor (1883). Indeed, as explained above, Cantor (1883) considers cardinal numbers to be represented by number-classes of well-ordered sets (or ordinals) having a strictly smaller power. While Cantor (1883) remains vague about the actual definition of cardinal numbers – claiming that number-classes only *represent* cardinal numbers – Hallett (1984) proposes to identify cardinal numbers with number-classes, thus stressing the similarities with the ZFC ordinal theory of cardinalities. Instead, Cantor (1887) explicitly states that cardinal numbers (or powers) are themself sets, as clearly expressed by the following example:

> For the formation of the general concept 'five' one needs only a set (for example all the fingers of my right hand) which corresponds to this cardinal number. (Cantor et al., 1991, p. 418)

In order to better understand Cantor's reduction of cardinal numbers to set-sized objects, it is useful to compare it with the Frege-Russell proposal – which will highlights also the similarities with the BK definition. Indeed, Hallett (1984) suggests to distinguish the definitions of cardinal numbers between the *representational* one of Cantor (as further develop by Zermelo-von Neumann) and the *non-representational* account proposed by Frege-Russell. Roughly, an account of the types of some kind is representational if each type of the given type is of that very type. More precisely, a definition of cardinal numbers is representational if and only if the number set or class is itself the cardinal number – and non-representational otherwise. Following Hallett (1984), this distinction can be better appreciated by pinning down the conditions for the definition of cardinal numbers. There are two minimal requirements:

i) The operation $|t|$ is defined for all sets t.

ii) $\forall x \forall y (x \approx y \leftrightarrow |x| = |y|)$.

Conditions (i) and (ii) are the ones adopted by the Frege-Russell account of cardinal number, where condition (ii) is clearly (CP)[10]. However, Hallett (1984) stresses that the Zermelo-von Neumann account – namely, the ZFC definition adopting the *von Neumann cardinal assignment* (see §2) – endorses two further conditions, which were first introduced by Cantor (1887):

iii) For every set t, $|t|$ is a set.

iv) For every set t, $|t| \approx t$.

Namely, not only cardinal numbers are sets, but they are also equinumerous with the set they number. Conditions (iii) and (iv) were informally stated by Cantor (1887), while discussing the abstractionist definition of cardinal numbers – see §2. Then, the comparison with the Frege-Russell proposal is straightforward. As explained above, according to the Frege-Russell account, the cardinal number of a set (or concept in Frege's terminology) is the class of all sets or concepts equinumerous to it. More precisely, while Frege (1884) takes the cardinal number belonging to a concept F to be the extension of the concept *equinumerous with* F, Russell and Whitehead (1910) takes the cardinal number associated with a set a to be the class of all sets equinumerous with a. In both cases, the whole equivalence class is taken as the *definiens* of cardinal number. Therefore, it is clear why the Frege-Russell account is non-representational: every cardinal number (except 0) is way bigger than the set they number. Instead, the account of cardinal numbers given by Zermelo-von Neumann is representational in this sense: each cardinal number is of that number. The merit of the *von Neumann cardinal assignment* is precisely to explain how the conditions (iii) and (iv) can be met without appealing to Cantor's abstractionist theory. Moreover, Hallett (1984) distinction between representational and non representational accounts of cardinal numbers further supports the similarity between the BK definition and Cantor (1887) proposal. Indeed, **Def.1** is clearly representational because it satisfies conditions (iii) and (iv). More precisely, **Def.1** is specified for any set t – as for condition (iii) – which is assumed to be equinumerous with the cardinal set $|t|$ – as for condition (iv). I take this as further evidence supporting the reassessment of Cantorian abstraction based on the BK framework.

Someone could then wonder why a representational theory of cardinal numbers is considered by Cantor as a *desideratum*. As explained below, this requirement is motivated

[10] Or, equivalently, HP if we assume that concepts are defined as sets.

by the set theoretic foundational program started by Cantor himself. More precisely, by defining powers as sets, Cantor aims to prove that the arithmetical operations on numbers can be represented as operations on sets, thus stressing also the uniformity between finite and infinite numbers. But if natural numbers can be represented as sets, then also all the other mathematical objects can be reduced to set theoretic constructions too. Indeed, all the domains which one normally uses in mathematics are embeddable in a higher domain, e.g. the natural numbers in the rational numbers, the rational numbers in the real numbers, the real line in the real plane, and so on. That is why Cantor (1885) assumes that every well-determined set has a power, either a finite number or an infinite cardinal[11]. Then, the representational assumption for cardinal numbers is part of the broader project endorsed by Cantor – and radically different from the Frege-Russell logicist program[12] – according to which pure mathematics should be reduced to set theory, as clearly expressed by himself while discussing his theory of ordinal numbers:

> It forms a large and important part of pure set theory, thus also of pure mathematics, since this latter according to my conception is nothing other than pure set theory. (Grattan-Guinness, 1970, p. 84)

Based on Cantor's reductionist program, the representational account of cardinal numbers has a heuristic advantage concerning the arithmetical operations on numbers. Indeed, for both Cantor (1887) and the Frege-Russell definition, the relation of precedence between numbers pins down to the relation of cardinal comparability between sets. However, as pointed out by Hallett (1984), in the Frege-Russell definition the whole class $|t|$ is detached from the set t, in the sense that $|t|$ is not a direct replacement for t. Instead, Cantor remarks that any set from the equivalence class would work as a representative of the cardinal number, since by definition it is equinumerous to any other set in the class. Therefore, according to the reductionist program, Cantor considers the representational account of cardinal numbers as a heuristic *desiderata*, by which the operations on cardinal numbers can be carried out on the representative sets themselves. The same requirement is met as well by the BK system, where the cardinal arithmetic is developed from the representative sets introduced by **Def.1**. Moreover, it is worth mentioning that Bourbaki (1968) is the first volume of a series of books entitled *'Elements of Mathematics'*, which lays down the logical and set theoretical framework required for the subsequent foundation of the whole body of mathematics, from arithmetic up to algebraic topology. That is why the work of Bourbaki (1968) might be considered as the fulfilment of the foundational project first advocated by Cantor himself, according to which all mathematics should be reduced to set theory.

3 Frege's Objections

After presenting Cantor (1887) abstractionist theory of cardinal and ordinal numbers, I have pointed out two assumptions deeply rooted in Cantor's work, namely the logical status of the well-ordering principle and the representational account of numbers required by his foundational project. Based on the reconstruction of Cantor's 1887 work, I have

[11] I set aside the issue of what, according to Cantor, distinguishes determinate from indeterminate sets. While Hallett (1984) argues that the theological beliefs of Cantor brought him to distinguishes sets from collections that are too big to form a set, Ebert and Rossberg (2009) complains that this an *ex post* interpretation of Cantor's remarks, based on the later development of the set theoretic paradoxes.

[12] In the most general terms, the logicist program aims to ground mathematical knowledge on a purely logical basis only.

argued that the BK definition of cardinal numbers meets these two *desiderata*, proposing a representational account of cardinal numbers which relies on the logical resources of the ε-operator (equivalent to AGC). That is why **Def.1** represents a faithful reconstruction of Cantor (1887) insights on the definition of cardinal numbers.

I will now move to consider the objections made by Frege (1884) to Cantor (1887) abstractionism, which as explained above constitutes an essential part of his definition of ordinal and cardinal numbers. As mentioned in §1, Frege's objections have been considered in the literature – especially by Dummett (1991) – as a knock-out argument against Cantorian abstraction. Frege's objections can be summarized as a *dilemma*: either the abstractionist process is a psychological process which does not provide an objective definition of numbers, or the theory of pure units composing the set-sized numbers obtained by abstraction is incoherent. It should be remarked that Frege does not state the objections as a dilemma. However the author claims that, even if we grant the psychological account of abstraction, the theory of pure units would still be incoherent:

> We cannot succeed in making different things identical simply by operating with concepts. But even if we did, we should then no longer have things in the plural, but only one thing [...]. (Frege, 1884, p. 46)

Therefore, Frege seems to suggest that the opponent has to choose between two unfavourable alternatives. I will consider both horns of the dilemma in turn. The reconstruction and defence of Cantorian abstraction will rely on the notion of arbitrary reference described in §1.2 – adopted for the interpretation of ε-terms. More precisely, I will argue that the abstractionist process should be replaced by the arbitrary reference to representative sets as formalized by the ε-operator in **Def.1**. Moreover, I will account for Cantor's compelling theory of pure units based on the featureless elements composing the representative set denoted by the ε-operator in **Def.1**. However, it should be remarked that my defence of Cantorian abstraction will partially depart from Cantor (1887) philosophical insights. Indeed, both the ε-operator and the notion of arbitrary reference were clearly extraneous to Cantor's work. Nevertheless, I have argued above that the BK framework is a faithful reconstruction of Cantor's leading ideas. That is why the reassessment of Cantorian abstraction based on BK will be evaluated according to its coherence and plausibility, rather than its fidelity to Cantor's philosophical insights. For instance, given that Frege (1884) objections concern solely Cantor's definition of cardinal numbers, I will disregard the distinction between the first act of abstraction $\bar{M}$ from the properties of the elements of the set and the second act of abstraction $\bar{\bar{M}}$ from the order of the so-obtained pure-units – see §2.

3.1 The Psychologism Objection

Frege (1884) sections §34-35 and Frege (1890) review of Cantor (1887) mainly focuses on the process of abstraction as a mental activity, which claims to provide the cardinal number starting from a given set of objects – as presented in §2. Before evaluating Frege's complaints, it should be remarked that abstractionist theories of numbers were widespread among philosophers and mathematicians of the 19^{th} century – such as Dedekind and Husserl. The leading idea is that through abstraction from the properties of a set of objects we get the number corresponding to its size. Frege's psychologism objection can be summarized as follows: abstraction from the properties of the elements of a set does not provide the cardinal of the set, but only a more general property characterizing the elements of the set. Therefore, abstractionism is a psychological process which fails to provide an objective definition of numbers. As stressed by Dauben (2020), Cantor held an ambivalent stance over the psychological characterization of mathematics. On the one hand, he praises Frege (1884) for omitting psychological considerations from the foundation of arithmetic. On the other hand, he explicitly describes the abstraction process presented in §2 in psychological terms, asking:

> Are not a set and the cardinal numbering belonging to it quite different things? Does not the first stand to us as an object, whereas the latter is an abstract image in our intellect? (Cantor, 1887, p. 416)

That is why, in order to resist Frege's psychologism objection, I will depart from Cantor's characterization of abstraction as a psychological process, relying instead on the logical resources of the ε-operator. Frege starts by questioning whether removing the properties from a set of objects through abstraction would end up with a set of pure units corresponding to its size:

> For suppose that we do, as Thomae demands, "abstract from the peculiarities of the individual members of a set of items", or "disregard, in considering separate things, those characteristics which serve to distinguish them". In that event we are not left, as Lipschitz maintains, with "the concept of the Number of the things considered"; what we get is rather a general concept under which the things in question fall. The things themselves do not in the process lose any of their special characteristics. If, for example, given a white cat and a black cat, I disregard the properties which serve to distinguish them, then I get presumably the concept 'cat'. Even if I proceed to bring them both under this concept and call them, I suppose, units, the white one still remains white just the same, and the black black. (Frege, 1884, p. 45)

Therefore, abstraction fails to transform a set of objects into a set of pure units, corresponding to the cardinal number. Instead, according to Frege, abstraction from the characteristic properties of a set of objects would only result in a more general concept under which these objects fall. Moreover, abstractionism as a mental process does not meet the intersubjective standard required for an objective definition of numbers. More precisely, Frege points out that, given a set of objects, there is no guarantee that the abstraction process as carried out by different subjects would end up with the same general concept:

> So let us get a number of men together and ask them to exert themselves to the utmost in abstracting from the nature of the pencil and the order in which its elements are given. After we have allowed them sufficient time for this difficult task, we ask the first "What general concept have you arrived at?" Non-mathematician that he is, he answers "Pure Being". The second thinks rather "Pure nothingness", the third - I suspect a pupil of Cantor's - "The cardinal number one". [...] Now why shouldn't one man come out with the answer and another with another? (Frege, 1890, p. 71)

The standard of intersubjectivity for mathematical knowledge is dictated by Frege's logicist program, according to which a theory can be considered as part of mathematics only by explicit definitions of the basic concepts and step-by-step logical deductions. Given that abstraction rests on the subject's mental abilities, Frege concludes that Cantorian abstraction jeopardizes the objectivity of the definition of numbers.

Ultimately, I will claim that Frege's objection does not apply to the BK framework. Indeed, by replacing Cantor's psychological account of abstraction with the logical one of **Def.1**, the BK framework is immune from Frege's *psychologism* objection because it dispenses with the very aspects of Cantor's theory that Frege objected to[13]. However, before continuing, it is worth questioning Frege's interpretation of Cantorian abstraction. These clarifications will be adopted below to characterize the semantic conception of abstraction involved with **Def.1** – as opposed to both Cantor's and Frege's proposals. Indeed, as remarked by Hallett (1984), Cantor does not conceive abstraction as a gradual process by which each object is separately removed from its characteristic properties and then assembled into a set of pure units. Rather, Cantor considers abstraction as an immediate shift from the given set to the cardinal set of pure units. This point is clearly expressed by Cantor himself, who criticizes Leibniz's definition of numbers by stating that:

> The addition of ones, however, can never serve for a definition of a number, since here the specification of the main thing, namely how often the ones must be added, cannot be achieved without using the number itself. This proves that the number is to be explained only as an organic unity of ones achieved by a *single act of abstraction.* Cantor (1887, p. 381) [Italics added]

We see that for Cantor the pure units composing the cardinal set $|t|$ are obtained by a single act of abstraction over the set t[14]. It should be remarked that this seems much more like the use of a classically conceived functional operation on sets rather than mental reflection. However, bearing on the interpretation of abstraction as a single act, a further question remains: are we abstracting from the property determining the set or simultaneously from the members of the set? Cantor neither answers this question, nor provides an explanation of the grounds on which we should accept without demonstration the existence of a set equinumerous with the given set t. Resembling Frege's objection above, Russell notes that Cantor's description of abstraction "is merely a phrasing indicating what is to be spoken of, not a true definition" (Russell and Whitehead, 1910, pp. 304-305).

Both concerns can be addressed in the BK framework as follows. More precisely, I will argue that the BK framework is immune from Frege's objections and **Def.1** answers the ambiguities left by Cantor's work. First of all, there is a straightforward difference between the BK framework and Cantor's abstractionist theory. While the latter conceives abstraction as a psychological process, according to the former abstraction is logically defined by **Def.1**. That is why the reassessment of Cantorian abstraction based on BK will be evaluated according to its coherence and plausibility, rather than the fidelity to Cantor's philosophical insights – as remarked in §3. Then, it is clear why Frege's psychologism objection does not apply to the BK framework: **Def.1** is an *explicit* definition of cardinal numbers by which the fundamental law of CP can be deduced. The cardinal set $|t|$ is the definiendum specified by the definiens containing the ε-operator. The explicit definition guarantees the existence of a representative set $|t|$ equinumerous with the given set t. Moreover, CP and the ε-axiom of *Extensionality* (**Ax.2**) determine the identity condition for cardinal sets. That is why the BK definition of cardinal numbers meets the desiderata of logical foundation advocated by Frege. Therefore, if the BK system is considered as the foundation of mathematics, as argued at length by Bourbaki (1968), then no ambiguities could arise concerning the definition of cardinal numbers – thus debunking Frege's psychologism objection.

[13]Thanks to one of the anonymous reviewer for stressing this point.

[14]As explained above, I am here ignoring the distinction between the two steps of abstraction introduced by Cantor (1887) – namely $\bar{M}$ and $\bar{\bar{M}}$ – see §2.

Even if the BK framework provides a logical definition of cardinal numbers according to Frege's guidelines, **Def.1** should not be regarded as a definition by abstraction in the sense of the *Hume's Principle* (HP) mentioned in §1.2. The recent resurgence of interest in definition by abstraction follows the Neologicist program, which attempts to establish the analicity of arithmetic by deriving the Peano axioms from Second-order logic extended with the abstraction principle of HP – see Wright (1983). According to the Neologicist program, abstraction principles implicitly define mathematical objects by specifying their identity conditions – for instance, HP is taken to implicitly define cardinal numbers. Instead, **Def.1** explicitly defines cardinal sets by means of the ε-operator, which is axiomatically defined by **Ax.1** and **Ax.2**. If **Def.1** does not correspond to a definition by abstraction[15], then we are left to explain what corresponds to the abstraction process in the BK framework. I suggest that we should understand the process of abstraction in terms of the semantic function of the ε-operator. More precisely, the ε-operator is a variable-binding operator which forms terms from open sentences. For instance in **Def.1**, given any set t, $\varepsilon_x(x \approx t)$ assigns the representative set equinumerous with t. That is why the ε-term '$\varepsilon_x(x \approx t)$' is functional in the sense hinted by Cantor's single act of abstraction – namely, a direct operation moving from the base set to its cardinal set. Moreover, when evaluating the denotation of $\varepsilon_x(x \approx t)$, we disregard from all the properties of the elements of t except from the property *'being equinumerous with t'*. Indeed, the equivalence relation can be thought of as capturing a common feature among the entities standing in the $\approx$ relation – as in $(x \approx t)$. Then, one ignores any individual features of the objects except whether they stand in the $\approx$ relation, thereby abstracting away from any other feature distinguishing the elements so related – as in $\varepsilon_x(x \approx t)$. Therefore, while deriving $\varepsilon_x(x \approx t)$ from $(x \approx t)$, we are 'abstracting' from the elements composing t rather than from the property *'being equinumerous with t'* – thus answering the aforementioned question concerning Cantor's proposal. That is why in the context of BK the process of abstraction should be understood in terms of the semantic function of the ε-operator in **Def.1** – i.e. arbitrary reference to the representative set equinumerous with the given set.

3.2 The Incoherence Objection

Frege (1884) sections §36-39 focuses on the result of the abstractionist process, namely the theory of *pure units* which is supposed to compose cardinal sets. The leading motivation of the theory is that pure units, by lacking any characteristic properties apart from their unicity, contribute only to the size of sets - thus representing cardinal numbers. The theory of numbers as composed by units has a long-standing pedigree. Indeed, it should be remarked that the objections of Frege (1884) are actually directed to Leibniz and Locke - among the early writers - and Thomae, Lipshitz, Jevons and Schröder among his contemporaries. Cantor (1887) is not directly mentioned because he presented his abstractionist theory only after the publication of Frege's work. However, Hallett (1984), Dummett (1991) and Dauben (2020) unanimously agree that Frege's objections apply as well to Cantor (1887). Frege (1884) advances two main objections to the theory of units, which undermine the consistency of Cantor's definition of cardinal numbers. After presenting Frege's objections, I will claim that the sets of pure units corresponding to cardinal numbers should be understood as the representative sets denoted by the ε-operator in **Def.1**. Indeed, the representative set is arbitrarily picked out, namely any other set

[15] Woods (2014) adopts the Second-order ε-operator in order to define an abstraction operator like 'the number of' according to the equivalence relation of equinumerosity. Boccuni and Woods (2020) further discuss how to make use of arbitrary reference in the context of the Neologicist program.

from the equivalence class of $\approx$ would work as well. That is why the elements of the representative sets could be conceived as featureless, contributing only to the size of the set – as for Cantor's *pure units*.

Frege (1884) first objection amounts to a proof that according to Cantor (1887) abstractionist account there can be only two numbers, namely 0 and 1. If cardinal sets are composed by pure units which correspond to the members of the given set, and if two identical cardinal sets have identical units, then the units of any cardinal set bigger than 2 would be equal to that of 1! Frege sums up the objection by saying that Cantor ascribes two contradictory properties to units, namely identity and distinguishability. Therefore, as clearly summarized by Fine (1998), we must explain how a cardinal set can result from abstracting on the elements of set even though no units of it can be uniquely associated with each element of the base set.

However, by making use of the notion of arbitrary reference introduced in §1.2, it is possible to resist Frege's incoherence objection as follows. The ε-term $\varepsilon_x(x \approx t)$ in **Def.1** refers to a representative set x which is arbitrarily picked out from the equivalence class. But if the cardinal set $|t|$ is arbitrarily determined, then there is no fact of the matter regarding the internal composition of $|t|$, namely what are the elements of $|t|$. More precisely, being arbitrarily determined means that any other set from the equivalence class would work as well as a representative for the definition of cardinal sets. Therefore, asking for the internal composition of the cardinal set $|t|$ is beside the point, as noted by Tait: "The cardinal set corresponding to $\bar{\bar{M}}$ should not be a set of points or of numbers or of apples or of sets (as in the case of the initial von Neumann ordinals)" (Tait, 1996, p. 27). That is why, according to the BK framework, the cardinal set of pure units described by Cantor should be replaced by the representative set arbitrarily picked out from the equivalence class $\approx$ in **Def.1**. As pointed out by Leisenring:

> This definition of cardinal number is an essentially indeterminate one since nothing can be said about the set $|t|$ in [**Def.1**] except that it is equivalent to t and that it equals the cardinal number of any set which is equivalent to t. (Leisenring, 1969, p. 105)

As remarked in §1.2, arbitrary reference can be modelled – though not reduced – in terms of a supervaluational semantics where properties had by all individual choices of referent are had by the denotation of the ε-term. Then, the supervaluational semantics clarifies why the elements of the representative set $|t|$ are identical to each other and yet distinguishable – as for Cantor's *pure units*. Indeed, every precisification of a model assigning a particular set to $\varepsilon_x(x \approx t)$ agree that the elements of the set have a definite self-identity. This follows from the fact that sets are extensional entities – more on this below. Then, the elements of the representative set $|t|$ in **Def.1** have also a definite self-identity. However, the precisifications would not agree on any other property characterizing the elements of a particular set. Indeed, the property would fail to be shared by all the elements of the individual sets in the equivalence class of $\approx$. That is why the elements of the representative set $|t|$ in **Def.1** are qualitatively identical in all respects and yet distinguishable – *contra* Frege's incoherence objection. Moreover, the supervaluational interpretation of ε-terms highlights why the epistemic account of arbitrariness advocated by Breckenridge and Magidor (2012) fails to account for Cantor's theory of pure units[16]. According to the epistemic interpretation, the ε-term '$\varepsilon_x(x \approx t)$' refers to a specific set from the equivalence class $\approx$, even if we do not and cannot know which one. However,

[16] Thanks to the anonymous reviewers for stressing this point.

if this was the case, then the elements of $|t|$ would retain their specific characteristics – contrary to the featureless pure units. That is why the epistemic interpretation of arbitrary terms provides an incoherent reconstruction of Cantor's pure units.

However, one could object that this reassessment of *pure units* diverges in a fundamental aspect from Cantor's insights on abstraction. More precisely, someone could argue that while for Cantor (1887) abstraction is a process which introduces a new kind of entities – namely the *pure units* composing the cardinal sets – according to the BK definition a cardinal number is a representative set arbitrarily denoted by the ε-operator, which is on a par with any other set from the equivalence class $\approx$. Indeed, some of Cantor's scholars have interpreted abstraction as ontologically innovative – see Fine's 1998 quote in §1. Roughly, Fine (1998) defines pure units as arbitrary objects, which are introduced through extension of the domain and ordered by a dependence relation. However, it should be pointed out that the interpretation advocated by our objector is not straightforward. For instance, Ternullo and Zanetti (2021) argue that Cantor's theory of pure units should be considered as a purely descriptive, ontologically non-committal, medium to convey the explanation of what numerical abstraction consists in. More precisely, the authors point out that the key notion of CP (see §1.2) is the one-to-one correspondence between the elements of two sets. Given that any bijective function is as good as any other for the abstraction process, then the nature of the elements of the sets involved is entirely irrelevant. So, according to Ternullo and Zanetti (2021), Cantor's theory of pure units can be paraphrased away: whenever the set M is paired with any other set equinumerous to it, then it may practically come to be 'seen' as a set composed of units. This interpretation debunks the claim made by Fine (1998) concerning Cantor's ontological commitment to pure units. I will not take a stance in this debate. Indeed, in order to resist the objection, it is sufficient to show that the ontological interpretation of Cantor's pure units is not straightforward. That is why I will here restrict myself to the coherence of the interpretation of pure units as the elements of the representative sets arbitrarily picked out by **Def.1**, without questioning whether this interpretation is faithful to Cantor's philosophical insights.

Frege (1884) second objection concerns the internal composition of cardinal sets, which according to Cantor (1887) are made up by pure units. More precisely, we must explain how the units within a cardinal set are indistinguishable from one another even though the units from different cardinal sets are not. Based on the interpretation of *pure units* as the members of the representative set denoted by the ε-operator in **Def.1**, it is possible to face this second objection as follows.

I have already explained above why the members of a cardinal set are identical to each other. This bears on the fact that according to **Def.1**, the cardinal set $|t|$ is a representative set arbitrarily picked out from the equivalence class of $\approx$. This means that there is no fact of the matter concerning what are the elements of $|t|$, except that they are equinumerous with the given set t. That is why the members of $|t|$ contribute only to the size of the cardinal set and they can be thus regarded as featureless. Therefore, we are left to explain how the units of different cardinal sets can be nonetheless distinguished from one another. This issue can be addressed based on the **Ax.2** of Extensionality. More precisely, pure units are individuated by the only property of being members of the representative set picked out by the ε-operator. Moreover, co-extensional sets have the same members. Then, based on **Def.1**, a representative set x equinumerous to t is co-extensional with a representative set y equinumerous to s, if and only if t and s are equinumerous. But this is not the case for different cardinal sets, which explains why the elements of their respective representative sets are distinguishable.

Someone could object to the assumption of **Ax.2**, on which this second argument rests. According to **Ax.2**, the ε-terms of co-extensional sets denote the same representative object. Roughly, the objector complains that if I imagine picking an arbitrary creature with a heart, and an arbitrary creature with a kidney, why must I pick the same one each time? Dropping **Ax.2** would require an intensional interpretation for the ε-operator, by which ε-terms containing co-extensional sets might refer to different objects – see Zach (2017). However, this concern should be addressed in the BK framework here at issue, which is an axiomatic theory of sets. Indeed, once the predicates and relations of the logical language are interpreted as sets – as done in the BK system – then the assumption of **Ax.2** is beyond doubts. For instance, the identity conditions of sets are specified by the Axiom of *Extensionality*, by which two sets are identical if and only if their members are the same. This is a very uncontroversial assumption, deeply rooted in our understanding of set theory, as expressed by Boolos: "A theory that did not affirm that the objects with which it dealt were identical if they had the same members would only by charity be called a theory of sets alone" (Boolos, 1971, p. 28). Therefore, if the predicates and relations of the logical language are defined as sets, then also the indefinite descriptions formalized by the ε-operator will be conceived extensionally. More precisely, as sets are determined uniquely by their members, so the meaning of ε-terms is determined solely by the extension of their descriptive component. That is why the extensional interpretation of the ε-operator is forced upon us by the theory of which is part, namely BK. This argument should convince the objector of the plausibility of **Ax.2** within the BK framework, without undermining the usefulness of intensional interpretation of the ε-operator for natural language semantics – see Meyer Viol (1995).

4 Conclusion

The present paper aimed to achieve two separate goals. On the one hand, I argued that the BK framework and the ε-operator provide a faithful reconstruction of Cantor's insights on ordinal and cardinal numbers. I pointed out two assumptions underlying Cantor's (1887) abstractionist theory. First, the implicit assumption of the well-ordering principle, which constitutes the structure providing the arithmetical scale for the comparability of powers. Secondly, the assumption that cardinal numbers have to be represented as sets, rather than classes. As explained above, these assumptions are deeply rooted in, respectively, Cantor's conception of sets and his set theoretic foundational program – so they are the *desiderata* for a correct interpretation of Cantorian abstraction. That is why, using a comparison with the Zermelo-von Neumann and Frege-Russell definitions of cardinal numbers, I argued that both desiderata are met by the BK framework. Specifically, concerning the ordinal assumption, I proved that the ε-operator is logically equivalent to the AGC and thus to the well-ordering of the set theoretic universe V, which is the assumption required by Cantor (1887) definition of cardinal numbers. Therefore, I concluded that, even if the ε-operator is extraneous to the Cantorian framework, **Def.1** is an accurate reconstruction of Cantor's ordinal theory of powers – as for the assumption of the Axiom of *Choice* in the Zermelo-von Neumann definition of cardinal numbers. Instead, concerning the representational assumption, I spelled out the conditions for a representational account of cardinal numbers, arguing that they are met by the BK definition, contrary to the Frege-Russell one. Given that both Cantor and Bourbaki pursue the same foundational project, by which all mathematics should be reduced to set theory, I concluded that the BK system is a faithful reconstruction of the Cantorian framework.

On the other hand, I resisted Frege's (1884) objections, by proposing a reassessment of

Cantorian abstraction based upon the BK framework and the notion of arbitrary reference as formalized by the ε-operator. More precisely, Frege's objections – which have been considered by Dummett (1991) as a knock-out argument for the Cantor's proposal – are divided into two types. On the one hand, the psychologism objection complains the subjectivity of the abstractionist process, which fails to provide an objective definition of numbers. I resisted this objection by pointing out that **Def.1** is an explicit definition of cardinal numbers, by which the fundamental law of CP is deduced. Moreover, I proposed to replace the process of abstraction with the particular semantic function of the ε-operator in **Def.1**, namely that of arbitrary reference to a representative set equinumerous with the given set. Indeed, by arbitrarily picking out a representative set x, we are abstracting from the properties of the elements of t, except from the one contained in the ε-term of **Def.1**, namely *'being equinumerous with t'* – as for Cantor's single act of abstraction. On the other hand, the incoherence objection focuses on the *pure units* composing the cardinal set, claiming that Cantor ascribes two contradictory properties to pure units, namely identity and distinguishability. I suggested to understand pure units as the elements of the representative set arbitrarily picked out by the ε-operator in **Def.1**. Indeed, given that any other set from the equivalence class of $\approx$ would work as well, there is no fact of the matter about the nature of the elements composing $|t|$, which contribute only to the size of the cardinal set. I supported my claim by making use of the supervaluational interpretation of arbitrary reference, which clarifies why the elements of the representative set $|t|$ are identical to each other and yet distinguishable – as for Cantor's pure units.

References

Ackermann, W. (1938). Mengentheoretische begründung der logik. *Journal of Symbolic Logic*, 3(2):85–85.

Anacona, M., Arboleda, L. C., and Pérez-Fernández, F. J. (2014). On Bourbaki's axiomatic system for set theory. *Synthese*, 191(17):4069–4098.

Bernays, P. (1991). *Axiomatic Set Theory.* Dover Books on Mathematics. Dover Publications.

Boccuni, F. and Woods, J. (2020). Structuralist neologicism. *Philosophia Mathematica*, 28(3):296–316.

Boolos, G. (1971). The iterative conception of set. *The Journal of Philosophy*, 68(8):215–231.

Bourbaki, N. (1968). *Theory of Sets.* Springer.

Breckenridge, W. and Magidor, O. (2012). Arbitrary reference. *Philosophical Studies*, 158(3):377–400.

Cantor, G. (1883). *Grundlagen einer allgemeinen Mannigfaltigkeitslehre. Ein mathematisch-philosophischer Versuch in der Lehre des Unendlichen.* B. G. Teubner, Leipzig.

Cantor, G. (1885). Review of Frege's Die Grundlagen der Arithmetik. *Deutsche Litteraturzeitung, VI*, (6/20):728–729.

Cantor, G. (1887). *Mitteilungen zur Lehre vom Transfiniten I, II.* Number 91-92. Zeitschrift fur Philosophie und philosophische Kritik.

Cantor, G. (1895). Beiträge zur begründung der transfiniten mengenlehre. *Mathematische Annalen*, 46(4):481–512.

Cantor, G., Meschkowski, H., and Nilson, W. (1991). *Briefe*. Berlin: Springer-Verlag.

Dauben, J. W. (2020). *Georg Cantor: His mathematics and philosophy of the infinite*. Princeton University Press.

Dummett, M. (1991). *Frege: Philosophy of Mathematics*. Harvard University Press.

Ebert, P. A. and Rossberg, M. (2009). Cantor on Frege's foundations of arithmetic: Cantor's 1885 review of Frege's Die Grundlagen der Arithmetik. *History and Philosophy of Logic*, 30(4):341–348.

Fine, K. (1985). *Reasoning with Arbitrary Objects*. Blackwell.

Fine, K. (1998). Cantorian abstraction: A reconstruction and defense. *The Journal of Philosophy*, 95(12):599–634.

Frege, G. (1884). *Die Grundlagen der Arithmetik: eine logisch mathematische Untersuchung über den Begriff der Zahl*. w. Koebner.

Frege, G. (1890). Entwurf zu einer besprechung von Cantor's Gesammelten Abhandlungen zur Lehre vom Transfiniten. *Manuscript*.

Grattan-Guinness, I. (1970). An unpublished paper by Georg Cantor: Principien einer theorie der ordnungstypen erste mittheilung. *Acta mathematica*, 124(1):65–107.

Grimm, J. (2010). Implementation of Bourbaki's elements of mathematics in coq: Part one, theory of sets. *Journal of Formalized Reasoning*, 3(1):79–126.

Hallett, M. (1984). *Cantorian Set Theory and Limitation of Size*. Clarendon Press.

Heinzmann, G. and Petitot, J. (2020). The Functional Role of Structures in Bourbaki. In Reck, H., E., Schiemer, and G., editors, *The Prehistory of Mathematical Structuralism*, pages 187–214. Oxford University Press.

Hilbert, D. (1922). The logical foundations of mathematics. *Mathematical Annals*, 88(1-2):151–165.

Hill, C. O. (1997). Did Georg Cantor influence Edmund Husserl? *Synthese*, 113(1):145–170.

Husserl, E. (1891). *Philosophie der Arithmetik: psychologische und logische Untersuchungen*, volume 1. CEM Pfeffer.

Incurvati, L. (2020). *Conceptions of Set and the Foundations of Mathematics*. Cambridge University Press.

Jech, T. (2013). *Set theory*. Springer Science & Business Media.

Leisenring, A. C. (1969). *Mathematical Logic and Hilbert's ε-Symbol*. London: Macdonald Technical & Scientific.

Leitgeb, H. (2022). Ramsification and semantic indeterminacy. *The Review of Symbolic Logic*, pages 1–51.

Mathias, A. R. (1992). The ignorance of Bourbaki. *The Mathematical Intelligencer*, 14(3):4–13.

Meyer Viol, W. (1995). Instantial logic. Master's thesis, ILLC, Amsterdam.

Russell, B. and Whitehead, A. N. (1910). *Principia Mathematica Vol. I.* Cambridge University Press.

Tait, W. (1996). Frege versus Cantor and Dedekind: On the concept of number. *Manuscript.*

Ternullo, C. and Zanetti, L. (2021). Cantor's abstractionism and Hume's principle. *History and Philosophy of Logic*, page Forthcoming.

Wang, H. (1957). On denumerable bases of formal systems. *Journal of Symbolic Logic*, 22(3):292–293.

Woods, J. (2014). Logical indefinites. *Logique Et Analyse – Special Issue Edited by Julien Murzi and Massimiliano Carrara*, 227:277–307.

Wright, C. (1983). *Frege's Conception of Numbers as Objects.* Aberdeen University Press.

Zach, R. (2017). Semantics and proof theory of the epsilon calculus. In Ghosh, S. and Prasad, S., editors, *Logic and Its Applications. ICLA 2017*, pages 27–47. Berlin, Heidelberg: Springer.

Zermelo, E. (1908). Untersuchungen über die grundlagen der mengenlehre. i. *Mathematische Annalen*, 65(2):261–281.